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An Attempt at Axiomatization of Extending Mechanism of Solutions to the Fluid Dynamical Systems*

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Abstract Note that some classic fluid dynamical systems such as the Navier-Stokes equations, Magnetohydrodynamics (MHD for short), Boussinesq equations and etc., are observably different from each other but obey some energy inequalities of the similar type. In this paper, the authors attempt to axiomatize the extending mechanism of solutions to these systems, merely starting from several basic axiomatized conditions such as the local existence, joint property of solutions and some energy inequalities. The results established have nothing to do with the concrete forms of the systems and, thus, give the extending mechanisms in a unified way to all systems obeying the axiomatized conditions. The key tools are several new multiplicative interpolation inequalities of Besov type, which have their own interests.

Keywords Interpolation inequalities of Besov type, Blow-up criteria, Navier-Stokes equations

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1 Introduction

The fluid dynamical models, just mention some of them, include the Navier-Stokes equations, Magnetohydrodynamics (MHD for short) and the liquid crystals. Generally speaking, for those models with viscosities, the local well-posedness can be established in somewhat standard way; however, recalling that it is a well-known open question to prove the global existence of smooth solutions to the three-dimensional Navier-Stokes equations, though a lot of attentions have been made, we are still far from the complete mathematical understanding of the the global well-posedness of these systems.

To understand the possible singularity or global regularity, it may be helpful for us to study some blow-up criteria. The aim of establishing the blow-up criteria is to find such conditions as weak as possible that ensure the global regularity of the solutions. There have been a lot of

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works on establishing the blow-up criteria for the classic fluid dynamical systems, in particular for the systems mentioned above, in the existing literatures. There are two well-known blow-up criteria for the Navier-Stokes equations, the Ladyzhenskaya-Prodi-Serrin type (see, e.g., [14–15]) and the Beale-Kato-Majda type (see [1]). The Ladyzhenskaya-Prodi-Serrin blow-up criteria imply that if the following Serrin condition

$$u \in L^q(0,T;L^p(\mathbb{R}^3)) \quad \text{with } \frac{2}{q}+\frac{3}{p}\le 1,\ 3<p\le\infty$$

holds for a positive time T and a pair of (p,q), then the solution to the Navier-Stokes equations will not blow-up at time T; while the Beale-Kato-Majda blow up criteria tell us that as long as the $L^1(0,T;BMO(\mathbb{R}^3))$ norm of the vorticity is finite, then the solution to the Navier-Stokes equations can be extended beyond the time T. Both cases can be proved by the energy method with Gagliardo-Nirenberg interpolation inequality or some embedding inequality of logarithmic type. Note that the endpoint case of the Serrin condition $u \in L^\infty(0,T;L^3(\mathbb{R}^3))$ is different, which was proved by Escauriaza, Seregin and Šverák [6] by employing blow-up analysis and the backward uniqueness property of the parabolic operator.

Many generalizations and extensions to other systems of these two kinds of blow-up criteria have been made. For the Navier-Stokes and MHD equations, Ladyzhenskaya-Prodi-Serrin's and Beale-Kato-Majda's criteria were considered in Sobolev space or Besov space, for example see [3–4, 7–8, 9–10, 13, 17–19] and the references therein.

We note that though the systems from the fluid dynamics may be observably different from each other, they may obey some energy inequalities of the same type. For example, one can easily check that the Navier-Stokes equations, MHD and Boussineq equations satisfy the same type energy inequalities stated in (H4), (H4$'$) and (H5), below. Therefore, there should be some common features for these systems, such as the well-posedness under the same assumptions on the initial data, and the same type blow-up criteria under the same conditions on the solutions. Keeping this in mind, it will be very interesting to reveal all these common features determined by the same characteristics of the systems from the fluid dynamics, while this paper is employed as the first attempt to exploit a tip of the iceberg of these common features.

Specifically, we list some axiomatized conditions, by which we get a unified proof for blow-up criteria of Ladyzhenskaya-Prodi-Serrin type and Beale-Kato-Majda type in Besov space of the systems satisfying these conditions. Without such restrictions, these systems may have some blow-up solutions. Recently Tao [16] proved that in the averaged Navier-Stokes equations, it is assumed that the nonlinear term satisfies all the classical harmonic analysis estimates and the fundamental cancellation property, so as to construct the exploding solution. Throughout this paper, we use $(\mathcal{P})$ to denote the Cauchy problem of an arbitrary PDE system in $\mathbb{R}^d$. The problem $(\mathcal{P})$ considered in this paper is supposed to satisfies (H1)–(H3) and at least one of (H4) and (H4$'$):

(H1) (Local existence) For any $u_0 \in H^1(\mathbb{R}^d)$ (if some other conditions, such as $\operatorname{div} u = 0$, is included in $(\mathcal{P})$, then u_0 is required to satisfy the same conditions), there exists a unique solution u to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T_*)$, with initial data u_0, and for any $T < T_*$, there hold

$$u \in C([0,T]; H^1(\mathbb{R}^d)) \cap L^2(0,T; H^2(\mathbb{R}^d)), \quad \partial_t u \in L^2(0,T; L^2(\mathbb{R}^d)),$$

where T_* is a positive number depending only on the upper bound of the initial norm $\|u_0\|_{H^1(\mathbb{R}^d)}$.

(H2) (Joint property) For any two solutions u_1 and u_2 to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T_1)$ and on $\mathbb{R}^d \times (T_1, T_2)$, respectively, with initial data u_0 and $u_1(T_1)$, respectively, such that $u_1 \in C([0, T_1]; H^1(\mathbb{R}^d))$ and $u_2 \in C([T_1, T_2); H^1(\mathbb{R}^d))$, then the joint function u defined as

$$u := \begin{cases} u_1(x,t), & t \in [0, T_1), \\ u_2(x,t), & t \in [T_1, T_2) \end{cases}$$

is a solution to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T_2)$, with initial data u_0.

(H3) (Basic energy) For any solution u to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T)$, it satisfies

$$\sup_{0 \le s \le t} \|u\|_{L^2(\mathbb{R}^d)} \le \mathcal{M}(t)$$

for any $t \in (0, T)$, where $\mathcal{M}$ is a continuous nondecreaing function on $[0, \infty)$, determined by the initial data.

(H4) (First energy inequality) For any solution u to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T)$, it satisfies the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + c_1 \|\Delta u\|_{L^2(\mathbb{R}^d)}^2 \le c_1' \int_{\mathbb{R}^d} |u|^2 |\nabla u|^2 \mathrm{d}x$$

for $t \in (0, T)$, where c_1 and c_1' are two positive constants.

(H4′) (First energy inequality′) For any solution u to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T)$, it satisfies the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + c_2 \|\Delta u\|_{L^2(\mathbb{R}^d)}^2 \le c_2' \|\nabla u\|_{L^3(\mathbb{R}^d)}^3$$

for $t \in (0, T)$, where c_2 and c_2' are two positive constants.

The following hypothesis on the second energy may also be used in some specific case (the case that $s = 1$ in Theorem 1.2, below):

(H5) (Second energy inequality) For any solution u to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T)$, it satisfies the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2(\mathbb{R}^d)}^2 + c_3 \|\nabla \Delta u\|_{L^2(\mathbb{R}^d)}^2 \le c_3' \int_{\mathbb{R}^d} (|\nabla u||\nabla^2 u|^2 + |\nabla u|^4) \mathrm{d}x$$

for $t \in (0, T)$, where c_3 and c_3' are two positive constants.

Remark 1.1 It can be verified that both the Navier-Stokes system and the MHD system meet the above conditions (H1)–(H4), (H4′) and (H5).

The first result of this paper is the following.

Theorem 1.1 *Given a positive time T_*, let $(\mathcal{P})$ be the Cauchy problem of an arbitrary PDE system, such that it satisfies the hypothesises* (H1)–(H4). *Let u be a solution to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T_*)$, satisfying*

$$u \in L^q(0, T_*; B^{-s}_{p,\infty}(\mathbb{R}^d))$$

for some constants s, p and q, such that

$$\frac{2}{q}+\frac{d}{p}=1-s \quad \text{with } p\in\Big(\frac{d}{1-s},\infty\Big] \text{ and } s\in(0,1).$$

Then the solution u can be extended uniquely beyond T_*.

Note that the hypothesis (H4′) is stronger than (H4). Therefore, one can expect that if the system ($\mathcal{P}$) satisfies the stronger hypothesis (H4′) instead of (H4), then a better result than Theorem 1.1 should hold. In fact, we have the following theorem.

Theorem 1.2 *Given a positive time T_*, let ($\mathcal{P}$) be the Cauchy problem of an arbitrary PDE system, such that it satisfies the hypothesises* (H1)–(H3) *and* (H4’). *Let u be a solution to ($\mathcal{P}$) on $\mathbb{R}^d\times(0,T_*)$, satisfying*

$$u\in L^q(0,T_*;B^s_{p,\infty}(\mathbb{R}^d))$$

for some constants s, p and q, such that

$$\frac{2}{q}+\frac{d}{p}=1+s \quad \text{with } p\in\Big(\frac{d}{1+s},\infty\Big] \text{ and } s\in(-1,1].$$

Then, for the case that $(p,s)\neq(\infty,1)$, the solution u can be extended uniquely beyond T_*. While for the case that $(p,s)=(\infty,1)$, the solution u can also be extended uniquely beyond T_*, if we have further that (H5) *holds, and $d=2,3$.*

Remark 1.2 (1) It should be noticed that if considering the Cauchy problem to the MHD equations in three dimensions, Theorem 1.2 has already been obtained in [4]. However, comparing with their work, one of our biggest features here is that we need neither the concrete structure of the equations nor the Bony decomposition and commutator estimates.

(2) Since $L^p(\mathbb{R}^d)\subsetneq B^0_{p,\infty}(\mathbb{R}^d)$ for $s=0$, the classic Ladyzhenskaya-Prodi-Serrin criterion is included in Theorem 1.2. Theorems 1.1–1.2 still hold if replacing the inhomogeneous Besov spaces by the corresponding homogeneous ones. In fact, by checking the proof, it suffices to verify that Lemmas 1.2–1.3 continue to hold if replacing the inhomogeneous Besov spaces there by the homogeneous ones. Thanks to this and noticing that $\dot B^s_{p,\infty}(\mathbb{R}^d)\subset B^s_{p,\infty}(\mathbb{R}^d)$ for $s<0$ and $\dot B^s_{p,\infty}(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)=B^s_{p,\infty}(\mathbb{R}^d)$ for $s>0$, the condition for u in Theorem 1.2 can be relaxed to

$$\begin{cases} u\in L^q(0,T_*;B^s_{p,\infty}(\mathbb{R}^d)), & -1<s\le 0,\\ u\in L^q(0,T_*;\dot B^s_{p,\infty}(\mathbb{R}^d)), & \quad 0<s\le 1\end{cases}$$

with p, q, and s obeying the relations stated in Theorem 1.2. In particular, in the case that $(p,q,s)=(\infty,1,1)$, the relaxed condition reduces to $\nabla u\in L^1(0,T_*;\dot B^0_{\infty,\infty}(\mathbb{R}^d))$, which meets the well-known Beale-Kato-Majda type criteria.

(3) Note that the endpoint case $(p,q,s)=(\infty,\infty,-1)$, i.e., the case that $u\in L^\infty(0,T;B^{-1}_{\infty,\infty})$ is excluded in both Theorems 1.1–1.2. In fact, through the regularity of the Leray-Hopf weak solutions to the three dimensional Navier-Stoke equation were obtained under the condition that $u\in C((0,T];B^{-1}_{\infty,\infty})$ (see [5]), it is still unknown if it can be relaxed to $u\in L^\infty(0,T;B^{-1}_{\infty,\infty})$.

The rest of this paper is organized as follows. In Section 2, we introduce the definition of Besov space and several new interpolation inequalities of Besov type, which are our main technical tools. In Section 3, we prove Theorems 1.1–1.2 under the abstract assumptions (H1)–(H5).

1.1 Littlewood-Paley decomposition

Let us recall some basic facts on Littlewood-Paley theory (see [2] for more details). Choose two nonnegative radial functions $\chi, \phi \in \mathcal{S}(R^n)$ supported respectively in $\left\{\xi \in R^n, |\xi| \le \frac{4}{3}\right\}$ and $\left\{\xi \in R^n, \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\}$ such that for any $\xi \in R^n$,

$$\chi(\xi) + \sum_{j\ge 0} \phi(2^{-j}\xi) = 1.$$

The frequency localization operators Δ_j and S_j are defined by

$$\begin{aligned}
&\Delta_j f = \phi(2^{-j}D)f = 2^{nj}\int_{R^n} h(2^j y) f(x-y)\mathrm{d}y \quad \text{for } j \ge 0,\\
&S_j f = \chi(2^{-j}D)f = \sum_{-1\le k\le j-1} \Delta_k f = 2^{nj}\int_{R^n} \widetilde{h}(2^j y) f(x-y)\mathrm{d}y,\\
&\Delta_{-1} f = S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \le -2,
\end{aligned}$$

where $h = \mathcal{F}^{-1}\phi$, $\widetilde{h} = \mathcal{F}^{-1}\chi$. With this choice of ϕ, it is easy to verify that

$$\Delta_j \Delta_k f = 0 \quad \text{if } |j-k| \ge 2; \quad \Delta_j(S_{k-1} g \Delta_k f) = 0 \quad \text{if } |j-k| \ge 5. \tag{1.1}$$

In terms of Δ_j, the norm of the inhomogeneous Besov space $B^s_{p,q}$ for $s \in R$ and $p, q \ge 1$ is defined by

$$\|f\|_{B^s_{p,q}} \doteq \left\|\{2^{js}\|\Delta_j f\|_p\}_{j\ge -1}\right\|_{\ell^q}$$

and

$$\|f\|_{B^s_{p,\infty}} \doteq \sup_{j\ge -1}\{2^{js}\|\Delta_j f\|_p\}.$$

We will constantly use the following Bernstein's inequality (see [2]).

Lemma 1.1 (Bernstein inequality) *Let $c \in (0,1)$ and $R > 0$. Assume that $1 \le p \le q \le \infty$ and $f \in L^p(R^n)$. Then*

$$\begin{aligned}
&\mathrm{supp}\widehat{f} \subset \{|\xi| \le R\} \Rightarrow \|\partial^\alpha f\|_q \le C R^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})}\|f\|_p,\\
&\mathrm{supp}\widehat{f} \subset \{cR \le |\xi| \le R\} \Rightarrow \|f\|_p \le C R^{-|\alpha|} \sup_{|\beta| = |\alpha|}\|\partial^\beta f\|_p,
\end{aligned}$$

where the constant C is independent of f and R.

1.2 New interpolation inequalities of Besov type

Lemma 1.2 *Let $s\in(0,\infty)$. Then, for any nonzero function $f\in H^2\cap B^{-s}_{\infty,\infty}$, the following interpolation inequality holds*

$$\|f\|_p\le C_{n,p,s}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\|f\|_{H^1}^{\frac{4}{p}-s(1-\frac{2}{p})}\|f\|_{H^2}^{s(1-\frac{2}{p})-\frac{2}{p}}$$

for any $p\in\left(2+\frac{2}{s},2+\frac{4}{s}\right)$, where

$$C_{n,p,s}=C_n\max\left\{\frac{1}{2^{s(1-\frac{2}{p})-\frac{2}{p}}-1},\frac{1}{1-2^{s(1-\frac{2}{p})-\frac{4}{p}}}\right\}$$

for a positive constant C_n depending only on n.

Proof For any integer $k\ge0$ and $p\in[2,\infty]$, it follows from the Hölder and Bernstein inequalities that

$$\begin{aligned}\|\Delta_kf\|_p&\le\|\Delta_kf\|_2^{\frac{2}{p}}\|\Delta_kf\|_\infty^{1-\frac{2}{p}}\le C_n2^{-\frac{2k}{p}}\|\nabla\Delta_kf\|_2^{\frac{2}{p}}(2^{ks}2^{-ks}\|\Delta_kf\|_\infty)^{1-\frac{2}{p}}\\&\le C_n2^{k[s(1-\frac{2}{p})-\frac{2}{p}]}\|\nabla\Delta_kf\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}},\end{aligned}$$

from which, using the Bernstein inequality again, one arrives at

$$\|\Delta_kf\|_p\le C_n2^{k[s(1-\frac{2}{p})-\frac{4}{p}]}\|\Delta\Delta_kf\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}$$

for any integer $k\ge0$. Thus, noticing that $\|\Delta_kg\|_2\le C_n\|g\|_2$, we obtain

$$\|\Delta_kf\|_p\le C_n2^{k[s(1-\frac{2}{p})-\frac{2}{p}]}\|\nabla f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}},\tag{1.2}$$

$$\|\Delta_kf\|_p\le C_n2^{-k[\frac{4}{p}-s(1-\frac{2}{p})]}\|\Delta f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\tag{1.3}$$

for any integer $k\ge0$ and $p\in[2,\infty]$. For $k=-1$, by the Hölder inequality, and noticing that $\|\Delta_{-1}f\|_2\le C_n\|f\|_2$, one deduces

$$\begin{aligned}\|\Delta_{-1}f\|_p&\le\|\Delta_{-1}f\|_2^{\frac{2}{p}}\|\Delta_{-1}f\|_\infty^{1-\frac{2}{p}}\le C_n2^{-s(1-\frac{2}{p})}\|f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\\&\le C_n2^{-[s(1-\frac{2}{p})-\frac{2}{p}]}\|f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\end{aligned}\tag{1.4}$$

for any $p\in[2,\infty]$.

For any $p\in\left(2+\frac{2}{s},2+\frac{4}{s}\right)$, one can easily verify

$$s\Big(1-\frac{2}{p}\Big)-\frac{2}{p}>0\quad\text{and}\quad\frac{4}{p}-s\Big(1-\frac{2}{p}\Big)>0.\tag{1.5}$$

On account of this, for any integer $k_0\ge0$, it follows from (1.2)–(1.4) that

$$\begin{aligned}\|f\|_p&=\Big\|\sum_{k=-1}^{\infty}\Delta_kf\Big\|_p\le\|\Delta_{-1}f\|_p+\sum_{k=0}^{k_0}\|\Delta_kf\|_p+\sum_{k=k_0+1}^{\infty}\|\Delta_kf\|_p\\&\le C_n2^{-[s(1-\frac{2}{p})-\frac{2}{p}]}\|f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}+C_n\sum_{k=0}^{k_0}2^{k[s(1-\frac{2}{p})-\frac{2}{p}]}\|\nabla f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\end{aligned}$$

$$
\begin{aligned}
&+ C_n \sum_{k=k_0+1}^{\infty} 2^{-k[\frac{4}{p}-s(1-\frac{2}{p})]}\|\Delta f\|_2^{\frac{2}{p}}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\\
&\le C_n\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\Big(\sum_{k=-1}^{k_0} 2^{k[s(1-\frac{2}{p})-\frac{2}{p}]}\|f\|_{H^1}^{\frac{2}{p}} + \sum_{k=k_0+1}^{\infty} 2^{-k[\frac{4}{p}-s(1-\frac{2}{p})]}\|\Delta f\|_2^{\frac{2}{p}}\Big)\\
&\le C_{n,p,s}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\Big(2^{k_0[s(1-\frac{2}{p})-\frac{2}{p}]}\|f\|_{H^1}^{\frac{2}{p}} + 2^{-(k_0+1)[\frac{4}{p}-s(1-\frac{2}{p})]}\|\Delta f\|_2^{\frac{2}{p}}\Big)
\end{aligned}
\tag{1.6}
$$

for any $p \in \big(2+\frac{2}{s}, 2+\frac{4}{s}\big)$.

Noting that $\frac{\|\Delta f\|_2+\|f\|_{H^1}}{\|f\|_{H^1}} \ge 1$, there is a unique integer $k_0 \ge 0$, such that

$$
2^{k_0} \le \frac{\|\Delta f\|_2 + \|f\|_{H^1}}{\|f\|_{H^1}} < 2^{k_0+1}.
$$

Choosing such k_0 in (1.6), and recalling (1.5), we obtain

$$
\begin{aligned}
\|f\|_p &\le C_{n,p,s}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\Big(\frac{\|\Delta f\|_2+\|f\|_{H^1}}{\|f\|_{H^1}}\Big)^{s(1-\frac{2}{p})-\frac{2}{p}}\|f\|_{H^1}^{\frac{2}{p}}\\
&\quad + C_{n,p,s}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\Big(\frac{\|f\|_{H^1}}{\|\Delta f\|_2+\|f\|_{H^1}}\Big)^{\frac{4}{p}-s(1-\frac{2}{p})}\|\Delta f\|_2^{\frac{2}{p}}\\
&\le C_{n,p,s}\|f\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p}}\|f\|_{H^1}^{\frac{4}{p}-s(1-\frac{2}{p})}\|f\|_{H^2}^{s(1-\frac{2}{p})-\frac{2}{p}}
\end{aligned}
$$

for any $p \in \big(2+\frac{2}{s}, 2+\frac{4}{s}\big)$. This completes the proof.

Lemma 1.3 *Let $s \in (-\infty, 1)$. Then, for any nonzero function $f \in B^s_{\infty,\infty}$ such that $\nabla f \in H^1$, we have*

$$
\|\nabla f\|_q \le C_{n,q,s}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\|\nabla f\|_2^{\frac{2}{q}-(1-s)(1-\frac{2}{q})}\|\nabla f\|_{H^1}^{(1-s)(1-\frac{2}{q})}
$$

for any $q \in \big(2, 2+\frac{2}{1-s}\big)$, where

$$
C_{n,q,s} = C_n \max\Big\{\frac{1}{2^{(1-s)(1-\frac{2}{q})}-1}, \frac{1}{1-2^{-[\frac{2}{q}-(1-s)(1-\frac{2}{q})]}}\Big\}
$$

for a positive constant C_n depending only on n.

Proof For any $q \in [2,\infty]$ and any integer $k \ge 0$, it follows from the Hölder and Bernstein inequalities that

$$
\|\nabla\Delta_k f\|_q \le \|\nabla\Delta_k f\|_2^{\frac{2}{q}}\|\nabla\Delta_k f\|_\infty^{1-\frac{2}{q}} \le C_n 2^{k(1-s)(1-\frac{2}{q})}\|\nabla\Delta_k f\|_2^{\frac{2}{q}}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}},
$$

from which, using the Bernstein inequality again, we have

$$
\|\nabla\Delta_k f\|_q \le C_n 2^{-k[\frac{2}{q}-(1-s)(1-\frac{2}{q})]}\|\Delta\Delta_k f\|_2^{\frac{2}{q}}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}.
$$

Thanks to the above two inequalities, noticing that $\|\Delta_k g\|_2 \le C_n\|g\|_2$, we get

$$
\|\nabla\Delta_k f\|_q \le C_n 2^{k(1-s)(1-\frac{2}{q})}\|\nabla f\|_2^{\frac{2}{q}}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}, \tag{1.7}
$$

$$
\|\nabla\Delta_k f\|_q \le C_n 2^{-k[\frac{2}{q}-(1-s)(1-\frac{2}{q})]}\|\Delta f\|_2^{\frac{2}{q}}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}} \tag{1.8}
$$

for any $q\in[2,\infty]$ and integer $k\geq 0$. For $k=-1$, by the Hölder and Bernstein inequalities, we have

$$\begin{aligned}\|\nabla\Delta_{-1}f\|_q&\leq\|\nabla\Delta_{-1}f\|_2^{\frac{2}{q}}\|\nabla\Delta_{-1}f\|_\infty^{1-\frac{2}{q}}\leq C_n2^{s(1-\frac{2}{q})}\|\nabla f\|_2^{\frac{2}{q}}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\\&\leq C_n2^{-(1-s)(1-\frac{2}{q})}\|\nabla f\|_2^{\frac{2}{q}}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\end{aligned}\tag{1.9}$$

for any $q\in[2,\infty]$.

For any $q\in\big(2,2+\frac{2}{1-s}\big)$, one can easily check that

$$(1-s)\Big(1-\frac{2}{q}\Big)>0\quad\text{and}\quad\frac{2}{q}-(1-s)\Big(1-\frac{2}{q}\Big)>0.$$

Thus, it follows from (1.7)–(1.9) that

$$\begin{aligned}\|\nabla f\|_q&=\Big\|\sum_{k=-1}^{\infty}\nabla\Delta_kf\Big\|_q\leq\sum_{k=-1}^{k_0}\|\nabla\Delta_kf\|_q+\sum_{k=k_0+1}^{\infty}\|\nabla\Delta_kf\|_q\\&\leq C_n\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\Big(\sum_{k=-1}^{k_0}2^{k(1-s)(1-\frac{2}{q})}\|\nabla f\|_2^{\frac{2}{q}}+\sum_{k=k_0+1}^{\infty}2^{-k[\frac{2}{q}-(1-s)(1-\frac{2}{q})]}\|\Delta f\|_2^{\frac{2}{q}}\Big)\\&\leq C_{n,p,s}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}(2^{k_0(1-s)(1-\frac{2}{q})}\|\nabla f\|_2^{\frac{2}{q}}+2^{-(k_0+1)[\frac{2}{q}-(1-s)(1-\frac{2}{q})]}\|\Delta f\|_2^{\frac{2}{q}})\end{aligned}\tag{1.10}$$

for any $k_0\geq 0$. Take $k_0\geq 0$ be the unique integer such that

$$2^{k_0}\leq\frac{\|\Delta f\|_2+\|\nabla f\|_2}{\|\nabla f\|_2}<2^{k_0+1},$$

then it follows from (1.10) that

$$\begin{aligned}\|\nabla f\|_q&\leq C_{n,p,s}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\Big(\frac{\|\Delta f\|_2+\|\nabla f\|_2}{\|\nabla f\|_2}\Big)^{(1-s)(1-\frac{2}{q})}\|\nabla f\|_2^{\frac{2}{q}}\\&\quad+C_{n,p,s}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\Big(\frac{\|\nabla f\|_2}{\|\Delta f\|_2+\|\nabla f\|_2}\Big)^{\frac{2}{q}-(1-s)(1-\frac{2}{q})}\|\Delta f\|_2^{\frac{2}{q}}\\&\leq C_{n,p,s}\|f\|_{B^s_{\infty,\infty}}^{1-\frac{2}{q}}\|\nabla f\|_2^{\frac{2}{q}-(1-s)(1-\frac{2}{q})}\|\nabla f\|_{H^1}^{(1-s)(1-\frac{2}{q})}\end{aligned}$$

for any $q\in\big(2,2+\frac{2}{1-s}\big)$, proving the conclusion.

We also will use an integral in time version of the logarithmic type inequality stated in the next lemma. Some similar inequalities can be found in Huang-Wang [12] and Hong-Li-Xin [11].

Lemma 1.4 *Suppose that $n=2,3$. Then the following inequality holds*

$$\int_{t_1}^{t_2}\|\nabla f\|_\infty\mathrm{d}t\leq C\Big[\Big(\int_{t_1}^{t_2}\|f\|_{B^1_{\infty,\infty}}\mathrm{d}t\Big)\log\Big(\int_{t_1}^{t_2}\|\nabla\Delta f\|_2\mathrm{d}t+\mathrm{e}\Big)+1\Big]$$

for any f such that the quantities in the formulas make sense and are finite, where C is a positive constant independent of t_1 and t_2.

Proof By the definition of $B^1_{\infty,\infty}$, and the Bernstein inequality, it is obvious that

$$\|\nabla\Delta_kf\|_\infty\leq C2^k\|\Delta_kf\|_\infty\leq C\|f\|_{B^1_{\infty,\infty}}\tag{1.11}$$

for any integer $k \geq -1$. Using the Bernstein inequality again, and noticing that $\|\Delta_k g\|_2 \leq C_n \|g\|_2$, we deduce

$$\|\nabla\Delta_k f\|_\infty \leq C2^{(\frac{n}{2}-2)k}\|\nabla\Delta\Delta_k f\|_2 \leq C2^{-k(2-\frac{n}{2})}\|\nabla\Delta f\|_2 \tag{1.12}$$

for any integer $k \geq 0$. With the aid of the above two inequalities, we have

$$\begin{aligned}\|\nabla f\|_\infty &= \Big\|\sum_{k=-1}^{\infty}\nabla\Delta_k f\Big\|_\infty \leq \sum_{k=-1}^{k_0}\|\nabla\Delta_k f\|_\infty + \sum_{k=k_0+1}^{\infty}\|\nabla\Delta_k f\|_\infty \\ &\leq C(k_0+1)\|f\|_{B^1_{\infty,\infty}} + C\sum_{k=k_0+1}^{\infty} 2^{-k(2-\frac{n}{2})}\|\nabla\Delta f\|_2 \\ &\leq C[(k_0+1)\|f\|_{B^1_{\infty,\infty}} + 2^{-(k_0+1)(2-\frac{n}{2})}\|\nabla\Delta f\|_2]\end{aligned}$$

for any integer $k_0 \geq 0$. Integrating the above inequality with respect to t over the interval (t_1, t_2), and choosing $k_0 \geq 0$ be the unique integer such that

$$k_0 \leq \Big[\Big(2-\frac{n}{2}\Big)\log 2\Big]^{-1}\log\Big(\int_{t_1}^{t_2}\|\nabla\Delta f\|_2 + 1\Big) < k_0 + 1,$$

we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\|\nabla f\|_\infty \mathrm{d}t &\leq C(k_0+1)\int_{t_1}^{t_2}\|f\|_{B^1_{\infty,\infty}}\mathrm{d}t + C2^{-(k_0+1)(2-\frac{n}{2})}\int_{t_1}^{t_2}\|\nabla\Delta f\|_2\mathrm{d}t \\ &\leq C\Big(\int_{t_1}^{t_2}\|f\|_{B^1_{\infty,\infty}}\mathrm{d}t\Big)\Big[\log\Big(\int_{t_1}^{t_2}\|\nabla\Delta f\|_2 + 1\Big) + 1\Big] \\ &\quad + C\mathrm{e}^{-[(2-\frac{n}{2})\log 2](k_0+1)}\int_{t_1}^{t_2}\|\nabla\Delta f\|_2\mathrm{d}t \\ &\leq C\Big[\Big(\int_{t_1}^{t_2}\|f\|_{B^1_{\infty,\infty}}\mathrm{d}t\Big)\log\Big(\int_{t_1}^{t_2}\|\nabla\Delta f\|_2 + \mathrm{e}\Big) + 1\Big],\end{aligned}$$

proving the inequality.

2 Proofs of Theorem 1.1 and Theorem 1.2

In this section, we give the proof of Theorems 1.1–1.2.

Proof of Theorem 1.1 Noticing that $B^{-s}_{p,\infty} \hookrightarrow B^{-s-\frac{n}{p}}_{\infty,\infty}$, it follows that

$$L^q(0,T;B^{-s}_{p,\infty}) \hookrightarrow L^q(0,T;B^{-s_1}_{\infty,\infty}), \quad s_1 := s + \frac{n}{p}.$$

By assumption

$$\frac{2}{q} + \frac{n}{p} = 1 - s \quad \text{with } p \in \Big(\frac{n}{1-s}, \infty\Big] \quad \text{and } s \in (0,1),$$

it has

$$s_1 := s + \frac{n}{p} = 1 - \frac{2}{q} \in [s, 1) \subseteq (0,1).$$

Thus, by the aid of the assumption in Theorem 1.1, we always has $u \in L^q(0,T;B^{-s_1}_{\infty,\infty})$ for some $s_1 \in (0,1)$. As a result, to prove Theorem 1.1, it suffice to prove the case that $p = \infty$. Because of this, without loss of generality, we suppose that $u \in L^q(0,T;B^{-s}_{\infty,\infty})$ for some $s \in (0,1)$ in the following proof.

We first note that for any $s \in (0,1)$, one can choose the numbers $p_s \in \big(2+\frac{2}{s}, 2+\frac{4}{s}\big)$ and $q_s \in \big(2, 2+\frac{2}{s+1}\big)$, such that

$$\frac{2}{p_s} + \frac{2}{q_s} = 1.$$

In fact, when p and q run in the intervals $\big(2+\frac{2}{s}, 2+\frac{4}{s}\big)$ and $\big(2, 2+\frac{2}{s+1}\big)$, respectively, then the quantity $\frac{2}{p}+\frac{2}{q}$ runs in the interval $\big(\frac{1+2s}{2+s}, 1+\frac{s}{s+1}\big)$, and each point in this interval can be arrived at by $\frac{2}{p}+\frac{2}{q}$. The validity of the above equality then follows from the observation that $\frac{1+2s}{2+s} < 1 < 1+\frac{s}{s+1}$ for any $s \in (0,1)$. By Lemmas 1.2–1.3, we have the following estimates

$$\begin{aligned}
\|u\|_{p_s} &\le C\|u\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{p_s}} \|u\|_{H^1}^{\frac{4}{p_s}-s(1-\frac{2}{p_s})} \|u\|_{H^2}^{s(1-\frac{2}{p_s})-\frac{2}{p_s}},\\
\|\nabla u\|_{q_s} &\le C\|u\|_{B^{-s}_{\infty,\infty}}^{1-\frac{2}{q_s}} \|u\|_{H^1}^{\frac{2}{q_s}-(1+s)(1-\frac{2}{q_s})} \|u\|_{H^2}^{(1+s)(1-\frac{2}{q_s})},
\end{aligned}$$

and thus

$$\begin{aligned}
\|u\|_{p_s}\|\nabla u\|_{q_s} &\le C\|u\|_{B^{-s}_{\infty,\infty}}^{2-\frac{2}{p_s}-\frac{2}{q_s}} \|u\|_{H^1}^{\frac{4}{p_s}-s(1-\frac{2}{p_s})+\frac{2}{q_s}-(1+s)(1-\frac{2}{q_s})}\\
&\quad \times \|u\|_{H^2}^{s(1-\frac{2}{p_s})-\frac{2}{p_s}+(1+s)(1-\frac{2}{q_s})} = C\|u\|_{B^{-s}_{\infty,\infty}} \|u\|_{H^1}^{1-s} \|u\|_{H^2}^{s},
\end{aligned}$$

where C is a positive constant.

With the aid of the above estimate, by hypothesis (H4), it follows from the Hölder and Young inequalities that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 + c_1\|\Delta u\|_2^2 &\le c_1' \int_{\mathbb{R}^n} |u|^2 |\nabla u|^2 \mathrm{d}x\\
&\le c_1'\|u\|_{p_s}^2 \|\nabla u\|_{q_s}^2 \le C\|u\|_{B^{-s}_{\infty,\infty}}^2 \|u\|_{H^1}^{2(1-s)} \|u\|_{H^2}^{2s}\\
&\le C\|u\|_{B^{-s}_{\infty,\infty}}^2 (\|u\|_{H^1}^{2(1-s)} \|\Delta u\|_2^{2s} + \|u\|_{H^1}^2)\\
&\le \frac{c_1}{2}\|\Delta u\|_2^2 + C(\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}} + \|u\|_{B^{-s}_{\infty,\infty}})\|u\|_{H^1}^2\\
&\le \frac{c_1}{2}\|\Delta u\|_2^2 + C(\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}} + 1)\|u\|_{H^1}^2,
\end{aligned}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 + \frac{c_1}{2}\|\Delta u\|_2^2 \le C(\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}} + 1)(\|\nabla u\|_2^2 + \|u\|_2^2)$$

for any $t \in (0,T)$. Applying the Gronwall inequality to the above inequality and recalling the hypothesis (H3), we obtain

$$\begin{aligned}
&\sup_{0\le\tau\le t} \|\nabla u\|_2^2 + \frac{c_1}{2}\int_0^t \|\Delta u\|_2^2 \mathrm{d}\tau\\
&\le \mathrm{e}^{C\int_0^t (1+\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}})\mathrm{d}\tau} \Big(\|\nabla u_0\|_2^2 + \mathcal{M}(t)\int_0^t (1+\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}})\mathrm{d}\tau\Big)\\
&\le \mathrm{e}^{C\int_0^T (1+\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}})\mathrm{d}\tau} \Big(\|\nabla u_0\|_2^2 + \mathcal{M}(T)\int_0^T (1+\|u\|_{B^{-s}_{\infty,\infty}}^{\frac{2}{1-s}})\mathrm{d}\tau\Big)
\end{aligned}$$

for any $t \in (0,T)$. On account of this, and recalling the hypothesis (H3), we obtain the a priori estimate

$$\sup_{0 \le t < T} \|u\|_{H^1}^2 + \int_0^T \|u\|_{H^2}^2 \mathrm{d}t \le \mathcal{G}_1 < \infty$$

for any $t \in (0,T)$, where $\mathcal{G}_1$ is a positive constant.

Choose a positive time $T_{**} \in (0, T_*)$. By the hypothesis (H1) and recalling the above a priori estimate, starting from the time T_{**}, there is a unique solution u_{**} to $(\mathcal{P})$ on $\mathbb{R}^n \times (T_{**}, T_{**} + \mathcal{T})$, with initial data $u(T_{**})$, such that $u_{**} \in C([T_{**}, T_{**} + \mathcal{T}]; H^1)$, where $\mathcal{T} = \mathcal{T}(\mathcal{G}_1)$. Define

$$\widetilde{u} = \begin{cases} u(x,t), & t \in [0, T_{**}), \\ u_{**}(x,t), & t \in [T_{**}, T_{**} + \mathcal{T}(\mathcal{G}_1)], \end{cases}$$

then, by hypothesis (H2), $\widetilde{u}$ is a solution to $(\mathcal{P})$ on $\mathbb{R}^n \times (0, T_{**} + \mathcal{T}(\mathcal{G}_1))$. Note that $\mathcal{T}(\mathcal{G}_1)$ is independent of T_{**}, therefore, one can choose T_{**} close enough to T_*, such that $T_{**} + \mathcal{T} > T_*$, and as a result $\widetilde{u}$ is an extension of u to a time beyond T_*. This completes the proof.

Proof of Theorem 1.2 Similarly to the situation encountered in the proof of Theorem 1.1, it suffices to consider the case $p = \infty$. Besides, without loss of generality, we suppose that

$$\sup_{0 \le t \le T} \|\nabla u\|_2^2 \ge 1,$$

otherwise, we then already have the a priori estimate $\sup\limits_{0 \le t \le T} \|\nabla u\|_2^2 \le 1$, on account of which, by the same argument as that in the last paragraph of the proof of Theorem 1.1, one can easily prove the conclusion.

We first consider the case that $s \in (-1, 1)$. By Lemma 1.3, it follows from the Young inequality that

$$\begin{aligned} \|\nabla u\|_3^3 &\le C\|u\|_{B^s_{\infty,\infty}} \|\nabla u\|_2^{1+s} \|\nabla u\|_{H^1}^{1-s} \\ &\le C\|u\|_{B^s_{\infty,\infty}} (\|\nabla u\|_2^2 + \|\nabla u\|_2^{1+s} \|\Delta u\|_2^{1-s}) \\ &\le \frac{c_2}{2}\|\Delta u\|_2^2 + C(\|u\|_{B^s_{\infty,\infty}} + \|u\|_{B^s_{\infty,\infty}}^{\frac{2}{1+s}})\|\nabla u\|_2^2 \\ &\le \frac{c_2}{2}\|\Delta u\|_2^2 + C(1 + \|u\|_{B^s_{\infty,\infty}}^{\frac{2}{1+s}})\|\nabla u\|_2^2, \end{aligned} \tag{2.1}$$

and thus it follows from the hypothesis (H4') that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 + \frac{c_2}{2}\|\Delta u\|_2^2 \le C(1 + \|u\|_{B^s_{\infty,\infty}}^{\frac{2}{1+s}})\|\nabla u\|_2^2$$

for any $t \in (0,T)$. Applying the Gronwall inequality to the above inequality yields

$$\begin{aligned} &\sup_{0 \le \tau \le t} \|\nabla u\|_2^2 + \frac{c_2}{2}\int_0^t \|\Delta u\|_2^2 \mathrm{d}\tau \\ &\le \mathrm{e}^{C\int_0^t (1 + \|u\|_{B^s_{\infty,\infty}}^{\frac{2}{1+s}})\mathrm{d}\tau} \|\nabla u_0\|_2^2 \le \mathrm{e}^{C\int_0^T (1 + \|u\|_{B^s_{\infty,\infty}}^{\frac{2}{1+s}})\mathrm{d}\tau} \|\nabla u_0\|_2^2 \end{aligned}$$

for any $t \in (0,T)$. With the aid of this a priori estimate and recalling the hypothesis (H3), we have the a priori estimate

$$\sup_{0 \le \tau \le t} \|u\|_{H^1}^2 + \int_0^t \|u\|_{H^2}^2 \mathrm{d}\tau \le \mathcal{G}_2 < \infty$$

for any $t \in (0, T)$, where $\mathcal{G}_2$ is a positive constant. Thanks to this a priori estimate, the same argument as that for the proof of Theorem 1.1 yields the conclusion for the case $s \in (-1, 1)$.

Now we consider the case that $(p, s) = (\infty, 1)$. Let ε be a sufficiently small positive number which will be determined later. Since $u \in L^1(0, T_*; B^1_{\infty,\infty}(\mathbb{R}^d))$, by the absolutely continuity of the integrals, there is a positive constant $\delta > 0$, such that

$$\int_{T_*-\delta}^{T_*} \|u\|_{B^1_{\infty,\infty}} \mathrm{d}t \le \varepsilon,$$

from which, by Lemma 1.4, we have

$$\int_{T_*-\delta}^{t} \|\nabla u\|_\infty \mathrm{d}s \le C\Big[\varepsilon \log\Big(\int_{T_*-\delta}^{t} \|\nabla\Delta u\|_2 \mathrm{d}s + \mathrm{e}\Big) + 1\Big] \tag{2.2}$$

for any $t \in [T - \delta, T)$. We set

$$f_1(t) = \|\nabla u\|_2^2(t) + c_2 \int_{T_*-\delta}^{t} \|\Delta u\|_2^2 \mathrm{d}s,$$
$$f_2(t) = \|\Delta u\|_2^2(t) + c_3 \int_{T_*-\delta}^{t} \|\nabla\Delta u\|_2^2 \mathrm{d}s$$

for any $t \in [T_* - \delta, T_*)$.

By the Ladyzhenskaya and Sobolev embedding inequalities, we deduce

$$\begin{aligned}\int_{T_*-\delta}^{t} \|\nabla u\|_4^4 \mathrm{d}s &\le C \int_{T_*-\delta}^{t} \|\nabla u\|_2^2 \|\Delta u\|_2^2 \mathrm{d}s \\ &\le C \sup_{T_*-\delta \le s \le t} f_1^2(s), \quad t \in [T_* - \delta, T_*)\end{aligned} \tag{2.3}$$

for $d = 2$, and

$$\begin{aligned}\int_{T_*-\delta}^{t} \|\nabla u\|_4^4 \mathrm{d}s &\le C \int_{T_*-\delta}^{t} \|\nabla u\|_2 \|\Delta u\|_2^3 \mathrm{d}s \le C \int_{T-\delta}^{t} f_1^{\frac12}(s) f_2^{\frac12}(s) \|\Delta u\|_2^2 \mathrm{d}s \\ &\le C f_1(t) \sup_{T_*-\delta \le s \le t} f_1^{\frac12}(s) f_2^{\frac12}(s), \quad t \in [T_* - \delta, T_*)\end{aligned} \tag{2.4}$$

for $d = 3$. It follows from (H4′) and (H5) that

$$f_1'(t) \le C\|\nabla u\|_\infty f_1(t),$$
$$f_2'(t) \le C\|\nabla u\|_\infty f_2(t) + C\|\nabla u\|_4^4,$$

from which, by the Gronwall inequality, recalling (2.2)–(2.4), and choosing ε small enough, we have

$$\begin{aligned}f_1(t) &\le C\Big(\int_{T_*-\delta}^{t} \|\nabla\Delta u\|_2 \mathrm{d}s + 1\Big)^{C\varepsilon} f_1(T_* - \delta) \\ &\le C(f_2(t) + 1)^{C\varepsilon} \le C(f_2(t) + 1)^{\frac16}\end{aligned} \tag{2.5}$$

and

$$f_2(t) \le C\Big(\int_{T_*-\delta}^{t} \|\nabla\Delta u\|_2 \mathrm{d}s + 1\Big)^{C\varepsilon} \Big(f_2(T_* - \delta) + \int_{T_*-\delta}^{t} \|\nabla u\|_4^4 \mathrm{d}s\Big)$$

$$\leq \begin{cases} C(f_2(t)+1)^{\frac{1}{6}}\Big(1+\sup\limits_{T_*-\delta\leq s\leq t} f_1^2(s)\Big), & \delta=2, \\ C(f_2(t)+1)^{\frac{1}{6}}\Big(1+f_1(t)\sup\limits_{T_*-\delta\leq s\leq t} f_1^{\frac{1}{2}}(s)f_2^{\frac{1}{2}}(s)\Big), & \delta=3 \end{cases} \tag{2.6}$$

for any $t\in[T_*-\delta,T_*)$.

Define

$$F_1(t)=\sup_{T_*-\delta\leq s\leq t} f_1(s)+1,\quad F_2(t)=\sup_{T_*-\delta\leq s\leq t} f_2(s)+1$$

for any $t\in[T_*-\delta,T_*)$. Then, it follows from (2.5) that $F_1(t)\leq CF_2^{\frac{1}{6}}(t)$, and thus it follows from (2.6) that $F_2(t)\leq CF_2^{\frac{1}{2}}(t)$ for $d=2$; similarly, $F_2(t)\leq CF_2^{\frac{1}{6}}(t)F_2^{\frac{1}{4}}(t)F_2^{\frac{1}{2}}(t)=CF_2^{\frac{11}{12}}(t)$ for $d=3$. Hence $F_1(t),F_2(t)\leq C<\infty$ for any $t\in[T_*-\delta,T_*)$. On account of this uniform in time estimates, and recalling the hypothesis (H3), one can easily obtain the a priori estimate

$$\sup_{\delta\leq\tau\leq t}\|u\|_{H^2}^2+\int_\delta^t\|u\|_{H^3}^2\mathrm{d}\tau\leq\mathcal{G}_3<\infty$$

for any $t\in(T_*-\delta,T_*)$, where $\mathcal{G}_3$ is a positive constant. On account of this estimate, the same argument as before yields the conclusion.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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