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The Logarithmic Sobolev Inequality for a Submanifold in Manifolds with Nonnegative Sectional Curvature*

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Abstract The authors prove a sharp logarithmic Sobolev inequality which holds for compact submanifolds without boundary in Riemannian manifolds with nonnegative sectional curvature of arbitrary dimension and codimension. Like the Michael-Simon Sobolev inequality, this inequality includes a term involving the mean curvature. This extends a recent result of Brendle with Euclidean setting.

Keywords Logarithmic Sobolev inequality, Nonnegative sectional curvature, Submanifold
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1 Introduction

In 2019, Brendle [5] proved a Sobolev inequality which holds on submanifolds in Euclidean space of arbitrary dimension and codimension. The inequality is sharp if the codimension is at most 2. Soon, he (cf. [6]) proved a sharp logarithmic Sobolev inequality which holds on submanifolds in Euclidean space of arbitrary dimension and codimension at the same year. In 2020, he (cf. [3]) extended the result of the Sobolev inequality to Riemannnian manifolds with nonnegative curvature which gives the asymptotic volume ratio due to the Bishop-Gromov volume comparison theorem. Inspired by [3–4, 6], we extend the result of the logarithmic Sobolev inequality to ambient Riemannian manifolds with nonnegative sectional curvature under an assumption.

Let (M, g) be a complete noncompact Riemannian manifold of dimension k with nonnegative Ricci curvature. Define the asymptotic volume ratio of M:

$$\theta = \mathrm{AVR}(M, g) := \lim_{r\to\infty} \frac{|B_r(p)|}{\omega_k r^k}$$

for some (any) fixed point $p \in M$, where $B_r(p)$ denotes the geodesic ball in M, $|B_r(p)|$ denotes its volume and ω_k denotes the volume of the unit ball in $\mathbb{R}^k$. By Bishop-Gromov volume comparison theorem, the limit exists and $0 \le \theta \le 1$. Moreover, by L'Hospital's rule,

$$\lim_{r\to\infty} \frac{|\partial B_r(p)|}{k\omega_k r^{k-1}} = \theta.$$

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We have the following result.

Theorem 1.1 *Let M be a complete noncompact Riemannian manifold of dimension $n+m$ with nonnegative sectional curvature and Euclidean volume growth (i.e., $\theta > 0$). Let Σ be a compact n-dimension submanifold of M without boundary, and let f be a positive smooth function on Σ. Then*

$$\begin{aligned}&\int_\Sigma f\Big(\log f + n + \frac{n}{2}\log(4\pi) + \log\theta\Big)\mathrm{d\,vol} - \int_\Sigma \frac{|\nabla^\Sigma f|^2}{f}\mathrm{d\,vol} - \int_\Sigma f|H|^2\mathrm{d\,vol}\\ &\le \Big(\int_\Sigma f\mathrm{d\,vol}\Big)\log\Big(\int_\Sigma f\mathrm{d\,vol}\Big),\end{aligned}$$

where H denotes the mean curvature vector of Σ.

Minimal submanifolds have been studied for hundreds of years. We refer the readers to [11, 17]. One of the topic is the nonexistence of closed submanifolds. For codimension 1, Kasue [15] and Agostiniani, Fogagnolo and Mazzieri [1] have got the nonexistence of closed minimal hypersurfaces in complete noncompact Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth. For higher codimension, Chen [8] have obtained the nonexistence of closed minimal submanifolds of any co-dimension in $\mathbb{R}^n$. For complete noncompact Riemannian manifold with nonnegative sectional curvature, we have the following two equivalent results which have not been found in the literatures by us until now.

Corollary 1.1 *If (M, g) is a complete noncompact Riemannian manifold of dimension n (≥ 2) with nonnegative sectional curvature and Euclidean volume growth (i.e., $\theta > 0$), then there does not exist any closed minimal submanifold in M.*

Corollary 1.2 *Let (M, g) be a complete noncompact Riemannian manifold of dimension n (≥ 2) with nonnegative sectional curvature. If there exists some closed minimal submanifold of some co-dimension k in M, then M does not have maximum volume growth, i.e.,* $\mathrm{AVR}(M, g) = 0$.

Remark 1.1 The above two corollaries can be deduced from Brendle's paper [3]. Indeed, for co-dimension $m \geq 2$, assume that (M, g) is a complete noncompact Riemannian manifold of dimension $n+m$ with nonnegative sectional curvature and Euclidean volume growth (i.e., $\theta > 0$) and Σ is a closed minimal submanifold of M of dimension n, taking $f = 1$ in [3, Theorem 1.4], we obtain that $0 \geq n\big(\frac{(n+m)|B^{n+m}|}{m|B^m|}\big)^{\frac{1}{n}}\theta^{\frac{1}{n}}|\Sigma|^{\frac{n-1}{n}} > 0$ which is a contradiction. For co-dimension 1, by the same method, one can deduce the conclusion from [3, Corollary 1.5] since the corollary also holds in the co-dimension 1 setting which has been mentioned in the paragraph behind [3, Corollary 1.7]. We will give a proof of Corollary 1.1 in Section 4 by using Theorem 1.1.

The logarithmic Sobolev inequality has been studied by numerous authors (cf. [9, 12, 14, 16]). Our proof of main result, Theorem 1.1, is in the spirit of ABP-techniques in [3, 6]. ABP-techniques have been applied to various classes of linear and nonlinear elliptic equations in the Euclidean space for a long time. Due to some difficulties, it was not until 1997 that Cabré [7] developed them to Riemannnian manifolds.

This paper is organized as follows. In Section 2, we give some properties of the asymptotic volume ratio. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Corollary 1.1.

2 Preliminaries

In order to make it convenient to use the asymptotic volume ratio, we get the following two results.

Lemma 2.1 *Let M be a complete noncompact Riemannian manifold of dimension k with nonnegative Ricci curvature. Then*

$$\lim_{r\to\infty}\Big((4\pi)^{-\frac{k}{2}}r^{-k}\int_M \mathrm{e}^{-\frac{d(x,p)^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}(x)\Big)=\theta$$

for any point $p\in M$.

Proof Given a fixed point $p\in M$, let B_s^k denote the ball of radius s centered at the origin in $\mathbb{R}^k$. By Bishop-Gromov volume comparison theorem (cf. [10, Chapter 1, §11.2]), we have

$$\theta k\omega_k s^{k-1}=\theta|\partial B_s^k|\le|\partial B_s(p)|\le|\partial B_s^k|=k\omega_k s^{k-1}$$

for all $s>0$. By co-area formula, we get that

$$\begin{aligned}(4\pi)^{-\frac{k}{2}}r^{-k}\int_M \mathrm{e}^{-\frac{d(x,p)^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}(x)&=\int_0^\infty\Big(\int_{\partial B_s(p)}(4\pi)^{-\frac{k}{2}}r^{-k}\mathrm{e}^{-\frac{s^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}\Big)\mathrm{d}s\\&=\int_0^\infty(4\pi r^2)^{-\frac{k}{2}}\mathrm{e}^{-\frac{s^2}{4r^2}}|\partial B_s(p)|\mathrm{d}s.\end{aligned}$$

Since $\int_{\mathbb{R}^k}(4\pi)^{-\frac{k}{2}}\mathrm{e}^{-\frac{|y|^2}{4}}\mathrm{d}y=1$, we have

$$\begin{aligned}\theta&=\theta\int_{\mathbb{R}^k}(4\pi r^2)^{-\frac{k}{2}}\mathrm{e}^{-\frac{|y|^2}{4r^2}}\mathrm{d}y\\&=\theta\int_0^\infty(4\pi r^2)^{-\frac{k}{2}}\mathrm{e}^{-\frac{s^2}{4r^2}}|\partial B_s^k|\mathrm{d}s\\&\le\int_0^\infty(4\pi r^2)^{-\frac{k}{2}}\mathrm{e}^{-\frac{s^2}{4r^2}}|\partial B_s(p)|\mathrm{d}s\\&=(4\pi)^{-\frac{k}{2}}r^{-k}\int_M\mathrm{e}^{-\frac{d(x,p)^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}(x).\end{aligned}$$

We conclude that

$$\theta\le\liminf_{r\to\infty}\Big((4\pi)^{-\frac{k}{2}}r^{-k}\int_M\mathrm{e}^{-\frac{d(x,p)^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}(x)\Big).$$

On the other hand, since $\lim\limits_{s\to\infty}\frac{|\partial B_s(p)|}{k\omega_k s^{k-1}}=\theta$, for any $\varepsilon>0$, we can find a positive number $s_0=s_0(\varepsilon)>0$ such that

$$\frac{|\partial B_s(p)|}{k\omega_k s^{k-1}}<\theta+\varepsilon$$

for $s\ge s_0$. Thus

$$\begin{aligned}&(4\pi)^{-\frac{k}{2}}r^{-k}\int_M\mathrm{e}^{-\frac{d(x,p)^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}(x)\\&=\int_0^\infty\Big(\int_{\partial B_s(p)}(4\pi)^{-\frac{k}{2}}r^{-k}\mathrm{e}^{-\frac{s^2}{4r^2}}\,\mathrm{d}\,\mathrm{vol}\Big)\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&= \int_0^{s_0} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} |\partial B_s(p)| \mathrm{d}s + \int_{s_0}^{\infty} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} |\partial B_s(p)| \mathrm{d}s \\
&\le \int_0^{s_0} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} k\omega_k s^{k-1} \mathrm{d}s + (\theta+\varepsilon) \int_{s_0}^{\infty} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} k\omega_k s^{k-1} \mathrm{d}s \\
&\le \int_0^{s_0} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} k\omega_k s_0^{k-1} \mathrm{d}s + (\theta+\varepsilon) \int_0^{\infty} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} k\omega_k s^{k-1} \mathrm{d}s \\
&= \int_0^{s_0} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} k\omega_k s_0^{k-1} \mathrm{d}s + (\theta+\varepsilon) \int_{\mathbb{R}^k} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{|y|^2}{4r^2}} \mathrm{d}y \\
&= \int_0^{s_0} (4\pi r^2)^{-\frac{k}{2}} \mathrm{e}^{-\frac{s^2}{4r^2}} k\omega_k s_0^{k-1} \mathrm{d}s + (\theta+\varepsilon).
\end{aligned}$$

Taking $\limsup\limits_{r\to\infty}$ on both sides of the above inequalities, we have

$$\limsup_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \Big) \le \theta + \varepsilon.$$

Letting $\varepsilon \to 0$, we get that

$$\limsup_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \Big) \le \theta.$$

Thus, the limit $\lim\limits_{r\to\infty} \big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \big)$ exists, and

$$\lim_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \Big) = \theta.$$

Then, the lemma follows.

Lemma 2.2 *Let M be a complete noncompact Riemannian manifold of dimension k with nonnegative Ricci curvature. Then the limit $\lim\limits_{r\to\infty} \big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \big)$ exists, and*

$$\lim_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \Big) = \lim_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x) \Big)$$

for any point $p_0 \in M$, any compact subset $K \subset M$ and any Borel map $p : M \to K$.

Proof Given a fixed point $p_0 \in M$, a compact subset $K \subset M$ and a Borel map $p : M \to K$, define a nonnegative constant

$$C := \sup\{d(p_0, p(x)) : x \in M\}.$$

For any $\varepsilon > 0$, by triangle inequality we have

$$\begin{aligned}
-\frac{d(x,p(x))^2}{4r^2} &= -\frac{d(x,p_0)^2}{4r^2} \cdot \frac{d(x,p(x))^2}{d(x,p_0)^2} \\
&\ge -\frac{d(x,p_0)^2}{4r^2} \cdot \frac{|d(x,p_0) + d(p_0,p(x))|^2}{d(x,p_0)^2} \\
&= -\frac{d(x,p_0)^2}{4r^2} \Big(1 + \frac{d(p_0,p(x))}{d(x,p_0)} \Big)^2
\end{aligned}$$

$$\geq -\frac{d(x,p_0)^2}{4r^2}(1+C\varepsilon)^2$$

for all $x \in M \setminus B_{\varepsilon^{-1}}(p_0)$. Similarly, we have

$$-\frac{d(x,p(x))^2}{4r^2} \leq -\frac{d(x,p_0)^2}{4r^2}(1-C\varepsilon)^2$$

for all $x \in M \setminus B_{\varepsilon^{-1}}(p_0)$. Thus,

$$\begin{aligned}
&(1+C\varepsilon)^{-k} \liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}(1+C\varepsilon)^2} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_{M\setminus B_{\varepsilon^{-1}}(p_0)} \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}(1+C\varepsilon)^2} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
\leq&\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_{M\setminus B_{\varepsilon^{-1}}(p_0)} \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
\leq&\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_{M\setminus B_{\varepsilon^{-1}}(p_0)} \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}(1-C\varepsilon)^2} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}(1-C\varepsilon)^2} \mathrm{d}\,\mathrm{vol}(x)\Big)\\
=&(1-C\varepsilon)^{-k} \liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big).
\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that

$$\liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big) = \liminf_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big).$$

Similarly, we have

$$\limsup_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big) = \limsup_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big).$$

By Lemma 2.1, the limit $\lim_{r\to\infty} \big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\big)$ exists, and

$$\lim_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p(x))^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big) = \lim_{r\to\infty} \Big((4\pi)^{-\frac{k}{2}} r^{-k} \int_M \mathrm{e}^{-\frac{d(x,p_0)^2}{4r^2}} \mathrm{d}\,\mathrm{vol}(x)\Big).$$

Then, the lemma follows.

3 Proof of Theorem 1.1

Recall the definition of the second fundamental form h of Σ with respect to M:

$$\langle h(X,Y), V\rangle = \langle \overline{D}_X Y, V\rangle = -\langle \overline{D}_X V, Y\rangle,$$

where X, Y are tangent vector fields, V is a normal vector field and $\overline{D}$ denotes the connection on M. Moreover, the mean curvature vector H is defined as the trace of the second fundamental form h.

We now give the proof of Theorem 1.1. We first consider the special case that Σ is connected. By scaling, we may assume that

$$\int_\Sigma f\log f\,\mathrm{d}\,\mathrm{vol}-\int_\Sigma \frac{|\nabla^\Sigma f|^2}{f}\mathrm{d}\,\mathrm{vol}-\int_\Sigma f|H|^2\mathrm{d}\,\mathrm{vol}=0.$$

From functional analysis and standard elliptic theory, we can find a smooth function $u:\Sigma\to\mathbb{R}$ such that

$$\mathrm{div}_\Sigma(f\nabla^\Sigma u)=f\log f-\frac{|\nabla^\Sigma f|^2}{f}-f|H|^2.$$

In the following, we fix a positive number r. Define the contact set

$$\begin{aligned}A=\Big\{(\overline{x},\overline{y})\in T^\perp\Sigma: ru(x)+\frac{1}{2}\mathrm{d}(x,\exp_{\overline{x}}(r\nabla^\Sigma u(\overline{x})+r\overline{y}))^2\\ \geq ru(\overline{x})+\frac{1}{2}r^2(|\nabla^\Sigma u(\overline{x})|^2+|\overline{y}|^2),\forall x\in\Sigma\Big\}.\end{aligned}$$

Moreover, we define a map $\Phi: T^\perp\Sigma\to M$ by

$$\Phi(x,y)=\exp_x(r\nabla^\Sigma u(x)+ry)$$

for all $(x,y)\in T^\perp\Sigma$.

Lemma 3.1 *Suppose that* $(\overline{x},\overline{y})\in A$. *Then*

$$\mathrm{d}(\overline{x},\Phi(\overline{x},\overline{y}))^2=r^2(|\nabla^\Sigma u(\overline{x})|^2+|\overline{y}|^2).$$

Proof Let $\overline{\gamma}(t):=\exp_{\overline{x}}(rt\nabla^\Sigma u(\overline{x})+rt\overline{y})$ for $t\in[0,1]$. From the definition of A, we have

$$ru(\overline{x})+\frac{1}{2}\mathrm{d}(\overline{x},\exp_{\overline{x}}(r\nabla^\Sigma u(\overline{x})+r\overline{y}))^2\geq ru(\overline{x})+\frac{1}{2}r^2(|\nabla^\Sigma u(\overline{x})|^2+|\overline{y}|^2).$$

Thus, $\mathrm{d}(\overline{x},\Phi(\overline{x},\overline{y}))^2\geq r^2(|\nabla^\Sigma u(\overline{x})|^2+|\overline{y}|^2)$. On the other hand,

$$\begin{aligned}r^2(|\nabla^\Sigma u(\overline{x})|^2+|\overline{y}|^2)&=|\overline{\gamma}'(0)|^2\\&=\Big(\int_0^1|\overline{\gamma}'(t)|\mathrm{d}t\Big)^2\\&\geq \mathrm{d}(\overline{x},\Phi(\overline{x},\overline{y}))^2.\end{aligned}$$

Then, the lemma follows.

Lemma 3.2 $\Phi(A)=M$.

Proof Fix a point $p\in M$. Since Σ is compact without boundary, the function $x\mapsto ru(x)+\frac{1}{2}d(x,p)^2$ must attain its minimum at some point denoted by $\overline{x}$ on Σ. Moreover, we can find a minimizing geodesic $\overline{\gamma}:[0,1]\to M$ such that $\overline{\gamma}(0)=\overline{x}$ and $\overline{\gamma}(1)=p$. For every path $\gamma:[0,1]\to M$ satisfying $\gamma(0)\in\Sigma$ and $\gamma(1)=p$, we obtain

$$\begin{aligned}ru(\gamma(0))+E(\gamma)&\geq ru(\gamma(0))+\frac{1}{2}d(\gamma(0),p)^2\\&\geq ru(\overline{x})+\frac{1}{2}d(\overline{x},p)^2\end{aligned}$$

$$= ru(\overline{\gamma}(0)) + \frac{1}{2}|\overline{\gamma}'(0)|^2$$
$$= ru(\overline{\gamma}(0)) + E(\overline{\gamma}),$$

where $E(\gamma)$ denotes the energy of γ. In other words, the path γ minimizes the functional $ru(\gamma(0)) + E(\gamma)$ among all paths $\overline{\gamma} : [0,1] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(1) = p$. Hence, the formula for the first variation implies

$$\overline{\gamma}'(0) - r\nabla^{\Sigma}u(\overline{x}) \in T^{\perp}_{\overline{x}}\Sigma.$$

Consequently, we can find a vector $\overline{y} \in T^{\perp}_{\overline{x}}\Sigma$ such that

$$\overline{\gamma}'(0) = r\nabla^{\Sigma}u(\overline{x}) + r\overline{y}.$$

It remains to show $(\overline{x}, \overline{y}) \in A$. For each point $x \in \Sigma$, we have

$$\begin{aligned} ru(x) + \frac{1}{2}\mathrm{d}(x, \exp_{\overline{x}}(r\nabla^{\Sigma}u(\overline{x}) + r\overline{y}))^2 &= ru(x) + \frac{1}{2}d(x,p)^2 \\ &\geq ru(\overline{x}) + \frac{1}{2}d(\overline{x},p)^2 \\ &= ru(\overline{\gamma}(0)) + \frac{1}{2}|\overline{\gamma}'(0)|^2 \\ &= ru(\overline{x}) + \frac{1}{2}r^2(|\nabla^{\Sigma}u(\overline{x})|^2 + |\overline{y}|^2). \end{aligned}$$

The lemma follows.

Lemma 3.3 *Suppose that $(\overline{x}, \overline{y}) \in A$, and let $\overline{\gamma}(t) := \exp_{\overline{x}}(rt\nabla^{\Sigma}u(\overline{x}) + rt\overline{y})$ for $t \in [0,1]$. If Z is a vector field along $\overline{\gamma}$ satisfying $Z(0) \in T_{\overline{x}}\Sigma$ and $Z(1) = 0$, then*

$$\begin{aligned} &r(D^2_{\Sigma}u)(Z(0), Z(0)) - r\langle h(Z(0), Z(0)), \overline{y}\rangle \\ &+ \int_0^1 (|\overline{D}_t Z(t)|^2 - \overline{R}(\overline{\gamma}'(t), Z(t), \overline{\gamma}'(t), Z(t)))\mathrm{d}t \geq 0. \end{aligned}$$

Lemma 3.4 *Suppose that $(\overline{x}, \overline{y}) \in A$. Then $g + rD^2_{\Sigma}u(\overline{x}) - r\langle h(\overline{x}), \overline{y}\rangle \geq 0$.*

Lemma 3.5 *Suppose that $(\overline{x}, \overline{y}) \in A$, and let $\overline{\gamma}(t) := \exp_{\overline{x}}(rt\nabla^{\Sigma}u(\overline{x}) + rt\overline{y})$ for $t \in [0,1]$. Moreover, let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_{\overline{x}}\Sigma$. Suppose that W is a Jacobi field along $\overline{\gamma}$ satisfying $W(0) \in T_{\overline{x}}\Sigma$ and $\langle \overline{D}_t W(0), e_j\rangle = r(D^2_{\Sigma}u)(W(0), e_j) - r\langle h(W(0), e_j), \overline{y}\rangle$ for each $1 \leq j \leq n$. If $W(\tau) = 0$ for some $0 < \tau < 1$, then W vanishes identically.*

Lemma 3.6 *The Jacobian determinant of Φ satisfies*

$$|\mathrm{det}D\Phi(x,y)| \leq r^m \mathrm{det}(g + rD^2_{\Sigma}u(x) - r\langle h(x), y\rangle)$$

for all $(x,y) \in A$.

The proofs of Lemmas 3.3–3.6 are identical to Lemmas 3.1–3.3 and Lemma 3.6 in [2], respectively. We omit them. Lemma 3.3 is needed to proof Lemma 3.4, and Lemma 3.5 is needed to proof Lemma 3.6. The assumption of nonnegative sectional curvature is necessary.

Lemma 3.7 *The Jacobian determinant of* Φ *satisfies*

$$\mathrm{e}^{-\frac{d(x,\Phi(x,y))^2}{4r^2}}|\det D\Phi(x,y)| \leq r^{n+m} f(x)\mathrm{e}^{\frac{n}{r}-n}\mathrm{e}^{-\frac{|2H(x)+y|^2}{4}}$$

for all $(x,y)\in A$.

Proof Given a point $(x,y)\in A$, using the identity $\mathrm{div}_\Sigma(f\nabla^\Sigma u) = f\log f - \frac{|\nabla^\Sigma f|^2}{f} - f|H|^2$, we have

$$\begin{aligned}\Delta_\Sigma u(x) - \langle H(x),y\rangle &= \log f(x) - \frac{|\nabla^\Sigma f(x)|^2}{f(x)^2} - |H(x)|^2 \\ &\quad - \frac{\langle\nabla^\Sigma f(x),\nabla^\Sigma u(x)\rangle}{f(x)} - \langle H(x),y\rangle \\ &= \log f(x) + \frac{|\nabla^\Sigma u(x)|^2+|y|^2}{4} \\ &\quad - \frac{|2\nabla^\Sigma f(x)+f(x)\nabla^\Sigma u(x)|^2}{4f(x)^2} - \frac{|2H(x)+y|^2}{4} \\ &\leq \log f(x) + \frac{|\nabla^\Sigma u(x)|^2+|y|^2}{4} - \frac{|2H(x)+y|^2}{4}.\end{aligned}$$

Using Lemmas 3.4, 3.6 and the elementary inequality $\lambda\leq \mathrm{e}^{\lambda-1}$, we have

$$\begin{aligned}|\det D\Phi(x,y)| &\leq r^m\det(g + rD^2_\Sigma u(x) - r\langle h(x),y\rangle) \\ &= r^{n+m}\det\Big(\frac{g}{r} + D^2_\Sigma u(x) - \langle h(x),y\rangle\Big) \\ &\leq r^{n+m}\mathrm{e}^{\frac{n}{r}+\Delta_\Sigma u(x)-\langle H(x),y\rangle - n} \\ &\leq r^{n+m}\mathrm{e}^{\frac{n}{r}+\log f(x)+\frac{|\nabla^\Sigma u(x)|^2+|y|^2}{4}-\frac{|2H(x)+y|^2}{4}-n} \\ &= r^{n+m}f(x)\mathrm{e}^{\frac{n}{r}-n}\mathrm{e}^{-\frac{|2H(x)+y|^2}{4}}\mathrm{e}^{\frac{d(x,\Phi(x,y))^2}{4r^2}}.\end{aligned}$$

The lemma follows.

By Lemma 3.2, for any $p\in M$, we choose some point $(x_p,y_p)\in A$ arbitrarily such that $\Phi(x_p,y_p)=p$. Using Lemmas 3.2, 3.7 and area formula in geometric measure theory (cf. [13]), we have

$$\begin{aligned}\int_M \mathrm{e}^{-\frac{d(x_p,p)^2}{4r^2}}\mathrm{d\,vol}(p) &\leq \int_M\Big(\int_{\{(x,y)\in A:\Phi(x,y)=p\}}\mathrm{e}^{-\frac{d(x,\Phi(x,y))^2}{4r^2}}\mathrm{d}\mathscr{H}^0\Big)\mathrm{d\,vol}(p) \\ &= \int_\Sigma\Big(\int_{T_x^\perp\Sigma}\mathrm{e}^{-\frac{d(x,\Phi(x,y))^2}{4r^2}}|\det D\Phi(x,y)|1_A(x,y)\mathrm{d}y\Big)\mathrm{d\,vol}(x) \\ &\leq \int_\Sigma\Big(\int_{T_x^\perp\Sigma} r^{n+m}f(x)\mathrm{e}^{\frac{n}{r}-n}\mathrm{e}^{-\frac{|2H(x)+y|^2}{4}}1_A(x,y)\mathrm{d}y\Big)\mathrm{d\,vol}(x) \\ &\leq \int_\Sigma\Big(\int_{T_x^\perp\Sigma} r^{n+m}f(x)\mathrm{e}^{\frac{n}{r}-n}\mathrm{e}^{-\frac{|2H(x)+y|^2}{4}}\mathrm{d}y\Big)\mathrm{d\,vol}(x) \\ &= r^{n+m}\mathrm{e}^{\frac{n}{r}-n}(4\pi)^{\frac{m}{2}}\int_\Sigma f(x)\mathrm{d\,vol}(x),\end{aligned}$$

where $\mathscr{H}^0$ denotes the counting measure. After dividing by r^{n+m}, by Lemmas 2.1–2.2 , letting $r\to\infty$ on both sides of the above inequalities, we have

$$(4\pi)^{\frac{n+m}{2}}\theta \leq \mathrm{e}^{-n}(4\pi)^{\frac{m}{2}}\int_\Sigma f(x)\mathrm{d\,vol}(x).$$

Consequently,

$$n + \frac{n}{2}\log(4\pi) + \log\theta \le \log\Big(\int_\Sigma f \mathrm{d}\,\mathrm{vol}\Big).$$

Combining this inequality with the normalization

$$\int_\Sigma f\log f \mathrm{d}\,\mathrm{vol} - \int_\Sigma \frac{|\nabla^\Sigma f|^2}{f}\mathrm{d}\,\mathrm{vol} - \int_\Sigma f|H|^2 \mathrm{d}\,\mathrm{vol} = 0$$

gives

$$\begin{aligned}&\int_\Sigma f\Big(\log f + n + \frac{n}{2}\log(4\pi) + \log\theta\Big)\mathrm{d}\,\mathrm{vol} - \int_\Sigma \frac{|\nabla^\Sigma f|^2}{f}\mathrm{d}\,\mathrm{vol} - \int_\Sigma f|H|^2\mathrm{d}\,\mathrm{vol}\\ =&\int_\Sigma f\Big(n + \frac{n}{2}\log(4\pi) + \log\theta\Big)\mathrm{d}\,\mathrm{vol}\\ \le&\Big(\int_\Sigma f\mathrm{d}\,\mathrm{vol}\Big)\log\Big(\int_\Sigma f\mathrm{d}\,\mathrm{vol}\Big).\end{aligned}$$

It remains to consider the case when Σ is disconnected. For completeness, we list Brendle's proof (cf. [6]). In that case, we apply the inequality to each individual connected component of Σ, and sum over all connected components. Since

$$a\log a + b\log b < a\log(a+b) + b\log(a+b) = (a+b)\log(a+b)$$

for $a, b > 0$, we conclude that

$$\begin{aligned}&\int_\Sigma f\Big(\log f + n + \frac{n}{2}\log(4\pi) + \log\theta\Big)\mathrm{d}\,\mathrm{vol} - \int_\Sigma \frac{|\nabla^\Sigma f|^2}{f}\mathrm{d}\,\mathrm{vol} - \int_\Sigma f|H|^2\mathrm{d}\,\mathrm{vol}\\ <&\Big(\int_\Sigma f\mathrm{d}\,\mathrm{vol}\Big)\log\Big(\int_\Sigma f\mathrm{d}\,\mathrm{vol}\Big),\end{aligned}$$

if Σ is disconnected. This completes the proof of Theorem 1.1.

4 Proof of Corollary 1.1

We prove the corollary by contradiction. Assume that $(\Sigma, g|_\Sigma)$ is a closed minimal k-submanifold of (M, g). For any positive number ε, we choose the rescaled metric $\varepsilon^2 g$ on M. Then $(M, \varepsilon^2 g)$ is also a complete noncompact Riemannian manifold of dimension n with nonnegative sectional curvature. Moreover, the asymptotic volume ratio of $(M, \varepsilon^2 g)$,

$$\mathrm{AVR}(M, \varepsilon^2 g) = \lim_{r\to\infty}\frac{|B_r^{\varepsilon^2 g}(p)|_{\varepsilon^2 g}}{\omega_n r^n} = \lim_{r\to\infty}\frac{\varepsilon^n |B^g_{r\varepsilon^{-1}}(p)|_g}{\omega_n r^n} = \mathrm{AVR}(M, g),$$

where $B_r^{g_0}(p)$ denotes the geodesic ball of radius r centered at the point p with respect to some metric g_0 on M and $|\cdot|_{g_0}$ denotes the volume with respect to some metric g_0 on M. So we can denote both $\mathrm{AVR}(M, \varepsilon^2 g)$ and $\mathrm{AVR}(M, g)$ by θ. Note that $(\Sigma, \varepsilon^2 g|_\Sigma)$ is also a closed minimal k-submanifold of $(M, \varepsilon^2 g)$. By Theorem 1.1, choosing $f = 1$ for $(M, \varepsilon^2 g)$, we have

$$\Big(n + \frac{n}{2}\log(4\pi) + \log\theta\Big)|\Sigma|_{\varepsilon^2 g|_\Sigma} \le |\Sigma|_{\varepsilon^2 g|_\Sigma}\log(|\Sigma|_{\varepsilon^2 g|_\Sigma}),$$

i.e., $|\Sigma|_{\varepsilon^2 g|_\Sigma} \ge (4\pi)^{\frac{n}{2}}\theta \mathrm{e}^n (> 0)$, where $|\Sigma|_{\varepsilon^2 g|_\Sigma}$ denotes the area of Σ with respect to the metric $\varepsilon^2 g|_\Sigma$ on Σ. Note that $|\Sigma|_{\varepsilon^2 g|_\Sigma} = \varepsilon^k |\Sigma|_{g|_\Sigma}$. Thus $\varepsilon^k|\Sigma|_{g|_\Sigma} \ge (4\pi)^{\frac{n}{2}}\theta\mathrm{e}^n > 0$ which is a contradiction provided that ε is chosen sufficiently small. This contradiction completes the proof.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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