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Representation Functions on the Additive Group of Residue Classes*

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Abstract For any positive integer m, let $\mathbb{Z}_m$ be the additive group of residue classes modulo m. For $A \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let the representation function $R_A(\overline{n})$ denote the number of solutions of the equation $\overline{n} = \overline{a} + \overline{a'}$ with unordered pairs $(\overline{a}, \overline{a'}) \in A \times A$. Let $m = 2^\alpha M > 2$, where α is a positive integer and M is a positive odd integer. In this paper, the author proves that if $M \geq 3$, then there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m - 2, A \cap B = \emptyset$ and $B \neq \overline{\frac{m}{2}} + A$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. The author also proves that if $M = 1$ and $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m - 2$ and $A \cap B = \emptyset$, then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $B = \overline{\frac{m}{2}} + A$.

Keywords Sárközy's problem, Representation function, Residue class
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1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, the representation function $R'_S(n)$ is the number of solutions of the equation $s + s' = n$ with $s, s' \in S$ and $s \leq s'$. Sárközy asked whether there exist two subsets A, B of $\mathbb{N}$ with $|(A \cup B) \setminus (A \cap B)| = \infty$ such that $R'_A(n) = R'_B(n)$ for all sufficiently large integers n. In the last few years, the partitions of $\mathbb{N}$ with the same representation functions have been widely studied (see [3–10, 12–16]).

For any positive integer m, let $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \cdots, \overline{m-1}\}$ be the additive group of residue classes modulo m. We define the ordering as $\overline{0} < \overline{1} < \cdots < \overline{m-1}$, and $\overline{a} \leq \overline{b}$ if and only if $\overline{a} = \overline{b}$ or $\overline{a} < \overline{b}$. For $A \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let $R_A(\overline{n})$ denote the number of solutions of $\overline{n} = \overline{a} + \overline{a'}$ with $\overline{a}, \overline{a'} \in A$ and $\overline{a} \leq \overline{a'}$. For $\overline{n} \in \mathbb{Z}_m$ and $A \subseteq \mathbb{Z}_m$, let $\overline{n} + A = \{\overline{n} + \overline{a} : \overline{a} \in A\}$.

In 2012, Yang and Chen [17] studied the analogue of Sárközy's problem in $\mathbb{Z}_m$. They determined the structure of $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.

Theorem A *The equality $R_A(\overline{n}) = R_{\mathbb{Z}_m \setminus A}(\overline{n})$ holds for all $\overline{n} \in \mathbb{Z}_m$ if and only if m is even and $\overline{t} \in A \Leftrightarrow \overline{t} + \overline{\frac{m}{2}} \notin A$ for $\overline{t} = \overline{0}, \overline{1}, \cdots, \overline{\frac{m}{2} - 1}$.*

In 2017, Yang and Tang [18] determined all sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = 2$ or $m - 1$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.

Theorem B *Let $m \geq 2$ be an integer and $A, B, T \subseteq \mathbb{Z}_m$ satisfy $A = T \cup \{\overline{a}\}, B = T \cup \{\overline{b}\}$,*

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where $\overline{a}, \overline{b} \notin T$ *and* $\overline{a} \neq \overline{b}$. *Let* $d = \frac{m}{\gcd(a-b,m)}$. *Then* $R_A(\overline{n}) = R_B(\overline{n})$ *for all* $\overline{n} \in \mathbb{Z}_m$ *if and only if the following two conditions hold*:

(i) $\{\overline{a} + j \cdot \overline{a-b} : j = 1, 2, \cdots, d-2\} \subseteq T$.

(ii) *For any integer* a' *with* $a' \not\equiv a \pmod{\gcd(a-b, m)}$, *we have* $\overline{a'} + i \cdot \overline{a-b} \in T \Leftrightarrow \overline{a'} + j \cdot \overline{a-b} \in T$ *for all integers* i, j *with* $0 \leq i \leq j \leq d-1$.

Theorem C *Let* $m \geq 2$ *be an odd integer and* $A, B \subseteq \mathbb{Z}_m$ *satisfying* $A \cup B = \mathbb{Z}_m, A \cap B = \{\overline{c}\}$ *and* $|A| = |B|$. *Then* $R_A(\overline{n}) = R_B(\overline{n})$ *for all* $\overline{n} \in \mathbb{Z}_m$ *if and only if* $\overline{t} \in A \Leftrightarrow \overline{2t} - \overline{c} \in B$ *for all* $\overline{t} \in \mathbb{Z}_m$.

Yang and Tang [18] also posed the following problem for further research.

Problem 1.1 Given a positive even integer m and an integer k with $2 \leq k \leq m-1$. Do there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $|A| = |B| = k$ and $B \neq \overline{\frac{m}{2}} + A$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$?

For other related results about partitions of $\mathbb{Z}_m$ with the same representation functions, please see [1–2] and the references therein.

In this paper, we consider for which positive even integers m there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m-2$ and $A \cap B = \emptyset$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ and obtain the following results.

Theorem 1.1 *Let* $m = 2^\alpha M$, *where* α *is a positive integer and* M *is an odd integer with* $M \geq 3$. *Then there exist two distinct sets* $A, B \subseteq \mathbb{Z}_m$ *with* $|A \cup B| = m-2, A \cap B = \emptyset$ *and* $B \neq \overline{\frac{m}{2}} + A$ *such that* $R_A(\overline{n}) = R_B(\overline{n})$ *for all* $\overline{n} \in \mathbb{Z}_m$.

Theorem 1.2 *Let* α *be an integer with* $\alpha \geq 2$ *and* $m = 2^\alpha$. *Let* $A, B \subseteq \mathbb{Z}_m$ *with* $|A \cup B| = m-2$ *and* $A \cap B = \emptyset$. *Then* $R_A(\overline{n}) = R_B(\overline{n})$ *for all* $\overline{n} \in \mathbb{Z}_m$ *if and only if* $B = \overline{\frac{m}{2}} + A$.

Throughout this paper, for a property P, we define $\theta(P) = 1$ if P is true, otherwise $\theta(P) = 0$. For any integer k and $A \subseteq \mathbb{Z}_m$, let $kA = \{k \cdot \overline{a} : \overline{a} \in A\}$. For $A, B \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let $R_{A,B}(\overline{n})$ be the number of solutions of $\overline{n} = \overline{a} + \overline{b}$ with $\overline{a} \in A$ and $\overline{b} \in B$. The characteristic function of $A \subseteq \mathbb{Z}_m$ is denoted by

$$\chi_A(n) = \begin{cases} 1, & \overline{n} \in A, \\ 0, & \overline{n} \notin A. \end{cases}$$

2 Lemmas

Lemma 2.1 (see [11, Lemma 3]) *Let* m *be a positive even integer and* $A \subseteq \mathbb{Z}_m$. *Then*

$$R_{\mathbb{Z}_m \setminus A}(\overline{n}) = \frac{m}{2} - |A| + R_A(\overline{n}), \qquad \text{if } 2 \nmid n$$

and

$$R_{\mathbb{Z}_m \setminus A}(\overline{n}) = \frac{m}{2} + 1 - |A| - \chi_A\Big(\frac{n}{2}\Big) - \chi_A\Big(\frac{n+m}{2}\Big) + R_A(\overline{n}), \quad \text{if } 2 \mid n.$$

Lemma 2.2 *Let* m *be a positive even integer. Let* $A, B \subseteq \mathbb{Z}_m$ *with* $A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}, \overline{r_2}\}$, $A \cap B = \emptyset$ *and* $|A| = |B|$. *For all* $\overline{n} \in \mathbb{Z}_m$, *we have*

$$R_B(\overline{n}) = -1 + \chi_A(n - r_1) + \chi_A(n - r_2) + \theta(\overline{n} = \overline{r_1} + \overline{r_2}) + R_A(\overline{n}), \quad \text{if } 2 \nmid n$$

and

$$R_B(\overline{n}) = \chi_A(n-r_1)+\chi_A(n-r_2)-\chi_A\Big(\frac{n}{2}\Big)-\chi_A\Big(\frac{n+m}{2}\Big)+\theta(\overline{n}=\overline{r_1}+\overline{r_2})+R_A(\overline{n}), \quad \text{if } 2\mid n.$$

Proof By $A\cup B=\mathbb{Z}_m\backslash\{\overline{r_1},\overline{r_2}\}$, $A\cap B=\emptyset$ and $|A|=|B|$, we have $B=\mathbb{Z}_m\backslash(A\cup\{\overline{r_1},\overline{r_2}\})$ and $|A\cup\{\overline{r_1},\overline{r_2}\}|=\frac{m}{2}+1$.

If $2\nmid n$, then $\theta(\overline{n}=\overline{2r_1})=0$ and $\theta(\overline{n}=\overline{2r_2})=0$. By Lemma 2.1, we have

$$\begin{aligned}
R_B(\overline{n}) &= R_{\mathbb{Z}_m\backslash(A\cup\{\overline{r_1},\overline{r_2}\})}(\overline{n})\\
&= \frac{m}{2}-|A\cup\{\overline{r_1},\overline{r_2}\}|+R_{A\cup\{\overline{r_1},\overline{r_2}\}}(\overline{n})\\
&= -1+R_{A,\{\overline{r_1},\overline{r_2}\}}(\overline{n})+R_{\{\overline{r_1},\overline{r_2}\}}(\overline{n})+R_A(\overline{n})\\
&= -1+\chi_A(n-r_1)+\chi_A(n-r_2)+\theta(\overline{n}=\overline{r_1}+\overline{r_2})+R_A(\overline{n}).
\end{aligned}$$

If $2\mid n$, then

$$\theta(\overline{n}=\overline{2r_i})-\theta\Big(\overline{\frac{n}{2}}=\overline{r_i}\Big)-\theta\Big(\overline{\frac{n+m}{2}}=\overline{r_i}\Big)=0$$

for $i\in\{1,2\}$. By Lemma 2.1, we have

$$\begin{aligned}
R_B(\overline{n}) &= R_{\mathbb{Z}_m\backslash(A\cup\{\overline{r_1},\overline{r_2}\})}(\overline{n})\\
&= \frac{m}{2}+1-|A\cup\{\overline{r_1},\overline{r_2}\}|-\chi_{A\cup\{\overline{r_1},\overline{r_2}\}}\Big(\frac{n}{2}\Big)-\chi_{A\cup\{\overline{r_1},\overline{r_2}\}}\Big(\frac{n+m}{2}\Big)+R_{A\cup\{\overline{r_1},\overline{r_2}\}}(\overline{n})\\
&= -\chi_A\Big(\frac{n}{2}\Big)-\theta\Big(\overline{\frac{n}{2}}=\overline{r_1}\Big)-\theta\Big(\overline{\frac{n}{2}}=\overline{r_2}\Big)-\chi_A\Big(\frac{n+m}{2}\Big)-\theta\Big(\overline{\frac{n+m}{2}}=\overline{r_1}\Big)\\
&\quad -\theta\Big(\overline{\frac{n+m}{2}}=\overline{r_2}\Big)+\chi_A(n-r_1)+\chi_A(n-r_2)+\theta(\overline{n}=\overline{2r_1})+\theta(\overline{n}=\overline{2r_2})\\
&\quad +\theta(\overline{n}=\overline{r_1}+\overline{r_2})+R_A(\overline{n})\\
&= \chi_A(n-r_1)+\chi_A(n-r_2)-\chi_A\Big(\frac{n}{2}\Big)-\chi_A\Big(\frac{n+m}{2}\Big)+\theta(\overline{n}=\overline{r_1}+\overline{r_2})+R_A(\overline{n}).
\end{aligned}$$

This completes the proof of Lemma 2.2.

3 Proof of Theorem 1.1

If $\alpha=1$, then $m=2M$. Let $\overline{r_1}=\overline{0}$, $\overline{r_2}=\overline{1}$ and

$$A=\{\overline{2},\overline{4},\cdots,\overline{2M-2}\},\quad B=\{\overline{3},\overline{5},\cdots,\overline{2M-1}\}.$$

Clearly, $A\cup B=\mathbb{Z}_m\backslash\{\overline{r_1},\overline{r_2}\}$, $A\cap B=\emptyset$, $|A|=|B|=\frac{m}{2}-1$ and $B\neq\overline{\frac{m}{2}}+A$. If $2\nmid n$, then

$$\chi_A(n)+\chi_A(n-1)+\theta(\overline{n}=\overline{1})=1.$$

By Lemma 2.2, we have

$$R_B(\overline{n})=-1+\chi_A(n)+\chi_A(n-1)+\theta(\overline{n}=\overline{1})+R_A(\overline{n})=R_A(\overline{n}).$$

If $2\mid n$, then $\theta(\overline{n}=\overline{1})=0$ and

$$\chi_A(n)+\chi_A(n-1)-\chi_A\Big(\frac{n}{2}\Big)-\chi_A\Big(\frac{n+m}{2}\Big)+\theta(\overline{n}=\overline{1})=0.$$

By Lemma 2.2, we have

$$R_B(\overline{n}) = \chi_A(n) + \chi_A(n-1) - \chi_A\Big(\frac{n}{2}\Big) - \chi_A\Big(\frac{n+m}{2}\Big) + \theta(\overline{n} = \overline{1}) + R_A(\overline{n}) = R_A(\overline{n}).$$

Now we consider the case of $\alpha \ge 2$. Let $\overline{r_1} = \overline{0}$, $\overline{r_2} = \overline{2^{\alpha-1}}$ and

$$A = \bigcup_{k=0}^{M-1} \{\overline{2^\alpha k},\ \overline{2^\alpha k+1},\ \cdots,\ \overline{2^\alpha k + 2^{\alpha-1}-1}\}\backslash\{\overline{r_1}\}, \tag{3.1}$$

$$B = \bigcup_{k=0}^{M-1} \{\overline{2^\alpha k + 2^{\alpha-1}},\ \overline{2^\alpha k + 2^{\alpha-1}+1},\ \cdots,\ \overline{2^\alpha k + 2^{\alpha}-1}\}\backslash\{\overline{r_2}\}. \tag{3.2}$$

It is clear that $A \cup B = \mathbb{Z}_m \backslash \{\overline{r_1}, \overline{r_2}\}$, $A \cap B = \emptyset$, $|A| = |B| = \frac{m}{2} - 1$ and $B \ne \overline{\frac{m}{2}} + A$. By (3.1) and (3.2), we have $\chi_A(0) = \chi_A(2^{\alpha-1}) = 0$ and

$$\chi_A(1) = \chi_A(2) = \cdots = \chi_A(2^{\alpha-1}-1) = 1,$$
$$\chi_A(2^{\alpha-1}+1) = \chi_A(2^{\alpha-1}+2) = \cdots = \chi_A(2^{\alpha}-1) = 0$$

and

$$\chi_A(2^\alpha k) = \chi_A(2^\alpha k+1) = \cdots = \chi_A(2^\alpha k + 2^{\alpha-1}-1) = 1,$$
$$\chi_A(2^\alpha k + 2^{\alpha-1}) = \chi_A(2^\alpha k + 2^{\alpha-1}+1) = \cdots = \chi_A(2^\alpha k + 2^{\alpha}-1) = 0$$

for $k = 1, \cdots, M-1$. If $2^\alpha \mid n$, then $\overline{n} = \overline{2^\alpha s}$ for $s \in \{0, 1, \cdots, M-1\}$. Thus $\theta(\overline{n} = \overline{2^{\alpha-1}}) = 0$ and

$$\begin{aligned}
&\chi_A(n) + \chi_A(n - 2^{\alpha-1}) - \chi_A\Big(\frac{n}{2}\Big) - \chi_A\Big(\frac{n+m}{2}\Big) + \theta(\overline{n} = \overline{2^{\alpha-1}})\\
&= \chi_A(2^\alpha s) + \chi_A(2^\alpha s - 2^{\alpha-1}) - \chi_A(2^{\alpha-1}s) - \chi_A(2^{\alpha-1}M + 2^{\alpha-1}s)\\
&= \chi_A(2^\alpha s) + \chi_A(2^\alpha (s-1) + 2^{\alpha-1}) - \chi_A(2^{\alpha-1}s) - \chi_A(2^{\alpha-1}(M+s))\\
&= 0.
\end{aligned} \tag{3.3}$$

If $2^\alpha \nmid n$, then $\overline{n} = \overline{2^\alpha s + 2^l t_l}$ for $s \in \{0, 1, \cdots, M-1\}$, $l \in \{0, 1, \cdots, \alpha-1\}$ and $t_l \in \{2i-1 : i = 1, 2, \cdots, 2^{\alpha-l-1}\}$. If $l = 0$, then $t_0 \in \{1, 3, \cdots, 2^\alpha - 1\}$. Thus $2 \nmid n$, $\theta(\overline{n} = \overline{2^{\alpha-1}}) = 0$ and

$$\chi_A(n) + \chi_A(n - 2^{\alpha-1}) + \theta(\overline{n} = \overline{2^{\alpha-1}}) = 1. \tag{3.4}$$

By Lemma 2.2 and (3.4), if $2 \nmid n$, then

$$R_B(\overline{n}) = -1 + \chi_A(n) + \chi_A(n - 2^{\alpha-1}) + \theta(\overline{n} = \overline{2^{\alpha-1}}) + R_A(\overline{n}) = R_A(\overline{n}).$$

If $l \in \{1, \cdots, \alpha-1\}$, then

$$\begin{aligned}
&\chi_A(n) + \chi_A(n - 2^{\alpha-1}) - \chi_A\Big(\frac{n}{2}\Big) - \chi_A\Big(\frac{n+m}{2}\Big) + \theta(\overline{n} = \overline{2^{\alpha-1}})\\
&= \chi_A(2^\alpha s + 2^l t_l) + \chi_A(2^\alpha s + 2^l t_l - 2^{\alpha-1}) - \chi_A(2^{\alpha-1}s + 2^{l-1}t_l)\\
&\quad - \chi_A(2^{\alpha-1}(M+s) + 2^{l-1}t_l) + \theta(\overline{2^\alpha s + 2^l t_l} = \overline{2^{\alpha-1}})\\
&= 0.
\end{aligned} \tag{3.5}$$

By Lemma 2.2, (3.3) and (3.5), if $2 \mid n$, then

$$R_B(\overline{n}) = \chi_A(n) + \chi_A(n - 2^{\alpha-1}) - \chi_A\Big(\frac{n}{2}\Big) - \chi_A\Big(\frac{n+m}{2}\Big) + \theta(\overline{n} = \overline{2^{\alpha-1}}) + R_A(\overline{n}) = R_A(\overline{n}).$$

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

If $B = \overline{\frac{m}{2}} + A$, then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. Now we suppose that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. Then

$$\binom{|A|}{2} + |A| = \sum_{\overline{n} \in \mathbb{Z}_m} R_A(\overline{n}) = \sum_{\overline{n} \in \mathbb{Z}_m} R_B(\overline{n}) = \binom{|B|}{2} + |B|.$$

Thus $|A| = |B|$. Noting that

$$|A| + |B| = |A \cup B| + |A \cap B| = m - 2,$$

we have $|A| = |B| = \frac{m}{2} - 1$.

By $|A \cup B| = m - 2$, we may suppose that $A \cup B = \mathbb{Z}_m \backslash \{\overline{r_1}, \overline{r_2}\}$. It is clear that $(-\overline{r_1} + A) \cup (-\overline{r_1} + B) = \mathbb{Z}_m \backslash \{\overline{0}, \overline{r_2 - r_1}\}, (-\overline{r_1} + A) \cap (-\overline{r_1} + B) = \emptyset$ and $R_{-\overline{r_1}+A}(\overline{n}) = R_{-\overline{r_1}+B}(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. This allows us to consider $\overline{r_1} = \overline{0}$. Moreover, if x is an odd integer, then $(xA) \cup (xB) = \mathbb{Z}_m \backslash \{\overline{0}, \overline{xr_2}\}, (xA) \cap (xB) = \emptyset$ and $R_{xA}(\overline{n}) = R_{xB}(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. Thus we can suppose that $r_2 \mid m$. Clearly, the result is true for $\alpha = 2$. Now we may assume that $\alpha \geq 3$. For all $\overline{n} \in \mathbb{Z}_m$, by Lemma 2.2, we have

$$\chi_A(n) + \chi_A(n - r_2) + \theta(\overline{n} = \overline{r_2}) = 1, \quad \text{if } 2 \nmid n \tag{4.1}$$

and

$$\chi_A(n) + \chi_A(n - r_2) - \chi_A\Big(\frac{n}{2}\Big) - \chi_A\Big(\frac{n+m}{2}\Big) + \theta(\overline{n} = \overline{r_2}) = 0, \quad \text{if } 2 \mid n. \tag{4.2}$$

Case 1 $r_2 = 1$. By choosing $n = 2k + 1$ for $k \in \{1, 2, \cdots, 2^{\alpha-1} - 1\}$ in (4.1), we have $\theta(\overline{n} = \overline{1}) = 0$ and

$$\chi_A(2k + 1) + \chi_A(2k) = 1. \tag{4.3}$$

By choosing $n = 4l, 4l+2$ for $l \in \{1, 2, \cdots, 2^{\alpha-2} - 1\}$ in (4.2) respectively, we have $\theta(\overline{n} = \overline{1}) = 0$ and

$$\chi_A(4l) + \chi_A(4l - 1) - \chi_A(2l) - \chi_A(2^{\alpha-1} + 2l) = 0, \tag{4.4}$$

$$\chi_A(4l + 2) + \chi_A(4l + 1) - \chi_A(2l + 1) - \chi_A(2^{\alpha-1} + 2l + 1) = 0. \tag{4.5}$$

By (4.3)–(4.5), we have

$$\chi_A(4l - 1) + \chi_A(4l + 2) = 1.$$

Again, by (4.3), we have

$$\chi_A(4l - 1) + \chi_A(4l - 2) = 1.$$

Then

$$\chi_A(4l - 2) = \chi_A(4l + 2).$$

Thus

$$\chi_A(2) = \chi_A(6) = \cdots = \chi_A(2^{\alpha-1} + 2) = \cdots = \chi_A(m - 2). \tag{4.6}$$

By choosing $l = 1$ in (4.4), we have

$$\chi_A(4) + \chi_A(3) - \chi_A(2) - \chi_A(2^{\alpha-1} + 2) = 0.$$

By (4.6), we have $\chi_A(4) + \chi_A(3) = 2\chi_A(2)$. Then $\chi_A(4) = \chi_A(3) = \chi_A(2)$. However, by (4.3), we have $\chi_A(3) + \chi_A(2) = 1$, a contradiction.

Case 2 $r_2 = 2^\beta$ with $1 \le \beta \le \alpha - 2$. By choosing $n = 2^\beta k + t_0$ for $k \in \{0, 1, \cdots, 2^{\alpha-\beta} - 1\}$ and $t_0 \in \{2i - 1 : i = 1, \cdots, 2^{\beta-1}\}$ in (4.1), we have $\theta(\overline{n} = \overline{2^\beta}) = 0$ and

$$\chi_A(2^\beta k + t_0) + \chi_A(2^\beta(k - 1) + t_0) = 1. \tag{4.7}$$

By (4.7), we have

$$\chi_A(t_0) = \chi_A(2r_2 + t_0) = \cdots = \chi_A((2^{\alpha-\beta} - 2)r_2 + t_0) \tag{4.8}$$

and

$$\chi_A(r_2 + t_0) = \chi_A(3r_2 + t_0) = \cdots = \chi_A((2^{\alpha-\beta} - 1)r_2 + t_0) = 1 - \chi_A(t_0). \tag{4.9}$$

If $\beta = 1$, then $\alpha \ge 3$ and $t_0 = 1$. By choosing $n = 2$ in (4.2), we have

$$\chi_A(2) + \chi_A(0) - \chi_A(1) - \chi_A(2^{\alpha-1} + 1) + 1 = 0.$$

Noting that $\chi_A(2) = \chi_A(0) = 0$, we have $\chi_A(1) + \chi_A(2^{\alpha-1} + 1) = 1$. However, by (4.8), we have $\chi_A(1) = \chi_A(2^{\alpha-1} + 1)$, a contradiction.

If $\beta \ge 2$, then $\alpha \ge 4$. By choosing $n = 2, 2^{\beta+1} + 2, 2^\beta + 2$ in (4.2) respectively, we have $\theta(\overline{n} = \overline{2^\beta}) = 0$ and

$$\chi_A(2) + \chi_A(2^\alpha + 2 - 2^\beta) - \chi_A(1) - \chi_A(2^{\alpha-1} + 1) = 0, \tag{4.10}$$

$$\chi_A(2^{\beta+1} + 2) + \chi_A(2^\beta + 2) - \chi_A(2^\beta + 1) - \chi_A(2^{\alpha-1} + 2^\beta + 1) = 0, \tag{4.11}$$

$$\chi_A(2^\beta + 2) + \chi_A(2) - \chi_A(2^{\beta-1} + 1) - \chi_A(2^{\alpha-1} + 2^{\beta-1} + 1) = 0. \tag{4.12}$$

By (4.8)–(4.9), we have

$$\chi_A(1) = \chi_A(2^{\alpha-1} + 1),\ \chi_A(2^\beta + 1) = \chi_A(2^{\alpha-1} + 2^\beta + 1),\ \chi_A(2^{\beta-1} + 1) = \chi_A(2^{\alpha-1} + 2^{\beta-1} + 1).$$

By (4.10), we have $\chi_A(2) = \chi_A(1)$. By (4.9) and (4.11), we have

$$\chi_A(2^\beta + 2) = \chi_A(2^\beta + 1) = 1 - \chi_A(1).$$

Then $\chi_A(2) + \chi_A(2^\beta + 2) = 1$. However, by (4.12), we have $\chi_A(2) = \chi_A(2^\beta + 2)$, a contradiction.

Case 3 $r_2 = 2^{\alpha-1}$. Noting that

$$\mathbb{Z}_{2^\alpha} \backslash \{\overline{0}, \overline{2^{\alpha-1}}\} = \{\overline{2^k t_k} : k \in \{0, 1, \cdots, \alpha - 2\},\ t_k \in \{2i - 1 : i = 1, 2, \cdots, 2^{\alpha-k-1}\}\},$$

we should prove

$$\chi_A(2^k t_k) + \chi_A(2^{\alpha-1} + 2^k t_k) = 1 \tag{4.13}$$

for $k \in \{0, 1, \cdots, \alpha - 2\}$ and $t_k \in \{2i - 1 : i = 1, 2, \cdots, 2^{\alpha-k-1}\}$.

By choosing $n = t_0$ in (4.1), we have $\theta(\overline{n} = \overline{2^{\alpha-1}}) = 0$ and

$$\chi_A(t_0) + \chi_A(2^{\alpha-1} + t_0) = 1,$$

which is $k = 0$ in (4.13). Assume that $1 \leq k \leq \alpha - 2$ and

$$\chi_A(2^{k-1}t_{k-1}) + \chi_A(2^{\alpha-1} + 2^{k-1}t_{k-1}) = 1. \tag{4.14}$$

By choosing $n = 2^k t_k$ in (4.2), we have $\theta(\overline{n} = \overline{2^{\alpha-1}}) = 0$ and

$$\chi_A(2^k t_k) + \chi_A(2^{\alpha-1} + 2^k t_k) - \chi_A(2^{k-1}t_k) - \chi_A(2^{\alpha-1} + 2^{k-1}t_k) = 0.$$

By (4.14), we have

$$\chi_A(2^k t_k) + \chi_A(2^{\alpha-1} + 2^k t_k) = 1.$$

Thus $B = \overline{\frac{m}{2}} + A$.

This completes the proof of Theorem 1.2.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Chen, S. Q., Wang, R. J. and Yu, W. X., On the structure of sets in a residue class ring with the same representation function, *Adv. in Appl. Math.*, **148**, 2023, 102533.

[2] Chen, S. Q. and Yan, X. H., On certain properties of partitions of $\mathbb{Z}_m$ with the same representation function, *Discrete Math.*, **343**, 2020, 111981.

[3] Chen, Y. G. and Lev, V. F., Integer sets with identical representation functions, *Integers*, **16**, 2016, A36.

[4] Chen, Y. G. and Tang, M., Partitions of natural numbers with the same representation functions, *J. Number Theory*, **129**, 2009, 2689–2695.

[5] Chen, Y. G. and Wang, B., On additive properties of two special sequences, *Acta Arith.*, **110**, 2003, 299–303.

[6] Dombi, G., Additive properties of certain sets, *Acta Arith.*, **103**, 2002, 137–146.

[7] Kiss, S. Z. and Sándor, C., Partitions of the set of nonnegative integers with the same representation functions, *Discrete Math.*, **340**, 2017, 1154–1161.

[8] Lev, V. F., Reconstructing integer sets from their representation functions, *Electron. J. Combin.*, **11**, 2004, R78.

[9] Li, J. W. and Tang, M., Partitions of the set of nonnegative integers with the same representation functions, *Bull. Aust. Math. Soc.*, **97**, 2018, 200–206.

[10] Sándor, C., Partitions of natural numbers and their representation functions, *Integers*, **4**, 2004, A18.

[11] Sun, C. F., Xiong, M. C. and Yang, Q. H., Constructing finite sets from their representation functions, *Acta Math. Hungar*, **165**, 2021, 134–145.

[12] Tang, M., Partitions of the set of natural numbers and their representation functions, *Discrete Math.*, **308**, 2008, 2614–2616.

[13] Tang, M., Partitions of natural numbers and their representation functions, *Chinese Ann. Math. Ser A*, **37**, 2016, 41–46.

[14] Tang, M. and Chen, S. Q., On a problem of partitions of the set of nonnegative integers with the same representation functions, *Discrete Math.*, **341**, 2018, 3075–3078.

[15] Tang, M. and Li, J. W., On the structure of some sets which have the same representation functions, *Period. Math. Hungar.*, **77**, 2018, 232–236.

[16] Yan, X. H., On partitions of nonnegative integers and representation functions, *Bull. Aust. Math. Soc.*, **99**, 2018, 1–3.

[17] Yang, Q. H. and Chen, F. J., Partitions of $\mathbb{Z}_m$ with the same representation functions, *Australas. J. Combin.*, **53**, 2012, 257–262.

[18] Yang, Q. H. and Tang, M., Representation functions on finite sets with extreme symmetric differences, *J. Number Theory*, **180**, 2017, 73–85.