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Conjugate Surfaces of a Family of Minimal Surfaces of Genus 1 with 4 Planar Ends in $\mathbb{R}^3$*

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Abstract Costa first constructed a family of complete minimal surfaces which have genus 1 and 4 planar ends by use of Weierstrass-$\wp$ functions. They are Willmore tori of Willmore energy 16π. In this paper, the authors consider the geometry of conjugate surfaces of these surfaces. It turns out that these conjugate surfaces are doubly periodic minimal surfaces with flat ends in $\mathbb{R}^3$. Moreover, the authors can also perform a Lorentzian deformation on these Costa's minimal tori, which produce a family of complete space-like stationary surfaces (i.e., of zero mean curvature) with genus 1 and 4 planar ends in 4-dimensional Lorentz-Minkowski space $\mathbb{R}^4_1$.

Keywords Conjugate surfaces, Weierstrass representation, Elliptic functions, Doubly periodic minimal surfaces
2000 MR Subject Classification 53A10, 53C50, 53C42, 33E05

1 Introduction

Minimal surface in $\mathbb{R}^3$, as critical surfaces of area functional, plays important roles in the development of differential geometry. In 1984, Costa [7] first succeed in the construction of a complete minimal surface of genus 1 with 3 embedded ends by use of Weierstrass$-\wp$ functions. Hoffman and Meeks [10] soon proved that Costa surface is embedded in $\mathbb{R}^3$. Its discovery has led to an enormous amount of progress. We refer to the book by Colding and Minicozzi [5–6] for recent progress. On the other hand, concerning the famous Willmore conjecture, Bryant considered Willmore surfaces and proved surprisingly that a Willmore 2-sphere in $\mathbb{R}^3$ is conformally equivalent to a complete minimal surface of genus 0 with planar ends (flat embedded ends). This means that minimal surfaces with planar ends are of importance in view of the Willmore problem. The existence of such surfaces are non-trivial: For obvious reasons, there are no complete minimal surfaces (see [1-2, 15–16]) of genus 1 with 1 or 2 planar ends. Kusner

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and Schmitt [12] proved that there are no complete minimal surface of genus 1 with 3 planar ends. Costa [9] constructed a family of complete minimal surfaces of genus 1 with 4 planar ends in $\mathbb{R}^3$, which are conformally equivalent to Willmore tori with Willmore energy 16π. In 2005, Shamaev [16] constructed a complete minimal surface of genus 1 with n flat ends, where n is even and $n \geq 6$.

In this note, we want to consider furthermore the properties of the family of complete minimal surfaces of genus 1 with 4 planar ends by Costa in 1993. In particular, it is of interest and of importance to consider the geometry of conjugate surfaces of Costa's examples. A careful discussion indicates that the conjugate surfaces are in fact doubly-periodic in $\mathbb{R}^3$ and hence we obtain new examples of doubly periodic minimal surfaces with infinitely many planar ends. Moreover, it turns out that after suitable transformation, one can perform Lorentzian deformations introduced in [4, 13, 17]. In this way we obtain a 2-parameter family of complete space-like stationary surfaces of genus 1 with 4 planar ends in four-dimensional Minkowski space $\mathbb{R}^4_1$.

We first recall Costa's examples (see [9]). Consider the lattice $L(\mathrm{i}\tau)$ (here $\tau \geq 1$):

$$L(\mathrm{i}\tau) = \{m + n\mathrm{i}\tau \in \mathbb{C},\ m, n \in \mathbb{Z}\}. \tag{1.1}$$

Set

$$w_0 = 0, \quad w_1 = \frac{1}{2}, \quad w_2 = \frac{1+\mathrm{i}\tau}{2}, \quad w_3 = \frac{\mathrm{i}\tau}{2}. \tag{1.2}$$

Theorem 1.1 (see [9]) *Let $T_\tau = \mathbb{C}/L(\mathrm{i}\tau)$ be the torus with its canonical projection $\pi : \mathbb{C} \to T_\tau = \mathbb{C}/L(\mathrm{i}\tau)$. Set $M_\tau = T_\tau \setminus \{P_0, P_1, P_2, P_3\}$ with $P_k = \pi(w_k)$ for $0 \leq k \leq 3$ (see Figure 1). There exist $\mathbb{C}^\infty$-functions $\alpha(\tau), \beta(\tau), c(\tau) : [1, +\infty) \to \mathbb{R}\backslash\{0\}$, such that for any $\tau \geq 1$, the following Weierstrass representation defines complete minimal surfaces $I_\tau : M_\tau \to \mathbb{R}^3$ $(\tau \geq 1)$ with 4 planar ends :*

$$I_\tau = \frac{1}{2}\,\mathrm{Re}\int(\Phi_1, \Phi_2, \Phi_3)\mathrm{d}z = \frac{1}{2}\,\mathrm{Re}\int(f(1-g^2), \mathrm{i}f(1+g^2), 2fg)\mathrm{d}z. \tag{1.3}$$

Here, f and g are chosen to be

$$\begin{cases} g = g_\tau = a(\tau)\wp(z) + b(\tau)\wp(z - w_3) + e_3 c(\tau), \\ f\mathrm{d}z = f_\tau \mathrm{d}z = (\wp(z - w_1) + \wp(z - w_2) + e_3)\mathrm{d}z, \end{cases} \tag{1.4}$$

where $a(\tau) = \overline{b(\tau)} = \alpha(\tau) + \mathrm{i}\beta(\tau)$, $e_k = \wp(w_k)$, $k = 1, 2, 3$.

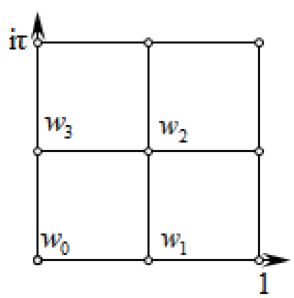


Figure 1 Fundamental domain of M_τ.

To prove Theorem 1.1, Costa [9] needs only to consider the real parts of the integrals appeared in (1.3). Recall that a conjugate surface of I_τ is

$$\widetilde{I_\tau} = \frac{1}{2}\,\mathrm{Re}\int(\mathrm{i}f(1-g^2), -f(1+g^2), 2\mathrm{i}fg)\mathrm{d}z.$$

The study of the conjugate surfaces involve the imaginary parts of the integrals appeared in (1.3). It turns out that the imaginary parts are not as good as the real parts. By discussions on the imaginary parts, we obtain the following theorem.

Theorem 1.2 *We retain the notion of Theorem* 1.1*. Then the conjugate surface* $\widetilde{I_\tau} : M \to \mathbb{R}^3$ *of* I_τ *is a family of complete, doubly-periodic, minimal surfaces with infinitely many planar ends in* $\mathbb{R}^3$*. Here* $M = \mathbb{C} \setminus \Lambda$*, where* $\Lambda = \{\omega_k + m + n\mathrm{i}\tau \mid m, n \in \mathbb{Z}, 0 \le k \le 3\}$ *(see Figure* 2*). In particular, the periodiciy of* $\widetilde{I_\tau}$ *can be represented as follows*:

$$\begin{cases} \widetilde{I_\tau}(z+1) = \widetilde{I_\tau}(z) + (0, T_2, 0), \\ \widetilde{I_\tau}(z+\mathrm{i}\tau) = \widetilde{I_\tau}(z) + (T_1, 0, T_3). \end{cases} \tag{1.5}$$

Here T_1*,* T_2 *and* T_3 *are some non-zero constants determined by* I_τ *(see* (3.11)*).*

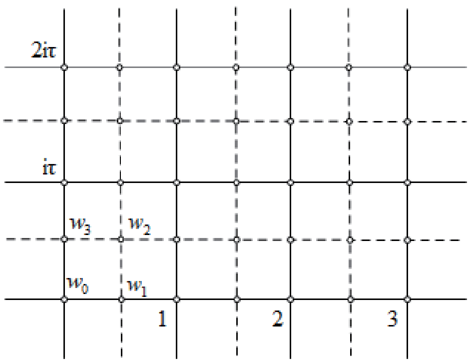


Figure 2 $M \subset \mathbb{C}$.

This paper is organized as follows: In Section 2, we recall the Weierstrass representation of minimal surfaces, basic properties of elliptic functions and basic notations of the Costa's minimal surfaces with 4 planar ends of [7]. Then in Section 3, we derive the proof of Theorem 1.2. In Section 4, by Lorentz deformations, we obtain a 2-parameter family of complete space-like stationary surfaces of genus 1 with 4 flat embedded ends in $\mathbb{R}^4_1$.

2 Weierstrass Representation for Minimal Surfaces and Weierstrass $\wp$ Functions

2.1 Weierstrass representation of minimal surfaces in $\mathbb{R}^3$

Definition 2.1 (see [14]) *Let* $\overline{M}$ *be a Riemann surface, and* $M = \overline{M}\backslash\{p_1, \cdots, p_n\}$*. Suppose that* f *is a holomorphic function on* M *and* g *is a meromorphic function on* M*, satisfying*:

(1) *A point $z \in M$ is a pole of g of degree m if and only if z is a zero of f of degree $2m$.*

(2) *Every divergent path γ in M has infinite length.*

(3) *For any closed path γ in M,*

$$\oint_\gamma fg^2\mathrm{d}z = \overline{\oint_\gamma f\mathrm{d}z}, \quad \mathrm{Re}\oint_\gamma fg\mathrm{d}z = 0. \tag{2.1}$$

Then the following Weierstrass representation

$$X = \frac{1}{2}\mathrm{Re}\int((1-g^2)f, \mathrm{i}(1+g^2)f, 2fg)\mathrm{d}z \tag{2.2}$$

defines a complete minimal surface $X : M \to \mathbb{R}^3$. Moreover, its conjugate surface $\widetilde{X} \subset \mathbb{R}^3$ is defined to be

$$\widetilde{X} = \frac{1}{2}\mathrm{Re}\int(\mathrm{i}(1-g^2)f, -(1+g^2)f, 2\,\mathrm{i}fg)\mathrm{d}z. \tag{2.3}$$

Note that $\widetilde{X}$ may not be defined on M in general. Moreover, the associated family X_θ of X is defined by

$$X_\theta = \frac{1}{2}\mathrm{Re}\int \mathrm{e}^{\mathrm{i}\theta}((1-g^2)f, \mathrm{i}(1+g^2)f, 2fg)\mathrm{d}z. \tag{2.4}$$

Here θ is a constant taking values in $[0,\pi]$. Note that $X_{\frac{\pi}{2}} = \widetilde{X}$.

2.2 Weierstrass representation of a space-like stationary surface in $\mathbb{R}^4_1$

The 4-dimensional Minkowski space $\mathbb{R}^4_1$ is the space $\mathbb{R}^4$ equipped with the Lorentzian inner product

$$\langle X, X\rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

Let $X : M \to \mathbb{R}^4_1$ be a complete space-like stationary surface. Let $z = u + \mathrm{i}v$ be a local complex coordinate of M such that the induced Riemannian metric in Σ is $\mathrm{d}s^2 = \mathrm{e}^{2\omega}|\mathrm{d}z|^2$. The Weierstrass representation of the space-like stationary surface in $\mathbb{R}^4_1$ is given by [13],

$$X = 2\mathrm{Re}\int(\phi+\psi, -\mathrm{i}(\phi-\psi), 1-\phi\psi, 1+\phi\psi)\mathrm{d}h, \tag{2.5}$$

where $\mathrm{d}h$ is a holomorphic 1-form and ϕ, ψ are meromorphic functions. From this we see

$$X_z\mathrm{d}z = (\phi+\psi, -\mathrm{i}(\phi-\psi), 1-\phi\psi, 1+\phi\psi)\mathrm{d}h. \tag{2.6}$$

Theorem 2.1 (see [13]) *Given a holomorphic 1-form $\mathrm{d}h$ and two meromorphic functions ϕ, ψ on a Riemann surface M. If $\mathrm{d}h$, ϕ and ψ satisfy the following conditions*

(1) *$\phi \neq \bar{\psi}$ and they do not have same poles;*

(2) *a point is a pole of ϕ or ψ of degree m if and only if it is a zero of $\mathrm{d}h$ of degree m;*

(3) *every divergent path γ in M has infinite length;*

(4) *for any closed path γ in M,*

$$\oint_\gamma \phi\mathrm{d}h = -\overline{\oint_\gamma \psi\mathrm{d}h}, \quad \mathrm{Re}\oint_\gamma \mathrm{d}h = 0 = \mathrm{Re}\oint_\gamma \phi\psi\mathrm{d}h. \tag{2.7}$$

Then (2.5) determines a complete space-like stationary surface $X : M \to \mathbb{R}^4_1$. Conversely, if $X : M \to \mathbb{R}^4_1$ is a complete space-like stationary surface, then there exists a holomorphic 1-form $\mathrm{d}h$ and two meromorphic functions ϕ, ψ on M such that the above conditions and (2.5) hold.

2.3 On Elliptic functions

Here we will collect some basic properties of the Weierstrass-$\wp$ function. We refer to [3, 7] for more details.

Definition 2.2 (see [3]) *Consider the torus $T^2(\nu_1,\nu_2)=\mathbb{C}/L(\nu_1,\nu_2)$, with its lattice $L(\nu_1, \nu_2)\subset\mathbb{C}$ being*

$$L=L(\nu_1,\nu_2)=\{m\nu_1+n\nu_2\in\mathbb{C}, m,n\in\mathbb{Z}\}\quad \textit{with } \mathrm{Im}\Big(\frac{\nu_2}{\nu_1}\Big)>0.$$

The Weierstrass-$\wp$ function is a doubly periodic meromorphic function of $T^2(\nu_1,\nu_2)$, defined by

$$\wp(z)=\frac{1}{z^2}+\sum_{\Omega\in L(\nu_1,\nu_2)\backslash\{0\}}\Big(\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\Big). \tag{2.8}$$

It is well-known that the Weierstrass-$\wp$ function satisfies (see [3])

$$\wp'(z)^2=4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3) \tag{2.9}$$

with

$$e_1+e_2+e_3=0,\qquad e_1>e_2>e_3, \tag{2.10}$$

where

$$e_j=\wp(W_j),\quad j=1,2,3\quad \text{with } W_1=\frac{\nu_1}{2},\quad W_2=\frac{\nu_1+\nu_2}{2},\quad W_3=\frac{\nu_2}{2}. \tag{2.11}$$

In particular, when $(\nu_1,\nu_2)=(1,\mathrm{i})$, we have $e_1=-e_3\approx 6.875185$, $e_2=0$.

2.4 On Costa's minimal surfaces with 4 planar ends

This subsection is mainly a summary of Costa's notations and results in [9]. Now we will have a detailed discussion on Costa's minimal surfaces I_τ with 4 planar ends, where the lattice will be $L(1,\mathrm{i}\tau)$. First we recall the Weierstrass data of I_τ in [9] as follows:

$$\begin{cases} g=g_\tau=a(\tau)\wp(z)+b(\tau)\wp(z-w_3)+e_3c(\tau),\\ f\mathrm{d}z=f_\tau\mathrm{d}z=(\wp(z-w_1)+\wp(z-w_2)+e_3)\mathrm{d}z.\end{cases}$$

Here (see [9, (2.12) to (2.17)])

$$a(\tau)=\overline{b(\tau)}=\alpha(\tau)+\mathrm{i}\beta(\tau). \tag{2.12}$$

In fact, $\alpha(\tau),\beta(\tau)\in\mathbb{R}^+$ are positive real functions of τ, $\tau\geq 1$, which will be determined in (2.18) (see Remark 2 for the details on [9, p. 1286]).
Set

$$\eta_1=-\frac{1}{2}\oint_{l_1}\wp(z)\mathrm{d}z,\quad \eta_3=-\frac{1}{2}\oint_{l_3}\wp(z)\mathrm{d}z, \tag{2.13}$$

where the paths l_1 and l_3: $[0,1]\to\mathbb{C}/L(1,\mathrm{i}\tau)$ are

$$l_1(t)=t+\frac{\mathrm{i}\tau}{3},\quad l_3(t)=\frac{1}{3}+\mathrm{i}\tau t. \tag{2.14}$$

Definition 2.3 (see [9]) *Set*

$$S_j = 2\eta_1 + e_j, \quad e_j = \wp(w_j), \quad j = 1,2,3, \quad S = S_1 + S_2, \quad b_4 = -S + 2\eta_1\Big(\frac{\tau S}{\pi} - 2\Big). \tag{2.15}$$

By [9, p. 1285], *all of* $\{\eta_1,\ b_4,\ e_j,\ S_j,\ j = 1,2,3\}$ *are real functions about* τ. *We define furthermore the real-valued functions* $c, \alpha_j, \beta_j, \gamma_j$ *as follows*:

$$c = c(\tau) = -\frac{2(e_1S_2 + e_2S_1)}{e_3S}\alpha \in \mathbb{R}\backslash\{0\}, \tag{2.16}$$

$$\begin{cases} \alpha_1 = 4(S_1^2 + S_2^2 + S_1S_2), \\ \alpha_2 = 2(e_1S_1^2 + e_2S_2^2 - 2e_3S_1S_2), \\ \beta_1 = -4S_1S_2, \\ \beta_2 = 2((e_3 - e_2)S_1^2 + (e_3 - e_1)S_2^2 + 2e_3S_1S_2), \\ \gamma_1 = \tau\dfrac{S}{\pi} - 2, \\ \gamma_2 = -S + 2\eta_1\Big(\tau\dfrac{S}{\pi} - 2\Big). \end{cases} \tag{2.17}$$

Lemma 2.1 (see [9, Lemma 1]) *The positive functions* α *and* β *are determined as follows*:

$$\begin{cases} \dfrac{(e_1 - e_2)^2}{S^2}(\alpha^2 - \beta^2) = \dfrac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \\ \dfrac{(e_1 - e_2)^2}{S^2}(\alpha^2 + \beta^2) = \dfrac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}. \end{cases} \tag{2.18}$$

In particular, when $\tau = 1$, *we have*

$$\begin{cases} \alpha = \sqrt{\dfrac{(e_1 + 2\pi)^4(e_1^2 + 2\pi^2)}{4e_1^3(e_1^4 + 8e_1^3\pi + 23e_1^2\pi^2 + 26\mathrm{e}\pi^3 + 12\pi^4)}} \approx 0.132915, \\ \beta = \sqrt{\dfrac{\pi(e_1 + 2\pi)^4}{4e_1^3(e_1^4 + 8e_1^3\pi + 23e_1^2\pi^2 + 26\mathrm{e}\pi^3 + 12\pi^4)}} \approx 0.0510109. \end{cases} \tag{2.19}$$

Lemma 2.2 (see [9, (15)–(16)]) *The functions* f, g *in* (1.4) *satisfy the following equations*:

$$\begin{aligned} gf &= a_1\wp(z - w_1) + a_2\wp(z - w_2) + a_0, \\ g^2f &= b_0\wp(z) + \sum_{j=1}^{3} b_j\wp(z - w_j) + b_4, \end{aligned} \tag{2.20}$$

where a_j, b_j *are defined as follows* $(recall a(\tau) = \overline{b(\tau)} = \alpha(\tau) + \mathrm{i}\beta(\tau))$:

$$\begin{cases} a_0 = -2e_1e_2(a + b) + e_3^2c, \quad a_1 = e_1a + e_2b + e_3c, \quad a_2 = e_2a + e_1b + e_3c, \\ b_0 = (e_1 - e_2)^2a^2, \quad b_1 = a_1^2, \quad b_2 = a_2^2, \quad b_3 = (e_1 - e_2)^2b^2, \\ b_4 = -S + 2\eta_1\Big(\dfrac{\tau S}{\pi} - 2\Big). \end{cases} \tag{2.21}$$

Proof See [9, pp. 1283–1285] for a detailed proof, the Zeros and poles of $\{f, g, fg, fg^2\}$ is illustrated in Figure 3.

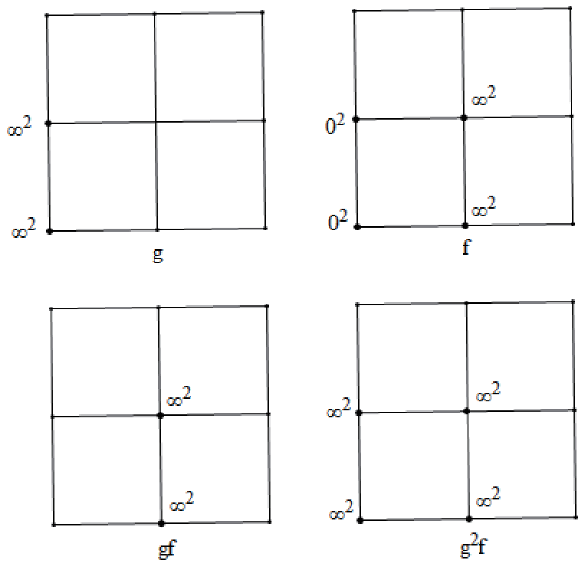
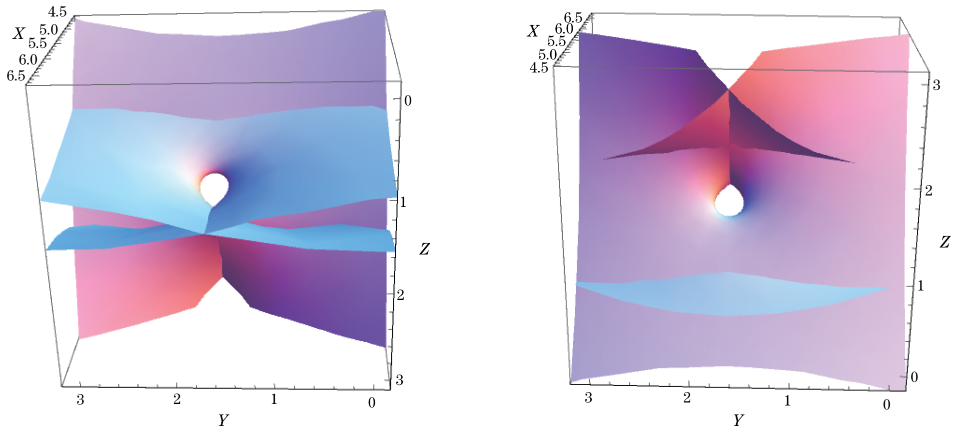


Figure 3 Zeros and poles of $\{f, g, fg, fg^2\}$ (see [9, (13)–(14)]).

3 Geometry of the Conjugate Surface of I_τ

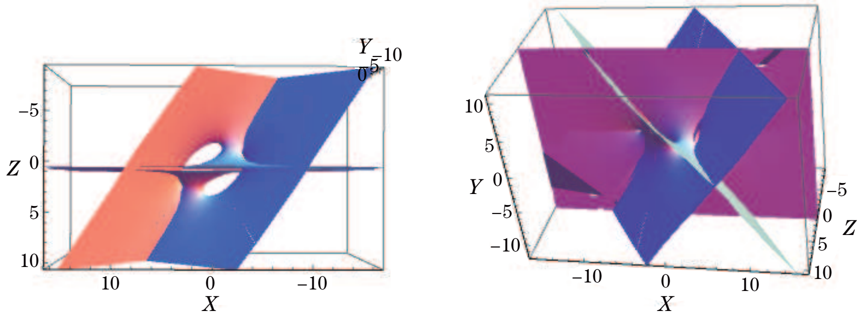
In this section, we will first consider the periodic problems of the conjugate surfaces $\widetilde{I_\tau}$ of the Costa's minimal surfaces I_τ (see Figure 4). Then we will give the proof of Theorem 1.2. We will retain the notion of Subsection 2.4.



(a) Main view of I_τ (b) Rear view of I_τ

Figure 4 Picture of I_τ, where $\tau = 1$ and $z \in [0.01, 0.49] \times [0.01, 0.49]$.

The following picture (see Figure 5) shows the behaviour of asymptotic planes Σ_k of I_τ.



(a) Main view of I_τ (b) Left view of I_τ

Figure 5 The asymptotic planes Σ_k of I_τ, $\tau = 1$.

3.1 Periodic property of the conjugate surfaces $\widetilde{I_\tau}$

The Weierstrass representation of the conjugate surfaces $\widetilde{I_\tau}$ is (here f, g are given by (1.4))

$$\widetilde{I_\tau} = \frac{1}{2}\,\mathrm{Re}\int(\Phi_1', \Phi_2', \Phi_3')\mathrm{d}z = \frac{1}{2}\,\mathrm{Re}\int(\mathrm{i}f(1-g^2), -f(1+g^2), 2\,\mathrm{i}fg)\mathrm{d}z \tag{3.1}$$

from [9], the ends P_0, P_1, P_2 and P_3 are all planar ends. So we have the following lemma.

Lemma 3.1 (see [9]) *The residue of Φ_j' at each end P_k is 0 for $1 \le j \le 3$ and $0 \le k \le 3$.*

Now concerning the periodic properties, we have the following lemma.

Lemma 3.2 *On the two closed paths l_1 and l_3 of* (2.14)*, the following equations hold:*

$$\int_{l_1}(1-g^2)f\mathrm{d}z = 0, \quad \int_{l_1}(1+g^2)f\mathrm{d}z = 2e_3 - 8\eta_1, \quad \int_{l_1} gf\mathrm{d}z = 0 \tag{3.2}$$

and

$$\begin{cases} \displaystyle\int_{l_3}(1-g^2)f\mathrm{d}z = -2\mathrm{i}(\tau(4\eta_1 - e_3) - 4\pi), \\ \displaystyle\int_{l_3}(1+g^2)f\mathrm{d}z = 0, \\ \displaystyle\int_{l_3} gf\mathrm{d}z = \mathrm{i}(4e_3(c-\alpha)(\pi - \tau\eta_1) + \tau(e_3^2c - 4e_1e_2\alpha)). \end{cases} \tag{3.3}$$

Proof By (1.4) and (2.20), we can obtain (see [9])

$$\int_{l_k} gf\mathrm{d}z = -(a_1 + a_2)2\eta_k + 2w_k a_0, \quad k = 1, 3, \tag{3.4}$$

$$\int_{l_k} g^2 f\mathrm{d}z = -2\Big(\sum_{j=0}^{3} b_j\Big)\eta_k + 2w_k b_4, \quad k = 1, 3, \tag{3.5}$$

$$\int_{l_k} f\mathrm{d}z = -4\eta_k + 2w_k e_3, \quad k = 1, 3. \tag{3.6}$$

Substituting (2.16) and (2.21) into (3.4), we have

$$\int_{l_1} gf\mathrm{d}z = -2(a_1 + a_2)\eta_1 + 2w_1 a_0$$

$$\begin{aligned}&= -4(e_3(c-\alpha)\eta_1 + e_1e_2\alpha) + e_3^2c\\ &= 0.\end{aligned} \tag{3.7}$$

Here the second equality can be obtained by using the first equality, of (2.21) the third equality can be obtained by using the second equality of (2.16).

Similarly, we have

$$\begin{aligned}\int_{l_3} gf\mathrm{d}z &= -2(a_1+a_2)\eta_3 + 2w_3a_0\\ &= \mathrm{i}(4e_3(c-\alpha)(\pi-\tau\eta_1) + \tau(e_3^2c - 4e_1e_2\alpha)),\end{aligned} \tag{3.8}$$

where $2\eta_3 = \mathrm{i}\tau(2\eta_1) - 2\pi\mathrm{i}$ by the Legendre's relation. Substituting (1.2), (2.15) and (2.21) into (3.5), we have

$$\begin{aligned}&\int_{l_1} g^2f\mathrm{d}z = -2\Big(\sum_{j=0}^{3} b_j\Big)\eta_1 + 2w_1b_4 = e_3 - 4\eta_1,\\ &\int_{l_3} g^2f\mathrm{d}z = -2\Big(\sum_{j=0}^{3} b_j\Big)\eta_3 + 2w_3b_4 = \mathrm{i}(\tau(4\eta_1 - e_3) - 4\pi).\end{aligned} \tag{3.9}$$

Similarly, substituting (1.2), (2.15) and (2.21) into (3.6), we have

$$\begin{aligned}&\int_{l_1} f\mathrm{d}z = e_3 - 4\eta_1,\\ &\int_{l_3} f\mathrm{d}z = -\mathrm{i}(\tau(4\eta_1 - e_3) - 4\pi).\end{aligned} \tag{3.10}$$

Summing up we finish the proof of the lemma.

Lemma 3.3 *Set*

$$\begin{cases} T_1 = -\tau(4\eta_1 - e_3) + 4\pi,\\ T_2 = 4\eta_1 - e_3,\\ T_3 = 4e_3(c-\alpha)(\pi - \tau\eta_1) + \tau(e_3^2c - 4e_1e_2\alpha),\end{cases} \tag{3.11}$$

when $\tau \geq 1$, $T_j \in \mathbb{R}\backslash\{0\}$, $j = 1, 2, 3$. *Moreover, the conjugate surface* $\widetilde{I_\tau}$ *is doubly periodic.*

Proof By [9, Proposition 1(10)], we get $S_k = 2\eta_1 + e_k > 0$, $k = 1, 2$ and $e_k \in \mathbb{R}$. Together with $e_1 + e_2 + e_3 = 0$, we obtain $T_2 \in \mathbb{R}^+$.

By [9, pp. 1281–1282, p. 1285], when $\tau \geq 1$, α, c and e_j, $j = 1, 2, 3$ are all real. So we have $T_3 \in \mathbb{R}$. We show by contradiction that $T_3 \neq 0$. Suppose that $T_3 = 4e_3(c-\alpha)(\pi - \tau\eta_1) + \tau(e_3^2c - 4e_1e_2\alpha) = 0$, i.e.,

$$c = \frac{4e_3\alpha(\pi - \tau\eta_1) + 4e_1e_2\alpha\tau}{e_3^2\tau + 4e_3(\pi - \tau\eta_1)}.$$

By (2.15)–(2.16), we have

$$\frac{4e_3\alpha(\pi - \tau\eta_1) + 4e_1e_2\alpha\tau}{e_3^2\tau + 4e_3(\pi - \tau\eta_1)} = c = \frac{4\alpha(\eta_1e_3 - e_1e_2)}{e_3(4\eta_1 - e_3)}. \tag{3.12}$$

i.e,

$$\frac{\eta_1 e_3 - e_1 e_2}{4\eta_1 - e_3} = \frac{e_1 e_2 \tau + e_3 \pi - e_3 \tau \eta_1}{4\pi - 4\tau\eta_1 + e_3\tau} = \frac{(e_1 e_2 - \eta_1 e_3)\tau + e_3 \pi}{4\pi - \tau(4\eta_1 - e_3)}. \tag{3.13}$$

Then (3.13) is equivalent to $\mathrm{e}_3^2 = 4e_1e_2$. Since $e_1 + e_2 + e_3 = 0$, this is equivalent to $e_1 = e_2$, contradicting to the fact that $e_1 > e_2$. So $T_3 \neq 0$.

Finally, by [9, p. 1285], η_1 and e_3 are real functions when $\tau \geq 1$. So $T_1 \in \mathbb{R}$. Moreover, we claim that $T_1 < 0$ for any τ. See Figure 6 for the picture of $T_1(\tau)$.

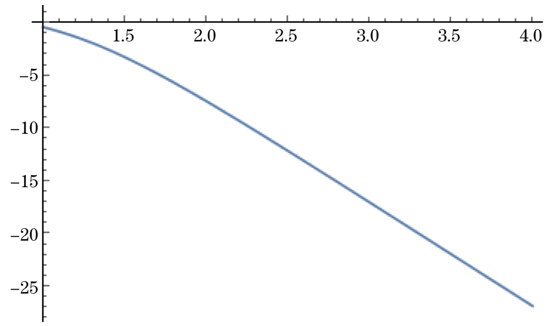


Figure 6 Picture of $T_1(\tau)$, here $\tau \geq 1$.

To prove this, first, by [8, proof of Proposition 5, pp. 613–614], we have

$$2\eta_1 + e_1 \geq \pi^2, \quad 2\eta_1 + e_2 = 8\pi^2 \sum_{n=1}^{\infty} R_n(\tau) \quad \text{with } R_n(\tau) = \frac{(-1)^{n+1} n \mathrm{e}^{-\pi n \tau}}{1 - \mathrm{e}^{-2\pi n \tau}}.$$

So

$$\begin{aligned} T_1 &= -\tau(4\eta_1 - e_3) + 4\pi \\ &= 4\pi - \tau(2\eta_1 + e_1 + 2\eta_1 + e_2) \\ &= 4\pi - \tau(2\eta_1 + e_1) - \tau\Big(8\pi^2 \sum_{n=1}^{\infty} R_n(\tau)\Big) \end{aligned}$$

with $T_1(1) = 2\pi - e_1 < 0$. From (28)–(32) appearing on [8, p. 614], we obtain

$$|R_n(\tau)| > |R_{n+1}(\tau)| > 0 \quad R_{2k}(\tau) < 0 \quad \text{and} \quad R_{2k+1}(\tau) > 0, \quad n, k \in \mathbb{Z}^+, \tag{3.14}$$

i.e., $-R_{2k}(\tau) > R_{2k+1}(\tau) > -R_{2k+2}(\tau) > 0$ for $k \in \mathbb{Z}^+$. Together with $2\eta_1 + e_1 \geq \pi^2$, we obtain

$$\begin{aligned} T_1(\tau) &\leq -\pi^2\tau + 4\pi + \tau\Big(8\pi^2 \sum_{n=1}^{\infty}(-R_n(\tau))\Big) \\ &= -\pi^2\tau + 4\pi + \tau\Big(8\pi^2\Big(-\frac{\mathrm{e}^{-\pi\tau}}{1 - \mathrm{e}^{-2\pi\tau}} + \frac{2\mathrm{e}^{-2\pi\tau}}{1 - \mathrm{e}^{-4\pi\tau}} + \sum_{n=3}^{\infty}(-R_n(\tau))\Big)\Big) \\ &\leq -\pi^2\tau + 4\pi + 8\pi^2\tau\Big(-\frac{\mathrm{e}^{-\pi\tau}}{1 - \mathrm{e}^{-2\pi\tau}} + \frac{2\mathrm{e}^{-2\pi\tau}}{1 - \mathrm{e}^{-4\pi\tau}}\Big). \end{aligned}$$

Set

$$F(\tau) = -\pi^2\tau + 4\pi + 8\pi^2\tau\Big(-\frac{\mathrm{e}^{-\pi\tau}}{1 - \mathrm{e}^{-2\pi\tau}} + \frac{2\mathrm{e}^{-2\pi\tau}}{1 - \mathrm{e}^{-4\pi\tau}}\Big).$$

We have

$$F'(\tau) = -\pi^2 + 8\pi^2\Big(-\frac{e^{-\pi\tau}}{1-e^{-2\pi\tau}} + \frac{2e^{-2\pi\tau}}{1-e^{-4\pi\tau}}\Big) + 8\pi^3\tau\Big(\frac{-8e^{-6\pi\tau}}{(1-e^{-4\pi\tau})^2} + \frac{-4e^{-2\pi\tau}}{1-e^{-4\pi\tau}} + \frac{2e^{-3\pi\tau}}{(1-e^{-2\pi\tau})^2} + \frac{e^{-\pi\tau}}{1-e^{-2\pi\tau}}\Big). \tag{3.15}$$

Let

$$G(\tau) = 8\pi^3\tau\Big(\frac{-8e^{-6\pi\tau}}{(1-e^{-4\pi\tau})^2} + \frac{-4e^{-2\pi\tau}}{1-e^{-4\pi\tau}} + \frac{2e^{-3\pi\tau}}{(1-e^{-2\pi\tau})^2} + \frac{e^{-\pi\tau}}{1-e^{-2\pi\tau}}\Big).$$

When $\tau \geq 1$, $G(\tau) < \pi^2$ by Lemma 5.1 of Appendix 5. Then we have

$$F'(\tau) < -\pi^2 + 8\pi^2\Big(-\frac{e^{-\pi\tau}}{1-e^{-2\pi\tau}} + \frac{2e^{-2\pi\tau}}{1-e^{-4\pi\tau}}\Big) + \pi^2 < 0. \tag{3.16}$$

So $F(\tau)$ is a decreasing function and hence $F(\tau) \leq F(1) \approx -0.426756 < 0$. As a consequence, $T_1(\tau) \leq F(\tau) < 0$.

3.2 The proof of Theorem 1.2

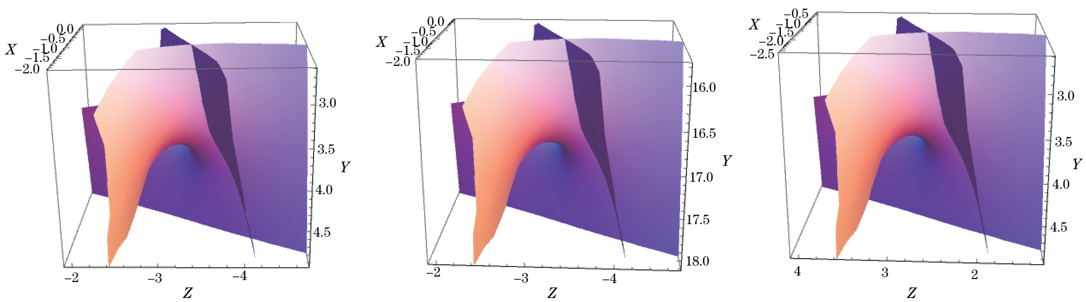
From Lemma 3.3, we have

$$\begin{cases} T_1 = -\tau(4\eta_1 - e_3) + 4\pi \in \mathbb{R}^-, \\ T_2 = 4\eta_1 - e_3 \in \mathbb{R}^+, \\ T_3 = 4e_3(c-\alpha)(\pi - \tau\eta_1) + \tau(e_3^2 c - 4e_1e_2\alpha) \in \mathbb{R}\backslash\{0\}. \end{cases}$$

By (3.2)–(3.3) of Lemma 3.2, we have

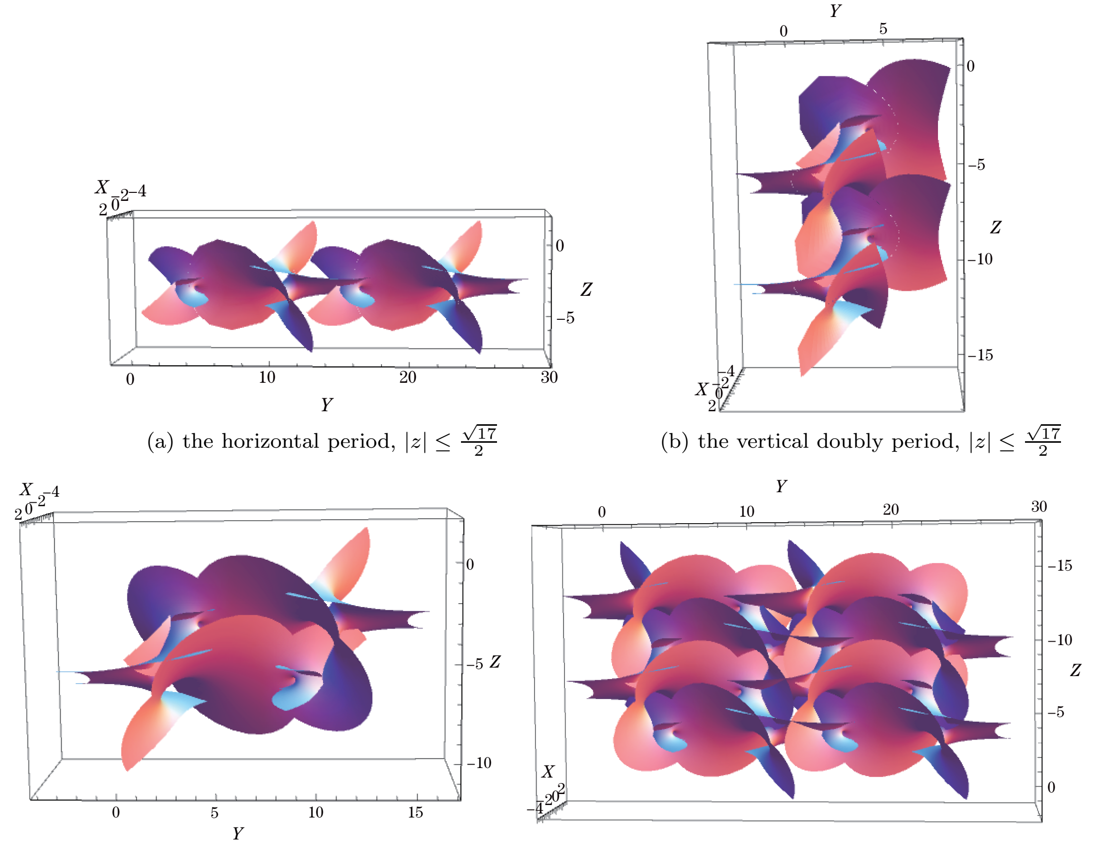
$$\begin{cases} \widetilde{I}_\tau(z+1) = \widetilde{I}_\tau(z) + (0, T_2, 0), \\ \widetilde{I}_\tau(z+\mathrm{i}\tau) = \widetilde{I}_\tau(z) + (T_1, 0, T_3). \end{cases} \tag{3.17}$$

In particular, when $\tau = 1$, $(T_1, T_2, T_3) \approx (-0.5920, 13.1584, 5.9999)$. The pictures of $\widetilde{I}_1$ are as follows (see Figures 7–8):



(a) $\widetilde{I}_1(z)$ (b) $\widetilde{I}_1(z+1)$ (c) $\widetilde{I}_1(z+\mathrm{i})$

Figure 7 $|z| \leq \frac{2\sqrt{2}}{5}$, $\tau = 1$.



(a) the horizontal period, $|z| \leq \frac{\sqrt{17}}{2}$ (b) the vertical doubly period, $|z| \leq \frac{\sqrt{17}}{2}$

(c) $|z| \leq \sqrt{2}$ (d) $|z| \leq 2\sqrt{2}$

Figure 8 The double periodic conjugate surfaces $\widetilde{I}_1(z)$.

4 Lorentz Deformation of Costa's Minimal Surfaces I_τ in $\mathbb{R}^4_1$

In this section, by orthogonal transformation and Lorentz deformation of Costa's minimal surfaces I_τ, we can get a 2-parameters family of complete space-like stationary surfaces of genus 1 with 4 planar ends in $\mathbb{R}^4_1$.

4.1 Lorentz deformation of minimal surfaces in $\mathbb{R}^3$

First we recall the Lorentz deformation for minimal surfaces in $\mathbb{R}^3$.

Definition 4.1 (see [17]) *Let $X_z\mathrm{d}z = (\Theta_1, \Theta_2, \Theta_3)\mathrm{d}z$ be the holomorphic differential of the minimal surface $X : M \to \mathbb{R}^3$ in $\mathbb{R}^3$. The Lorentz deformation $\widetilde{X}$ in $\mathbb{R}^4_1$ of the minimal surface X is defined by the following equation*

$$\widetilde{X}_z\mathrm{d}z = \Big(\Theta_1, \Theta_2, \frac{\zeta+\zeta^{-1}}{2}\Theta_3, \frac{\zeta-\zeta^{-1}}{2}\Theta_3\Big)\mathrm{d}z \tag{4.1}$$

with

$$\phi_t = \mathrm{e}^{-\mathrm{i}t}\phi, \quad \psi_t = -\mathrm{e}^{-\mathrm{i}t}\frac{1}{\phi}, \quad \mathrm{d}h_t = \mathrm{e}^{\mathrm{i}t}\mathrm{d}h, \quad \mathrm{e}^{\mathrm{i}t} \in \mathbb{C}\backslash \mathrm{i}\mathbb{R}. \tag{4.2}$$

Here $\zeta = \mathrm{e}^{\mathrm{i}t} \in \mathbb{C}\backslash\mathrm{i}\mathbb{R}$ *is the parameter. Note that when* $\zeta = 1$, $\widetilde{X}$ *reduces to* X.

Theorem 4.1 (see [17]) *Let* $X : M \to \mathbb{R}^3$ *be a complete minimal surface with no real or imaginary vertical period, i.e.,*

$$\int_\gamma \Theta_3 \mathrm{d}z = 0 \quad \text{for any closed curve } \gamma \text{ on } M.$$

Then for any $\zeta = \mathrm{e}^{\mathrm{i}t} \in \mathbb{C}\backslash\mathrm{i}\mathbb{R}$, $\{\phi_t, \psi_t, \mathrm{d}h_t\}$ *in* (4.2)*, we define a* 2*-parameter family of complete stationary surfaces* $\widetilde{X}$ *fully immersed in* $\mathbb{R}^4_1$.

4.2 Lorentz deformation of Costa's minimal surfaces I_τ

We retain the notion of Section 3. Recall that (see (1.3))

$$(I_\tau)_z \mathrm{d}z = (\Phi_1, \Phi_2, \Phi_3)\mathrm{d}z$$

with f and g defined in (1.4). Set

$$A = \begin{pmatrix} \mu_1 & 0 & -\mu_2 \\ 0 & 1 & 0 \\ \mu_2 & 0 & \mu_1 \end{pmatrix} \quad \text{with } \mu_1 = \frac{T_1}{\sqrt{T_1^2 + T_3^2}},\ \mu_2 = \frac{T_3}{\sqrt{T_1^2 + T_3^2}}, \tag{4.3}$$

where T_1, T_2, T_3 are non-zero real constants given by (3.11). Set

$$(\mathrm{II}_\tau)_z = (\widetilde{\Phi_1}, \widetilde{\Phi_2}, \widetilde{\Phi_3}) = (\Phi_1, \Phi_2,\ \Phi_3)A = (\mu_1\Phi_1 + \mu_2\Phi_3, \Phi_2, -\mu_2\Phi_1 + \mu_1\Phi_3). \tag{4.4}$$

By (3.2)–(3.3), we get

$$\frac{1}{2}\int_{l_1} (\widetilde{\Phi_1}, \widetilde{\Phi_2},\ \widetilde{\Phi_3})\mathrm{d}z = \mathrm{i}(0, T_2, 0), \quad \frac{1}{2}\int_{l_3} (\widetilde{\Phi_1}, \widetilde{\Phi_2},\ \widetilde{\Phi_3})\mathrm{d}z = \mathrm{i}\Big(\sqrt{T_1^2 + T_3^2}, 0, 0\Big). \tag{4.5}$$

Now consider the surface II_τ in $\mathbb{R}^3 \subset \mathbb{R}^4_1$. We have

$$\begin{aligned}(\mathrm{II}_\tau)_z \mathrm{d}z &= (\mu_1\Phi_1 + \mu_2\Phi_3, \Phi_2, -\mu_2\Phi_1 + \mu_1\Phi_3, 0)\mathrm{d}z \\ &= (\phi + \psi, -\mathrm{i}(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi)\mathrm{d}h \end{aligned} \tag{4.6}$$

with

$$\phi = \frac{(\mu_1 + 2)g^2 - 2\mu_2 g - (\mu_1 - 2)}{\mu_2 g^2 + 2\mu_1 g - \mu_2}, \quad \psi = -\frac{1}{\phi}, \quad \mathrm{d}h = \frac{-f(\mu_2 g^2 + 2\mu_1 g - \mu_2)}{2}\mathrm{d}z, \tag{4.7}$$

Proposition 4.1 *Set*

$$\phi_t = \mathrm{e}^{-\mathrm{i}t}\phi, \quad \psi_t = \mathrm{e}^{-\mathrm{i}t}\psi, \quad \mathrm{d}h_t = \mathrm{e}^{\mathrm{i}t}\mathrm{d}h \quad \textit{with } \mathrm{e}^{\mathrm{i}t} \in \mathbb{C}\backslash\mathrm{i}\mathbb{R}, \tag{4.8}$$

where $\{\phi, \psi, \mathrm{d}h\}$ *of* II_τ *is given by* (4.7)*. Then the Weirestrass representation data* $\{\phi_t, \psi_t, \mathrm{d}h_t\}$ *defines a complete stationary surface* $\mathrm{II}_{\tau,t}: M_\tau \to \mathbb{R}^4_1$*, which has genus* 1 *and* 4 *planar ends. Moreover when* $\mathrm{e}^{\mathrm{i}t} \notin \mathbb{R}$*,* $\mathrm{II}_{\tau,t}$ *is full in* $\mathbb{R}^4_1$ *for all* $\tau \geq 1$*.*

Proof First note that the Lorentz deformation (4.2) does not change the shape of the end. By (4.5) and the properties of ends of II_τ, $\int_\gamma \widetilde{\Phi_3}\mathrm{d}z = 0$ for any closed curve γ on M. By Theorem 4.1, the proposition holds.

Remark 4.1 By the Willmore energy formula in [1] and [11], we can obtain the Willmore energy of the surface $\mathrm{II}_{\tau,t}$, as follows:

$$W(\mathrm{II}_{\tau,t}) = \int_{M_\tau} (H^2 - K)\mathrm{d}M_\tau = -\int_{M_\tau} K\mathrm{d}M_\tau = -2\pi(2 - 2g - 2r) = 16\pi, \tag{4.9}$$

where g is the genus of M_τ, and r is the number of ends.

5 Appendix

Lemma 5.1 *The inequality* $G(\tau) = 8\pi^3\tau\big(\frac{-8\mathrm{e}^{-6\pi\tau}}{(1-\mathrm{e}^{-4\pi\tau})^2} + \frac{-4\mathrm{e}^{-2\pi\tau}}{1-\mathrm{e}^{-4\pi\tau}} + \frac{2\mathrm{e}^{-3\pi\tau}}{(1-\mathrm{e}^{-2\pi\tau})^2} + \frac{\mathrm{e}^{-\pi\tau}}{1-\mathrm{e}^{-2\pi\tau}}\big) < \pi^2$ *holds for all* $\tau \geq 1$*.*

Proof Letting $x = \mathrm{e}^{\pi\tau}$, we have

$$\begin{aligned}
G'(\tau) &= 8\pi^3\Big(\frac{-8\mathrm{e}^{-6\pi\tau}}{(1-\mathrm{e}^{-4\pi\tau})^2} + \frac{-4\mathrm{e}^{-2\pi\tau}}{1-\mathrm{e}^{-4\pi\tau}} + \frac{2\mathrm{e}^{-3\pi\tau}}{(1-\mathrm{e}^{-2\pi\tau})^2} + \frac{\mathrm{e}^{-\pi\tau}}{1-\mathrm{e}^{-2\pi\tau}}\Big) + 8\pi^4\tau\Big(\frac{64\mathrm{e}^{-10\pi\tau}}{(1-\mathrm{e}^{-4\pi\tau})^3} \\
&\quad + \frac{64\mathrm{e}^{-6\pi\tau}}{(1-\mathrm{e}^{-4\pi\tau})^2} + \frac{8\mathrm{e}^{-2\pi\tau}}{1-\mathrm{e}^{-4\pi\tau}} - \frac{8\mathrm{e}^{-5\pi\tau}}{(1-\mathrm{e}^{-2\pi\tau})^3} - \frac{8\mathrm{e}^{-3\pi\tau}}{(1-\mathrm{e}^{-2\pi\tau})^2} - \frac{\mathrm{e}^{-\pi\tau}}{1-\mathrm{e}^{-2\pi\tau}}\Big) \\
&= -8\pi^3\mathrm{e}^{\pi\tau}\frac{(-1+\mathrm{e}^{\pi\tau}+3\mathrm{e}^{2\pi\tau}+5\mathrm{e}^{3\pi\tau}+5\mathrm{e}^{4\pi\tau}+3\mathrm{e}^{5\pi\tau})}{(1+\mathrm{e}^{\pi\tau})^3(1+\mathrm{e}^{2\pi\tau})^3} \\
&\quad - 8\pi^3\mathrm{e}^{\pi\tau}\frac{\mathrm{e}^{6\pi\tau} - \pi\tau + 5\mathrm{e}^{\pi\tau}\pi\tau + 9\mathrm{e}^{2\pi\tau}\pi\tau + 11\mathrm{e}^{3\pi\tau}\pi\tau - 11\mathrm{e}^{4\pi\tau}\pi\tau}{(1+\mathrm{e}^{\pi\tau})^3(1+\mathrm{e}^{2\pi\tau})^3} \\
&\quad - 8\pi^3\mathrm{e}^{\pi\tau}\frac{(-9\mathrm{e}^{5\pi\tau}\pi\tau - 5\mathrm{e}^{6\pi\tau}\pi\tau + \mathrm{e}^{7\pi\tau}\pi\tau - \mathrm{e}^{7\pi\tau})}{(1+\mathrm{e}^{\pi\tau})^3(1+\mathrm{e}^{2\pi\tau})^3} \\
&= -8\pi^3 x\frac{(-1+x+3x^2+5x^3+5x^4+3x^5)}{(1+x)^3(1+x^2)^3} \\
&\quad - 8\pi^3 x\frac{x^6 - \pi\tau + 5x\pi\tau + 9x^2\pi\tau + 11x^3\pi\tau - 11x^4\pi\tau}{(1+x)^3(1+x^2)^3} \\
&\quad - 8\pi^3 x\frac{(-9x^5\pi\tau - 5x^6\pi\tau + x^7\pi\tau - x^7)}{(1+x)^3(1+x^2)^3}.
\end{aligned}$$

When $\tau \geq 1$, by $\pi\tau \geq \pi > 3$ and $x = \mathrm{e}^{\pi\tau} \geq \mathrm{e}^{\pi} > 23$, we have

$$\begin{aligned} & -9x^5\pi\tau - 5x^6\pi\tau + x^7\pi\tau - x^7 \\ & \geq -9x^5\pi\tau - 5x^6\pi\tau + 2x^7 \\ & = (x^7 - 9x^5\pi\tau) + (x^7 - 5x^6\pi\tau) \\ & = x^5(x^2 - 9\pi\tau) + x^6(x - 5\pi\tau) \\ & > 0, \end{aligned} \tag{5.1}$$

where the functions $x^2 - 9\pi\tau$ and $x - 5\pi\tau$ are increasing about τ. Then we have

$$x^2 - 9\pi\tau \geq \mathrm{e}^{2\pi} - 9\pi > 0, \quad x - 5\pi\tau \geq \mathrm{e}^{\pi} - 5\pi > 0.$$

Similarly, we have

$$\begin{aligned} & x^6 - \pi\tau + 5x\pi\tau + 9x^2\pi\tau + 11x^3\pi\tau - 11x^4\pi\tau \\ & = (x^6 - 11x^4\pi\tau) + (5x\pi\tau - \pi\tau) + 9x^2\pi\tau + 11x^3\pi\tau \\ & = x^4(x^2 - 11\pi\tau) + (5x - 1)\pi\tau + 9x^2\pi\tau + 11x^3\pi\tau \\ & > 0 \end{aligned} \tag{5.2}$$

and

$$-1 + x + 3x^2 + 5x^3 + 5x^4 + 3x^5 > 0. \tag{5.3}$$

Combing $(1+x)^3(1+x^2)^3 > 0$, $-8\pi^3 x < 0$ with (5.1)–(5.3) , we get that $G'(\tau) < 0$ for any $\tau \geq 1$, i.e, $G(\tau)$ is a decreasing function and $G(\tau) < G(1) \approx 8.92656 < \pi^2$.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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