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Remarks on the Global Existence for Incompressible Navier-Stokes Equations*

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Abstract In this article, the authors use the special structure of helicity for the three-dimensional incompressible Navier-Stokes equations to construct a family of finite energy smooth solutions to the Navier-Stokes equations which critical norms can be arbitrarily large.

Keywords Navier-Stokes euqations, Helicity, Global existence, Critical norm
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1 Introduction

Whether the solutions to three-dimensional incompressible Navier-Stokes equations (NSE for short) can develop finite time singularities from regular initial data remains a question of central importance in the theory of partial differential equations. This problem is also called the Millennium Prize problems by Clay Mathematics Institute. The only known coercive a priori estimate is the Leray-Hopf energy estimate which implies that the three-dimensional Navier-Stokes equations are supercritical with respect to its natural scalings. The latter may capture the essence of difficulties of this long standing open problem.

Here, we recall the incompressible Navier-Stokes equations in three dimensions are

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u, \\ \nabla \cdot u = 0, \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{NSE}$$

where u is the velocity field of the fluid, p is the scalar pressure. To solve the NSE in $\mathbb{R}_+ \times \mathbb{R}^3$, one assumes that the initial datum

$$u(0,x) = u_0(x)$$

is divergence-free and possesses certain regularity.

As well-known, if (u,p) solves NSE, so does (u^λ, p^λ) for any $\lambda > 0$, where

$$u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad p^\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x). \tag{1.1}$$

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From above scalings, we usually assign each x_i a positive dimension 1, t a positive dimension 2, u a negative dimension -1 and p a negative dimension -2.

In fact, the known a priori Leray-Hopf energy estimate satisfied by classical solutions of NSE is as follows

$$\sup_{t>0} \|u(t,\cdot)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \int_0^\infty \|\nabla u(t,\cdot)\|_{L^2} \mathrm{d}t \leq \|u_0\|_{L^2}^2. \tag{1.2}$$

By the standard dimensional analysis, we show that all energy norms in (1.2) have positive dimensions, and thus the Navier-Stokes equations are supercritical with respect to the natural scalings (1.1).

In addition, under the natural scalings (1.1), we know the critical space as follows

$$\dot{H}^{\frac{1}{2}}\left(\mathbb{R}^3\right) \hookrightarrow L^3\left(\mathbb{R}^3\right) \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{3}{p}}\left(\mathbb{R}^3\right) \hookrightarrow BMO^{-1}\left(\mathbb{R}^3\right) \hookrightarrow \dot{B}_{\infty,\infty}^{-1}\left(\mathbb{R}^3\right), \tag{1.3}$$

where $p \geq 3$. And the existence of global-in-time smooth solutions arising from small initial data in this functional spaces has been established up to $BMO^{-1}\left(\mathbb{R}^3\right)$ (some details can be seen in [2, 4–7, 10]. All these results are obtained by looking at fixed points of the functional

$$u = \mathrm{e}^{t\Delta} u_0 - \int_0^t \mathrm{e}^{(t-s)\Delta} \mathbb{P}\left(u \cdot \nabla u\right) \mathrm{d}s, \tag{1.4}$$

which is an integral reformulation of the differential problem of NSE, where $\mathrm{e}^{t\Delta}$ denotes the heat kernel and $\mathbb{P}$ is the projection on the divergence-free vector field subspace. It is important to point out that the space $\mathrm{BMO}^{-1}\left(\mathbb{R}^3\right)$ is actually the largest scaling invariant critical space for the Navier-Stokes equations. However, the Navier-Stokes equations are ill-posed in $\dot{B}_{\infty,\infty}^{-1}\left(\mathbb{R}^3\right)$ as shown in [1].

For the three-dimensional incompressible Navier-Stokes equations, the most important quantity is the vorticity

$$\omega := \nabla \times u. \tag{1.5}$$

Applying the curl operator for NSE, we can eliminate nonlocal term pressure p to obtain the equations for vorticity

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega. \tag{1.6}$$

From (1.2), we know that the energy is supercritical, but we can find a quantity called helicity

$$H(u) := \frac{1}{2} \int_{\mathbb{R}^3} u \cdot \omega \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \nabla u \cdot \nabla \omega \mathrm{d}x \mathrm{d}s$$

being critical and conserved. Here, we recall some structure of helicity developed in the paper of [8].

Noting

$$\langle \nabla \times u, v \rangle_{L^2(\mathbb{R}^3)} = \langle u, \nabla \times v \rangle_{L^2(\mathbb{R}^3)}, \tag{1.7}$$

we know that the curl operator is a symmetric operator. So it spectral is real. If $\nabla \cdot u = 0$, its zero spectrum projection is zero. Let u_+ be the projection to positive spectrum, u_- be the projection to negative spectrum, then

$$\begin{aligned} \nabla \times u_+ &= \Lambda u_+, \\ \nabla \times u_- &= -\Lambda u_-, \end{aligned}$$

where $\Lambda = \sqrt{-\Delta}$ and $u = u_+ + u_-$.

To study the regularity of three-dimensional incompressible Navier-Stokes equations, we define the following energy

$$E_c\left(u(t)\right) := \frac{1}{2}\|\Lambda^{\frac{1}{2}} u(t)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla \Lambda^{\frac{1}{2}} u\left(s\right)\|_{L^2(\mathbb{R}^3)} \mathrm{d}s, \tag{1.8}$$

which is dimension 0 respect to Navier-Stokes scalings (1.1). So this energy is also called critical energy.

Since u_+ and u_- are strongly orthogonal to each other, we know

$$E_c\left(u(t)\right) = E_c\left(u_+(t)\right) + E_c\left(u_-(t)\right), \tag{1.9}$$

and from the helicity conservation law, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_c(u_+) = \frac{\mathrm{d}}{\mathrm{d}t} E_c(u_-). \tag{1.10}$$

For more detials about the helicity structure, we refer the readers to [8].

We focus on the $\dot{H}^{\frac{1}{2}}\left(\mathbb{R}^3\right)$-regularity for the NSE. The aim of this paper is to gain a suitable improvement of this classical result. We construct a class of initial data, such that critical norm can be arbitrary large, and we can obtain the solutions with global regularity.

We now claim our main theorem.

Theorem 1.1 *Consider the Cauchy problem of* NSE. Suppose that

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \le M, \tag{1.11}$$

where M can be arbitrarily large. There exists a small constant $\varepsilon_0(M)$ such that, if

$$\varepsilon = \|\Lambda^{-\frac{1}{2}} \omega_0\|^2_{L^2(\mathbb{R}^3)} - \frac{\langle \Lambda^{-1}\omega_0, u_0\rangle^2_{L^2(\mathbb{R}^3)}}{\|\Lambda^{-\frac{1}{2}} u_0\|^2_{L^2(\mathbb{R}^3)}} < \varepsilon_0(M), \tag{1.12}$$

then there exists a global regular solution of NSE, where $\omega_0 = \nabla \times u_0$ and $\Lambda = \sqrt{-\Delta}$.

Remark 1.1 Particularly, in the case

$$\begin{aligned} &u_0 = M v_0, \\ &\|\Lambda^{\frac{1}{2}} v_0\|_{L^2} \le 1, \quad \operatorname{supp} \widehat{v}_0 \subseteq \{x|\ 1-\delta \le |x| \le 1+\delta\}, \\ &\nabla \times v_0 = \Lambda v_0, \end{aligned}$$

we have

$$\varepsilon = \frac{\|\Lambda^{-\frac{1}{2}} u_0\|^2_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{1}{2}} u_0\|^2_{L^2(\mathbb{R}^3)} - \|u_0\|^4_{L^2(\mathbb{R}^3)}}{\|\Lambda^{-\frac{1}{2}} u_0\|^2_{L^2(\mathbb{R}^3)}}$$

$$
\begin{aligned}
&= M^2 \frac{\|\Lambda^{-\frac{1}{2}} v_0\|^2_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{1}{2}} v_0\|^2_{L^2(\mathbb{R}^3)} - \|v_0\|^4_{L^2(\mathbb{R}^3)}}{\|\Lambda^{-\frac{1}{2}} v_0\|^2_{L^2(\mathbb{R}^3)}} \\
&\le M^2[(1-\delta)^{-1} - (1+\delta)^{-2}] \|\Lambda^{\frac{1}{2}} v_0\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim M^2 \frac{\delta(3+\delta)}{(1-\delta)(1+\delta)}.
\end{aligned}
$$

Now choose δ sufficiently small such that $\varepsilon \le \varepsilon_0(M)$, then there exists a global regular solution by Theorem 1.1. This give a simple and directly proof of the similar result of [8] and also of [9].

2 Preliminaries

We conclude the introduction by giving some notations which will be used throughout this paper. We always use $X \lesssim Y$ to denote $X \le CY$ for some constant $C > 0$. Similarly, $X \lesssim_u Y$ indicates that there exists a constant $C := C(u)$ depending on u such that $X \le C(u)Y$. We also use the notation $X \sim Y$ to denote $X \lesssim Y \lesssim X$.

Let $\psi(\xi)$ be a radial smooth function supported in the ball $\{\xi \in \mathbb{R}^3 : |\xi| \le \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^3 : |\xi| \le 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$
\begin{aligned}
\widehat{P_{\le N}\mathrm{g}}(\xi) &:= \psi\Big(\frac{\xi}{N}\Big)\widehat{\mathrm{g}}(\xi), \\
\widehat{P_{> N}\mathrm{g}}(\xi) &:= \Big(1 - \psi\Big(\frac{\xi}{N}\Big)\Big)\widehat{\mathrm{g}}(\xi), \\
\widehat{P_{N}\mathrm{g}}(\xi) &:= \Big(\psi\Big(\frac{\xi}{N}\Big) - \psi\Big(\frac{2\xi}{N}\Big)\Big)\widehat{\mathrm{g}}(\xi)
\end{aligned}
$$

and similarly define $P_{<N}$ and $P_{\ge N}$. We also define

$$
P_{M<\cdot\le N} := P_{\le N} - P_{\le M} = \sum_{M<N'\le N} P_{N'}
$$

whenever $M < N$. We usually use this multipliers when M and N are dyadic numbers.

As some applications of the Littlewood-Paley theory, we have the following lemma.

Lemma 2.1 *Suppose that $a(D)$ is s-order pseudo-differential operator satisfying $\widehat{a}\,(\mu\cdot) = \mu^s \widehat{a}\,(\cdot)$. Then we have*

$$
\|[a\,(D)\,, u]f\|_{L^2} \lesssim \|\nabla u\|_{L^3} \|\Lambda^s f\|_{L^2}, \tag{2.1}
$$

where $\Lambda = \sqrt{-\Delta}$.

Proof By Littlewood-Paley theory, we know

$$
\begin{aligned}
\|[a\,(D)\,, u]f\|^2_{L^2} &\lesssim \sum_{\mu} \|[a\,(D)\,, P_{\le C^{-1}\mu} u] P_\mu f\|^2_{L^2} + \sum_{C^{-1}\mu\le\sigma, \sigma\sim\sigma'} \|[a\,(D)\,, P_{\sigma'} u] P_\sigma f\|^2_{L^2} \\
&\quad + \sum_{\mu} \|[a\,(D)\,, P_\mu u] P_{\le C^{-1}\mu} f\|^2_{L^2} \\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned} \tag{2.2}
$$

where C is a large fixed constant. We only estimate I_1, the rest terms can be estimated similarly.

From the frequency support property, we see

$$[a(D), P_{\leq C^{-1}\mu}u]P_\mu f \sim [P_{\leq C\mu}a(D), P_{\leq C^{-1}\mu}u]P_\mu f.$$

We use the notation χ to denote the kernel of $P_{\leq C\mu}a(D)$, and $\widehat{a}(\xi)\chi_1\big(\frac{\xi}{\mu}\big)$ to denote the Fourier transform of $P_{\leq C\mu}a(D)$.

Then, we have

$$\begin{aligned}
&[P_{\leq C\mu}a(D), P_{\leq C^{-1}\mu}u]P_\mu f = P_{\leq C\mu}a(D)(P_{\leq C^{-1}\mu}uP_\mu f) - P_{\leq C^{-1}\mu}uP_{\leq C\mu}a(D)P_\mu f\\
&= \int_{\mathbb{R}^3}\chi(x-y)P_{\leq C^{-1}\mu}u(y)P_\mu f(y)\,\mathrm{d}y - \int_{\mathbb{R}^3}P_{\leq C^{-1}\mu}u(x)\chi(x-y)P_\mu f(y)\,\mathrm{d}y\\
&= \int_0^1\int_{\mathbb{R}^3}\chi(x-y)(x-y)\cdot P_{\leq C^{-1}\mu}\nabla u(sx+(1-s)y)P_\mu f(y)\,\mathrm{d}y\mathrm{d}s\\
&\lesssim \int_0^1\int_{\mathbb{R}^3}|z\chi(z)||P_{\leq C^{-1}\mu}\nabla u(x+(s-1)z)||P_\mu f(x-z)|\mathrm{d}z\mathrm{d}s.
\end{aligned} \tag{2.3}$$

By Minkowski inequality, we have

$$\begin{aligned}
\|[a(D), P_{\leq C^{-1}\mu}u]P_\mu f\|_{L^2}^2 &\lesssim \|z\chi(z)\|_{L^1}^2\||P_{\leq C^{-1}\mu}\nabla u\|_{L^3}^2\||P_\mu f\|_{L^6}^2\\
&\lesssim \|z\chi(z)\|_{L^1}^2\||P_{\leq C^{-1}\mu}\nabla u\|_{L^3}^2\||P_\mu \nabla f\|_{L^2}^2.
\end{aligned} \tag{2.4}$$

In fact,

$$\chi(z) = \mathcal{F}^{-1}\Big(\widehat{a}(\xi)\chi_1\Big(\frac{\xi}{\mu}\Big)\Big) = \mu^s\mathcal{F}^{-1}\Big(\widehat{a}\Big(\frac{\xi}{\mu}\Big)\chi_1\Big(\frac{\xi}{\mu}\Big)\Big) := \mu^s\mu^3\widetilde{\chi}(\mu z).$$

Therefore,

$$\|z\chi(z)\|_{L^1} = \mu^{s-1}\mu^3\|\mu z\widetilde{\chi}(\mu z)\|_{L^1} \lesssim \mu^{s-1}. \tag{2.5}$$

Combining above, we obtain

$$\mathrm{I}_1 \lesssim \|\nabla u\|_{L^3}^2\|\Lambda^s f\|_{L^2}^2. \tag{2.6}$$

3 Proof of the Main Results

Proof Let λ be a constant depending only on the initial data which is to be determined. We will use a bootstrap argument to prove (1.1).

We first assume that

$$\|\omega-\lambda u\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} + \int_0^t\|\omega-\lambda u\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}\mathrm{d}\tau \leq \varepsilon^{\frac{1}{2}}, \quad \text{where } \omega = \nabla\times u. \tag{3.1}$$

If the bootstrap assumption holds, we can prove the theorem as follows. From the identity

$$u\cdot\nabla u = \omega\times u + \nabla\frac{|u|^2}{2},$$

we have

$$\partial_t u + \omega\times u + \nabla\Big(p+\frac{|u|^2}{2}\Big) - \Delta u = 0, \tag{3.2}$$

which leads to

$$\partial_t u + (\omega - \lambda u) \times u + \nabla\Big(p + \frac{|u|^2}{2}\Big) - \Delta u = 0.$$

Thus, we obtain

$$\langle \Lambda u, \partial_t u + (\omega - \lambda u) \times u - \Delta u \rangle_{L^2(\mathbb{R}^3)} = 0.$$

Making direct energy estimate, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} + \|\nabla \Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} \\
&\le \|\Lambda u\|_{L^3(\mathbb{R}^3)} \|u\|_{L^3(\mathbb{R}^3)} \|\omega - \lambda u\|_{L^3(\mathbb{R}^3)} \\
&\lesssim \|\Lambda u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\Lambda^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{1}{2}}(\omega - \lambda u)\|_{L^2(\mathbb{R}^3)} \\
&\le \frac{1}{2}\|\Lambda u\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + C\|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} \|\nabla \Lambda^{\frac{1}{2}}(\omega - \lambda u)\|^2_{L^2(\mathbb{R}^3)} \\
&\le \frac{1}{2}\|\Lambda u\|^2_{L^2(\mathbb{R}^3)} + C\|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{1}{2}}(\omega - \lambda u)\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Integrating in t and using the Young inequality, we get

$$\begin{aligned}
&\|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla \Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}\tau \\
&\lesssim \|\Lambda^{\frac{1}{2}} u_0\|^2_{L^2(\mathbb{R}^3)} + C\int_0^t \|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{1}{2}}(\omega - \lambda u)\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}\tau.
\end{aligned} \tag{3.3}$$

Thus, by bootstrap assumption (3.1) and Gronwall's inequality,

$$\|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\Lambda^{\frac{3}{2}} u\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}\tau \le \mathrm{e}^{C\varepsilon^{\frac{1}{2}}} M \le 2M, \tag{3.4}$$

if $\varepsilon < \frac{\ln^2 2}{C^2}$.

Now, we prove the bootstrap assumption. We recall the equations for vorticity ω and velocity u

$$\begin{aligned}
&\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega, \\
&\partial_t u + u \cdot \nabla u + \nabla p = \Delta u.
\end{aligned}$$

So

$$\begin{aligned}
&\partial_t(\omega - \lambda u) + u \cdot \nabla(\omega - \lambda u) - \Delta(\omega - \lambda u) \\
&= \omega \cdot \nabla u - \lambda \nabla p \\
&= (\omega - \lambda u) \cdot \nabla u + \lambda u \cdot \nabla u - \lambda \nabla p \\
&= (\omega - \lambda u) \cdot \nabla u + \omega \times \lambda u + \lambda \nabla\Big(\frac{|u|^2}{2} - p\Big) \\
&= (\omega - \lambda u) \cdot \nabla u + \omega \times (\lambda u - \omega) + \lambda \nabla\Big(\frac{|u|^2}{2} - p\Big).
\end{aligned}$$

Taking inner product with $\Lambda^{-1}(\omega - \lambda u)$ and making an intergration by parts, we have energy estimate as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{-\frac{1}{2}}(\omega - \lambda u)\|^2_{L^2(\mathbb{R}^3)} + \|\Lambda^{\frac{1}{2}}(\omega - \lambda u)\|^2_{L^2(\mathbb{R}^3)}$$

$$= -\langle\Lambda^{-1}(\omega-\lambda u), u\cdot\nabla(\omega-\lambda u)\rangle_{L^2(\mathbb{R}^3)} + \langle\Lambda^{-1}(\omega-\lambda u), (\omega-\lambda u)\cdot\nabla u\rangle_{L^2(\mathbb{R}^3)}$$
$$+ \langle\Lambda^{-1}(\omega-\lambda u), \omega\times(\lambda u-\omega)\rangle_{L^2(\mathbb{R}^3)}.$$

Noting that

$$\langle\Lambda^{-1}(\omega-\lambda u), u\cdot\nabla(\omega-\lambda u)\rangle_{L^2(\mathbb{R}^3)} = \langle\Lambda^{-\frac12}(\omega-\lambda u), [\Lambda^{-\frac12}, u\cdot\nabla](\omega-\lambda u)\rangle_{L^2(\mathbb{R}^3)} \tag{3.5}$$

by Hölder inequality, Sobolev embedding and Lemma 2.1, we have

$$\begin{aligned}&\langle\Lambda^{-\frac12}(\omega-\lambda u), [\Lambda^{-\frac12}, u\cdot\nabla](\omega-\lambda u)\rangle_{L^2(\mathbb{R}^3)}\\ &\lesssim \|\Lambda^{-\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}\|[\Lambda^{-\frac12}, u\cdot\nabla](\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \|\Lambda^{-\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}\|\Lambda^{\frac12}\omega\|_{L^2(\mathbb{R}^3)}\|\Lambda^{\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{3.6}$$

Similarly, we also obtain

$$\begin{aligned}&\langle\Lambda^{-1}(\omega-\lambda u), (\omega-\lambda u)\cdot\nabla u\rangle_{L^2(\mathbb{R}^3)}\\ &\lesssim \|\Lambda^{-1}(\omega-\lambda u)\|_{L^3(\mathbb{R}^3)}\|(\omega-\lambda u)\|_{L^3(\mathbb{R}^3)}\|\nabla u\|_{L^3(\mathbb{R}^3)}\\ &\lesssim \|\Lambda^{-\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}\|\Lambda^{\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}\|\Lambda^{\frac12}\omega\|_{L^2(\mathbb{R}^3)}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\langle\Lambda^{-1}(\omega-\lambda u), \omega\times(\lambda u-\omega)\rangle_{L^2(\mathbb{R}^3)}\\ &\lesssim \|\Lambda^{-1}(\omega-\lambda u)\|_{L^3(\mathbb{R}^3)}\|\omega\|_{L^3(\mathbb{R}^3)}\|(\omega-\lambda u)\|_{L^3(\mathbb{R}^3)}\\ &\lesssim \|\Lambda^{-\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}\|\Lambda^{\frac12}\omega\|_{L^2(\mathbb{R}^3)}\|\Lambda^{\frac12}(\omega-\lambda u)\|_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{3.8}$$

Combining (3.6)–(3.8) and using Hölder inequality and Young inequality, we obtain the energy estimates as follows:

$$\begin{aligned}&\|\Lambda^{-\frac12}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\Lambda^{\frac12}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau\\ &\lesssim \|\Lambda^{-\frac12}(\omega_0-\lambda u_0)\|^2_{L^2(\mathbb{R}^3)}\\ &\quad + \Big(\int_0^t \|\Lambda^{\frac12}\omega\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau\|\Lambda^{-\frac12}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau\Big)^{\frac12}\Big(\int_0^t \|\Lambda^{\frac12}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau\Big)^{\frac12}\\ &\lesssim \|\Lambda^{-\frac12}(\omega_0-\lambda u_0)\|^2_{L^2(\mathbb{R}^3)}\\ &\quad + \frac12\int_0^t \|\Lambda^{\frac12}\omega\|^2_{L^2(\mathbb{R}^3)}\|\Lambda^{-\frac12}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau + \frac12\int_0^t \|\Lambda^{\frac12}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau.\end{aligned}$$

Noting the fact

$$\begin{aligned}\|\Lambda^{-\frac12}(\omega_0-\lambda u_0)\|^2_{L^2(\mathbb{R}^3)} &= \|\Lambda^{-\frac12}\omega_0\|^2_{L^2(\mathbb{R}^3)} - 2\lambda\langle\Lambda^{-\frac12}\omega_0, \Lambda^{-\frac12}u_0\rangle_{L^2(\mathbb{R}^3)}\\ &\quad + \lambda^2\|\Lambda^{-\frac12}u_0\|^2_{L^2(\mathbb{R}^3)},\end{aligned} \tag{3.9}$$

we choose λ to minimize (3.9), which is

$$\lambda = \frac{\langle\Lambda^{-1}\omega_0, u_0\rangle_{L^2(\mathbb{R}^3)}}{\|\Lambda^{-\frac12}u_0\|^2_{L^2(\mathbb{R}^3)}}.$$

Therefore,

$$\|\Lambda^{-\frac{1}{2}}(\omega_0-\lambda u_0)\|^2_{L^2(\mathbb{R}^3)}=\|\Lambda^{-\frac{1}{2}}\omega_0\|^2_{L^2(\mathbb{R}^3)}-\frac{\langle\Lambda^{-1}\omega_0,u_0\rangle^2_{L^2(\mathbb{R}^3)}}{\|\Lambda^{-\frac{1}{2}}u_0\|^2_{L^2(\mathbb{R}^3)}}.$$

Thus, by Gronwall's inequality, we get

$$\begin{aligned}\|\Lambda^{-\frac{1}{2}}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}+\int_0^t\|\Lambda^{\frac{1}{2}}(\omega-\lambda u)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau&\lesssim\varepsilon\exp\Big(\int_0^t\|\Lambda^{\frac{1}{2}}\omega\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}\tau\Big)\\&\lesssim\mathrm{e}^{CM}\varepsilon.\end{aligned}\tag{3.10}$$

Now, we take $\varepsilon_0(M)=\min\big\{\frac{\ln^2 2}{C^2},\frac{1}{4}\mathrm{e}^{-4CM}\big\}$, then (3.10) improves (3.1). By continuous induction we finish our proof.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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