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A Brief Approach to a Riemann-Hilbert Problem on Quasi-circles*

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Abstract The authors introduce the Sobolev space $H^{\frac{1}{2}}(\Gamma)$ on a quasi-circle Γ and give a fast approach to the jump formula which gives a decomposition of an element in $H^{\frac{1}{2}}(\Gamma)$ as the boundary values of two Dirichlet functions in the complementary domains of Γ.

Keywords Sobolev space, Dirichlet space, Quasi-circle, Quasiconformal mapping
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1 Introduction

The paper deals with a special case of the Riemann-Hilbert problem or the jump problem: Given a function f on a closed Jordan curve Γ in the extended complex plane $\widehat{\mathbb{C}}$, find functions F_1 and F_2 holomorphic respectively in the complementary domains Ω^+ and Ω^- of Γ so that the difference of their boundary values is exactly f. We are mainly concerned with the existence and uniqueness of the pair F_1, F_2 and its continuous dependence on f.

The main issue in the jump problem is the regularity of the Jordan curve Γ and of the boundary function f. When both Γ and f have nice properties, the classical Plemelj-Sokhotski jump formula provides an affirmative answer to this question (see [7–9] for related materials). To make this precise, let f be integrable on a locally rectifiable closed Jordan curve Γ. Then the Cauchy integral

$$C_\Gamma f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \quad z \notin \Gamma \tag{1.1}$$

defines a holomorphic function off Γ. Set $F^+ = C_\Gamma f|_{\Omega^+}$, $F^- = C_\Gamma f|_{\Omega^-}$. When Γ is smooth and f is Hölder continuous, F^+ and F^- can be continuously extended to $\overline{\Omega^+}$ and $\overline{\Omega^-}$ respectively such that the Plemelj-Sokhotski jump formula

$$F^+ - F^- = f \tag{1.2}$$

holds on Γ. In general, it follows from the singular Cauchy integral theory (see [4]) that the functions F^+ and F^- have non-tangential limit values almost everywhere in Γ (with respect to the arc-length measure) and the jump formula (1.2) still holds on Γ when f is integrable on a general locally rectifiable Jordan curve Γ. David [5] showed that both boundary value functions

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F^+ and F^- depend continuously on $f \in L^p(\Gamma)$ $(1 < p < \infty)$ precisely when Γ is AD-regular, which means that there is a constant $C(\Gamma) > 0$ such that for all $z \in \mathbb{C}$ and $r > 0$ the arc-length of Γ contained in the disk with center z and radius r is at most $C(\Gamma)r$. Later, Semmes [20] gave a new approach to David's result for un-bounded chord-arc curves with small constants by means of quasiconformal mappings. Recall that a locally rectifiable closed Jordan curve Γ is called a chord-arc curve with constant k if length$(\widetilde{\zeta z}) \le (1+k)|\zeta - z|$ for the smaller (i.e., with less length) subarc $\widetilde{\zeta z}$ of Γ joining any two finite points z and ζ of Γ.

In this paper, we will discuss the jump problem on quasi-circles. A closed Jordan curve Γ is called a quasi-circle if there is a constant $C(\Gamma) > 0$ such that diameter$(\widetilde{\zeta z}) \le C(\Gamma)|\zeta - z|$ for the smaller (i.e., with less diameter) subarc $\widetilde{\zeta z}$ of Γ joining any two finite points z and ζ of Γ. It is obvious that a chord-arc curve must be a quasi-circle. In fact, a closed Jordan curve is a chord-arc curve if and only if it is an AD-regular quasi-circle (see [17]). In general a quasi-circle might not be rectifiable (see [12]), and Gehring and Väisälä [10] even proved that the Hausdorff dimension of a quasi-circle can take any value in [1, 2). Therefore, the Cauchy integral (1.1) may not be available for a quasi-circle Γ. On the other hand, Schippers and Staubach [19] have discussed the jump problem on quasi-circles by generalizing the Cauchy integral (1.1) using a limiting process. The purpose of the paper is to give a fast and alternative approach to the jump problem on quasi-circles based on Semmes' idea (see [20–21]). The results turn to be useful in our forthcoming paper [13] on chord-arc curves.

In the paper, C, C_1, C_2, $\cdots$ will denote universal constants that might change from one line to another, while $C(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$, $\cdots$ will denote constants that depend only on the elements put in the brackets. The notation $A \lesssim B$ $(A \gtrsim B)$ means that there is a constant C such that $A \le CB$ $(A \ge CB)$. The notation $A \asymp B$ means both $A \lesssim B$ and $A \gtrsim B$.

2 Basic Facts

A sense-preserving homeomorphism ρ of the complex plane $\mathbb{C}$ is called quasiconformal if it has locally integrable distributional derivatives $\overline{\partial}\rho$, $\partial\rho$ which satisfy the Beltrami equation

$$\overline{\partial}\rho = \mu\partial\rho, \tag{2.1}$$

where $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$ is called the Beltrami coefficient or complex dilatation of ρ. Conversely, the measurable Riemann mapping theorem (see [1]) for quasiconformal mappings says that for each $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$, there is a quasiconformal self-mapping ρ of $\mathbb{C}$ with Beltrami coefficient μ, and ρ is unique up to a Möbius transformation.

A sense-preserving self-homeomorphism h of the unit circle S^1 is said to be quasisymmetric and belongs to the class $\mathrm{QS}(S^1)$ if there exists a (least) positive constant $C(h)$, called the quasisymmetric constant of h, such that

$$\frac{1}{C(h)} \le \frac{|h(I_1)|}{|h(I_2)|} \le C(h) \tag{2.2}$$

for all pairs of adjacent arcs I_1 and I_2 on S^1 with the same arc-length $|I_1| = |I_2| (\le \pi)$. Beurling and Ahlfors [3] proved that a sense-preserving self-homeomorphism h is quasisymmetric if and only if there exists some quasiconformal homeomorphism of the unit disk $\Delta^+ = \{z : |z| < 1\}$

onto itself which has boundary values h. Later Douady and Earle [6] gave a quasiconformal extension of h to the unit disk which is conformally invariant.

It is known that a closed Jordan curve is a quasi-circle if and only if it is the image of the extended real line $\widehat{\mathbb{R}}$ or the unit circle S^1 under a global quasiconformal self-mapping of $\mathbb{C}$ (see [1]). A Jordan domain is called a quasidisk if it is bounded by a quasi-circle. Let Γ be a closed Jordan curve with complementary domains Ω^+ and Ω^-, ϕ_+ and ϕ_- map Δ^+ and $\Delta^- \doteq \widehat{\mathbb{C}} - \overline{\Delta^+}$ conformally onto Ω^+ and Ω^-, respectively. Since ϕ_+ and ϕ_- can be continuously extended to the unit circle, we can form $h_\Gamma = \phi_-^{-1} \circ \phi_+$, which is known to be a conformal sewing for Γ. It is well known that h_Γ is quasisymmetric if and only if Γ is a quasi-circle (see [1]).

In quasiconformal mapping theory, especially in the study of the Beltrami equation (2.1), there are two operators that play fundamental roles (see [1]): The Cauchy operator on the plane

$$Tf(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(w)}{w - z} \mathrm{d}u\mathrm{d}v, \tag{2.3}$$

where $f \in L^p(\mathbb{C})$, $p > 2$, with compact support. The other one is the Beurling operator S defined by

$$Sf(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(w)}{(w - z)^2} \mathrm{d}u\mathrm{d}v, \tag{2.4}$$

where $f \in L^p(\mathbb{C})$, $p > 1$. It is known that S is norm-preserving on $L^2(\mathbb{C})$, bounded on $L^p(\mathbb{C})$ for general $p > 1$. It is also known that $I - \mu S$ is invertible on $L^p(\mathbb{C})$ for $2 \le p < 1 + \|\mu\|_\infty^{-1}$ (see [2]). Finally, the following relations

$$\overline{\partial}(Tf) = f, \quad \partial(Tf) = Sf \tag{2.5}$$

hold in the distributional sense.

3 Dirichlet and Sobolev Spaces Revisited

This is mostly a review section. It contains a description of old and recent results concerning Dirichlet and Sobolev spaces on general quasi-circles. For more details (see [16, 19]).

3.1 Sobolev space

The Sobolev class $H^{\frac{1}{2}}(S^1)$ on the unit circle S^1 is the set of all functions u on the unit circle such that

$$\|u\|^2_{H^{\frac{1}{2}}} = \frac{1}{4\pi^2} \int_{S^1} \int_{S^1} \frac{|u(\zeta) - u(z)|^2}{|\zeta - z|^2} |\mathrm{d}\zeta||\mathrm{d}z| < \infty. \tag{3.1}$$

$H^{\frac{1}{2}}(S^1)$ has Möbius invariance. In fact, for each Möbius transformation $\gamma \in$Möb(S^1) keeping the unit circle, it holds that

$$(\gamma(\zeta) - \gamma(z))^2 = \gamma'(\zeta)\gamma'(z)(\zeta - z)^2 \tag{3.2}$$

from which it follows immediately that $\|u \circ \gamma\|_{H^{\frac{1}{2}}} = \|u\|_{H^{\frac{1}{2}}}$. More generally, we have the following result which shows that $H^{\frac{1}{2}}(S^1)$ can be used to characterize the quasi-symmetry of a homeomorphism.

Proposition 3.1 (see [3, 15]) *Let h be a sense-preserving self-homeomorphism of the unit circle. Then the pull-back operator P_h defined by $P_h u = u \circ h$ is a bounded operator from $H^{\frac{1}{2}}(S^1)$ into itself if and only if h is quasisymmetric.*

Let Γ be a bounded quasi-circle with complementary domains Ω^+ and $\Omega^- \ni \infty$, and as before ϕ_+ and ϕ_- with $\phi_-(\infty) = \infty$ map Δ^+ and Δ^- conformally onto Ω^+ and Ω^-, respectively. We denote by $H^+(\Gamma)$ the set of all functions f on Γ such that $f \circ \phi^+ \in H^{\frac{1}{2}}(S^1)$. By the conformal invariance of harmonic measure, $f \in H^+(\Gamma)$ is defined almost everywhere in Γ with respect to the harmonic measure. $H^-(\Gamma)$ can be defined in the same way. Noting that $h_\Gamma = \phi_-^{-1} \circ \phi_+$ is quasisymmetric, we conclude by Proposition 3.1 that $H^+(\Gamma) = H^-(\Gamma)$, which will be denoted by $H^{\frac{1}{2}}(\Gamma)$ later. Each element f in $H^{\frac{1}{2}}(\Gamma)$ can be assigned a norm $\|f\|_{H^{\frac{1}{2}}}$ equal to $\|f\|_+ \doteq \|f \circ \phi_+\|_{H^{\frac{1}{2}}}$ or $\|f\|_- \doteq \|f \circ \phi_-\|_{H^{\frac{1}{2}}}$, which are equivalent by Proposition 3.1 again.

3.2 Dirichlet space

Let Ω be a domain in the Riemann sphere $\widehat{\mathbb{C}}$. The harmonic Dirichlet space $\mathcal{H}(\Omega)$ is the set of all harmonic functions F on Ω such that the Dirichlet integral

$$\mathcal{D}_\Omega(F) = \frac{1}{\pi} \iint_\Omega (|\partial F(z)|^2 + |\overline{\partial} F(z)|^2) \mathrm{d}x\mathrm{d}y < \infty. \tag{3.3}$$

We can assign a semi-norm on $\mathcal{H}(\Omega)$ by $\|F\|^2_{\mathcal{H}(\Omega)} = \mathcal{D}_\Omega(F)$. The (analytic) Dirichlet space $\mathcal{D}(\Omega)$ consists of the analytic functions in $\mathcal{H}(\Omega)$. Both $\mathcal{H}(\Omega)$ and $\mathcal{D}(\Omega)$ are conformally invariant, namely, a conformal mapping ϕ between two domains Ω_1 and Ω_2 induces a norm-preserving isomorphism $F \mapsto F \circ \phi$ from $\mathcal{H}(\Omega_2)$ onto $\mathcal{H}(\Omega_1)$ and from $\mathcal{D}(\Omega_2)$ onto $\mathcal{D}(\Omega_1)$.

It is well known that the Dirichlet problem has a unique solution on the unit circle. Precisely, a harmonic function F in $\mathcal{H}(\Delta^+)$ (or $\mathcal{H}(\Delta^-)$) has non-tangential limit values almost everywhere in S^1 (with respect to the arc-length measure) such that $u \doteq F|_{S^1} \in H^{\frac{1}{2}}(S^1)$ satisfies $\|u\|_{H^{\frac{1}{2}}} = \|F\|_{\mathcal{H}(\Delta^+)}$ (or $\|F\|_{\mathcal{H}(\Delta^-)}$). Conversely, the usual Poisson extension operator P takes each element $u \in H^{\frac{1}{2}}(S^1)$ to $F \doteq Pu$ in $\mathcal{H}(\Delta^+)$ such that $\|F\|_{\mathcal{H}(\Delta^+)} = \|u\|_{H^{\frac{1}{2}}}$. Similarly, there exists an extension operator also denoted by P which takes each element $u \in H^{\frac{1}{2}}(S^1)$ to $F \doteq Pu$ in $\mathcal{H}(\Delta^-)$ such that $\|F\|_{\mathcal{H}(\Delta^-)} = \|u\|_{H^{\frac{1}{2}}}$.

Similarly, the Dirichlet problem has a unique solution on a quasi-circle Γ with complementary domains Ω^+ and $\Omega^- \ni \infty$. Let F be a harmonic function in $\mathcal{H}(\Omega^+)$. Then $\widetilde{F} \doteq F \circ \phi_+ \in \mathcal{H}(\Delta^+)$ and has non-tangential limit values almost everywhere in S^1 (with respect to the arc-length measure) such that $u \doteq \widetilde{F}|_{S^1} \in H^{\frac{1}{2}}(S^1)$ with $\|u\|_{H^{\frac{1}{2}}} = \|\widetilde{F}\|_{\mathcal{H}(\Delta^+)} = \|F\|_{\mathcal{H}(\Omega^+)}$. Since Γ is a quasi-circle, ϕ_+ can be extended a quasiconformal mapping to the whole plane. Then ϕ_+ takes a non-tangential limit value to a non-tangential limit value so that F has non-tangential limit values almost everywhere in Γ (with respect to the harmonic measure) such that $f \doteq F|_\Gamma$ satisfies $f \circ \phi_+ = u$. Consequently, $f \in H^{\frac{1}{2}}(\Gamma)$ with $\|f\|_+ = \|F\|_{\mathcal{H}(\Omega^+)}$. Each element F in $\mathcal{H}(\Omega^-)$ also has non-tangential limit values almost everywhere in Γ (with respect to the harmonic measure) such that $f \doteq F|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ with $\|f\|_- = \|F\|_{\mathcal{H}(\Omega^-)}$. Conversely, let $f \in H^{\frac{1}{2}}(\Gamma)$ so that $u \doteq f \circ \phi_+ \in H^{\frac{1}{2}}(S^1)$. Then $\widetilde{F} \doteq Pu \in \mathcal{H}(\Delta^+)$ and $F \doteq \widetilde{F} \circ (\phi_+)^{-1} \in \mathcal{H}(\Omega^+)$ with $\|F\|_{\mathcal{H}(\Omega^+)} = \|\widetilde{F}\|_{\mathcal{H}(\Delta^+)} = \|u\|_{H^{\frac{1}{2}}} = \|f\|_+$. F has non-tangential limit values almost everywhere in Γ (with respect to the harmonic measure) such that $f = F|_\Gamma$. Therefore, the correspondence $f \mapsto F$ induces a linear operator P_+ from $H^{\frac{1}{2}}(\Gamma)$ onto $\mathcal{H}(\Omega^+)$ such that $\|P_+ f\|_{\mathcal{H}(\Omega^+)} = \|f\|_+$. By the same way, there exists a linear operator P_- from $H^{\frac{1}{2}}(\Gamma)$ onto $\mathcal{H}(\Omega^-)$ such that $\|P_- f\|_{\mathcal{H}(\Omega^-)} = \|f\|_-$.

4 Jump Decomposition for $H^{\frac{1}{2}}(\Gamma)$

Now we can state and prove the main result of the paper.

Theorem 4.1 *Let Γ be a quasi-circle in the complex plane with complementary domains Ω^+ and $\Omega^- \ni \infty$. Then for each $f \in H^{\frac{1}{2}}(\Gamma)$ there exists a pair of functions F_1, F_2 such that $F_1 \in \mathcal{D}(\Omega^+)$, $F_2 \in \mathcal{D}(\Omega^-)$, $\|F_1\|_+ \lesssim \|f\|_{H^{\frac{1}{2}}}$, $\|F_2\|_- \lesssim \|f\|_{H^{\frac{1}{2}}}$, and the jump formula $F_1 - F_2 = f$ holds on Γ. The pair F_1, F_2 is unique up to a constant.*

Proof Consider the conformal mapping ϕ_- which maps Δ^- conformally onto Ω^- with $\phi_-(\infty) = \infty$. ϕ_- can be extended a quasiconformal mapping ρ to the whole plane whose Beltrami coefficient μ is supported in Δ^+. Given $f \in H^{\frac{1}{2}}(\Gamma)$, we set $u = f \circ \rho$. Then $u \in H^{\frac{1}{2}}(S^1)$ so that $C'_u \in L^2(\mathbb{C})$ with $\|C'_u\|_{L^2(\mathbb{C})} \lesssim \|u\|_{H^{\frac{1}{2}}} \asymp \|f\|_{H^{\frac{1}{2}}}$, where $C_u \doteq C_{S^1} u$ is the Cauchy integral of u on the unit circle and $C'_u(z) \doteq \frac{\mathrm{d}(C_u(z))}{\mathrm{d}z}$ is defined on $\mathbb{C}$ in the distributional sense (see [23]). We will prove the result by a standard density argument.

Choose $p > 2$ such that $p < 1 + \|\mu\|_\infty^{-1}$. Set

$$H_0^{\frac{1}{2}}(\Gamma) = \{f \in H^{\frac{1}{2}}(\Gamma) : C'_u \in L^p(\mathbb{C})\}.$$

Clearly, $H_0^{\frac{1}{2}}(\Gamma)$ is dense in $H^{\frac{1}{2}}(\Gamma)$. We first assume that $f \in H_0^{\frac{1}{2}}(\Gamma)$. Set

$$g = (I - \mu S)^{-1}(\mu C'_u) = \mu(I - S\mu)^{-1}(C'_u)$$

and $G = Tg$. Noting that $I - \mu S$ is invertible on $L^p(\mathbb{C})$ we conclude that $g \in L^p(\mathbb{C})$, and has support in Δ^+, so G is continuous on the whole plane with $\overline{\partial} G = g$, $\partial G = Sg$. Thus

$$\overline{\partial} G - \mu \partial G = g - \mu S g = (I - \mu S) g = \mu C'_u.$$

Letting $H = G + C_u$, we conclude that H has jump u across the unit circle S^1, and $\overline{\partial} H = g$, $\partial H = Sg + C'_u$. Thus H satisfies the Beltrami equation $\overline{\partial} H = \mu \partial H$. Then $F \doteq H \circ \rho^{-1}$ is holomorphic off Γ. Set $F_1 = F|_{\Omega^+}$, $F_2 = F|_{\Omega^-}$. Now we show that F_1, F_2 is the desired pair. It is clear that both F_1 and F_2 have non-tangential limit values almost everywhere in Γ (with respect to the harmonic measure) such that the jump formula $F_1 - F_2 = f$ holds on Γ. It remains to prove the norm estimates. Actually, by the quasi-invariance of the Dirichlet integral under quasiconformal mappings, we have

$$\begin{aligned}
\|F_1\|_+^2 = \|F_1\|^2_{\mathcal{H}(\Omega^+)} &= \mathcal{D}_{\Omega^+}(H \circ \rho^{-1}) \lesssim C_1(\|\mu\|_\infty) \mathcal{D}_{\Delta^+}(H) \\
&\le 2C_1(\|\mu\|_\infty)(\mathcal{D}_{\Delta^+}(G) + \mathcal{D}_{\Delta^+}(C_u)) \\
&\lesssim C_2(\|\mu\|_\infty)(\|g\|^2_{L^2(\mathbb{C})} + \|C'_u\|^2_{L^2(\mathbb{C})}) \\
&\le C_2(\|\mu\|_\infty)(\|(I - \mu S)^{-1}\|^2 + 1)\|C'_u\|^2_{L^2(\mathbb{C})} \\
&\lesssim C_3(\|\mu\|_\infty)\|u\|^2_{H^{\frac{1}{2}}} \asymp C_3(\|\mu\|_\infty)\|f\|^2_{H^{\frac{1}{2}}}.
\end{aligned}$$

A similar estimate can be obtained for $\|F_2\|_-$. Having these estimates for $\|F_1\|_+$ and $\|F_2\|_-$, we may obtain the jump decomposition for a general $f \in H^{\frac{1}{2}}(\Gamma)$ by the density of $H_0^{\frac{1}{2}}(\Gamma)$ in $H^{\frac{1}{2}}(\Gamma)$.

To prove the uniqueness part, we need a further result on the pull-back operator. Recall that $\mathcal{H}(\Delta^+) = \mathcal{D}(\Delta^+) \oplus \overline{\mathcal{D}(\Delta^+)}$, or precisely, for each $F \in \mathcal{H}(\Delta^+)$, there exists a unique pair of holomorphic functions ϕ and ψ in $\mathcal{D}(\Delta^+)$ with $\phi(0) - F(0) = \psi(0) = 0$ such that $F = \phi + \overline{\psi}$. Thus we may form two basic operators on the harmonic Dirichlet space $\mathcal{H}(\Delta^+)$. They are P^+ and P^-, defined respectively by $P^+F = \phi$ and $P^-F = \overline{\psi(\overline{z})}$ for $F = \phi + \overline{\psi}$. On the other hand, for each quasisymmetric homeomorphism $h \in \mathrm{QS}(S^1)$, we consider the pull-back operator P_h in the statement of Proposition 3.1. P_h (or more precisely, $P \circ P_h$) can also be considered as a linear operator from $\mathcal{H}(\Delta^+)$ into itself. Setting $P_h^+ = P^+ \circ P_h$ and $P_h^- = P^- \circ P_h$, we conclude that both P_h^+ and P_h^- can be considered as bounded-linear operators from $\mathcal{D}(\Delta^+)$ into itself, and

$$\|P_h^+\phi\|^2_{\mathcal{H}(\Delta^+)} = \|P_h^-\phi\|^2_{\mathcal{H}(\Delta^+)} + \|\phi\|^2_{\mathcal{H}(\Delta^+)}, \quad \phi \in \mathcal{D}(\Delta^+). \tag{4.1}$$

A proof of (4.1) can be found in our paper [22, Proposition 4.3] .

We proceed to prove the theorem. Let $F_1 \in \mathcal{D}(\Omega^+)$, $F_2 \in \mathcal{D}(\Omega^-)$ and $\widehat{F}_1 \in \mathcal{D}(\Omega^+)$, $\widehat{F}_2 \in \mathcal{D}(\Omega^-)$ be two pairs such that $\widehat{F}_1 - \widehat{F}_2 = F_1 - F_2 = f$ holds on Γ. Then $\widehat{F}_1 - F_1 = \widehat{F}_2 - F_2$ on Γ. Set $F = \widehat{F}_1 - F_1$, $\widehat{F} = \widehat{F}_2 - F_2$. Then $F \in \mathcal{D}(\Omega^+)$, $\widehat{F} \in \mathcal{D}(\Omega^-)$ and $F|_\Gamma = \widehat{F}|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$. We need to show that F and $\widehat{F}$ are the same constant. Consider the conformal sewing map $h_\Gamma = \phi_-^{-1} \circ \phi_+$ and the corresponding operator $P_{h_\Gamma}^+$. Set $\phi = F \circ \phi_+$, $\widehat{\psi} = \widehat{F} \circ \phi_-$. Then $\phi \in \mathcal{D}(\Delta^+)$, $\widehat{\psi} \in \mathcal{D}(\Delta^-)$. Letting $\psi(z) = \overline{\widehat{\psi}(\overline{z}^{-1})}$, then $\psi \in \mathcal{D}(\Delta^+)$. Noting that on Γ we have

$$P_{h_\Gamma}\psi = \psi \circ h_\Gamma = \overline{\widehat{\psi}} \circ h_\Gamma = \overline{\widehat{F} \circ \phi_-} \circ \phi_-^{-1} \circ \phi_+ = \overline{F} \circ \phi_+ = \overline{\phi},$$

which implies that $P_{h_\Gamma}^+\psi = \overline{\phi}(0)$. We conclude by (4.1) that ψ is a constant, which implies that F and $\widehat{F}$ are the same constant. This completes the proof of Theorem 4.1.

5 Chord-Arc Curve Case

A natural question is how to define the Sobolev space $H^{\frac{1}{2}}(\Gamma)$ on a quasi-circle Γ without using the Riemann mapping ϕ_+ (or ϕ_-). To answer this question, we assume that Γ is a locally rectifiable closed Jordan curve. Inspired by the definition of $H^{\frac{1}{2}}(S^1)$, we denote by $H(\Gamma)$ the set of all functions f on Γ such that

$$\|f\|^2_{H(\Gamma)} = \frac{1}{4\pi^2}\int_\Gamma\int_\Gamma \frac{|f(\zeta) - f(z)|^2}{|\zeta - z|^2}|\mathrm{d}\zeta||\mathrm{d}z| < \infty. \tag{5.1}$$

Just like $H^{\frac{1}{2}}(S^1)$, $H(\Gamma)$ also has Möbius invariance, namely, for each Möbius transformation $\gamma \in$Möb$(\mathbb{C})$ on the complex plane, it holds that $\|f \circ \gamma\|_{H(\Gamma)} = \|f\|_{H(\gamma(\Gamma))}$.

We recall the following classical result of Lavrentiev [11].

Proposition 5.1 (see [11]) *Let Γ be a bounded chord-arc curve with complementary domains Ω^+ and $\Omega^- \ni \infty$, ϕ_+ and ϕ_- with $\phi_-(\infty) = \infty$ map Δ^+ and Δ^- conformally onto Ω^+ and Ω^-, respectively. Then both $|\phi_+'|$ and $|\phi_-'|$ are A^∞-weights introduced by Muckenhoupt* [14] (*see also* [9]). *In particular, the harmonic measure and arc-length measure on* Γ *are absolutely continuous to each other.*

Now we prove the following proposition.

Proposition 5.2 *Let* Γ *be a bounded chord-arc curve. Then* $H^{\frac{1}{2}}(\Gamma) = H(\Gamma) \subset L^2(\Gamma)$.

Proof First we point out by Proposition 5.1 that each element in $H^{\frac{1}{2}}(\Gamma)$ or $H(\Gamma)$ is well-defined almost everywhere in Γ with respect to the harmonic measure, or equivalently, with respect to the arc-length measure. It is known that there exists some bi-Lipschitz map ϕ of the complex plane onto itself such that $\phi(S^1) = \Gamma$ (see [17]). By definition (5.1) it is clear that $f \in H(\Gamma)$ if and only if $f \circ \phi \in H^{\frac{1}{2}}(S^1)$. On the other hand, since $h = \phi^{-1} \circ \phi_+$ is quasisymmetric and $f \circ \phi_+ = f \circ \phi \circ h = P_h(f \circ \phi)$, we conclude by Proposition 3.1 that $f \circ \phi \in H^{\frac{1}{2}}(S^1)$ if and only if $f \circ \phi_+ \in H^{\frac{1}{2}}(S^1)$, or equivalently, $f \in H^{\frac{1}{2}}(\Gamma)$. Consequently, $H^{\frac{1}{2}}(\Gamma) = H(\Gamma)$. It is also clear that $f \in L^2(\Gamma)$ if and only if $f \circ \phi \in L^2(S^1)$. Since $H^{\frac{1}{2}}(S^1) \subset L^2(S^1)$, we find out that $H(\Gamma) \subset L^2(\Gamma)$.

A consequence of Theorem 4.1 and Proposition 5.2 is in the following.

Theorem 5.1 *Let* Γ *be a bounded chord-arc curve with complementary domains* Ω^+ *and* $\Omega^- \ni \infty$. *Let* $f \in H^{\frac{1}{2}}(\Gamma)$ *and* F^+, F^- *be defined by the Cauchy integral as in Section* 1. *Then* $F^+ \in \mathcal{D}(\Omega^+)$, $F^- \in \mathcal{D}(\Omega^-)$, $\|F^+\|_+ \lesssim \|f\|_{H^{\frac{1}{2}}}$, $\|F^-\|_- \lesssim \|f\|_{H^{\frac{1}{2}}}$ *and the jump formula* $F^+ - F^- = f$ *holds on* Γ. *The jump decomposition is unique.*

Proof Given $f \in H^{\frac{1}{2}}(\Gamma)$, we have a pair F_1, F_2 satisfying the conditions in Theorem 4.1, in particular, we have the jump formula $F_1 - F_2 = f$ on Γ. On the other hand, since $H(\Gamma) \subset L^2(\Gamma)$, we also have the jump formula $F^+ - F^- = f$ on Γ. By the uniqueness of such a pair of functions in both cases, we obtain that $F^+ = F_1$, $F^- = F_2$ up to a constant. The proof is completed.

Remark 5.1 Radnell, Schippers and Staubach [18] obtained a jump formula for $H(\Gamma)$ on a so-called Weil-Petersson quasi-circle Γ by a different approach. A Weil-Petersson quasi-circle must be a chord-arc curve, but not the converse.

Problem 5.1 Proposition 5.2 says that $H^{\frac{1}{2}}(\Gamma) = H(\Gamma)$ for a chord-arc curve Γ. It is not known whether the same result holds for other locally rectifiable quasi-circles.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Ahlfors, L.V., Lectures on Quasiconformal Mappings, Van Nostrand, Princeton, New York, 1966.

[2] Astala, K., Iwaniec, T. and Saksman, E., Beltrami operators in the plane, *Duke Math. J.*, **107**, 2001, 27–56.

[3] Beurling, A. and Ahlfors, L. V., The boundary correspondence under quasiconformal mappings, *Acta Math.*, **96**, 1956, 125–142.

[4] Calderón, A. P., Calderon, C. P., Fabes, E., et al., Applications of the Cauchy integral on Lipschitz curves, *Bull. Amer. Math. Soc.*, **84**, 1978, 287–290.

[5] David, G., Opérateurs intégraux singuliers sur certaines courbes du plan complexe, *Ann. Sci. École Norm. Sup.*, **17**, 1984, 157–189.

[6] Douady, A. and Earle, C. J., Conformally natural extension of homeomorphisms of the circle, *Acta Math.*, **157**, 1986, 23–48.

[7] Duren, P., Theory of H^p Spaces, Academic Press, New York-London, 1970.

[8] Gakhov, F., Boundary Value Problems, Pergamon Press, Oxford, 1966.

[9] Garnett, J. B., Bounded Analytic Functions, Academic Press, New York, 1981.

[10] Gehring, F. W. and Väisälä, J., Hausdorff dimension and quasiconformal mappings, *J. London Math. Soc.* (2), **6**, 1973, 504–512.

[11] Lavrentiev, M., Boundary problems in the theory of univalent functions, *Mat. Sb.* (N.S.), **1**, 1936, 815–844; *Amer. Math. Soc. Transl. Ser.* 2, **32**, 1963, 1–35.

[12] Lehto, O. and Virtanen, K. I., Quasiconformal Mappings in the Plane, Springer-Verlag, New York-Heidelberg, 1973.

[13] Liu, T. and Shen,Y., The Faber operator acting on BMOA, BMO-Teichmüller space and chord-arc curves, *Acta Math. Sin.* (*Engl. Ser.*), **40**, 2024, 2359–2387.

[14] Muckenhoupt, B., Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.*, **165**, 1972, 207–226.

[15] Nag, S. and Sullivan, D., Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, *Osaka J. Math.*, **32**, 1995, 1–34.

[16] Osborn, H., The Dirichlet functional I, *J. Math. Anal. Appl.*, **1**, 1960, 61–112.

[17] Pommerenke, Ch., Boundary Behaviour of Conformal Maps, Springer-Verlag, Berlin, 1992.

[18] Radnell, D., Schippers, E. and Staubach, W., Dirichlet's problem and Sokhotski-Plemelj's jump formula on Weil-Petersson-class quasidisks, *Ann. Acad. Sci. Fenn. Math.*, **41**, 2016, 119–127.

[19] Schippers, E. and Staubach, W., Harmonic reflection in quasicircles and well-posedness of a Riemann-Hilbert problem on quasidisks, *J. Math. Anal. Appl.*, **448**, 2016, 864–884.

[20] Semmes, S., Estimates for $(\bar{\partial}-\mu\partial)^{-1}$ and Calderón's theorem on the Cauchy integral, *Trans. Amer. Math. Soc.*, **306**, 1988, 191–232.

[21] Semmes, S., Quasiconformal mappings and chord-arc curves, *Tran. Amer. Math. Soc.*, **306**, 1988, 233–263.

[22] Shen, Y., Weil-Peterssen Teichmüller space, *Amer. J. Math.*, **140**, 2018, 1041–1074.

[23] Zygmund, A., Trigonometric Series, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.