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Existence of Global Solutions to the Nonlocal mKdV Equation on the Line*

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Abstract In this paper, the authors address the existence of global solutions to the Cauchy problem for the integrable nonlocal modified Korteweg-de Vries (nonlocal mKdV for short) equation under the initial data $u_0 \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ with the $L^1(\mathbb{R})$ small-norm assumption. A Lipschitz L^2-bijection map between potential and reflection coefficient is established by using inverse scattering method based on a Riemann-Hilbert problem associated with the Cauchy problem. The map from initial potential to reflection coefficient is obtained in direct scattering transform. The inverse scattering transform goes back to the map from scattering coefficient to potential by applying the reconstruction formula and Cauchy integral operator. The bijective relation naturally yields the existence of global solutions in a Sobolev space $H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ to the Cauchy problem.

Keywords Nonlocal mKdV equation, Riemann-Hilbert problem, Plemelj projection operator, Lipschitz continuous, Global solutions
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1 Introduction and Main Results

In this paper, we establish the global existence of solutions to the Cauchy problem for the nonlocal mKdV equation

$$u_t(t,x) + u_{xxx}(t,x) + 6\sigma u(t,x)u(-t,-x)u_x(t,x) = 0, \tag{1.1}$$
$$u(0,x) = u_0(x), \tag{1.2}$$

where $u_0 \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ and $\sigma = \pm 1$ denote the focusing and defocusing cases, respectively.

The nonlocal mKdV equation (1.1), introduced in [1–2], can be regarded as the integrable nonlocal extension of the classical mKdV equation

$$u_t(t,x) + u_{xxx}(t,x) + 6\sigma u^2(t,x)u_x(t,x) = 0. \tag{1.3}$$

By replacing $u^2(t,x)$ with the PT-symmetric term $u(t,x)u(-t,-x)$ (see [3]). In physical application, the nonlocal mKdV equation (1.1) possesses delayed time reversal symmetry, and thus it can be related to the Alice-Bob system (see [4]). For instance, a special approximate solution of the nonlocal mKdV was applied to theoretically capture the salient features of two correlated dipole blocking events in atmospheric dynamical systems (see [5]).

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There is much work on the study of various mathematical properties for the nonlocal mKdV equation (1.1). The N-soliton solutions to the nonlocal mKdV equation (1.1) with zero boundary conditions were constructed by using the Darboux transformation and the inverse scattering transform, respectively (see [7–8]). Further the Riemann-Hilbert (RH for short) method was used to construct N-soliton solutions for the nonlocal mKdV equations (1.1) with nonzero boundary conditions (see [6]). The long-time asymptotics to the nonlocal mKdV equation (1.1) with decaying initial data was investigated in [9] via the nonlinear steepest-descent method developed by Deift and Zhou [10]. Recently, we obtained the long time asymptotic behavior for the Cauchy problem of the nonlocal mKdV equation (1.1) with nonzero initial data in the solitonic regions by using the $\bar{\partial}$-steepest-descent method (see [11–12]). This method, introduced by McLaughlin and Miller (see [13–14]), has been extensively used in the long-time asymptotic analysis and the soliton resolution conjecture of some integrable systems (see [15–24]). However, the existence of global solutions to the Cauchy problem (1.1)–(1.2) for the nonlocal mKdV equation is still unknown to our knowledge. A technical difficulty of proving global existence of the nonlocal mKdV equation (1.1) comes from the fact that the mass and energy conservation laws to (1.1) do not preserve any reasonable norm and may be negative. As we know, the mass and energy conservation laws of the classical mKdV equation (1.3) are the key point to obtain a priory estimates for establishing a unique global solution.

The main purpose in the present paper is to overcome such a difficulty and to establish the global existence of solutions to the Cauchy problem (1.1)–(1.2) in an appropriate Sobolev space by applying the inverse scattering theory. Our principal result is now stated as follows.

Theorem 1.1 *Let the initial data* $u_0 \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ *with* $L^1(\mathbb{R})$ *small-norm*

$$1 - \|u\|_{L^1(\mathbb{R})}(1 + 2e^{2\|u\|_{L^1(\mathbb{R})}}) > 0 \tag{1.4}$$

such that the spectral problem (2.1) *admits no eigenvalues or resonances. Then*

- *there exists an* L^2 *-bijection map between the potential* u *and reflection coefficients* $r_{1,2}$*,*

$$H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \ni u \mapsto r_{1,2} \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R}), \tag{1.5}$$

which is Lipschitz continuous.

- *There exists a unique global solution* $u \in C([0,\infty), H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}))$ *to the Cauchy problem* (1.1)–(1.2)*. Furthermore, the map*

$$H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \ni u_0 \mapsto u \in C([0,\infty); H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}))$$

is Lipschitz continuous.

A key in proving the above result is to establish a Lipschitz L^2-bijection (1.5) between solution and scattering coefficient by using inverse scattering method (see [25–29]). The L^2-bijection (1.5) implies that global well-posdeness of the Cauchy problem (1.1)–(1.2) in the space $H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.

Remark 1.1 Let's compare the differences between the nonlocal mKdV case and the local mKdV case during the proof of global well-posedness to the Cauchy problem. Firstly, the mass and energy conservation laws to the nonlocal equation (1.1) do not preserve any reasonable norm and may be negative, which prevent to obtain a priory estimates for establishing a unique global solution by using general analytical technique. Secondly, in nonlocal mKdV case, the $L^1(\mathbb{R})$ small-norm condition (1.4) not only ensures that the spectral problem (2.1) admits no eigenvalues or resonances, but also is used to prove the existence and uniqueness of the RH problem 3.1 via a vanishing lemma. While in the local mKdV case, the corresponding RH problem can be directly proved by a vanishing lemma without small norm condition on the reflection coefficient (see [25–26]).

The structure of the paper is as follows. In Section 2, we focus on the direct scattering transform to the Cauchy problem (1.1)–(1.2). We especially establish the Lipschitz continuous maps from the initial data to the Jost function and the reflection coefficient. In Section 3, we carry out the inverse scattering transform to set up an RH problem associated with the Cauchy problem (1.1)–(1.2), and the solvability of the RH problem is further shown. In Section 4, we reconstruct and estimate the potential from the solutions of the RH problem on positive half line $\mathbb{R}^+$ and negative half line $\mathbb{R}^-$, respectively. We further establish a Lipschitz continuous mapping from the reflection coefficients to the potentials. In Section 5, we perform the time evolution of the reflection coefficients and the RH problem. Then, we prove that there exists a unique global solution to the initial value problem (1.1)–(1.2) of the nonlocal mKdV equation in the space $H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.

2 Direct Scattering Transforms

In this section, we state some main results on the direct scattering transform associated with the Cauchy problem (1.1)–(1.2). The details can be found in [1–2].

2.1 Notations

To precisely state our main result, we first fix some notations used in this paper.

• Let I be an interval on the real line $\mathbb{R}$ and X be a Banach space. $C(I, X)$ denotes the space of continuous functions on I taking values in X. It is equipped with the norm

$$\|f(x)\|_{C(I,X)} = \sup_{x\in I} \|f(x)\|_X.$$

• For the spatial variable $x \in \mathbb{R}$, a weighted space $L^{2,s}(\mathbb{R})$ is specified by

$$L^{2,s}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid \langle\cdot\rangle^s f \in L^2(\mathbb{R}), s \in \mathbb{Z}^+\},$$

equipped with the norm

$$\|f\|_{L^{2,s}(\mathbb{R})} = \|\langle\cdot\rangle^s f\|_{L^2(\mathbb{R})},$$

where $\langle x\rangle = (1 + x^2)^{\frac{1}{2}}$. We further define a weighted Sobolev space by

$$H^{1,1}(\mathbb{R}) := \{f \mid \partial_x^j f \in L^{2,1}(\mathbb{R}),\ j = 0, 1\},$$

equipped with the norm

$$\|f\|_{H^{1,1}(\mathbb{R})} = (\|f\|^2_{L^{2,1}(\mathbb{R})} + \|f_x\|^2_{L^{2,1}(\mathbb{R})})^{\frac{1}{2}}.$$

- For the spectral parameter $k \in \mathbb{R}$, define the function space

$$L^{2,s}(\mathbb{R}) := \{r(k) \mid k^s r(k) \in L^2(\mathbb{R}),\ s \in \mathbb{Z}^+\},$$
$$H^{1,1}(\mathbb{R}) := \{r(k) \mid r(k), r'(k) \in L^{2,1}\mathbb{R})\},$$

equipped with the norm

$$\|r(k)\|_{H^{1,1}(\mathbb{R})} = (\|r(k)\|_{L^{2,1}(\mathbb{R})} + \|r'(k)\|_{L^{2,1}(\mathbb{R})})^{\frac{1}{2}}.$$

2.2 Lipschitz continuity of the Jost functions

The nonlocal mKdV equation (1.1) admits the Lax pair

$$\psi_x - \mathrm{i}z\sigma_3\psi = Q\psi, \tag{2.1}$$
$$\psi_t - 4\mathrm{i}z^3\sigma_3\psi = (4z^2Q - 2\mathrm{i}z(Q_x - Q^2)\sigma_3 + 2Q^3 - Q_{xx})\psi, \tag{2.2}$$

where

$$Q = \begin{pmatrix} 0 & u(t,x) \\ -\sigma u(-t,-x) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define the Jost functions $\psi^{\pm}(x,z)$ to the spectral problem (2.1) with the following boundary conditions

$$\psi^{\pm}(x,z) \sim \mathrm{e}^{\mathrm{i}zx\sigma_3}, \quad x \to \pm\infty.$$

Making a transformation

$$m^{\pm}(x,z) = \psi^{\pm}(x,z)\mathrm{e}^{-\mathrm{i}zx\sigma_3},$$

then

$$\lim_{x\to\pm\infty} m^{\pm}(x,z) = I$$

and $m^{\pm}(x,z)$ satisfies the Voterra integral equations

$$m^{\pm}(x,z) = I + \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}z(y-x)\,\mathrm{ad}\,\sigma_3} Q m^{\pm}(y,z)\mathrm{d}y, \tag{2.3}$$

where $\mathrm{e}^{\mathrm{ad}\,\sigma_3}A := \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3}$.

Denote $m^{\pm}(x,z) = [m_1^{\pm}(x,z), m_2^{\pm}(x,z)]$. From symmetry of Lax pair, we can get

$$m_1^{\pm}(x,z) = \sigma\Lambda\overline{m_2^{\mp}(-x,-\overline{z})}, \quad m_2^{\pm}(x,z) = \Lambda\overline{m_1^{\mp}(-x,-\overline{z})}, \tag{2.4}$$

where

$$\Lambda = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}.$$

It can be further shown that functions $m_1^+(x,z)$ and $m_2^-(x,z)$ are analytic in $z \in \mathbb{C}^+$, whereas the functions $m_2^+(x,z)$ and $m_1^-(x,z)$ are analytic in $z \in \mathbb{C}^-$. Moreover, there is a matrix $S(z)$ satisfying

$$m^+(x,z) = m^-(x,z)\mathrm{e}^{\mathrm{i}zx\mathrm{ad}\sigma_3}S(z), \tag{2.5}$$

where

$$S(z) = \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix}.$$

From (2.5), we deduce

$$a(z) = \det[m_1^+(x,z), m_2^-(x,z)], \tag{2.6}$$

$$d(z) = \det[m_1^-(x,z), m_2^+(x,z)], \tag{2.7}$$

$$b(z) = \det[m_1^-(x,z), m_1^+(x,z)]\mathrm{e}^{-2\mathrm{i}zx}. \tag{2.8}$$

It can be shown that $a(z)$ is analytic in $\mathbb{C}^+$ and $a(z) \to 1$ as $z \to \infty$ in $\overline{\mathbb{C}}^+$ while $d(z)$ is analytic in $\mathbb{C}^-$ and $d(z) \to 1$ as $z \to \infty$ in $\overline{\mathbb{C}}^-$.

From (2.4) we can get the scattering coefficients satisfying the symmetries

$$a(z) = \overline{a(-\bar{z})}, \quad d(z) = \overline{d(-\bar{z})}, \quad c(z) = -\sigma\overline{b(-\bar{z})}.$$

We define the reflection coefficients

$$r_1(z) = \frac{b(z)}{a(z)}, \quad r_2(z) = \frac{c(z)}{d(z)} = -\sigma\frac{\overline{b(-z)}}{d(z)}, \quad z \in \mathbb{R}.$$

The determinant of $S(z)$ is

$$a(z)d(z) + \sigma b(z)\overline{b(-\bar{z})} = 1.$$

In the follows, we prove the existence of $m^\pm(x,z)$. For

$$f(\cdot,z) = (f_1(\cdot,z), f_2(\cdot,z))^{\mathrm{T}} \in L^\infty(\mathbb{R}),$$

define

$$(K_u f)(x,z) := -\int_x^\infty \mathrm{diag}(1, \mathrm{e}^{2\mathrm{i}z(y-x)})Q(y)f(y,z)\mathrm{d}y, \tag{2.9}$$

then (2.3) can be written as

$$(I - K_u)m_1^+(x,z) = e_1. \tag{2.10}$$

Then the operator of K_u has the following property.

Lemma 2.1 *Let $u \in L^1(\mathbb{R})$, for fixed $z \in \overline{\mathbb{C}^+}$, $I - K_u$ is an invertible operator in $L^\infty(\mathbb{R})$.*

Proof Notice that $z \in \overline{\mathbb{C}^+}$ and

$$\begin{aligned} |K_u f(x,z)| &\le |(K_u f)_1(x,z)| + |(K_u f)_2(x,z)| \\ &= \Big|\int_x^\infty u(-y)f_1\mathrm{d}y\Big| + \Big|\int_x^\infty u(y)\mathrm{e}^{2\mathrm{i}z(y-x)}f_2\mathrm{d}y\Big| \\ &\le \|u\|_{L^1(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

which implies that K_u is a bounded operator in $L^\infty(\mathbb{R})$ for any fixed $z \in \overline{\mathbb{C}^+}$.

Similar to the analysis described above, we have for $n \ge 1$,

$$|K_u^n f| \le \Big(\frac{1}{n!}\Big)\|u\|_{L^1(\mathbb{R})}^n\|f\|_{L^\infty(\mathbb{R})}. \tag{2.11}$$

From the above analysis we know that for any fixed $z \in \overline{\mathbb{C}^+}$, K_u^n is a bounded operator in $L^\infty(\mathbb{R})$ and

$$\|K_u^n\|_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})} \le \Big(\frac{1}{n!}\Big)\|u\|_{L^1(\mathbb{R})}^n,$$

which yields $1-K_u$ is an invertible operator in $L^\infty(\mathbb{R})$. Moreover,

$$\|(1-K_u)^{-1}\|_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})} \le \mathrm{e}^{\|u\|_{L^1(\mathbb{R})}}.$$

Next, we study the asymptotics of the Jost functions $m^\pm(x,z)$.

Lemma 2.2 *If $u\in H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})$, then as $|\mathrm{Im}\,z|\to\infty$, for every $x\in\mathbb{R}$,*

$$\begin{aligned} m_1^\pm(x;z) &= e_1 + p_1^\pm(x)(2\mathrm{i}z)^{-1} + q_1^\pm(x)(2\mathrm{i}z)^{-2} + g_1^\pm(x)(2\mathrm{i}z)^{-3} + \mathcal{O}(z^{-4}),\\ m_2^\pm(x;z) &= e_2 + p_2^\pm(x)(2\mathrm{i}z)^{-1} + q_2^\pm(x)(2\mathrm{i}z)^{-2} + \mathcal{O}(z^{-3}), \end{aligned}$$

where

$$p_1^\pm(x) = \left[\sigma\int_x^{\pm\infty} u(-y)u(y)\mathrm{d}y,\ -\sigma u(-x)\right]^{\mathrm{T}}, \tag{2.12}$$

$$q_1^\pm(x) = \begin{bmatrix} \sigma\displaystyle\int_x^{\pm\infty}\partial u(-y)u(y)\mathrm{d}y + \int_x^{\pm\infty} u(-x_1)u(x_1)\int_{x_2}^{\pm\infty} u(-x_2)u(x_2) \\ -\sigma\partial u(-x) - u(-x)\displaystyle\int_x^{\pm\infty} u(-y)u(y)\mathrm{d}y \end{bmatrix},$$

$$g_1^\pm(x) = \begin{bmatrix} \sigma\displaystyle\int_x^{\pm\infty} u(y)\partial^2 u(-y)\mathrm{d}y + \int_x^{\pm\infty} u(x_1)\partial u(-x_1)\int_{x_1}^{\pm\infty} u(-x_2)u(x_2) \\ \quad +\displaystyle\int_x^{\pm\infty} u(-y)^2u(y)^2\mathrm{d}y \\ \quad +\sigma\displaystyle\int_x^{\pm\infty} u(-x_1)u(x_1)\int_{x_1}^{\pm\infty} u(-x_2)u(x_2)\int_{x_2}^{\pm\infty} u(-x_3)u(x_3) \\ \quad +\displaystyle\int_x^{\pm\infty} u(-x_1)u(x_1)\int_{x_1}^{\pm\infty} u(x_2)\partial u(-x_2) \\ -\partial u(-x)\displaystyle\int_x^{\pm\infty} u(-y)u(y)\mathrm{d}y - \sigma u(-x)\int_x^{\pm\infty} u(-x_1)u(x_1)\int_{x_1}^{\pm\infty} u(-x_2)u(x_2) \\ \quad -u(-x)\displaystyle\int_x^{\pm\infty}\partial u(-y)u(y)\mathrm{d}y - u^2(-x)u(x) - \sigma\partial^2 u(-x) \end{bmatrix},$$

$$p_2^\pm(x) = \left[-u(x),\sigma\int_x^{\pm\infty} u(y)u(-y)\mathrm{d}y\right]^{\mathrm{T}}, \tag{2.13}$$

$$q_2^\pm(x) = \begin{bmatrix} -\sigma\partial u(-x) - u(-x)\displaystyle\int_x^{\pm\infty} u(-y)u(y)\mathrm{d}y \\ \sigma\displaystyle\int_x^{\pm\infty}\partial u(-y)u(y)\mathrm{d}y + \int_x^{\pm\infty} u(-x_1)u(x_1)\int_{x_2}^{\pm\infty} u(-x_2)u(x_2) \end{bmatrix}.$$

Proof We will only to prove the statement for $m_1^\pm(x;z)$, while the proof of $m_2^\pm(x;z)$ is similar. Rewriting (2.3) as the component form

$$m_{11}^+(x;z) = 1 - \int_x^\infty u(y)m_{21}^+(y;z)\mathrm{d}y,$$

$$m_{21}^+(x;z) = \sigma\int_x^\infty \mathrm{e}^{2\mathrm{i}z(y-x)}u(-y)m_{11}^+(y;z)\mathrm{d}y.$$

We have proved that for every $x \in \mathbb{R}$, $m_1^{\pm}(x;z)$ is analytic in $z \in \mathbb{C}^+$. Noticing that $u \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \hookrightarrow L^\infty$ which yields (2.3) is bounded for every $z \in \mathbb{C}^+$ and the integrand converges to e_1 as $|\mathrm{Im}\, z| \to \infty$. Integrating by part and recalling $u \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, we can get

$$\begin{aligned} m_{11}^+(x;z) = 1 &+ \frac{1}{2\mathrm{i}z}\sigma \int_x^\infty u(y)u(-y)m_{11}^+ \mathrm{d}y + \frac{1}{(2\mathrm{i}z)^2}\sigma \int_x^\infty u(y)\partial u(-y)m_{11}^+ \mathrm{d}y \\ &+ \frac{1}{(2\mathrm{i}z)^2}\sigma \int_x^\infty u(y)u(-y)m_{21}^+ \mathrm{d}y + \mathcal{O}(z^{-3}) \end{aligned}$$

and

$$m_{21}^+(x;z) = -\sigma u(-x)\frac{m_{11}^+}{2\mathrm{i}z} - \frac{\sigma(\partial u(-x)m_{11}^+ - u(x)u(-x)m_{21}^+)}{(2\mathrm{i}z)^2} + \mathcal{O}(z^{-3}).$$

Letting $|z| \to \infty$ and noticing $m_1^+ \to e_1$ as $|\mathrm{Im}\, z| \to \infty$, we get the expanding formula of $m_1^{\pm}(x;z)$.

Similar to the above analysis, for a vector function

$$f(x,z) = (f_1(x;z), f_2(x;z))^{\mathrm{T}} \in L_x^\infty(\mathbb{R}) \otimes L_z^2(\mathbb{R}),$$

we define operators K_u and $\partial_z K_u$ as

$$K_u f(x;z) := -\int_x^\infty \mathrm{diag}(1, \mathrm{e}^{2\mathrm{i}z(y-x))})Q(y)f(y,z)\mathrm{d}y,$$

$$(\partial_z K_u f)(x,z) := -\int_x^\infty \mathrm{diag}(0, 2\mathrm{i}(y-x)\mathrm{e}^{2\mathrm{i}z(y-x)})Q(y)f(y,z)\mathrm{d}y,$$

respectively. Then we can prove the following lemma.

Lemma 2.3 *If $w \in L^2(\mathbb{R})$, then*

$$\sup_{x\in\mathbb{R}} \Big\| \int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y \Big\|_{L_z^2(\mathbb{R})} \le \sqrt{\pi}\|w\|_{L^2(\mathbb{R})}. \tag{2.14}$$

If $w \in H^3(\mathbb{R})$, then for every $n = 1, 2, 3$,

$$\sup_{x\in\mathbb{R}} \Big\|(2\mathrm{i}z)^n \int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y + \sum_{k=0}^{n-1}(-2\mathrm{i}z)^{k+1}\partial_x^k w(x)\Big\|_{L_z^2(\mathbb{R})} \le \sqrt{\pi}\|\partial_x^n w\|_{L^2(\mathbb{R})}. \tag{2.15}$$

If $w \in L^{2,1}(\mathbb{R})$, then for every $x_0 \in \mathbb{R}^+$,

$$\sup_{x\in(x_0,+\infty)} \Big\|\langle x\rangle \int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y\Big\|_{L_z^2(\mathbb{R})} \le \sqrt{\pi}\|w\|_{L^{2,1}(x_0,+\infty)}. \tag{2.16}$$

Furthermore, if $w \in H^{1,1}(\mathbb{R})$, then for every $x_0 \in \mathbb{R}^+$,

$$\sup_{x\in(x_0,+\infty)} \Big\|\langle x\rangle \Big[(2\mathrm{i}z) \int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y + w(x)\Big]\Big\|_{L_z^2(\mathbb{R})} \le \sqrt{\pi}\|w\|_{H^{1,1}(x_0,+\infty)} \tag{2.17}$$

and for every $x_0 \in \mathbb{R}$, we have

$$\sup_{x\in\mathbb{R}} \Big\|(2\mathrm{i}z) \int_x^\infty (y-x)\mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y\Big\|_{L_z^2(\mathbb{R})} \le \sqrt{\pi}\|w\|_{H^{1,1}(\mathbb{R})}, \tag{2.18}$$

where $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$.

Proof The bounds (2.14)–(2.15) (for $n=1$) and (2.16) were given in [27]. It remains to prove the estimate (2.15) (for $n=2,3$) and (2.17)–(2.18). For every $x\in\mathbb{R}$ and $z\in\mathbb{R}$, define

$$f(x;z)=\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y=\int_0^\infty \mathrm{e}^{2\mathrm{i}zy}w(x+y)\mathrm{d}y.$$

Using the Plancherel's theorem, we have

$$\|f(x;z)\|_{L_z^2}^2=\pi\int_0^\infty |w(x+y)|^2\mathrm{d}y=\pi\int_x^\infty |w(y)|^2\mathrm{d}y.$$

Furthermore, if $x\in\mathbb{R}^+$,we have

$$\|f(x;z)\|_{L_z^2}^2=\pi\int_0^\infty |w(x+y)|^2\mathrm{d}y=\pi\int_x^\infty |w(y)|^2\mathrm{d}y\le\pi\langle x\rangle^{-2}\int_x^\infty \langle y\rangle^2|w(y)|^2\mathrm{d}y,$$

which yields (2.16).

Integrate by part, we get

$$2\mathrm{i}zf(x;z)+w(x)=\int_{-\infty}^x \mathrm{e}^{-2\mathrm{i}z(x-y)}\partial_y w(y)\mathrm{d}y \tag{2.19}$$

and

$$(2\mathrm{i}z)^2f(x;z)+2\mathrm{i}zw(x)+\partial_x w(x)=\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}\partial_y^2 w(y)\mathrm{d}y.$$

We then obtain that

$$\Big\|2\mathrm{i}z\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w(y)\mathrm{d}y+w(x)\Big\|_{L_z^2}^2=\pi\int_x^\infty |\partial_y w(y)|^2\mathrm{d}y$$

and

$$\Big\|(2\mathrm{i}z)^2\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}w\mathrm{d}y+2\mathrm{i}zw(x)+\partial_x w(x)\Big\|_{L_z^2}^2=\pi\int_x^\infty |\partial_y^2 w(y)|^2\mathrm{d}y.$$

Finally we get (2.15) for $n=2$. The case when $n=3$ can be shown in a similar way.

Combine (2.16) with (2.19) we can get (2.17). Replacing w with $(y-x)w$ and repeating the above process, we can get (2.18).

Proposition 2.1 *Let $u\in H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})$. Then $1-K_u$ is an invertible operator in $L_x^\infty(\mathbb{R})\otimes L_z^2(\mathbb{R})$.*

Proof The result is easy to get from Lemma 2.1.

Proposition 2.2 *Letting $u\in H^3(\mathbb{R})$, then for every $x\in\mathbb{R}$, we have*

$$\begin{aligned}&m_1^\pm-e_1\in L_z^2(\mathbb{R}),\\&(2\mathrm{i}z)(m_1^\pm-e_1)-p_1^\pm\in L_z^2(\mathbb{R}),\\&(2\mathrm{i}z)^2(m_1^\pm-e_1)-(2\mathrm{i}z)p_1^\pm-q_1^\pm\in L_z^2(\mathbb{R}),\\&(2\mathrm{i}z)^3(m_1^\pm-e_1)-(2\mathrm{i}z)^2p_1^\pm-(2\mathrm{i}z)q_1^\pm-g_1^\pm\in L_z^2(\mathbb{R})\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}&m_2^\pm-e_2\in L_z^2(\mathbb{R}),\\&(2\mathrm{i}z)(m_2^\pm-e_2)-p_2^\pm\in L_z^2(\mathbb{R}),\\&(2\mathrm{i}z)^2(m_2^\pm-e_2)-(2\mathrm{i}z)p_2^\pm-q_2^\pm\in L_z^2(\mathbb{R}).\end{aligned} \tag{2.21}$$

If $u\in H^{1,1}(\mathbb{R})$*, then for every* $x\in\mathbb{R}^+$*, we have*

$$(2\mathrm{i}z)\partial_z m_1^{\pm}\in L_z^2(\mathbb{R}). \tag{2.22}$$

Proof We only prove (2.20) and (2.22), others can be shown in a similar way. Recalling that

$$(1-K_u)[m_1^+-e_1]=K_ue_1$$

and from Proposition 2.1, we know $1-K_u$ is an invertible operator in $L_x^\infty(\mathbb{R})\otimes L_z^2(\mathbb{R})$. Therefore $m_1^+(x;\cdot)-e_1\in L^2(\mathbb{R})$ if $K_ue_1(x;\cdot)\in L^2(\mathbb{R})$. We write K_ue_1 in the following form

$$(K_ue_1)(x;z)=\Big[0,\sigma\int_x^\infty u(-y)\mathrm{e}^{2\mathrm{i}z(y-x)\mathrm{d}y}\Big]^{\mathrm{T}}.$$

By Lemma 2.3, we know

$$\sup_{x\in\mathbb{R}}\Big\|\int_x^\infty \mathrm{e}^{2\mathrm{i}z(x-y)}\sigma u(-y)\mathrm{d}y\Big\|_{L^2}\le\sqrt{\pi}\|u\|_{L^2},$$

then

$$\|m_1^+-e_1\|_{L_x^\infty\otimes L_z^2}\le\|(1-K_u)^{-1}\|_{L_x^\infty\otimes L_z^2\to L_x^\infty\otimes L_z^2}\|K_ue_1\|_{L_x^\infty\otimes L_z^2}\le c\|u\|_{L^2}.$$

Similar to the above analysis, we can get

$$(I-K_u)[(2\mathrm{i}z)(m_1^+-e_1)-p_1^+]=(2\mathrm{i}z)K_ue_1-(I-K_u)p_1^+,$$

where

$$(I-K_u)p_1^+=\Big[0,-u(-x)-\int_x^\infty \mathrm{e}^{2\mathrm{i}z(x_1-x_2)}u(-x_1)\int_{x_1}^\infty u(-x_2)u(x_2)\mathrm{d}x_2\mathrm{d}x_1\Big]^{\mathrm{T}}.$$

Then, we can get

$$\begin{aligned}(I-K_u)[(2\mathrm{i}z)(m_1^+-e_1)-p_1^+]&=2\mathrm{i}z\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}\sigma u(-y)\mathrm{d}y+\sigma u(-x)\\&\quad+\int_x^\infty \mathrm{e}^{2\mathrm{i}z(x_1-x)}u(-x_1)\int_{x_1}^\infty u(x_2)u(-x_2)\mathrm{d}x_2\mathrm{d}x_1.\end{aligned}\tag{2.23}$$

From Lemma 2.3 again, notice that $u\in H^2(\mathbb{R})$, then

$$\sup_{x\in\mathrm{R}}\Big\|(2\mathrm{i}z)\sigma\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}u(-y)\mathrm{d}y+\sigma u(-x)\Big\|_{L^2}\le c\|\partial_xu\|_{L^2}$$

and

$$\sup_{x\in\mathrm{R}}\Big\|\int_x^\infty \mathrm{e}^{2\mathrm{i}z(x_1-x)}u(-x_1)\int_{x_1}^\infty u(-x_2)u(x_2)\mathrm{d}x_2\mathrm{d}x_1\Big\|_{L^2}\le c\|u\|_{L^2}^2,$$

where c is a constant.

Combining with (2.23), we have

$$\sup_{x\in\mathrm{R}}\Big\|\int_x^\infty(2\mathrm{i}z)\mathrm{e}^{2\mathrm{i}z(x-y)}\sigma u(-y)\mathrm{d}y-(I-K_u)p_1^+e_2\Big\|_{L^2}\le c(\|\partial_xu\|_{L^2}+\|u\|_{L^2}^2).$$

Inserting operator $I-K_u$ on $L_x^2(\mathbb{R})\otimes L_z^2(\mathbb{R})$, we get (2.20)–(2.21).

We derive on both side of (2.10),

$$(1-K_u)\partial_z m_1^+ = (\partial_z K_u)m_1^+,$$

then, we get

$$(1-K_u)\partial_z m_1^+ = (\partial_z K_u)[m_1^+ - e_1] + \partial_z K_u e_1 \tag{2.24}$$

and

$$(I-K_u)(2\mathrm{i}z)\partial_z m_1^+ = \partial_z K_u[(2\mathrm{i}z)(m_1^+ - e_1) - p_1^+] + (2\mathrm{i}z)\partial_z K_u e_1 + \partial_z K_u p_1^+. \tag{2.25}$$

By Lemma 2.3, we can get for every $x_0 \in \mathbb{R}^+$,

$$\begin{aligned}&\sup_{x\in(x_0,\infty)} \|\langle x\rangle(m_1^+ - e_1)\|_{L^2}\\ &\le c\Big\|\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}\sigma u(-y)\mathrm{d}y\Big\|_{L^2} \le c\|u\|_{L^{2,1}},\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\sup_{x\in(x_0,\infty)} \|\langle x\rangle(2\mathrm{i}z)[(m_1^+ - e_1) - p_1^+]\|_{L^2}\\ &\le c\Big\|\int_x^\infty \mathrm{e}^{-2\mathrm{i}z(x-y)}\partial\sigma u(-y)\mathrm{d}y\Big\|_{L^2} \le c\|u\|_{H^{1,1}}.\end{aligned} \tag{2.27}$$

Combining (2.24) with (2.26), using Lemma 2.3, we can get

$$\begin{aligned}\|(I-K_u)\partial_z m_1^+\|_{L^2} &\le \Big\|\int_x^\infty 2\mathrm{i}(y-x)\sigma u(-y)\mathrm{e}^{-2\mathrm{i}z(x-y)}(m_{11}^+ - 1)\mathrm{d}y\Big\|_{L^2}\\ &\quad + \Big\|\int_x^\infty 2\mathrm{i}(y-x)\mathrm{e}^{-2\mathrm{i}z(x-y)}\sigma u(-y)\mathrm{d}y\Big\|_{L^2}\\ &\le c\|u\|_{L^1}\sup_{x\in(x_0,\infty)}\|\langle x\rangle(m_1^+ - e_1)\|_{L^2} + c\|u\|_{L^{2,1}} \le c\|u\|_{H^{1,1}}.\end{aligned}$$

It follows that from (2.25), (2.27) and Lemma 2.3

$$\begin{aligned}\|(I-K_u)(2\mathrm{i}z)\partial_z m_1^+\|_{L^2} &\le \Big\|\int_x^\infty 2\mathrm{i}(y-x)\mathrm{e}^{-2\mathrm{i}z(x-y)}\sigma u(-y)[(2\mathrm{i}z)(m_{11}^+ - 1) - p_{11}^+]dy\Big\|_{L^2}\\ &\quad + \Big\|(2\mathrm{i}z)\int_x^\infty 2\mathrm{i}(y-x)\mathrm{e}^{-2\mathrm{i}z(x-y)}\sigma u(-y)\mathrm{d}y\Big\|_{L^2}\\ &\quad + \Big\|\int_x^\infty 2\mathrm{i}(y-x)\mathrm{e}^{-2\mathrm{i}z(x-y)}\sigma u(-y)p_{11}^+\mathrm{d}y\Big\|_{L^2}\\ &\le c\|u\|_{L^1}\sup_{x\in(x_0,\infty)}\|\langle x\rangle[(2\mathrm{i}z)(m_1^+ - e_1) - p_1^+]\|_{L^2} \le c\|u\|_{H^{1,1}}.\end{aligned}$$

Inserting operator $I-K_u$ on $L_x^2(\mathbb{R})\otimes L_z^2(\mathbb{R})$, we get (2.22). We finally complete the proof of the lemma.

We have constructed the following maps

$$\begin{aligned}
&H^{1,1} \ni u \to z\partial_z m_1^{\pm}(x;\cdot) \in L^2,\\
&H^{1,1} \ni u \to z\partial_z m_2^{\pm}(x;\cdot) \in L^2,\\
&H^3 \ni u \to z^3(m_1^{\pm}(x;\cdot) - e_1) - z^2 p_1^{\pm} - zq_1^{\pm} - g_1^{\pm} \in \text{Ł}^2,\\
&H^3 \ni u \to z^3(m_2^{\pm}(x;\cdot) - e_2) - z^2 p_2^{\pm} - zq_2^{\pm} - g_2^{\pm} \in \text{Ł}^2.
\end{aligned}$$

Next, we will show this maps and remainder of the Jost function in function space $L_x^\infty(\mathbb{R}) \otimes L_z^2(\mathbb{R})$ is Lipschitz continuous.

Corollary 2.1 *Let $u, \widetilde{u} \in H^{1,1}(\mathbb{R})$ satisfy $\|u\|_{H^{1,1}(\mathbb{R})} \le U$ and $\|\widetilde{u}\|_{H^{1,1}(\mathbb{R})} \le U$ for some $U > 0$, then there is a positive U-dependent on constant $C(U)$ such that for every $x \in \mathbb{R}$,*

$$\|\check{m}_1^{\pm}(x;\cdot) - \check{\widetilde{m}}_1^{\pm}(x;\cdot)\|_{L^2} + \|\check{m}_2^{\pm}(x;\cdot) - \check{\widetilde{m}}_2^{\pm}(x;\cdot)\|_{L^2} \le C(U)\|u - \widetilde{u}\|_{H^{1,1}},$$

where

$$\begin{aligned}
\check{m}_1^{\pm}(x;\cdot) &:= (2\mathrm{i}z)\partial_z m_1^{\pm}(x;\cdot),\\
\check{m}_2^{\pm}(x;\cdot) &:= (2\mathrm{i}z)\partial_z m_2^{\pm}(x;\cdot).
\end{aligned}$$

Moreover, if $u, \widetilde{u} \in H^3(\mathbb{R})$ satisfy $\|u\|_{H^3(\mathbb{R})} \le U$ and $\|\widetilde{u}\|_{H^3(\mathbb{R})} \le U$ for some $U > 0$ then there is a positive U-dependent on constant $C(U)$ such that for every $x \in \mathbb{R}$,

$$\left\|\widehat{m}_1^{\pm}(x;\cdot) - \widehat{\widetilde{m}}_1^{\pm}(x;\cdot)\right\|_{L^2} + \left\|\widehat{m}_2^{\pm}(x;\cdot) - \widehat{\widetilde{m}}_2^{\pm}(x;\cdot)\right\|_{L^2} \le C(U)\|u - \widetilde{u}\|_{H^3},$$

where

$$\begin{aligned}
\widehat{m}_1^{\pm}(x;\cdot) &:= (2\mathrm{i}z)^3(m_1^{\pm}(x;\cdot) - e_1) - (2\mathrm{i}z)^2 p_1^{\pm} - (2\mathrm{i}z)q_1^{\pm} - g_1^{\pm},\\
\widehat{m}_2^{\pm}(x;\cdot) &:= (2\mathrm{i}z)^3(m_2^{\pm}(x;\cdot) - e_2) - (2\mathrm{i}z)^2 p_2^{\pm} - (2\mathrm{i}z)q_2^{\pm} - g_2^{\pm}.
\end{aligned}$$

Proof From (2.10), we can get

$$\begin{aligned}
m_1^+ - \widetilde{m}_1^+ &= (I - K_u)^{-1}[K_u e_1 - \widetilde{K_u} e_1] + ((I - K_u)^{-1} - (I - \widetilde{K_u})^{-1})\widetilde{K_u} e_1\\
&= (I - K_u)^{-1}[K_u e_1 - \widetilde{K_u} e_1] + (I - K_u)^{-1}(\widetilde{K_u} - K_u)(I - \widetilde{K_u})^{-1}\widetilde{K_u} e_1,
\end{aligned}$$

where

$$\sup_{x\in\mathbb{R}} \|K_u e_1 - \widetilde{K_u} e_1\|_{L^2} = \sup_{x\in\mathbb{R}} \Big\| \int_x^\infty \mathrm{e}^{2\mathrm{i}z(y-x)}(u - \widetilde{u})\mathrm{d}y \Big\|_{L^2}.$$

Using Lemma 2.3 we can get

$$\sup_{x\in\mathbb{R}} \|K_u e_1 - \widetilde{K_u} e_1\|_{L^2} \le c \sup_{x\in\mathbb{R}} \|u - \widetilde{u}\|_{L^2}.$$

Furthermore, for every $f \in L_z^2(\mathbb{R}) \otimes L_x^\infty(\mathbb{R})$,

$$\|(K_u - \widetilde{K_u})f\|_{L_x^\infty \otimes L_z^2} \le C_2(U)\mathrm{e}^{\|u - \widetilde{u}\|_{L^1}} \|f\|_{L_x^\infty \otimes L_z^2}. \tag{2.28}$$

Then, we can get

$$\sup_{x\in\mathbb{R}} \|m_1^+ - \widetilde{m}_1^+\|_{L^2} \le c\|u - \widetilde{u}\|_{L^2}.$$

Using Lemma 2.3 again, we obtain that

$$\sup_{x\in(x_0,\infty)} \|\langle x\rangle(m_1^+ - \widetilde{m}_1^+)\|_{L^2} \le c\|u-\widetilde{u}\|_{L^{2,1}}. \tag{2.29}$$

Direct calculation yields

$$\begin{aligned}
\partial_z m_1^+ - \partial_z\widetilde{m}_1^+ &= (I-K_u)^{-1}\partial_z K_u(m_1^+ - e_1) - (I-\widetilde{K_u})^{-1}\partial_z\widetilde{K_u}(\widetilde{m}_1^+ - e_1)\\
&\quad - [(I-K_u)^{-1}\partial_z K_u e_1 - (I-\widetilde{K_u})^{-1}\partial_z\widetilde{K_u}e_1]\\
&= (I-K_u)^{-1}[\partial_z K_u(m_1^+ - e_1) - \partial_z\widetilde{K_u}(\widetilde{m}_1^+ - e_1)]\\
&\quad + (I-K_u)^{-1}(\widetilde{K_u}-K_u)(I-\widetilde{K_u})^{-1}\partial_z\widetilde{K_u}\widetilde{m}_1^+\\
&\quad - (I-K_u)^{-1}(\partial_z K_u e_1 - \partial_z\widetilde{K_u}e_1)\\
&\quad + (I-K_u)^{-1}(\widetilde{K_u}-K_u)(I-\widetilde{K_u})^{-1}\partial_z\widetilde{K_u}e_1.
\end{aligned} \tag{2.30}$$

Using Lemma 2.3 and (2.29), we get

$$\|(2\mathrm{i}z)\partial_z K_u m_1^+ - (2\mathrm{i}z)\partial_z\widetilde{K_u}\widetilde{m}_1^+\|_{L^2} \le C_1(U)\|u-\widetilde{u}\|_{H^{1,1}}, \tag{2.31}$$

where $C_1(U)$ is a U-dependent on positive constant. Noticing that $1-K_u$ is an invertible operator in $L_x^\infty(\mathbb{R})\otimes L_z^2(\mathbb{R})$ and (2.28), we get the bound (2.28). The others follow by repeating the same analysis as that above.

2.3 Lipschitz continuity of scattering data

In this section, we prove the Lipschitz continuity from initial value to scattering data.

Lemma 2.4 *If $u\in H^{1,1}(\mathbb{R})$, then the function $a(z)$ is continued analytically in $\mathbb{C}^+$. In addition, we have*

$$a(z)-1, d(z)-1, b(z)\in H^{1,1}(\mathbb{R}).$$

Moreover, if $u\in H^3(\mathbb{R})$, then

$$b(z)\in L^{2,3}(\mathbb{R}).$$

Proof From (2.3), (2.6) and (2.8), we get

$$b(z) = \sigma\int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy}u(-y)m_{11}^+\mathrm{d}y, \tag{2.32}$$

$$\partial_z b(z) = \sigma\int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy}u(-y)\partial_z m_{11}^+\mathrm{d}y + \sigma\int_{\mathbb{R}} 2\mathrm{i}y\mathrm{e}^{2\mathrm{i}zy}u(-y)m_{11}^+\mathrm{d}y, \tag{2.33}$$

$$\begin{aligned}
a(z)-1 &= -\int_{\mathbb{R}} u(y)m_{21}^+\mathrm{d}y,\\
d(z)-1 &= \sigma\int_{\mathbb{R}} u(-y)m_{12}^+\mathrm{d}y,
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\partial_z a(z) &= -\int_{\mathbb{R}} u(y)\partial_z m_{21}^+\mathrm{d}y,\\
\partial_z d(z) &= \sigma\int_{\mathbb{R}} u(-y)\partial_z m_{21}^+\mathrm{d}y.
\end{aligned} \tag{2.35}$$

We can easily obtain the following limit for the scattering coefficient $a(z)$ along a contour in $\mathbb{C}^+$ extended to

$$a(z) \to 1, \quad \operatorname{Im} z \to \infty.$$

In order to prove $b(z) \in L_z^{2,3}(\mathbb{R})$, we rewrite (2.8) as the following form

$$\begin{aligned}(2\mathrm{i}z)^3 b(z) = {} & \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u[(2\mathrm{i}z)^3(m_{11}^+ - 1) - (2\mathrm{i}z)^2 p_{11}^+ - (2\mathrm{i}z) q_{11}^+ - g_{11}^+]\mathrm{d}y \\ & + (2\mathrm{i}z)^3 \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u \mathrm{d}y + (2\mathrm{i}z)^2 \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u p_{11}^+ \mathrm{d}y + (2\mathrm{i}z)\sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u q_{11}^+ \mathrm{d}y \\ & + \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u g_{11}^+ \mathrm{d}y.\end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned}& \Big\| \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u[(2\mathrm{i}z)^3(m_{11}^+ - 1) - (2\mathrm{i}z)^2 p_{11}^+ - (2\mathrm{i}z) q_{11}^+ - g_{11}^+]\mathrm{d}y \Big\|_{L^2} \\ & \le c\|u\|_{L^1} \sup_{x\in\mathbb{R}} \|(2\mathrm{i}z)^3(m_{11}^+ - 1) - (2\mathrm{i}z)^2 p_{11}^+ - (2\mathrm{i}z) q_{11}^+ - g_{11}^+\|_{L^2} \\ & \le c\|u\|_{L^1}\|u\|_{H^3}.\end{aligned}$$

We proved that the first term of (2.36) is bounded in L^2 space. For the other terms of (2.36), using Plancherel's formula, we can get

$$\begin{aligned}& \Big\|(2\mathrm{i}z)^3 \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u \mathrm{d}y \Big\|_{L^2} \le c\|u\|_{H^3}, \\ & \Big\|(2\mathrm{i}z)^2 \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u p_{11}^+ \mathrm{d}y \Big\|_{L^2} \le c\|u\|_{H^3}, \\ & \Big\|(2\mathrm{i}z) \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u q_{11}^+ \mathrm{d}y \Big\|_{L^2} \le c\|u\|_{H^3}, \\ & \Big\| \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u g_{11}^+ \mathrm{d}y \Big\|_{L^2} \le c\|u\|_{H^3}.\end{aligned}$$

Then we get $b(z) \in L^{2,3}(\mathbb{R})$.

In order to prove $\partial_z b(z) \in L_z^{2,1}(\mathbb{R})$, we rewrite (2.33) as the following form

$$\begin{aligned}(2\mathrm{i}z)\partial_z b(z) = {} & \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u(2\mathrm{i}z)\partial_z m_{11}^+ + \sigma \int_{\mathbb{R}} (2\mathrm{i}y)\mathrm{e}^{2\mathrm{i}zy} u[(2\mathrm{i}z)(m_{11}^+ - 1) - p_{11}^+]\mathrm{d}y \\ & + (2\mathrm{i}z)\sigma \int_{\mathbb{R}} (2\mathrm{i}y)\mathrm{e}^{2\mathrm{i}zy} u \mathrm{d}y + \sigma \int_{\mathbb{R}} (2\mathrm{i}y)\mathrm{e}^{2\mathrm{i}zy} u p_{11}^+ \mathrm{d}y.\end{aligned} \tag{2.36}$$

Using Lemma 2.3 again, we get

$$\begin{aligned}\Big\| \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u(2\mathrm{i}z)\partial_z m_{11}^+ \mathrm{d}y \Big\|_{L^2} & \le c\|u\|_{L^1} \sup_{x\in\mathbb{R}} \|(2\mathrm{i}z)\partial_z m_{11}^+\|_{L^2} \\ & \le c\|u\|_{L^1}\|u\|_{H^{1,1}}.\end{aligned}$$

We proved that the first term of (2.36) is bounded in L^2 space. For the other terms of (2.36), using Plancherel's formula, we can get

$$\Big\|(2\mathrm{i}z) \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} (2\mathrm{i}y) u \mathrm{d}y \Big\|_{L^2} \le c\|u\|_{H^{1,1}},$$

$$\Big\| \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy}(2\mathrm{i}y)up_{11}^{+}\mathrm{d}y\Big\|_{L^2} \le c\|u\|_{H^{1,1}}.$$

We get $\partial_z b(z) \in L^{2,1}(\mathbb{R})$. Then we have proved the bounds of $b(z)$. The conclusion of $a(z), d(z)$ can be proved in the similar way.

Lemma 2.4 establishes the following two maps

$$H^{1,1}(\mathbb{R}) \ni u \to a(z)-1, d(z)-1, b(z) \in H^{1,1}(\mathbb{R})$$

and

$$H^3(\mathbb{R}) \ni u \to b(z) \in L^{2,3}(\mathbb{R}).$$

We will show that the two maps are Lipschitz continuous.

Corollary 2.2 *Let $u \in H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R})$, then*

$$H^{1,1}(\mathbb{R}) \ni u \to a(z)-1, d(z)-1, b(z) \in H_z^{1,1}(\mathbb{R})$$

and

$$H^3(\mathbb{R}) \ni u \to b(z) \in L_z^{2,3}(\mathbb{R})$$

are Lipschitz continuous.

Proof From the representations (2.32)–(2.35) and the Lipschitz continuity of the Jost function $m_1^{\pm}$ and $m_2^{\pm}$, we can obtain the Lipschitz continuity of the scattering coefficients.

For the reflection coefficients $r(z)$, we have the following results.

Lemma 2.5 *If $u \in H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R})$, then we have*

$$r_{1,2}(z) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R}).$$

As well, the mapping

$$H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R}) \ni u \mapsto r_{1,2} \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$$

is Lipschitz continuous.

Proof Let r_1 and $\widetilde{r}_1$ denote the reflection coefficients corresponding to u and $\widetilde{u}$, respectively. Owing to

$$r_1 - \widetilde{r}_1 = \frac{b-\widetilde{b}}{a} + \frac{\widetilde{b}((\widetilde{a}-1)-(a-1))}{a\widetilde{a}},$$

the Lipschitz continuity of r_1 follows from the Lipschitz continuity of $a-1$ and b, and we can get similar conclusion of r_2 .

Lemma 2.6 *If $u \in H^{1,1} \cap H^3$ with L^1-norm such that $\|u\|_{L^1}\mathrm{e}^{2\|u\|_{L^1}} < 1$, then the spectral problem admits no eigenvalues or resonances, and the scattering data $a(z)$ and $d(z)$ admit no zeros in $\mathbb{C}^+ \cup \mathbb{R}$ and $\mathbb{C}^- \cup \mathbb{R}$, respectively.*

Proof The small-norm condition implies that $\|u\|_{L^1(\mathbb{R})} < 1$. Recall that $m_1^+ = e_1 + K_u m_1^+$ in Lemma 2.1 and the operator $I - K_u$ is invertible and bounded from L_x^∞ to L_x^∞, then we reach that for every $z \in \mathbb{C}^+$,

$$\|m_1^+(\cdot; z) - e_1\|_{L^\infty} = \|(I - K_u)^{-1}\|\|K_u e_1(\cdot; z)\|_{L^\infty} \leq \|u\|_{L^1} \mathrm{e}^{2\|u\|_{L^1}}.$$

We derive for every $z \in \mathbb{C}^+$,

$$|a(z)| \geq 1 - \Big|\int_{\mathbb{R}} u(y) m_{21}^+(y; z) \mathrm{d}y\Big| > 1 - \|u\|_{L^1} \mathrm{e}^{2\|u\|_{L^1}}.$$

Due to the continuity of $a(z)$, we also obtain that

$$|a(z)| \geq 1 - \|u\|_{L^1(\mathbb{R})} \mathrm{e}^{2\|u\|_{L^1}} > 0, \quad z \in \mathbb{R},$$

then $a(z)$ admits no zeros in $\mathbb{C}^+ \cup \mathbb{R}$. Carrying out a similar manipulation for $d(z)$, we see that $d(z)$ admits no zeros in $\mathbb{C}^- \cup \mathbb{R}$.

Lemma 2.7 *If $u \in L^{2,1}(\mathbb{R})$ with $L^1(\mathbb{R})$ small-norm such that*

$$1 - \|u\|_{L^1(\mathbb{R})}(1 + 2\mathrm{e}^{2\|u\|_{L^1(\mathbb{R})}}) > 0, \tag{2.37}$$

then for every $z \in \mathbb{R}$, we have $|r_{1,2}(z)| < 1$.

Proof Rewrite (2.32) for $b(z)$ as

$$b(z) = \sigma \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}zy} u(-y) m_{11}^+ \mathrm{d}y, \tag{2.38}$$

under the condition (2.37). For every $z \in \mathbb{R}$, we obtain

$$\begin{aligned} |b(z)| &\leq \|m_{11}^+(\cdot; z) - 1\|_{L^\infty} \|u\|_{L^1} + \|u\|_{L^1} \leq \|u\|_{L^1(\mathbb{R})} + \|u\|_{L^1(\mathbb{R})} \mathrm{e}^{2\|u\|_{L^1(\mathbb{R})}} \\ &< 1 - \|u\|_{L^1(\mathbb{R})} \mathrm{e}^{2\|u\|_{L^1(\mathbb{R})}} \leq |a(z)|, \end{aligned}$$

which yields

$$|r_1(z)| = \frac{|b(z)|}{|a(z)|} < 1.$$

Similarly, we get $|r_2(z)| < 1$.

3 Inverse Scattering Transform

In this section, we will set up an RH problem and show the existence and uniqueness of its solution for the given data $r(z) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$.

3.1 Set-up of an RH problem

Define

$$M(x; z) = \begin{cases} \begin{pmatrix} m_1^{(+)} & m_2^{(-)} \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}, & \operatorname{Im} z > 0, \\ \begin{pmatrix} m_1^{(-)} & m_2^{(+)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & d^{-1} \end{pmatrix}, & \operatorname{Im} z < 0, \end{cases}$$

then $M(x;z)$ satisfies the following RH-problem.

Problem 3.1 Find a matrix function $M(x;z)$ satisfying

(i) $M(x;z) \to I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.

(ii) For $M(x;z)$ admits the following jump condition

$$M_+(x;z) = M_-(x;z)V_x(z), \tag{3.1}$$

where

$$V_x(z) := \begin{pmatrix} 1+\sigma r_1 r_2 & \sigma r_2(z)\mathrm{e}^{2\mathrm{i}xz} \\ r_1(z)\mathrm{e}^{-2\mathrm{i}xz} & 1 \end{pmatrix}, \quad z \in \mathbb{R}. \tag{3.2}$$

The reconstruction formula is given by

$$u(x) := 2\mathrm{i} \lim_{z\to\infty} zM_{12}(x,z), \qquad u(-x) := 2\mathrm{i}\sigma \lim_{z\to\infty} zM_{21}(x,z). \tag{3.3}$$

We write (5.7) in the form

$$M_+(x;z) - M_-(x;z) = M_-(x;z)S(x;z), \quad z \in \mathbb{R},$$

where

$$S(x;z) = \begin{pmatrix} \sigma r_1 r_2 & \sigma r_2(z)\mathrm{e}^{2\mathrm{i}xz} \\ r_1(z)\mathrm{e}^{-2\mathrm{i}xz} & 0 \end{pmatrix}.$$

Introduce a transformation

$$\Psi_\pm(x;z) = M_\pm(x;z) - I,$$

then we obtain a new RH problem for $\Psi(x;z)$,

$$\begin{aligned} &\Psi_+(x;z) - \Psi_-(x;z) = \Psi_-(x;z)S(x;z) + S(x;z), \quad z \in \mathbb{R}, \\ &\Psi_\pm(x;z) \to 0, \quad |z| \to \infty, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned}$$

3.2 Solvability of the RH problem

We introduce the Cauchy operator

$$\mathcal{C}(f)(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta, \quad z \in \mathbb{C}\backslash\mathbb{R}$$

and Plemelj projection operator

$$\mathcal{P}^\pm(f)(x;z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - (z \pm \mathrm{i}\varepsilon)}\mathrm{d}\zeta, \quad z \in \mathbb{R}, \tag{3.4}$$

where $f(z) \in L^2(\mathbb{R})$.

Proposition 3.1 (see [27]) *For every $f \in L^p(\mathbb{R})$ with $1 \le p < \infty$, the Cauchy operator $\mathcal{C}(f)$ is analytic off the real line, decays to zero as $|z| \to \infty$, and approaches to $\mathcal{P}^\pm(f)$ almost everywhere when a point $z \in C^\pm$ approaches to a point on the real axis by any non-tangential contour from $\mathbb{C}^\pm$. If $1 < p < \infty$, then there exists a positive constant C_p such that*

$$\|\mathcal{P}^\pm(f)\|_{L^p} \le C_p \|f\|_{L^p}. \tag{3.5}$$

If $f \in L^1(\mathbb{R})$, then the Cauchy operator admits the asymptotic

$$\lim_{|z|\to\infty} z\mathcal{C}(f)(z) = -\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} f(s)\mathrm{d}s. \tag{3.6}$$

Lemma 3.1 *Let $r_{1,2}(z) \in H^1(\mathbb{R})$ satisfying $|r_{1,2}(z)| \le 1$. Then there exist positive constants c_- and c_+ such that for every $x \in \mathbb{R}$ and every column-vector $g \in \mathbb{C}^2$, we have that*

$$\operatorname{Re} g^*(I + S(x;z))g \ge c_- g^* g, \quad z \in \mathbb{R} \tag{3.7}$$

and

$$\|(I + S(x;z))g\| \le c_+ \|g\|, \quad z \in \mathbb{R}, \tag{3.8}$$

where the asterisk denotes the Hermite conjugate.

Proof The original scattering matrix $S(x;z)$ is not Hermitian due to the fact that there is no relationship between $a(z)$ and $d(z)$. Therefore, we define Hermitian part of $S(x;z)$ by

$$\begin{aligned} S_H(x;z) &= \frac{1}{2}(S(x;z) + S^*(x;z)) \\ &= \begin{pmatrix} \sigma \operatorname{Re}(r_1 r_2) & \frac{1}{2}(\bar{r}_1 + \sigma r_2)\mathrm{e}^{2\mathrm{i}zx} \\ \frac{1}{2}(r_1 + \sigma \bar{r}_2)\mathrm{e}^{-2\mathrm{i}zx} & 0 \end{pmatrix}. \end{aligned} \tag{3.9}$$

Since $|r_{1,2}(z)| < 1$, the 2-order principle minor of the matrix $I + S_H$,

$$1 + \sigma \operatorname{Re}(r_1 r_2) - \frac{1}{4}|r_1 + \sigma \bar{r}_2|^2 = 1 - \frac{1}{4}|r_1 - \sigma \bar{r}_2|^2 > 0,$$

which indicates that the 1-order principle minor $1 + \sigma \operatorname{Re}(r_1 r_2) > 0$. Thus the matrix $I + S_H$ is positive definite.

In view of the algebra theory, for a Hermitian matrix, there exists a unitary matrix A such that

$$A^*(I + S_H)A = \operatorname{diag}(\mu_+, \mu_-), \tag{3.10}$$

where $\mu_\pm$ are the eigenvalues of the matrix $I + S_H$,

$$\mu_\pm(z) = \frac{2 + \sigma \operatorname{Re}(r_1 r_2) \pm \sqrt{\operatorname{Re}^2(r_1 r_2) + |r_1 + \sigma \bar{r}_2|^2}}{2}.$$

Noting $\mu_+(z) > \mu_-(z) > 0$ as $I + S_H$ is positive definite. And it follows from $r_{1,2}(z) \to 0$ that $\mu_-(z) \to 1$ as $|z| \to \infty$, $z \in \mathbb{R}$. Together with $r_{1,2}(z) \in H^1(\mathbb{R})$, there exists a positive constant c_- such that $\mu_- > c_-$.

Consequently, for every $g \in \mathbb{C}^2$, utilizing (3.10), we have

$$c_- g^* g < \mu_- g^* g \le \operatorname{Re} g^*(I + S(x;z))g = g^*(I + S_H)g,$$

which completes the proof of the bound (3.7).

Calculating $(I + S(x;z))g$ componentwise and utilizing $|r_{1,2}(z)| < 1$ give that

$$\begin{aligned} \|(I + S(x;z))g\|^2 &\le 2(1 + |r_1| + |r_2|)^2 \|g\|^2 \\ &\quad + 2\operatorname{Re}\{((\sigma + r_1 r_2)\bar{r}_2 + r_1)\mathrm{e}^{-2\mathrm{i}zx} g^{(1)} \overline{g^{(2)}}\} \\ &\le ((|r_1| + 1)^2 + (|r_2| + 1)^2 + (|r_1| + |r_2|)^2)\|g\|^2, \end{aligned} \tag{3.11}$$

here the norm for a 2-component vector f is $\|f\|^2 = |f^{(1)}|^2 + |f^{(2)}|^2$. Therefore, we take

$$c_+ = \sup_{z \in \mathbb{R}} \sqrt{(|r_1| + 1)^2 + (|r_2| + 1)^2 + (|r_1| + |r_2|)^2} < +\infty,$$

then one obtain the bound (3.8).

Lemma 3.2 *Let $r_{1,2}(z) \in H^1(\mathbb{R})$ satisfying $|r_{1,2}(z)| \le 1$. Then for every $F(z) \in L_z^2(\mathbb{R})$, there exists a unique solution $\Psi(z) \in L_z^2(\mathbb{R})$ of the equation*

$$(I - P_S^-)\Psi(z) = F(z), \quad z \in \mathbb{R}, \tag{3.12}$$

where $P_S^-\Psi = P^-(\Psi S)$.

Proof Since $I - P_S^-$ is a Fredholm operator of the index zero, by Fredholm's alternative theorem, there exists a unique solution of (3.12) if and only if the homogeneous equation

$$(I - P_S^-)g = 0, \quad z \in \mathbb{R} \tag{3.13}$$

admits zero solution in $L_z^2(\mathbb{R})$.

Assume that $g(z) \in L_z^2(\mathbb{R})$ and $g(z) \ne 0$ is a solution of (3.13). Define two analytic functions in $\mathbb{C}\backslash\mathbb{R}$,

$$g_1(z) = \mathcal{C}(gS)(z), \quad g_2(z) = \mathcal{C}(gS)^*(z).$$

The functions $g_1(z)$ and $g_2(z)$ are well-defined due to $S(z) \in L_z^2(\mathbb{R}) \cap L_z^\infty(\mathbb{R})$.

We integrate the function $g_1(z)g_2(z)$ along the semi-circle of radius R centered at zero in $\mathbb{C}^+$. It follows from Cauchy theorem that

$$\oint g_1(z)g_2(z)\mathrm{d}z = 0.$$

From $g(z)S(z) \in L_z^1(\mathbb{R})$, we have

$$g_1(z), g_2(z) = \mathcal{O}(z^{-1}), \quad |z| \to \infty.$$

Hence, the integral on the arc approaches to zero as the radius approaches to infinity. Therefore, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}} g_1(z)g_2(z)\mathrm{d}z = \int_R \mathcal{P}^+(gS)[\mathcal{P}^-(gS)]^*\mathrm{d}z \\ &= \int_{\mathbb{R}} [P^-(gS) + gS][\mathcal{P}^-(gS)]^*\mathrm{d}z. \end{aligned} \tag{3.14}$$

Utilizing the assumption $\mathcal{P}^-(gS) = g$, we have

$$\int_R g(I+S)g^*\mathrm{d}z = 0.$$

We get Re $g(I+S)g^* > c_-g^*g$ with c_- being a positive constant. Thus the function $g(z)$ has to be zero. This contradicts to the assumption $g \ne 0$. Therefore, $g = 0$ is a unique solution to the equation $(I - P_S^-)g = 0$ in $L_z^2(\mathbb{R})$. Finally there exists a unique solution to (3.12).

Lemma 3.3 *Let $r_{1,2}(z) \in H^1(\mathbb{R})$ satisfy $|r_{1,2}(z)| \le 1$. Then for every $x \in \mathbb{R}$, there exist unique solutions $\Psi_\pm(x;z) \in L_z^2(\mathbb{R})$ satisfying*

$$\Psi_+(x;z) - \Psi_-(x;z) = \Psi_-S(x;z) + S(x;z), \quad z \in \mathbb{R}.$$

Proof Owing to $S(x;z) \in L^2_z(\mathbb{R})$, we have $\mathcal{P}_{\mathcal{S}}{}^{-}(z) \in L^2_z(\mathbb{R})$ by (3.5). Then for every $x \in \mathbb{R}$, there exists a unique solution $\Psi_-(x;z) \in L^2_z(\mathbb{R})$ satisfying

$$\Psi_-(x;z) = \mathcal{P}^-(\Psi_-(x;z)S(x;z) + S(x;z)), \quad z \in \mathbb{R}. \tag{3.15}$$

Based on the existence of $\Psi_-(x;z)$, we define

$$\Psi_+(x;z) = \mathcal{P}^+(\Psi_-(x;z)S(x;z) + S(x;z)), \quad z \in \mathbb{R}. \tag{3.16}$$

Besides, analytic extensions of $\Psi_\pm(x;z)$ to $z \in \mathbb{C}^\pm$ are defined by Cauchy operator

$$\Psi_\pm(x;z) = \mathcal{C}(\Psi_-(x;z)S(x;z) + S(x;z)), \quad z \in \mathbb{C}^\pm. \tag{3.17}$$

Finally we obtain the solution $\Psi_\pm(x;z) \in L^2_z(\mathbb{R})$. Moreover, given the property of the Cauchy operator and the Plemelj projection operator, the solutions $\Psi_\pm(x;z)$ are analytic functions for $z \in \mathbb{C}^\pm$.

Lemma 3.4 *Let $r_{1,2} \in H^1(\mathbb{R})$ satisfying $|r_{1,2}(z)| \le 1$. Then the operator $(I - \mathcal{P}_S^-)^{-1} : L^2_z(\mathbb{R}) \to L^2_z(\mathbb{R})$ is bounded, that is, there exists a constant c that only depends on $\|r(z)\|_{L^\infty_z}$ such that*

$$\|(I - P_S^-)^{-1} f\|_{L^2_z} \le c\|f\|_{L^2_z}.$$

Proof For every $f(z) \in L^2_z(\mathbb{R})$, there exists a solution $\Psi(z) \in L^2_z(\mathbb{R})$ to the equation

$$(I - \mathcal{P}_S^-)\Psi(z) = f(z).$$

Note that $\mathcal{P}^+ - \mathcal{P}^- = I$, then we decompose the function into $\Psi = \Psi_+ - \Psi_-$ with

$$\Psi_- - \mathcal{P}^-(\Psi_- S) = \mathcal{P}^-(f), \quad \Psi_+ - \mathcal{P}^-(\Psi_+ S) = \mathcal{P}^+(f). \tag{3.18}$$

Since $\mathcal{P}^\pm(f) \in L^2_z(\mathbb{R})$, there exist unique solutions $\Psi_\pm(z) \in L^2_z(\mathbb{R})$ to (3.15) which implies the decomposition is unique. Therefore, we only need the estimates of $\Psi_\pm$ in $L^2_z(\mathbb{R})$.

To deal with Ψ_-, define two analytic functions in $\mathbb{C}\backslash\mathbb{R}$,

$$g_1(z) = \mathcal{C}(\Psi_- S)(z), \quad g_2(z) = \mathcal{C}(\Psi_- S + f)^*(z).$$

Analogous manipulation, we integrate on the semi-circle in the upper half-plane and have

$$\oint g_1(z)g_2(z)\mathrm{d}z = 0.$$

Since $g_1(z) = \mathcal{O}(z^{-1})$ and $g_2(z) \to 0$ as $|z| \to \infty$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \mathcal{P}^+(\Psi_- S)[\mathcal{P}^-(\Psi_- S + f)]^* \mathrm{d}z \\ &= \int_{\mathbb{R}} (\mathcal{P}^-(\Psi_- S) + \Psi_- S)[\mathcal{P}^-(\Psi_- S + f)]^* \mathrm{d}z \\ &= \int_{\mathbb{R}} (\Psi_- - \mathcal{P}^-(f) + \Psi_- S)\Psi_-^* \mathrm{d}z. \end{aligned} \tag{3.19}$$

Using the bounds (3.7)–(3.8) and the Hölder inequality, there exists a positive constant c_- such that

$$c_-\|\Psi_-\|_{L^2}^2 \le \operatorname{Re}\int_{\mathbb{R}} \Psi_-(I+S)\Psi_-^* \mathrm{d}z = \operatorname{Re}\int_{\mathbb{R}} \mathcal{P}^-(f)\Psi_-^* \mathrm{d}z \le \|f\|_{L^2}\|\Psi_-\|_{L^2},$$

which completes the estimates of Ψ_-,

$$\|(I-\mathcal{P}_S^-)^{-1}P^- f\|_{L_z^2} \le c_-^{-1}\|f\|_{L_z^2}. \tag{3.20}$$

To deal with Ψ_+, define two functions in $\mathbb{C}\backslash\mathbb{R}$,

$$g_1(z) = \mathcal{C}(\Psi_+ S)(z), \quad g_2(z) = \mathcal{C}(\Psi_+ S + f)^*(z).$$

Performing the similar procedure as that above leads to

$$\begin{aligned} 0 &= \oint g_1(z) g_2(z)\mathrm{d}z \\ &= \int_{\mathbb{R}} \mathcal{P}^-(\Psi_+ S)[\mathcal{P}^+(\Psi_+ S + f)]^* \mathrm{d}z \\ &= \int_{\mathbb{R}} [\Psi_+ - \mathcal{P}^+(f)][\Psi_+(I+S)]^* \mathrm{d}z, \end{aligned} \tag{3.21}$$

where we have used (3.15). Using the bounds (3.7)–(3.8), there are positive constants c_- and c_+ such that

$$c_-\|\Psi_+\|_{L^2}^2 \le \operatorname{Re}\int_{\mathbb{R}} \Psi_+(I+S)^*\Psi_+^* \mathrm{d}z = \operatorname{Re}\int_{\mathbb{R}} \mathcal{P}^+(f)(I+S)^*\Psi_+^* \mathrm{d}z \le c_+\|f\|_{L^2}\|\Psi_+\|_{L^2},$$

which means

$$\|(I-P_S^-)^{-1}P^+ f\|_{L_z^2} \le c_-^{-1}c_+\|f\|_{L_z^2}. \tag{3.22}$$

Combining (3.20) and (3.22), we obtain

$$\|(I-\mathcal{P}_S^-)^{-1} f\|_{L_z^2} \le c\|f\|_{L_z^2},$$

where c is a constant that only depends on $\|r(z)\|_{L_z^\infty}$.

3.3 Estimate on solutions to the RH problem

Next, we see solutions to the RH problem for $M(x;z)$. Denote the functions $M_\pm(x;z)$ column-wise

$$M_\pm(x;z) = [\mu_\pm(x;z), \nu_\pm(x;z)],$$

then the functions $\Psi_\pm$ can be written as

$$\Psi_\pm(x;z) = [\mu_\pm(x;z) - e_1, \nu_\pm(x;z) - e_2].$$

We have

$$\mu_\pm(x;z) - e_1 = \mathcal{P}^\pm(M_- S)^{(1)}(x;z), \quad z\in\mathbb{R} \tag{3.23}$$

and

$$\nu_\pm(x;z) - e_2 = \mathcal{P}^\pm(M_- S)^{(2)}(x;z), \quad z\in\mathbb{R}. \tag{3.24}$$

Combining (3.23) with (3.24), we obtain

$$M_{\pm}(x;z) = I + \mathcal{P}^{\pm}(M_{-}(x;\cdot)S(x;\cdot))(z), \quad z \in \mathbb{R}.$$

Further, analytic extensions of $M_{\pm}(x;z)$ to $z \in \mathbb{C}^{\pm}$ are

$$M_{\pm}(x;z) = I + \mathcal{C}(M_{-}(x;\cdot)S(x;\cdot))(z), \quad z \in \mathbb{C}^{\pm}. \tag{3.25}$$

Lemma 3.5 *Let $r_{1,2}(z) \in H^1(\mathbb{R})$ satisfying $|r_{1,2}(z)| \le 1$. Then there exists a constant c only depending on $\|r(z)\|_{L^\infty}$ such that for every $x \in \mathbb{R}$,*

$$\|M_{\pm}(x;\cdot) - I\|_{L^2} \le c(\|r_1\|_{L^2} + \|r_1\|_{L^2}). \tag{3.26}$$

Proof Due to $r_{1,2} \in H^1(\mathbb{R})$, we get $r_{1,2} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $S(x;\cdot) \in L^2(k)$. Moreover, there exists a constant c only depending on $\|r_{1,2}\|_{L^\infty}$ such that

$$\|S(x;\cdot)\|_{L^2_z} \le c(\|r_1\|_{L^2} + \|r_1\|_{L^2}),$$

we obtain

$$\|M_{\pm} - I\|_{L^2} = \|\psi_{\pm}\|_{L^2} \le c(\|r_1\|_{L^2} + \|r_1\|_{L^2}),$$

where we have used the equation $(I - \mathcal{P}^-_S)\Psi_{\pm} = \mathcal{P}^- S$, and c is another constant only depending on $\|r_{1,2}(z)\|_{L^\infty}$.

Proposition 3.2 *For every $x_0 \in \mathbb{R}^-$ and every $r_{1,2}(z) \in H^{1,1}(\mathbb{R})$, we have*

$$\sup_{x\in(-\infty,x_0)} \|\langle x\rangle \mathcal{P}^+(r_2 \mathrm{e}^{2\mathrm{i}zx})\|_{L^2_z} \le c\|r_2\|_{H^1}, \tag{3.27}$$

$$\sup_{x\in(-\infty,x_0)} \|\langle x\rangle \mathcal{P}^-(r_1 \mathrm{e}^{-2\mathrm{i}zx})\|_{L^2_z} \le c\|r_1\|_{H^1}, \tag{3.28}$$

$$\sup_{x\in(-\infty,x_0)} \|\langle x\rangle \mathcal{P}^+(z r_2 \mathrm{e}^{2\mathrm{i}zx})\|_{L^2_z} \le c\|r_2\|_{H^{1,1}},$$

$$\sup_{x\in(-\infty,x_0)} \|\langle x\rangle \mathcal{P}^-(z r_1 \mathrm{e}^{-2\mathrm{i}zx})\|_{L^2_z} \le c\|r_1\|_{H^{1,1}},$$

where $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$. Moreover, if $r_{1,2} \in L^{2,3}(\mathbb{R})$, then we have

$$\sup_{x\in\mathbb{R}} \|z^j \mathcal{P}^+(z^k r_2 \mathrm{e}^{2\mathrm{i}zx})\|_{L^2_z} \le \|z^{j+k} r_2\|_{L^2_z}, \tag{3.29}$$

$$\sup_{x\in\mathbb{R}} \|z^j \mathcal{P}^-(z^k r_1 \mathrm{e}^{-2\mathrm{i}zx})\|_{L^2_z} \le \|z^{j+k} r_1\|_{L^2_z}, \tag{3.30}$$

where $j,k = 0,1,2,3, j+k \le 3$ and c is a constant that depends on $\|r_{1,2}\|_{L^\infty}$.

Proof The proof can be completed by an analogous analysis as that in [27].

In order to obtain estimates on the vector columns $\mu_-(x;z) - e_1$ and $\nu_+(x;z) - e_2$ that will be needed in the subsequent section, we rewrite functions $\mu_-(x;z) - e_1$ and $\nu_+(x;z) - e_2$ by (3.25) as

$$\mu_-(x;z) - e_1 = \mathcal{P}^-(r_1 \mathrm{e}^{-2\mathrm{i}zx} \nu_+(x;z))(z), \quad z \in \mathbb{R} \tag{3.31}$$

and

$$\nu_+(x;z)-e_2=\mathcal{P}^+(\sigma r_2 \mathrm{e}^{2\mathrm{i}zx}\mu_-(x;z))(z), \quad z\in\mathbb{R},$$

where we have used the fact

$$M_-S=[\mu_-,\nu_-]\begin{pmatrix}\sigma r_1 r_2 & \sigma r_2 \mathrm{e}^{2\mathrm{i}zx}\\ r_1\mathrm{e}^{-2\mathrm{i}zx} & 0\end{pmatrix}=[r_1\mathrm{e}^{-2\mathrm{i}zx}\nu_+,\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\mu_-]. \tag{3.32}$$

Define a function

$$N(x;z)=[\mu_-(x;z)-e_1,\nu_+(x;z)-e_2],$$

which satisfies

$$N-\mathcal{P}^+(NS_+)-\mathcal{P}^-(NS_-)=F, \tag{3.33}$$

where

$$F(x;z)=[\mathcal{P}^-(r_1\mathrm{e}^{-2\mathrm{i}zx})e_2,\mathcal{P}^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})e_1],$$
$$S_+(x;z)=\begin{pmatrix}0 & \sigma r_2\mathrm{e}^{2\mathrm{i}zx}\\ 0 & 0\end{pmatrix},\quad S_-(x;z)=\begin{pmatrix}0 & 0\\ r_1\mathrm{e}^{-2\mathrm{i}zx} & 0\end{pmatrix}.$$

Lemma 3.6 *Let $r_{1,2}\in H^{1,1}(\mathbb{R})$, then for every $x_0\in\mathbb{R}^-$, we have*

$$\sup_{x\in(-\infty,x_0)}\|\langle x\rangle\mu_-^{(2)}(x;z)\|_{L^2_z}\le c\|r_1\|_{H^1}, \tag{3.34}$$

$$\sup_{x\in(-\infty,x_0)}\|\langle x\rangle\nu_+^{(1)}(x;z)\|_{L^2_z}\le c\|r_2\|_{H^1}, \tag{3.35}$$

$$\sup_{x\in(-\infty,x_0)}\|\langle x\rangle\partial_x\mu_-^{(2)}(x;z)\|_{L^2_z}\le c\|r_1\|_{H^{1,1}}, \tag{3.36}$$

$$\sup_{x\in(-\infty,x_0)}\|\langle x\rangle\partial_x\nu_+^{(1)}(x;z)\|_{L^2_z}\le c\|r_2\|_{H^{1,1}}, \tag{3.37}$$

where c is a constant that only depends on $\|r_{1,2}\|_{L^\infty}$. In addition, if $r\in L^{2,3}$, then we have

$$\sup_{x\in\mathbb{R}}\|z^k\partial_x^j\mu_-^{(2)}(x;z)\|_{L^2_z}\le c\|r_1\|_{L^{2,j+k}}, \tag{3.38}$$

$$\sup_{x\in\mathbb{R}}\|z^k\partial_x^j\nu_+^{(1)}(x;z)\|_{L^2_z}\le c\|r_2\|_{L^{2,j+k}}, \tag{3.39}$$

where $j,k=0,1,2,3, j+k\le 3$ and c is a constant that depends on $\|r_{1,2}\|_{L^\infty}$.

Proof Note $\mathcal{P}^+-\mathcal{P}^-=I$ and $S_++S_-=(I-S_+)S$, then (3.33) can be rewritten as

$$G-\mathcal{P}^-(GS)=F, \tag{3.40}$$

with $G=N(I-S_+)$. And the matrix $G(x;z)$ is written component-wise as

$$G(x;z)=\begin{pmatrix}\mu_-^{(1)}(x;z)-1 & \nu_+^{(1)}-\sigma r_2\mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z)-1)\\ \mu_-^{(2)}(x;z) & \nu_+^{(2)}-1-\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\mu_-^{(2)}(x;z)\end{pmatrix}.$$

Comparing the second row of $F(x;z)$ with $G(x;z)$ and utilizing the bound (3.28), we have

$$\begin{aligned}&\sup_{x\in(-\infty,x_0)}\|\langle x\rangle\mu_-^{(2)}\|_{L^2_z}\le c\sup_{x\in(-\infty,x_0)}\|\langle x\rangle\mathcal{P}^-(r_1\mathrm{e}^{-2\mathrm{i}zx})\|_{L^2_z},\\ &\|\nu_+^{(2)}-1-r_1\mathrm{e}^{-2\mathrm{i}zx}\mu_-^{(2)}(x;z)\|_{L^2_z}\le c\|\mathcal{P}^-(r_1\mathrm{e}^{-2\mathrm{i}zx})\|_{L^2_z},\end{aligned} \tag{3.41}$$

where c is a constant that depends on $\|r\|_{L^\infty}$. Substituting the bound (3.28) into (3.41), we obtain the estimate (3.34).

Similarly, comparing the first row of $F(x;z)$ and $G(x;z)$ yields

$$\begin{aligned}&\|\mu_-^{(1)}(x;z)-1\|_{L_z^2}\le c\|\mathcal{P}^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L_z^2},\\&\|\nu_+^{(1)}(x;z)-\sigma r_2\mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z)-1)\|_{L_z^2}\le c\|\mathcal{P}^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L_z^2}.\end{aligned}\tag{3.42}$$

Taking derivative in x of (3.33), we obtain

$$\partial_x N-\mathcal{P}^+(\partial_x N)S_+-\mathcal{P}^-(\partial_x N)S_-=F_1\tag{3.43}$$

with

$$\begin{aligned}F_1=&\partial_x F+\mathcal{P}^+N\partial_x S_++\mathcal{P}^-N\partial_x S_-\\=&2\mathrm{i}[e_2\mathcal{P}^-(-zr_1\mathrm{e}^{-2\mathrm{i}zx}),e_1\mathcal{P}^+(z\sigma r_2\mathrm{e}^{2\mathrm{i}zx})]\\&+2\mathrm{i}\begin{pmatrix}\mathcal{P}^-(-zr_1(z)\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(1)}(x;z)) & \mathcal{P}^+(z\sigma r_2\mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z)-1))\\ \mathcal{P}^-(-zr_1\mathrm{e}^{-2\mathrm{i}zx}(\nu_+^{(2)}(x;z)-1)) & \mathcal{P}^+(z\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\mu_-^{(2)}(x;z))\end{pmatrix}.\end{aligned}$$

Using the estimates (3.29)–(3.30), we obtain

$$\begin{aligned}&z(\mu_-(x;z)-e_1)\in L_x^\infty((-\infty,x_0);L_z^2(\mathbb{R})),\\&z(\nu_+(x;z)-e_2)\in L_x^\infty((-\infty,x_0);L_z^2(\mathbb{R})).\end{aligned}$$

On account of the bounds (3.27)–(3.28) and $r_{1,2}(z)\in L^\infty(\mathbb{R})$, we conclude that F_1 belongs to $L_x^\infty((-\infty,x_0);L_z^2(\mathbb{R}))$. By (3.34)–(3.35) and $r_{1,2}(z)\in L^\infty(\mathbb{R})$, we conclude that

$$F_1\in L_x^\infty((-\infty,x_0);L_z^2(\mathbb{R})),$$

which gives (3.36)–(3.37).

Taking j-order derivative of (3.33), we obtain

$$\partial_x^j N-\mathcal{P}^+(\partial_x^j N)S_+-\mathcal{P}^-(\partial_x^j N)S_-=F_j\tag{3.44}$$

with

$$F_j=\partial_x F_{j-1}+\mathcal{P}^+\partial_x^{j-1}N_x\partial_x S_++\mathcal{P}^-\partial_x^{j-1}N_x\partial_x S_-,\quad j=2,3.$$

Repeating the analysis for (3.40) and using (3.29)–(3.30), we derive the estimates (3.38)–(3.39).

4 Reconstruction and Estimates of the Potential

We shall now recover the potential u from the matrices $M_\pm(x;z)$, which satisfy the integral equations (3.25). This will give us the map

$$H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})\ni r_{1,2}\mapsto u\in H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R}).$$

4.1 Estimates on the negative half-line

It follows from (3.3) that

$$u(x)=2\mathrm{i}\lim_{z\to\infty}(zM_{\pm}(x;z))_{12}, \tag{4.1}$$

which can be used to get estimates of u on the negative half-line. Further from (3.25) and (3.3), we have

$$u(x)=2\mathrm{i}\lim_{|z|\to\infty}z\mathcal{C}((M_-S)_{12}). \tag{4.2}$$

Since $r\in H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})$, we have $S(x;\cdot)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Besides, the estimate (3.28) implies that

$$M_-(x;\cdot)-I\in L^2(\mathbb{R}).$$

Therefore, we arrive at $L^1(\mathbb{R})$. Subsequently, applying (3.6) to (4.2), we obtain

$$\begin{aligned}u(x)&=\frac{1}{\pi}\int_{\mathbb{R}}\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\mu_-^{(1)}(x;z)\mathrm{d}z\\&=\frac{1}{\pi}\int_{\mathbb{R}}\sigma r_2\mathrm{e}^{2\mathrm{i}zx}P^-(r\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(1)})\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\mathrm{d}z+\frac{1}{\pi}\int_R\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\mathrm{d}z,\end{aligned} \tag{4.3}$$

where we have used the identity

$$\mu_-^{(1)}(x;z)-1=P^-(r_1\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(1)}).$$

Lemma 4.1 *Let $r_{1,2}(z)\in H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})$. Then $u\in H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})$, moreover,*

$$\|u\|_{H^{1,1}(\mathbb{R}^-)\cap H^3(\mathbb{R}^-)}\le c(\|r_1\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})}+\|r_2\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})}), \tag{4.4}$$

where c is a constant that depends on $\|r_{1,2}\|_{L^\infty}$ and $\|zr_{1,2}\|_{L^\infty}$.

Proof For a function $r_{1,2}(z)\in L^2(\mathbb{R})$, by Parseval's equation,

$$\|r_{1,2}\|_{L^2}=\|\widehat{r}_{1,2}\|_{L^2},$$

where the function $\widehat{r}_{1,2}$ denotes the Fourier transform. Since $L^{2,3}(\mathbb{R})\cap H^{1,1}(\mathbb{R})$, the second term of (4.3) belongs to $H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})$ due to the property $\widehat{\partial_z r_{1,2}(z)}=x\widehat{r}_{1,2}(x)$.

Let

$$I(x)=\int_{\mathbb{R}}\sigma r_2\mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z)-1)\mathrm{d}z. \tag{4.5}$$

Substituting (3.32) into (4.5) and applying the Fubini's theorem yields

$$\begin{aligned}I(x)&=\int_{\mathbb{R}}\sigma r_2\mathrm{e}^{2\mathrm{i}zx}\lim_{\varepsilon\to0}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{r_1(s)\mathrm{e}^{-2\mathrm{i}sx}\nu_+^{(1)}(s)}{s-(z-i\varepsilon)}\mathrm{d}s\mathrm{d}z\\&=-\int_{\mathbb{R}}r_1(s)\mathrm{e}^{-2\mathrm{i}sx}\nu_+^{(1)}(s)\lim_{\varepsilon\to0}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\sigma r_2\mathrm{e}^{2\mathrm{i}zx}}{z-(s+i\varepsilon)}\mathrm{d}z\mathrm{d}s\\&=-\int_{\mathbb{R}}r_1(z)\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(1)}(z)P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}sx})(z)\mathrm{d}z.\end{aligned}$$

Therefore, for every $x_0 \in \mathbb{R}^-$, utilizing the Hölder's inequality and the estimates (3.27) and (3.35), we find

$$\sup_{x\in(-\infty,x_0)} |\langle x\rangle^2 I(x)| \le \|r_1\|_{L^\infty} \sup_{x\in(-\infty,x_0)} \|\langle x\rangle \nu_+^{(1)}\|_{L^2} \\ \times \sup_{x\in(-\infty,x_0)} \|\langle x\rangle P^+(\sigma r_2 \mathrm{e}^{2\mathrm{i}sx})\|_{L^2} \le c\|r_1\|_{H^1}\|r_2\|_{H^1}, \tag{4.6}$$

where c is a constant that only depends on $\|r\|_{L^\infty}$. Further, we obtain

$$\|\langle x\rangle I(x)\|_{L^2(\mathbb{R}^-)} \le c\|r_1\|_{H^1}\|r_2\|_{H^1},$$

where c is another constant that only depends on $\|r\|_{L^\infty}$. Combining the results of the two terms of (4.1) leads to

$$\|u(x)\|_{L^{2,1}(\mathbb{R}^-)} \le c(1 + \|r_1\|_{H^1} + \|r_2\|_{H^1})(\|r_1\|_{H^1} + \|r_2\|_{H^1}). \tag{4.7}$$

This completes the proof of $u \in L^{2,1}(\mathbb{R}^-)$.

By the Fourier theory, the derivative of the second term of (4.1) belongs to $L^2(\mathbb{R})$. For the second term $I(x)$, we differentiate $I(x)$ in x and obtain

$$\begin{aligned} I'(x) &= \partial \int_{\mathbb{R}} \sigma r_2 \mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z) - 1)\mathrm{d}z \\ &= -2\mathrm{i}\int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(1)}(x;z)P^+(s\sigma r_2\mathrm{e}^{2\mathrm{i}sx})(z)\mathrm{d}z \\ &\quad - 2\mathrm{i}\int_{\mathbb{R}} zr_1(z)\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(1)}(x;z)P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}sx})(z)\mathrm{d}z \\ &\quad - \int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}\partial_x\nu_+^{(1)}(x;z)P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}sx})(z)\mathrm{d}z, \end{aligned}$$

where we have used (3.31) and the Fubini's theorem.

Utilizing the estimates (3.27) and (3.35), we find that for every $x_0 \in \mathbb{R}^-$,

$$\begin{aligned} \sup_{x\in(-\infty,x_0)} |\langle x\rangle^2 I'(x)| &\le \|r\|_{L^\infty} \sup_{x\in(x_0,+\infty)} (2\|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(z\overline{r}\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} \\ &\quad + \|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(z\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} \\ &\quad + \|\langle x\rangle\partial_x\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}) \\ &\le c\|r_1\|_{H^{1,1}\cap L^{2,3}}\|r_2\|_{H^{1,1}\cap L^{2,3}}, \end{aligned}$$

which implies that

$$\|\langle x\rangle I'(x)\|_{L^2(\mathbb{R}^-)} \le c\|r_1\|_{H^{1,1}\cap L^{2,3}}\|r_2\|_{H^{1,1}\cap L^{2,3}}, \tag{4.8}$$

where c is another constant that depends on $\|r\|_{L^\infty}$. Subsequently, we obtain $I'(x) \in L^{2,1}(\mathbb{R}^-)$. We conclude that $u \in H^{1,1}(\mathbb{R}^-)$. The estimate (4.4) can be obtained from (4.7)–(4.8) with another constant c depending on $\|r\|_{L^\infty}$.

For $I''(x)$, we get

$$\sup_{x\in(-\infty,x_0)} |\langle x\rangle I''(x)|$$

$$\le \|r_1\|_{L^\infty} \sup_{x\in(x_0,+\infty)} (4\|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|z^2P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ \|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(z^2\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + \|\partial_x^2\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ 2\|z\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(z\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + \|\langle x\rangle\partial_x\nu_+^{(1)}\|_{L^2}\|zP^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ 2\|\partial_x\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(s\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}) \le c\|r_1\|_{H^{1,1}\cap L^{2,3}}\|r_2\|_{H^{1,1}\cap L^{2,3}}.$$

For $I'''(x)$, similar to the above analysis, we get

$$\sup_{x\in(-\infty,x_0)} |\langle x\rangle I'''(x)|$$
$$\le \|r_1\|_{L^\infty} \sup_{x\in(x_0,+\infty)} (8\|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|P^+(z^3\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ \|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|z^3P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + \|\partial_x^3\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ 4\|\langle x\rangle\partial_x\nu_+^{(1)}\|_{L^2}\|P^+(s^2\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + 2\|\langle x\rangle\partial_x^2\nu_+^{(1)}\|_{L^2}\|P^+(z^2\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ 2\|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|z^2P^+(s\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + 4\|\langle x\rangle\nu_+^{(1)}\|_{L^2}\|zP^+(z^2\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ \|\partial_x\nu_+^{(1)}\|_{L^2}\|\langle x\rangle s^2P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + \|z\partial_x^2\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2}$$
$$+ 4\|\langle x\rangle\nu_+^{(1)}\|_{L_z^2}\|P^+(s^2\sigma r_2\mathrm{e}^{2\mathrm{i}zx})\|_{L^2} + 2\|z\partial_x\nu_+^{(1)}\|_{L^2}\|\langle x\rangle P^+(z\overline{r}\mathrm{e}^{2\mathrm{i}zx})\|_{L^2})$$
$$\le c\|r_1\|_{H^{1,1}\cap L^{2,3}}\|r_2\|_{H^{1,1}\cap L^{2,3}}.$$

Then we conclude that $u \in H^3(\mathbb{R}^-)$. Finally we prove the conclusion.

We actually get the following map through the above analysis:

$$H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R}) \ni r_{1,2} \mapsto u \in H^3(\mathbb{R}^-) \cap H^{1,1}(\mathbb{R}^-).$$

We will prove that the map is Lipschitz continuous.

Lemma 4.2 *Let $r \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$, then the mapping*

$$H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R}) \ni r_{1,2} \mapsto u \in H^3(\mathbb{R}^-) \cap H^{1,1}(\mathbb{R}^-)$$

is Lipschitz continuous.

Proof Let $r_{1,2}, \widetilde{r}_{1,2} \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$. Let the functions u and $\widetilde{u}$ are the corresponding potentials respectively. We will show that there exists a constant c that depends on $\|r_{1,2}\|_{L^\infty}$ such that

$$\|u - \widetilde{u}\|_{H^3(\mathbb{R}^-)\cap H^{1,1}(\mathbb{R}^-)} \le c(\|r_1 - \widetilde{r}_1\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})} + \|r_2 - \widetilde{r}_2\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})}). \tag{4.9}$$

From (4.1), we have

$$u - \widetilde{u} = \frac{1}{\pi}\int_{\mathbb{R}} (\sigma r_2 - \sigma\widetilde{r}_2)\mathrm{e}^{2\mathrm{i}zx}\mathrm{d}z + \frac{1}{\pi}\int_{\mathbb{R}} (\sigma r_2 - \sigma\widetilde{r}_2)\mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z) - 1)\mathrm{d}z$$
$$+ \frac{1}{\pi}\int_R \sigma r_2\mathrm{e}^{2\mathrm{i}zx}(\mu_-^{(1)}(x;z) - \widetilde{\mu}_-^{(1)}(x;z))\mathrm{d}z.$$

Repeating the analysis in the proof of Lemma 4.1, we obtain the Lipschitz continuity of u.

4.2 Estimates on the positive half-line

Recalling (3.3) again, we can get

$$u(-x) = 2\mathrm{i}z\sigma \lim_{z\to\infty} (M_{\pm}(x;z))_{21}. \tag{4.10}$$

Performing the same manipulation for (4.1) yields

$$\begin{aligned} u(-x) &= -\frac{\sigma}{\pi}\int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}(\nu_-^{(2)}(x;z) + \sigma r_2(z)\mathrm{e}^{2\mathrm{i}kx}\mu_-^{(2)}(x;z))\mathrm{d}z \\ &= -\frac{\sigma}{\pi}\int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}\nu_+^{(2)}(x;z)\mathrm{d}z, \end{aligned} \tag{4.11}$$

where we have used the identity $\nu_+^{(2)} = \sigma r_2(z)\mathrm{e}^{2\mathrm{i}zx}\mu_-^{(2)} + \nu_-^{(2)}$.

Similar to condition $x_0 \in \mathbb{R}^-$, we summarize the above analysis as the following lemma.

Lemma 4.3 *Let $r_{1,2}(z) \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ satisfy $|r_{1,2}(z)| < 1$, then $u \in H^{1,1}(\mathbb{R}^+) \cap L^{2,3}(\mathbb{R}^+)$ with the following estimate*

$$\|u\|_{H^{1,1}(\mathbb{R}^+)\cap L^{2,3}(\mathbb{R}^+)} \le c(\|r_1\|_{H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})} + \|r_2\|_{H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})}), \tag{4.12}$$

where c is a constant that depends on $\|r_{1,2}\|_{L^\infty}$ and $\|zr_{1,2}\|_{L^\infty}$.

Proof We rewrite (4.2) as

$$u(-x) = -\frac{\sigma}{\pi}\int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}\mathrm{d}z - \frac{\sigma}{\pi}\int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}(\nu_+^{(2)}(x;z) - 1)\mathrm{d}z.$$

Let

$$\widehat{r}_1(-x) = \int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}z(-x)}\mathrm{d}z. \tag{4.13}$$

According to the Fourier theory, we have $-x\widehat{r}_1(-x) = \widehat{\partial_z r_1(z)}(x)$ and $\|x\widehat{r}_1(-x)\|_{L^2(\mathbb{R})} = \|\partial_z r_1(z)\|_{L^2(\mathbb{R})}$. Let

$$I_2(x) = -\frac{\sigma}{\pi}\int_{\mathbb{R}} r_1(z)\mathrm{e}^{-2\mathrm{i}zx}(\nu_+^{(2)}(x;z) - 1)\mathrm{d}z.$$

Repeating the analysis in the proof of Lemma 4.1, we obtain $u \in H^{1,1}(\mathbb{R}^+) \cap L^{2,3}(\mathbb{R}^+)$ and

$$\|u\|_{H^{1,1}(\mathbb{R}^+)\cap L^{2,3}(\mathbb{R}^+} \le c(\|r_1\|_{H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})} + \|r_2\|_{H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})}), \tag{4.14}$$

where c is a constant that depends on $\|r_{1,2}\|_{L^\infty}$ and $\|zr_{1,2}\|_{L^\infty}$.

Lemma 4.4 *Let $r_{1,2}(z) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$, then the following map*

$$H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R}) \ni r_{1,2} \mapsto u \in H^3(\mathbb{R}^+) \cap H^{1,1}(\mathbb{R}^+)$$

is Lipschitz continuous.

Summarize the results from Lemmas 4.2–4.4, we have the following proposition.

Proposition 4.1 *Let $r_{1,2}(z) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$, then we have $u \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ and*

$$\|u\|_{H^3(\mathbb{R})\cap H^{1,1}(\mathbb{R})} \le c(\|r_1\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})} + \|r_2\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})}).$$

Moreover, the mapping

$$H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R}) \ni r_{1,2} \mapsto u \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$$

is Lipschitz continuous.

5 Global Existence and Lipschitz Continuity

5.1 Time evolution of scattering data

From Sections 2–4, for the initial data $u_0 \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, we only consider the spatial spectral problem (2.1) and obtain its unique normalized solution

$$m_1^{\pm}(x,0;z) \to e_1, \quad m_2^{\pm}(x,0;z) \to e_2, \quad x \to \pm\infty, \tag{5.1}$$

which cannot satisfy the time spectral problem (2.2) since they are short of a function about the time t. For every $t \in [0,T]$, we define the normalized Jost functions of the Lax pair (2.1) and (2.2):

$$m_1^{\pm}(x,t;z) = m_1^{\pm}(x,0;z)\mathrm{e}^{4\mathrm{i}z^3 t}, \tag{5.2}$$
$$m_2^{\pm}(x,t;z) = m_2^{\pm}(x,0;z)\mathrm{e}^{-4\mathrm{i}z^3 t} \tag{5.3}$$

with the potential $u(\cdot,0) \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. It follows that for every $t \in [0,T]$, we have

$$m_1^{\pm}(x,t;z) \to \mathrm{e}^{4\mathrm{i}z^3 t} e_1, \quad x \to \pm\infty,$$

$$m_2^{\pm}(x,t;z) \to \mathrm{e}^{-4\mathrm{i}z^3 t} e_2, \quad x \to \pm\infty.$$

Repeating the analysis as the proof of Lemma2.1, we prove that there exist unique solution of the Volttera's integral equations for Jost functions $m_1^{\pm}(x,t;z)$ and $m_2^{\pm}(x,t;z)$, where the Jost functions $m_1^{\pm}(x,t;z)$ and $m_2^{\pm}(x,t;z)$ admit the same analytic property as $m_1^{\pm}(x,0;z)$ and $m_2^{\pm}(x,0;z)$. As well, for every $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ and every $z \in \mathbb{R}$, the Jost functions $m_1^{\pm}(x,t;z)$ and $m_2^{\pm}(x,t;z)$ are supposed to satisfy the scattering relation

$$\begin{aligned} m_1^{+}(x,t;z) &= a(t;z)m_1^{-}(t,x;z) + b(t;z)\mathrm{e}^{-2\mathrm{i}zx}m_2^{-}(t,x;z), \\ m_2^{+}(x,t;z) &= c(t;z)\mathrm{e}^{2\mathrm{i}zx}m_1^{-}(t,x;z) + d(t;z)m_2^{-}(t,x;z). \end{aligned}$$

By the Crammer's law and the evolution relation (5.2)–(5.3), we obtain the evolution of the scattering coefficients

$$\begin{aligned} a(t;z) &= W(m_1^{+}(0,0;z)\mathrm{e}^{4\mathrm{i}z^3 t}, m_2^{-}(0,0;z)\mathrm{e}^{-4\mathrm{i}z^3 t}) = a(0;z), \\ d(t;z) &= W(m_1^{-}(0,0;z)\mathrm{e}^{4\mathrm{i}z^3 t}, m_2^{+}(0,0;z)\mathrm{e}^{-4\mathrm{i}z^3 t}) = d(0;z), \\ b(t;z) &= W(m_1^{-}(0,0;z)\mathrm{e}^{-4\mathrm{i}z^3 t}, m_1^{+}(0,0;z)\mathrm{e}^{-4\mathrm{i}z^3 t}) = b(0;z)\mathrm{e}^{8\mathrm{i}z^3 t}. \end{aligned}$$

Direct calculation shows that the reflection coefficients are given by

$$r_1(t;z) = \frac{b(t;z)}{a(t;z)} = \frac{b(0;z)}{a(0;z)}\mathrm{e}^{8\mathrm{i}z^3 t} = r_1(0;z)\mathrm{e}^{8\mathrm{i}z^3 t}, \tag{5.4}$$

$$r_2(t;z) = \frac{\bar{b}(t;-z)}{d(t;z)} = \frac{\bar{b}(0;-z)}{d(0;z)}\mathrm{e}^{8\mathrm{i}z^3 t} = r_2(0;z)\mathrm{e}^{-8\mathrm{i}z^3 t}, \tag{5.5}$$

where $r_{1,2}(0;z)$ are initial reflection data founded from the initial data $u(x,0)$.

Proposition 5.1 *If $r_{1,2}(0;z) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$, then for any fixed $T > 0$ and $t \in [0,T]$, we have $r(\cdot;z) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$.*

Proof By (5.5), we obtain

$$\|r_1(t;\cdot)\|_{L^{2,3}(\mathbb{R})} = \|r_1(0;\cdot)\|_{L^{2,3}(\mathbb{R})}.$$

For every $t \in [0,T]$, we have

$$\begin{aligned}\|z\partial_z r_1(t;\cdot)\|_{L^2(\mathbb{R})} =& \|z\partial_z r_1(0;\cdot) + 24\mathrm{i}z^3 t r_1(t;z)\|_{L^2(\mathbb{R})} \\ \le& \|z\partial_z r_1(0;\cdot)\|_{L^2(\mathbb{R})} + 24T\|r_1(0;\cdot)\|_{L^{2,3}(\mathbb{R})}.\end{aligned} \tag{5.6}$$

Therefore, we infer that $r_1(t;\cdot) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$ for every $t \in [0,T]$ as $r_1(0;\cdot) \in H^{1,1}(\mathbb{R}) \cap L^{2,3}(\mathbb{R})$. We can get similar conclusion for $t \in [-T,0]$ and r_2.

Using the time-dependent data $r_{1,2}(t;z)$ we can construct a time-dependent RH-problem

Problem 5.1 Find a matrix function $M(t,x;z)$ satisfying
(i) $M(t,x;z) \to I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
(ii) For $M(t,x;z)$, we have the following jump condition

$$M_+(t,x;z) = M_-(t,x;z)V_{x,t}(z), \tag{5.7}$$

where

$$V_{x,t}(z) := \begin{pmatrix} 1+\sigma r_1 r_2 & \sigma r_2 \mathrm{e}^{2\mathrm{i}\theta(x,t;z)} \\ r_1 \mathrm{e}^{-2\mathrm{i}\theta(x,t;z)} & 1 \end{pmatrix}, \quad z \in \mathbb{R} \tag{5.8}$$

and $\theta(t,x;z) = zx + 4z^3 t$.

Proposition 5.2 *Assume that $M(t,x;z)$ is the solution of RH Problem* 5.1. *Then $M(t,x;z)$ satisfies the following system of linear differential equations*:

$$\begin{aligned}&M_x(t,x;z) = \mathrm{i}z[\sigma_3, M] + QM, \\ &M_t(t,x;z) = 4\mathrm{i}z^3[\sigma_3, M] + (4z^2 Q - 2\mathrm{i}z(Q_x - Q^2)\sigma_3 + 2Q^3 - Q_{xx})M,\end{aligned}$$

where

$$Q(t,x) = \begin{bmatrix} 0 & u(t,x) \\ -\sigma u(-t,-x) & 0 \end{bmatrix}$$

and

$$u(t,x) = 2\mathrm{i} \lim_{z\to\infty} zM_{12}(t,x;z). \tag{5.9}$$

Proof Define

$$LM := M_x - \mathrm{i}z[\sigma_3, M] - QM, \tag{5.10}$$

$$NM := M_t - 4\mathrm{i}z^3[\sigma_3, M] - (4z^2 Q - 2\mathrm{i}z(Q_x - Q^2)\sigma_3 + 2Q^3 - Q_{xx})M, \tag{5.11}$$

then direct calculation shows that

$$(LM)_+ = (LM)_- V_{x,t}, \quad (NM)_+ = (NM)_- V_{x,t}. \tag{5.12}$$

Substituting the asymptotic expansion

$$M(x,t;z) = I + \frac{M_1}{z} + \frac{M_2}{z^2} + \cdots + \frac{M_n}{z^n} + \cdots$$

into (5.10)–(5.11) and using (5.9), we obtain

$$LM \sim \mathcal{O}(z^{-1}),\ NM \sim \mathcal{O}(z^{-1}). \tag{5.13}$$

The equations (5.12)–(5.13) imply that both LM and NM satisfy the homogeneous RH Problem 5.1. The uniqueness of solution of the RH Problem yields $LM = NM = 0$.

So under the time evolution of the scattering data $r(t; z)$ in (5.5), the function reconstructed from the Problem 3.1 through the reconstruction formula (4.1) under time-dependent scattering data $r(t; z)$ is still a solution of the the nonlocal mkdV equation (1.1).

5.2 The proof of main results

In this section, we will prove the existence of the local and global solutions to the Cauchy problem. The scheme behind the proof can be described as below.

Lemma 5.1 *Let the initial data* $u_0(x) \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$*, then there exists a unique local solution to the Cauchy problem* (1.1)–(1.2)*,*

$$u \in C([0, T], H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})).$$

Furthermore, the map

$$H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \ni u_0 \mapsto u \in C([0, T], \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}))$$

is Lipschitz continuous.

Proof Performing a similar analysis as that in Lemmas 4.2 and 4.4, we can establish an RH problem for $r(t; z)$ for every $t \in [0, T]$ and address the existence and uniqueness of a solution to the RH problem. Further, the potential $u(t, x)$ can be recovered from the reflection coefficients $r(t; z)$. Moreover, the potential $u(t, \cdot) \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ for every $t \in [0, T]$ and is Lipschitz continuous of $r(t; z)$. Thus we have

$$\begin{aligned} \|u(t;\cdot)\|_{H^{1,1}\cap H^3} &\le c_1\|r(t;\cdot)\|_{H^{1,1}\cap L^{2,3}} \\ &\le c_2 r(0;\cdot)\|_{H^{1,1}\cap L^{2,3}} \le c_3\|u_0\|_{H^{1,1}\cap H^3}, \end{aligned} \tag{5.14}$$

where the positive constant c_1,c_2 and c_3 depend on $\|r\|_{L^\infty}$, $\|zr\|_{L^\infty}$ and $(T,\ \|u_0\|_{H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})})$, respectively.

Next we show that $u(x, t)$ is continuous with respect to $t \in [0, T]$ under the $H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})$ norm. Let $t \in [0, T]$ and $|\Delta t| < 1$ such that $t + \Delta t \in [0, T]$, then with the Lipschitz continuity from $u(t, x)$ to $r(t; z)$ in Proposition 4.1, we have

$$\begin{aligned} &\|u(t+\Delta t, x) - u(t, x)\|_{H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})} \\ &\le c(\|r_1(t+\Delta t; z) - r_1(t; z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})} + \|r_2(t+\Delta t; z) - r_2(t; z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})}) \\ &\le c|\Delta t|(\|r_1(0; z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})} + \|r_2(0; z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,3}(\mathbb{R})}) \le c|\Delta t| \to 0, \quad \Delta t \to 0, \end{aligned}$$

which together with the estimate (5.14) implies that there exists a unique local solution $u(x, t) \in C([0, T], H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R}))$ to the Cauchy problem (1.1)–(1.2) and the map

$$H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R}) \ni u_0(x) \mapsto u(t, x) \in C([0, T], H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R}))$$

is Lipschitz continuous.

Finally we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Suppose that the maximal time in which the local solution in Lemma 5.1 exists is $T_{\max}$.

If $T_{\max} = \infty$, then the local solution is global one.

If the local solution exists in the closed interval $[0, T_{\max}]$, we can use $u(T_{\max}, \cdot) \in H^3(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ as a new initial data. By a similar analysis as that in the previous sections, there exists a positive constant T_1 such that the solution $u \in C([T_{\max}, T_{\max} + T_1], H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R}))$ exists. This contradicts with the maximal time assumption.

If the local solution exists in the open interval $[0, T_{\max})$. According to (5.14), we have

$$\|u(t,x)\|_{H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})} \leq c_3(T_{\max})\|u_0\|_{H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})}, \quad t \in [0, T_{\max}).$$

Due to the continuity of $u(t,x)$ to the time t, the limit of $u(t,x)$ as t approaches to $T_{\max}$ exists. Let $u_{\max}(x) := \lim\limits_{t \to T_{\max}} u(t,x)$. Taking the limit by $t \to T_{\max}$ in (5.14), we have

$$\|u_{\max}\|_{H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})} \leq c_3(T_{\max})\|u_0\|_{H^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})},$$

which implies that we can extend the local solution $u \in C([0, T_{\max}), H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R}))$ to $u \in C([0, T_{\max}], H^{1,1}(\mathbb{R}) \cap H^3(\mathbb{R}))$, this contradicts with the premise that $[0, T_{\max})$ is the maximal open interval.

Data Availability Statements

The data that supports the findings of this study are available within the article.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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