

Uniqueness of Positive Solutions for m -Laplacian Equations with Polynomial Non-linearity

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Abstract The author considers the uniqueness of the following positive solutions of m -Laplacian equation:

$$\begin{cases} -\Delta_m u = \lambda u^{m-1} + u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (*)$$

where $m > 1$ is a constant. When $p \rightarrow m$, the uniqueness of positive solutions of $(*)$ is shown which is based on the essential uniqueness of first eigenfunction for m -Laplacian equation. Furthermore, it is proved that the uniqueness results when $(*)$ is a perturbation of Laplacian equation.

Keywords m -Laplacian equation, Uniqueness

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1 Introduction

In this paper, we study the uniqueness of positive solutions of m -Laplacian problem:

$$\begin{cases} -\Delta_m u = \lambda u^{m-1} + u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded C^2 domain in \mathbb{R}^N , $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2} \nabla u)$.

A solution $u \in W_0^{1,m}(\Omega)$ of (1.1) is called nondegenerate if the linearized problem

$$-\operatorname{div}(|\nabla u|^{m-2} \nabla v + (m-2)|\nabla u|^{m-4} \langle \nabla u, \nabla v \rangle \nabla u) = \lambda(m-1)u^{m-2}v + (p-1)u^{p-2}v$$

admits only the trivial solution in $W_0^{1,m}(\Omega)$. The uniqueness of non-trivial solutions of

$$\begin{cases} -\Delta_m u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

remains widely open. We denote by $\mathcal{D}_0^{1,m}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\phi \mapsto \|\nabla \phi\|_{L^m(\Omega)}.$$

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The space $\mathcal{D}_0^{1,m}(\Omega)$ is continuously embedded into $L^p(\Omega)$. There is a sharp constant for this embedding, i.e., the quantity defined by

$$\lambda_{m,p}(\Omega) = \inf_{u \in \mathcal{D}_0^{1,m}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^m dx : \int_{\Omega} |u|^p dx = 1 \right\}.$$

Any minimizer of the sharp constant solves the following quasilinear version of the Lane-Emden equation

$$-\Delta_m u = \lambda_{m,p}(\Omega) |u|^{p-2} u \quad \text{in } \Omega. \quad (1.3)$$

In Laplacian case, i.e., $m = 2$, extremals are unique when $1 \leq p < 2$. One can infer uniqueness of positive solutions to (1.3) by Brezis and Oswald [5]. By this result and the fact that extremal for $\lambda_{2,p}$ never changes sign (cf. [3, Proposition 2.3]), we obtain the simplicity property. For $p = 2$, $\lambda_{2,2}$ is nothing but the first eigenvalue of the Dirichlet-Laplacian. We can get the simplicity property by Linear Spectral Theory (cf. [16, Theorem 1.2.5]). In super-homogeneous regime, uniqueness of extremal is known to hold in balls (cf. [20, Theorem 2, Corollary 1]) and for planar convex sets (cf. [22, Theorem 1]). However, there are examples that extremal may not be unique. Nazarov constructed a counter-example in [25, Proposition 1.2]. Another counter-example can be found in a starshaped set consisting of two hypercubes overlapping in a small region near one corner in [3, Example 4.7].

For general $1 < m < \infty$, uniqueness of minimizers holds for subhomogeneous case in $1 \leq p < m$. This is a consequence of stronger uniqueness result for positive solutions of (1.3) contained in [18, Theorem 4] which is the quasilinear counterpart of the result by Brezis and Oswald. We can infer simplicity by combining this result with the fact that extremals for $\lambda_{m,p}$ (see for example the proof of [18, Theorem 1]). For $p = m$, the quantity $\lambda_{m,m}(\Omega)$ is the first eigenvalue of the m -Laplacian with Dirichlet conditions, then we can get the simplicity. In super-homogeneous regime, simplicity holds in a ball, but fails in general. The counter-example by Nazarov works in this case, too. We also refer to Kawohl's paper (cf. [19]) for the very same example.

For fractional Laplacian equation, Dieb, Ianni and Saldaña discussed positive solutions of the Dirichlet Lane-Emden-type fractional problem

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.4)$$

where $s \in (0, 1)$, $p > 1$, and $\lambda \in \mathbb{R}$ (cf. [9]). They obtain the uniqueness and nondegeneracy hold for the asymptotically linear problem. Furthermore, they prove that the uniqueness and nondegeneracy results when the fractional parameter sufficiently close to 1. But the uniqueness of solutions of (1.4) does not hold in general (cf. [2, 24]).

For fully nonlinear case,

$$\begin{cases} \det D^2 u = f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

there are few results. The uniqueness of solutions of (1.5) is known for $f = \Lambda_q t^q$ and $0 \leq q < N$ in [26]. When $q = N$, this is the eigenvalue problem of Monge-Ampère equation which was studied in [23]. For super-linear case, Huang studied the uniqueness results on smooth uniformly convex domain in [17]. Cheng, Huang, and Xu obtained the uniqueness results on symmetric convex domain in \mathbb{R}^2 in [6].

1.1 Main results

Let $\lambda_m(\Omega) > 0$ denote the first Dirichlet eigenvalue for $-\Delta_m$ in Ω . For $\lambda \in (0, \lambda_m)$ and $p \in (1, \frac{Nm}{N-m})$, (1.1) has a non-trivial solution in [11, Section 6]. In this paper, we study m -Laplacian equation and obtain two following theorems.

Theorem 1.1 *Let $N \geq 2$, $p \in (2, \frac{2N}{N-2})$, $\lambda \in (0, \lambda_2(\Omega))$, and Ω be such that the problem*

$$\begin{cases} -\Delta u = \lambda u + u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.6)$$

has a unique positive solution which is nondegenerate. Then, there is $\delta_1 = \delta_1(\Omega, \lambda, p) \in (0, 1)$ such that, for $m \in (2, 2 + \delta_1]$, (1.1) has a unique positive solution.

Theorem 1.2 *Let $m \in (2, +\infty)$, $N \geq 2$, and Ω be a bounded domain of class C^2 , and $\lambda \in (0, \lambda_m(\Omega))$. There is a $p_0 = p_0(\Omega, m, \lambda) > m$ such that (1.1) has a unique positive solution for every $p \in (m, p_0)$.*

1.2 Some comments on the proof

The proof of Theorem 1.2 is largely inspired by [22, Lemma 3], dealing with the case of Laplacian. Brasco and Lindgren used the method to deal with m -Laplacian in [4].

By assuming that simplicity fails, there exists a sequence $\{q_n\}_{n \in \mathbb{N}}$ such that $q_n \rightarrow q$ and $\lambda_{p,q}$ admits two linearly independent positive minimizers u_n and v_n . Let $\tilde{u}_n = \frac{u_n}{M_n}$, $\tilde{v}_n = \frac{v_n}{M_n}$, where M_n is L^∞ norm of u_n . Their difference $\tilde{u}_n - \tilde{v}_n$ must be sign-changing on the domain and solves the linear equation

$$-\operatorname{div}(A_n \nabla(\tilde{u}_n - \tilde{v}_n)) = (m-1)\lambda \int_0^1 z_n^{m-2}(\tilde{u}_n - \tilde{v}_n) dt + M_n^{p_n-m}(p_n-1) \int_0^1 z_n^{p_n-2}(\tilde{u}_n - \tilde{v}_n) dt,$$

where

$$z_n := (1-t)\tilde{v}_n + t\tilde{u}_n.$$

The coefficient matrix

$$A_n = \int_0^1 |\nabla z_n|^{m-2} \operatorname{Id} + (m-2)|\nabla z_n|^{m-4} \nabla z_n \otimes \nabla z_n dt$$

is degenerate elliptic. More precisely

$$\frac{1}{C}(|\nabla \tilde{u}_n|^{p-2} + |\nabla \tilde{v}_n|^{p-2})|\xi|^2 \leq \langle A_n \xi, \xi \rangle \leq C(|\nabla \tilde{u}_n|^{p-2} + |\nabla \tilde{v}_n|^{p-2})|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^N.$$

This permits to obtain a compact embedding in some L^t space, for weighted Sobolev spaces of functions such that

$$\int_{\Omega} (|\nabla \tilde{u}_n|^{p-2} + |\nabla \tilde{v}_n|^{p-2}) |\nabla \phi|^2 dx < +\infty.$$

It is possible to pass the limit in the linearized equation. The conclusion of the proof is similar to that of [22]: Obtain convergence to a non-trivial limit function φ , which by construction is a sign-changing first eigenvalue of a weighted linear eigenvalue problem.

In our paper, it is difficult for us to build a uniform boundary estimate as the domain grows. We can build a uniform estimate in local region in next section. Furthermore, we do not have a

uniform priori bound for all positive solutions. We need to learn from the method in [9, Section 4] and [22]. Let maximum value of the solution divides this equation. Then we can obtain the regularity in [10].

1.3 Plan of the paper

In Section 2, we prove some preliminary facts. In particular, we devote this section to proving weighted embeddings. Here we need to go through the proofs of [4] and [8]. In Section 3, we analyze the first eigenvalue of a weighted linear eigenvalue problem. This is a crucial ingredient for the proof of our main result. We prove the existence of the first eigenfunction and its uniqueness. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.1. Finally, we include appendice, contains the crucial regularity results in [4] and [8].

2 Preliminaries

In this section, we introduce some notations and preliminary results needed throughout this paper.

Let Ω be a bounded domain of class C^2 in \mathbb{R}^N , $N \geq 2$, $1 < m < \infty$, $p > 2$. Denote $x = (x_1, \dots, x_{N-1}, x_N) = (x', x_N)$ for $x \in \mathbb{R}^N$.

Set

$$u_+ := \max\{u, 0\} \quad \text{and} \quad u_- := \max\{-u, 0\}.$$

We have the following local gradient estimate for m -laplacian equation. For the case of Laplacian, i.e., $m = 2$, we can see Qing Han's book in [15, Lemma 1.1.11, Theorem 1.1.14].

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a domain satisfying a uniform exterior sphere condition, namely, there is $r_0 > 0$ such that, for every $x^0 \in \partial\Omega$, there is $y^0 \in \mathbb{R}^N \setminus \Omega$ with $\overline{B_{r_0}(y^0)} \cap \overline{\Omega} = x^0$. Let $u \in C(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$ be a solution of*

$$\begin{cases} -\Delta_m u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.1)$$

with $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$, $\alpha \in (0, 1)$. Let $M > 0$ satisfy that $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} < M$. Then for any $x \in \Omega \cap B_1(x^0)$, $|u(x) - u(x^0)| \leq C|x - x^0|$, where C is a positive constant depending only on m , N , r_0 and M .

Proof For the given $x^0 \in \partial\Omega$, let $d(x)$ be the distance from x to $\partial B_{r_0}(y^0)$, i.e., $d(x) := |x - y^0| - r_0$.

Claim: There exists a C^2 function $\omega = \psi(d)$ defined in $[0, 1)$, with $\psi(0) = 0$, $\psi' > 0$ in $(0, 1)$ and $\Delta_m \psi < -C(m, N, r_0) < 0$.

Proof of the Claim:

$$\begin{aligned} \partial_i d(x) &= \frac{x_i - y_i^0}{|x - y^0|}, \\ \partial_{ij} d(x) &= \frac{\delta_{ij}}{|x - y^0|} - \frac{(x_i - y_i^0)(x_j - y_j^0)}{|x - y^0|^3}. \end{aligned}$$

Then

$$\Delta_m \omega = \operatorname{div}(|\psi'|^{m-1} \nabla d) = (m-1)(\psi')^{m-2} \left(\psi'' + \frac{N-1}{m-1} \frac{\psi'}{r_0 + d} \right).$$

Set $a = \frac{N-1}{(m-1)r_0}$, then we get

$$\Delta_m \omega \leq (m-1)(\psi')^{m-2}(\psi'' + a\psi' + b) - (m-1)(\psi')^{m-2}b,$$

where b is a positive constant. We choose b and find a function ψ in $[0, 1)$ such that

$$\psi'' + a\psi' + b = 0 \quad \text{in } (0, 1).$$

First, a particular solution of the ordinary differential equation above is given by

$$\psi(d) = -\frac{b}{a}d + \frac{A}{a}(1 - e^{-ad})$$

for some constant A . Let $A = \frac{2b}{a}e^a$, then ψ satisfies all the requirements we imposed. We get

$$\begin{cases} \Delta_m(C\psi) \leq \Delta_m u & \text{in } \Omega \cap B_1(x^0), \\ C\psi \geq u & \text{on } \partial(\Omega \cap B_1(x^0)), \end{cases} \quad (2.2)$$

where C is a positive constant large enough which depends only on m , N , r_0 and M . Choose the test function $(u - C\psi)_+$ on the first equation of (2.2). Let

$$A_n(x) = \int_0^1 \nabla^2 \mathcal{H}(t\nabla(C\psi) + (1-t)\nabla u) dt,$$

where \mathcal{H} is defined in Section 3 for $p = m$. Thus

$$\int_{\Omega \cap B_1(x^0)} A_n(x) (\nabla(C\psi) - \nabla u) \cdot \nabla(u - C\psi)_+ dx \geq 0.$$

Then

$$\int_{\Omega \cap B_1(x^0)} \frac{1}{4^{m-1}} |\nabla(C\psi)|^{m-2} |\nabla((u - C\psi)_+)|^2 dx \leq 0.$$

Thus

$$\nabla(u - C\psi)_+ = 0 \quad \text{a.e. in } \Omega \cap B_1(x^0).$$

We get

$$u \leq C\psi \quad \text{in } \Omega \cap B_1(x^0).$$

Similarly,

$$-u \leq C\psi \quad \text{in } \Omega \cap B_1(x^0).$$

We obtain

$$|u - u(x^0)| \leq C|\psi - \psi(x^0)|.$$

This implies the desired result.

Remark 2.1 Let Ω be a domain satisfying uniform exterior sphere condition of r_0 . Note that, if $\mu > 1$, then the domain $\mu\Omega$ satisfies in particular the uniform exterior sphere condition with the same r_0 .

Remark 2.2 K is a compact subset in $(1, +\infty)$. $\forall m \in K$, the constant C in Lemma 2.1 remains bounded uniformly.

In [10], Fan considered the equation

$$-\operatorname{div} A(x, u, Du) = B(x, u, Du) \quad \text{in } \Omega \quad (2.3)$$

with Dirichlet boundary condition

$$u = g \quad \text{on } \partial\Omega. \quad (2.4)$$

$p(x) : \overline{\Omega} \rightarrow \mathbb{R}$ satisfies the condition

$$1 < p_- = p_-(\Omega) := \inf_{x \in \Omega} p(x) \leq p_+(\Omega) := \sup_{x \in \Omega} p(x) < \infty. \quad (2.5)$$

The symbols α, β, γ or $\alpha_i, \beta_i, \gamma_i$ denote positive constants in $(0, 1)$.

Assumption (p_H) p is Hölder continuous on $\overline{\Omega}$, which is denoted by $p \in C^{0, \beta_1}(\overline{\Omega})$, that is, there exists a positive constant L and an exponent $\beta_1 \in (0, 1)$ such that

$$|p(x_1) - p(x_2)| \leq L|x_1 - x_2|^{\beta_1} \quad \text{for } x_1, x_2 \in \overline{\Omega}.$$

Assumption (A^k) $A = (A_1, A_2, \dots, A_N) \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$. For every $(x, u) \in \overline{\Omega} \times \mathbb{R}$, $A(x, u, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$, and there exists a nonnegative constant $k \geq 0$, a non-increasing continuous function $\lambda : [0, +\infty) \rightarrow (0, +\infty)$ and a non-decreasing continuous function $\Lambda : [0, +\infty) \rightarrow (0, +\infty)$ such that for all $x, x_1, x_2 \in \overline{\Omega}, u, u_1, u_2 \in \mathbb{R}^N, \eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N$, the following conditions are satisfied:

$$A(x, u, 0) = 0, \quad (2.6)$$

$$\sum_{i,j} \frac{\partial A_j}{\partial \eta_i}(x, u, \eta) \xi_i \xi_j \geq \lambda(|u|)(k + \eta^2)^{\frac{p(x)-2}{2}} |\xi|^2, \quad (2.7)$$

$$\sum_{i,j} \left| \frac{\partial A_j}{\partial \eta_i}(x, u, \eta) \right| \leq \Lambda(|u|)(k + \eta^2)^{\frac{p(x)-2}{2}}, \quad (2.8)$$

$$\begin{aligned} & |A(x_1, u_1, \eta) - A(x_2, u_2, \eta)| \\ & \leq \Lambda(\max\{|u_1|, |u_2|\})(|x_1 - x_2|^{\beta_1} + |u_1 - u_2|^{\beta_2}) \\ & \quad \times [(k + |\eta|^2)^{\frac{p(x_1)-2}{2}} + (k + |\eta|^2)^{\frac{p(x_2)-2}{2}}] |\eta| (1 + |\log(k + |\eta|^2)|). \end{aligned} \quad (2.9)$$

Assumption (B) $B : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, the function $B(x, u, \eta)$ is measurable in x and is continuous in (u, η) , and

$$|B(x, u, \eta)| \leq \Lambda(|u|)(1 + |\eta|^{p(x)}), \quad \forall (x, u, \eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \quad (2.10)$$

where Λ is as in the assumption (A^k).

Theorem 2.1 (cf. [10, Theorem 1.1]) *Under Assumptions (p_H), (A^k) and (B), suppose that $u \in W^{1, p(x)}(\Omega) \cap L^\infty(\Omega)$ is a bounded generalized solution of (2.3) and*

$$\sup_{\Omega} |u(x)| := \operatorname{ess\,sup}_{\Omega} |u(x)| \leq M \quad \text{for some } M > 0. \quad (2.11)$$

Then $u \in C_{\operatorname{loc}}^{1, \alpha}(\Omega)$, where the Hölder exponent α depends only on $p_-, p_+, N, \lambda(M), M, \Lambda(M), L, \beta_1$ and β_2 , and for given $\Omega_0 \Subset \Omega$, $|u|_{C^{1, \alpha}(\overline{\Omega}_0)}$ -norm of u , depends on $p_-, p_+, N, \lambda(M), M, \Lambda(M), L, \beta_1, \beta_2$ and $\operatorname{dist}(\Omega_0, \partial\Omega)$.

Theorem 2.2 (cf. [10, Theorem 1.2]) *Under Assumptions (p_H) , (A^k) and (B) , also let the boundary $\partial\Omega$ of Ω be of class $C^{1,\gamma}$, and $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is a bounded generalized solution of (2.3) and satisfies (2.11). Then $u \in C^{1,\alpha}(\overline{\Omega})$, where the Hölder exponent α and $|u|_{C^{1,\alpha}(\overline{\Omega})}$ depends only on p_- , p_+ , N , $\lambda(M)$, M , $\Lambda(M)$, L , β_1 , β_2 , γ , $|g|_{C^{1,\gamma}(\partial\Omega)}$ and Ω .*

Remark 2.3 m -Laplacian satisfies the Assumptions (p_H) and (A^k) .

The L^∞ norm of the solution for (1.1) in $W_0^{1,m}(\Omega)$ can be found in [14, Corollary 1.1]. Then, we can obtain the global $C^{1,\alpha}$ regularity by Theorem 2.2.

The following two lemmas are inspired in [22], dealing with the case of Laplacian.

Lemma 2.2 *Let $m > 1$, $u_p \in W_0^{1,m}(\Omega)$ be a positive solution of (1.1). Let $M_p = \|u_p\|_{L^\infty(\Omega)}$. Then, there is a $\delta_0 = \delta_0(m, N, \Omega) > 0$, such that $M_p^{p-m} \leq C$ for $p \in [m, m + \delta_0]$, where $C = C(m, N, \Omega, \delta_0) > 0$.*

Proof By contradiction, assume that $\{p_k\} \rightarrow m$, $M_{p_k}^{p_k-m} = \|u_{p_k}\|_{L^\infty(\Omega)}^{p_k-m} \rightarrow +\infty$, where u_k is a sequence of positive solutions of (1.1) with $p = p_k$. Let $\{x^k\}_{k \in \mathbb{N}}$ be a sequence in Ω such that $u_k(x^k) = M_{p_k}$. Let

$$\tilde{u}_k(y) = \frac{u_k(\mu_k y + x^k)}{M_{p_k}}.$$

Then \tilde{u}_k is a function satisfying $0 \leq \tilde{u}_k \leq 1$, $\tilde{u}_k(0) = 1$ and

$$\begin{cases} -\Delta_m \tilde{u}_k = \lambda \mu_k^m \tilde{u}_k^{m-1} + M_{p_k}^{p_k-m} \mu_k^m \tilde{u}_k^{p_k-1} := f_k & \text{in } \Omega_k, \\ \tilde{u}_k = 0 & \text{on } \partial\Omega_k, \end{cases} \quad (2.12)$$

where $\Omega_k = \{y \in \mathbb{R}^N : \mu_k y + x^k \in \Omega\}$. Let $M_{p_k}^{p_k-m} \mu_k^m = 1$, then $\mu_k \rightarrow 0$ (as $k \rightarrow +\infty$). \tilde{u}_k satisfies

$$\begin{cases} -\Delta_m \tilde{u}_k = \lambda \mu_k^m \tilde{u}_k^{m-1} + \tilde{u}_k^{p_k-1} & \text{in } \Omega_k, \\ \tilde{u}_k = 0 & \text{on } \partial\Omega_k. \end{cases} \quad (2.13)$$

Up to some subsequences, two situations may occur:

$$\text{either } \text{dist}(x^k, \partial\Omega) \mu_k^{-1} \rightarrow +\infty \text{ or } \text{dist}(x^k, \partial\Omega) \mu_k^{-1} \rightarrow d \geq 0.$$

Case 1 $\lim_{k \rightarrow +\infty} \text{dist}(x^k, \partial\Omega) \mu_k^{-1} \rightarrow +\infty$. Then $\Omega_k \rightarrow \mathbb{R}^N$ as $k \rightarrow +\infty$ for any $r > 1$ with $B_r(0) \subset \Omega_k$, f_k satisfies (2.10). By applying Theorem 2.1, there exists an $\alpha \in (0, 1)$ and a constant $C = C(r) > 0$ which do not depend on k such that

$$\|\tilde{u}_k\|_{C^{1,\alpha}(B_r(0))} \leq C.$$

By choosing a subsequence, $\tilde{u}_k \rightarrow \tilde{u}$ in $C_{\text{loc}}^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0, \alpha)$. Moreover, $\tilde{u}(0) = 1$, $\tilde{u} \geq 0$ in \mathbb{R}^N , $f_k \rightarrow \tilde{u}^{m-1}$ in $C_{\text{loc}}^{1,\beta}(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}|^{m-2} \nabla \tilde{u} \nabla \psi \, dx = \int_{\mathbb{R}^N} \tilde{u}^{m-1} \psi \, dx \quad \text{for all } \psi \in C_c^1(\mathbb{R}^N),$$

namely, \tilde{u} is a distributional solution of

$$-\Delta_m \tilde{u} = \tilde{u}^{m-1} \quad \text{in } \mathbb{R}^N. \quad (2.14)$$

We claimed that $\tilde{u} = 0$. Suppose to the contrary that $\tilde{u} \not\equiv 0$. Let φ_R be the first eigenfunction of $-\Delta_m$ on $B_R(0)$, namely, φ_R satisfies

$$\begin{cases} -\Delta_m \varphi_R = \lambda_R \varphi_R^{m-1} & \text{in } B_R(0), \\ \varphi_R = 0 & \text{on } \partial B_R(0), \end{cases} \quad (2.15)$$

where λ_R is the first eigenvalue of $-\Delta_m$ on $B_R(0)$ and $\lambda_R \rightarrow 0$ as $R \rightarrow +\infty$. Testing $\psi_R = \varphi_R^m \tilde{u}^{1-m}$ on (2.14), we obtain

$$\int_{B_R(0)} |\nabla \tilde{u}|^{m-2} \nabla \tilde{u} \cdot \nabla (\varphi_R^m \tilde{u}^{1-m}) dx = \int_{B_R(0)} \varphi_R^m dx.$$

The left-hand side denotes by LHS. Then, by Young's inequality,

$$\begin{aligned} \text{LHS} &= (1-m) \int_{B_R(0)} \varphi_R^m |\nabla \tilde{u}|^m \tilde{u}^{-m} dx + m \int_{B_R(0)} |\nabla \tilde{u}|^{m-2} \tilde{u}^{1-m} \varphi_R^{m-1} \nabla \tilde{u} \cdot \nabla \varphi_R dx, \\ &\leq (1-m) \int_{B_R(0)} \varphi_R^m |\nabla \tilde{u}|^m \tilde{u}^{-m} dx + (m-1) \int_{B_R(0)} \varphi_R^m |\nabla \tilde{u}|^m \tilde{u}^{-m} dx \\ &\quad + C_m \int_{B_R(0)} |\nabla \varphi_R|^m dx. \end{aligned} \quad (2.16)$$

We obtain

$$\int_{B_R(0)} |\nabla \varphi_R|^m dx \geq C_m^{-1} \int_{B_R(0)} |\varphi_R|^m dx.$$

By (2.15),

$$\lambda_R \int_{B_R(0)} |\varphi_R|^m dx \geq C_m^{-1} \int_{B_R(0)} |\varphi_R|^m dx.$$

We let $R \rightarrow \infty$, and get the contradiction. Thus $\tilde{u} \equiv 0$. But this contradicts $\tilde{u}(0) = 1$.

Case 2 $\text{dist}(x^k, \partial\Omega) \mu_k^{-1} \rightarrow d \geq 0$. Then $x^k \rightarrow x^0 \in \partial\Omega$. Let d_k be the distance from x^k to $\partial\Omega$ and $d_k = \text{dist}(x^k, \tilde{x}^k)$, where $\tilde{x}^k \in \partial\Omega$. After translation, we may assume $\tilde{x}^k = 0$, $\nu(0) = -e_N$, where $\nu(0)$ is the outward unit normal vector at 0.

Claim: $d > 0$. By contradiction, assume that $d = 0$. We define the function $v_k(y) = \frac{u_k(\mu_k y)}{M_{p_k}}$ for $y \in \tilde{\Omega}_k$, where $\tilde{\Omega}_k = \{y \in \mathbb{R}^N \mid \mu_k y \in \Omega\}$. Then v_k satisfies that

$$\begin{cases} -\Delta_m v_k = \lambda \mu_k^m v_k^{m-1} + v_k^{p_k-1} & \text{in } \tilde{\Omega}_k, \\ v_k = 0 & \text{on } \partial\tilde{\Omega}_k. \end{cases} \quad (2.17)$$

Setting $z^k = \frac{x^k}{\mu_k}$, we obtain $v_k(z_k) = 1$ and

$$z^k \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Near the origin, $\partial\Omega$ can be represented by $x_N = \rho(x')$, $\rho(0) = 0$ and $\nabla \rho(0) = 0$. We have $\mu_k y_N = \rho(\mu_k y')$. We obtain $y_N = \frac{\rho(\mu_k y')}{\mu_k} \rightarrow 0$ locally uniformly as $k \rightarrow +\infty$. By Lemma 2.1, Remark 2.1, for any $y \in \tilde{\Omega}_k \cap B_1(0)$, $|v_k(y) - v_k(0)| \leq C|y|$, where C does not depend on k . Then

$$|v_k(z^k) - v_k(0)| \leq C|z^k| \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

But this contradicts $v_k(z^k) = 1$.

By choosing a subsequence, $v_k \rightarrow v$ locally uniformly $C^{1,\beta}(\{x_N > -d\})$, where \mathbb{R}_{d+}^N denotes $\{x \in \mathbb{R}^N | x_N > -d\}$. By a translation, v satisfies

$$\begin{cases} -\Delta_m v = v^{m-1} & \text{in } \mathbb{R}_+^N, \\ v = 0 & \text{on } x^N = 0, \\ v(de_N) = 1, \\ v > 0 & \text{in } \mathbb{R}_+^N. \end{cases} \quad (2.18)$$

Let φ_R be the first positive eigenfunction of $-\Delta_m$ on $B_R(\text{Re}_N)$. Then a similar argument as in Case 1 yields that $v \equiv 0$. But this contradicts $v(de_N) = 1$. The lemma is proved.

Lemma 2.3 *Let $m > 1$, $\{p_k\}_{k \in \mathbb{N}} \subset [m, m + \delta_0]$ be a sequence such that $p_k \rightarrow m$, where δ_0 is defined in Lemma 2.2. Let u_k be a sequence positive solutions of (1.1) with $p = p_k$. Let $M_{p_k} := \|u_k\|_{L^\infty(\Omega)}$. Then exists a subsequence denote by $\{u_k\}_{k \in \mathbb{N}}$ such that $\tilde{u}_k := \frac{u_k}{M_{p_k}} \rightarrow \tilde{u}$ in $C^{1,\beta}(\Omega)$ for some $\beta \in (0, 1)$, where \tilde{u} is the first eigenfunction of $-\Delta_m$.*

Proof \tilde{u}_k satisfies

$$\begin{cases} -\Delta_m \tilde{u}_k = \lambda \tilde{u}_k^{m-1} + M_{p_k}^{p_k-m} \tilde{u}_k^{p_k-1} := f_k & \text{in } \Omega, \\ \tilde{u}_k = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.19)$$

f_k has a uniform bound and satisfies (2.10). By Theorem 2.2, $\|\tilde{u}_k\|_{C^{1,\alpha}(\Omega)} \leq C$, where $\alpha \in (0, 1)$ and C depend on m, δ_0, N . Thus there exists $\beta \in (0, \alpha)$ and a subsequence $\tilde{u}_k \rightarrow \tilde{u} \geq 0$ in $C^{1,\beta}(\overline{\Omega})$ and $\|\tilde{u}\|_{L^\infty(\Omega)} = 1$. By Lemma 2.2, \tilde{u} satisfies

$$\begin{cases} -\Delta_m \tilde{u} = \lambda \tilde{u}^{m-1} + C_1 \tilde{u}^{m-1} = (\lambda + C_1) \tilde{u}^{m-1} & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.20)$$

By the uniqueness of the first eigenfunction of $-\Delta_m$, \tilde{u} is the first eigenfunction.

Due to Lemma 2.3, we consider the following equation

$$\begin{cases} -\Delta_m u = \lambda u^{m-1} + \mu u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.21)$$

where $\mu \in [0, \hat{\mu}]$, $\hat{\mu} > 0$, $p \in [m, m + \delta_0]$, δ_0 defined in Lemma 2.2.

The following lemma is due to Brasco and Lindgren [4, Lemma 2.2].

Lemma 2.4 *Let $1 < m < p < m + \delta_0$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set. Let $\lambda > 0$. Let $u, v \in W_0^{1,m}(\Omega)$ be two distinct positive solutions of (2.21). Then we must have*

$$|\{x \in \Omega : u(x) > v(x)\}| > 0 \quad \text{and} \quad |\{x \in \Omega : u(x) < v(x)\}| > 0.$$

In other words, the difference $u - v$ must change sign in Ω .

Proof We argue by contradiction and suppose, for example, that

$$|\{x \in \Omega : u(x) < v(x)\}| = 0.$$

Thus we assume that

$$u(x) \geq v(x) \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad |\{x \in \Omega : u(x) > v(x)\}| > 0.$$

Let $\{v_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\Omega)$ be a non-negative sequence such that

$$\lim_{k \rightarrow +\infty} \|\nabla v_k - \nabla v\|_{L^m(\Omega)} = 0.$$

For every $\varepsilon > 0$, we take the admissible test function $\varphi = \frac{v_k^m}{(u+\varepsilon)^{m-1}}$ in the weak formulation of the equation for u . This yields

$$\begin{aligned} & \int_{\Omega} (\lambda u^{m-1} + \mu u^{p-1}) \frac{v_k^m}{(u+\varepsilon)^{m-1}} dx \\ &= \int_{\Omega} \left\langle |\nabla u|^{m-2} \nabla u, \nabla \left(\frac{v_k^m}{(u+\varepsilon)^{m-1}} \right) \right\rangle dx \\ &= \int_{\Omega} \left\langle |\nabla(u+\varepsilon)|^{m-2} \nabla(u+\varepsilon), \nabla \left(\frac{v_k^m}{(u+\varepsilon)^{m-1}} \right) \right\rangle dx \\ &\leq \int_{\Omega} |\nabla v_k|^m dx. \end{aligned}$$

In the last inequality, we used Picone's inequality for the m -Laplacian (cf. [1, Theorem 1.1]). By taking the limit as k goes to ∞ , thus we get

$$\int_{\Omega} (\lambda u^{m-1} + \mu u^{p-1}) \frac{v^m}{(u+\varepsilon)^{m-1}} dx \leq \int_{\Omega} |\nabla v|^m dx = \int_{\Omega} \lambda v^m + \mu v^p dx.$$

By taking the limit as $\varepsilon \rightarrow 0$, using Fatou's lemma, we get

$$\int_{\Omega} (\lambda + \mu u^{p-m}) v^m dx \leq \int_{\Omega} |\nabla v|^m dx = \int_{\Omega} \lambda v^m + \mu v^p dx.$$

That is

$$\int_{\Omega} (u^{p-m} - v^{p-m}) v^m dx \leq 0.$$

We get a contradiction.

Theorem 2.3 *Let $\delta_1 > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. For every $m \in [1 + \delta_1, 1 + \frac{1}{\delta_1}]$, and $m \leq p \leq m + \delta_0$, let $u_{m,p} \in W_0^{1,m}(\Omega)$ be a positive solution of (2.21) with $\|u_{m,p}\|_{L^\infty(\Omega)} = 1$. Then there exist $\chi = \chi(\lambda, \hat{\mu}, \alpha, N, \delta_0, \delta_1, \Omega) \in (0, 1)$, $\tau = \tau(\lambda, \hat{\mu}, \alpha, N, \delta_0, \delta_1, \Omega) > 0$, and $\mu_0 = \mu_0(\lambda, \hat{\mu}, \alpha, N, \delta_0, \delta_1, \Omega) > 0$, $\mu_1 = \mu_1(\lambda, \hat{\mu}, \alpha, N, \delta_0, \delta_1, \Omega) > 0$ such that:*

(1) $u_{m,p} \in C^{1,\chi}(\overline{\Omega})$ with the uniform estimate

$$\|u_{m,p}\|_{C^{1,\chi}(\overline{\Omega})} \leq L$$

for some $L = L(\lambda, \hat{\mu}, \alpha, N, \delta_0, \delta_1, \Omega) > 0$;

(2) by defining $\Omega_\tau = \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \tau\}$, we have

$$|\nabla u_{m,p}| \geq \mu_0 \quad \text{in } \Omega_\tau$$

and

$$u_{m,p} \geq \mu_1 \quad \text{in } \overline{\Omega \setminus \Omega_\tau}.$$

Proof (1) is trivial. By Hopf's Lemma (cf. [25, Theorem 1]) and compactness argument. One knows

$$\min_{\partial\Omega} |\nabla u_{m,p}| > 0 \quad \text{for every } m \in \left[1 + \delta_1, 1 + \frac{1}{\delta_1}\right], \quad m \leq p \leq m + \delta_0.$$

By using that the family $\{|\nabla u_{m,p}|\}_{m \in [1+\delta_1, 1+\frac{1}{\delta_1}], m \leq p \leq m+\delta_0}$ has a uniform $C^{0,\chi}(\partial\Omega)$ estimate, an application of Arzelà-Ascoli Theorem gives that there exists a constant $\bar{\mu} > 0$ such that

$$\min_{\partial\Omega} |\nabla u_{m,p}| \geq \bar{\mu} \quad \text{for every } m \in \left[1 + \delta_1, 1 + \frac{1}{\delta_1}\right], \quad m \leq p \leq m + \delta_0.$$

We choose $\tau_0 > 0$ sufficiently small, such that each point $x \in \Omega_{\tau_0}$ can be uniquely written as

$$x = x' - |x' - x|\nu_\Omega(x') \quad \text{with } x' \in \partial\Omega.$$

Here ν_Ω stands for the normal outer vector. We then get for every $m \in [1 + \delta_1, 1 + \frac{1}{\delta_1}]$, every $m \leq p \leq m + \delta_0$, every $0 < \tau < \tau_0$ and every $x \in \Omega_\tau$,

$$|\nabla u_{m,p}(x)| \geq |\nabla u_{m,p}(x')| - \|\nabla u_{m,p}(x')\| - |\nabla u_{m,p}(x)| \geq \bar{\mu} - L|x' - x|^\chi \geq (\bar{\mu} - L\delta^\chi).$$

If we now choose

$$\tau = \min\left\{\left(\frac{\bar{\mu}}{2L}\right)^{\frac{1}{\chi}}, \tau_0\right\}$$

and set $\mu_0 = \frac{\bar{\mu}}{2}$, we obtain

$$|\nabla u_{m,p}(x)| \geq \mu_0 \quad \text{for every } x \in \Omega_\tau, \quad m \in \left[1 + \delta_1, 1 + \frac{1}{\delta_1}\right], \quad m \leq p \leq m + \delta_0.$$

Finally, the uniform lower bound on $u_{m,p}$ in $\overline{\Omega \setminus \Omega_\tau}$ can be proved by observing that

$$\min_{\overline{\Omega \setminus \Omega_\tau}} u_{m,p} > 0 \quad \text{for every } m \in \left[1 + \delta_1, 1 + \frac{1}{\delta_1}\right], \quad m \leq p \leq m + \delta_0, \quad (2.22)$$

thanks to the maximum principle. By compactness argument, one knows

$$\min_{\overline{\Omega \setminus \Omega_\tau}} u_{m,p} \geq \mu_1 \quad \text{for every } m \in \left[1 + \delta_1, 1 + \frac{1}{\delta_1}\right], \quad m \leq p \leq m + \delta_0. \quad (2.23)$$

This concludes the proof.

Theorem 2.4 (Uniform weighted Sobolev inequality) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. For every $m \in [1 + \delta_1, 1 + \frac{1}{\delta_1}]$ and $m \leq p \leq m + \delta_0$, let $u_p \in W_0^{1,m}(\Omega)$ be a positive solution of (2.21) with $\|u_p\|_{L^\infty(\Omega)} = 1$. We define*

$$\sigma_0 = 2\left(1 - \frac{\delta_1}{2N}\right)^{-1}, \quad (2.24)$$

then for every $2 < \sigma < \sigma_0$, there exists $\mathcal{T} = \mathcal{T}(\alpha, N, \lambda, \delta_0, \delta_1, \sigma, \Omega, \hat{\mu}) > 0$ such that

$$\begin{aligned} & \mathcal{T} \left(\int_{\Omega} |\phi|^\sigma dx \right)^{\frac{2}{\sigma}} \\ & \leq \int_{\Omega} |\nabla u_p|^{m-2} |\nabla \phi|^2 dx \quad \text{for every } \phi \in C^1(\overline{\Omega}) \cap W_0^{1,m}(\Omega), \quad p \in [m, m + \delta_0]. \end{aligned} \quad (2.25)$$

Proof For every $\phi \in C^1(\overline{\Omega}) \cap W_0^{1,m}(\Omega)$, we recall the classical representation formula

$$\phi(x) = C \int_{\mathbb{R}^N} \left\langle \nabla \phi(y), \frac{x-y}{|x-y|^N} \right\rangle dy,$$

where $C = C(N) > 0$ (cf. [13, Lemma 7.14]). This in turn implies that

$$|\phi(x)| \leq C \int_{\mathbb{R}^N} \frac{|\nabla \phi(y)|}{|x-y|^{N-1}} dy \quad \text{for every } x \in \Omega.$$

We use the notation

$$t = 1 + \delta_2, \quad \gamma = N - 1 - t = N - 2 - \delta_2,$$

where $\delta_2 > 0$, $\frac{\delta_2}{1+\delta_2} < \delta_1$. By suitably using Hölder's inequality with exponents

$$2t, \quad \frac{2t}{2t-1},$$

we obtain

$$|\phi(x)| \leq C \left(\int_{\Omega} \frac{1}{|\nabla u_p|^{t(m-2)} |x-y|^{\gamma}} dy \right)^{\frac{1}{2t}} \left(\int_{\Omega} \left(\frac{|\nabla \phi(y)| |\nabla u_p|^{\frac{m-2}{2}}}{|x-y|^{N-1-\frac{\gamma}{2t}}} \right)^{\frac{2t}{2t-1}} dy \right)^{\frac{2t-1}{2t}}.$$

Observe that by definition

$$t(m-2) < m-1 \quad \text{and} \quad \gamma < N-2,$$

thus we can apply Theorem A.5 with $r = t(m-2)$ and get

$$|\phi(x)| \leq C \mathcal{S}^{\frac{1}{2t}} \left(\int_{\Omega} \left(\frac{|\nabla \phi(y)| |\nabla u_p|^{\frac{m-2}{2}}}{|x-y|^{N-1-\frac{\gamma}{2t}}} \right)^{\frac{2t}{2t-1}} dy \right)^{\frac{2t-1}{2t}}. \quad (2.26)$$

For simplicity, we now set

$$F(y) = (|\nabla \phi(y)| |\nabla u_p|^{\frac{m-2}{2}})^{\frac{2t}{2t-1}},$$

and observe that

$$\|F\|_{L^{\frac{2t-1}{2t}}(\Omega)}^{\frac{2t-1}{2t}} = \left(\int_{\Omega} |F|^{\frac{2t-1}{t}} dy \right)^{\frac{1}{2}} = \left(\int_{\Omega} |\nabla \phi|^2 |\nabla u_p|^{m-2} dy \right)^{\frac{1}{2}}. \quad (2.27)$$

We introduce the exponent $0 < \Theta < N$ given by

$$\left(N - 1 - \frac{\gamma}{2t} \right) \frac{2t}{2t-1} = N - \Theta, \quad \text{that is,} \quad \Theta = N - \left(N - 1 - \frac{\gamma}{2t} \right) \frac{2t}{2t-1}.$$

Thanks to the choice of γ and t , it is not difficult to see that Θ is positive. More precisely, observe that this exponent is explicitly given by

$$\Theta = \frac{t-1}{2t-1} = \frac{\delta_2}{1+2\delta_2}. \quad (2.28)$$

In view of these definitions, we can rewrite (2.26) as

$$|\phi(x)| \leq C \mathcal{S}^{\frac{1}{2t}} \left(\int_{\Omega} \frac{|F(y)|}{|x-y|^{N-\Theta}} dy \right)^{\frac{2t-1}{2t}}, \quad (2.29)$$

so that one can recognize a suitable Riesz potential on the right-hand side. We then recall the classical potential estimate (cf. [13, Lemma 7.12])

$$\left\| \int_{\Omega} \frac{|F(y)|}{|\cdot - y|^{N-\Theta}} dy \right\|_{L^l(\Omega)} \leq \left(\frac{1-\delta}{\frac{\Theta}{N}-\delta} \right)^{1-\delta} \omega_N^{\frac{N-\Theta}{N}} |\Omega|^{\frac{\Theta}{N}-\delta} \|F\|_{L^s(\Omega)}, \quad (2.30)$$

where

$$0 \leq \delta := \frac{1}{s} - \frac{1}{l} < \frac{\Theta}{N}.$$

By direct computation, for $\sigma \in (2, \sigma_0)$, there holds

$$\frac{\sigma-2}{\sigma} \frac{t}{2t-1} < \frac{\Theta}{N}.$$

Then

$$\begin{aligned} \|\phi\|_{L^\sigma(\Omega)} &\leq C S^{\frac{1}{2t}} \left\| \left(\int_{\Omega} \frac{|F(y)|}{|\cdot - y|^{N-\Theta}} dy \right)^{\frac{2t-1}{2t}} \right\|_{L^\sigma(\Omega)} \\ &= C S^{\frac{1}{2t}} \left\| \int_{\Omega} \frac{|F(y)|}{|\cdot - y|^{N-\Theta}} dy \right\|_{L^{\sigma \frac{2t-1}{2t}}(\Omega)}^{\frac{2t-1}{2t}}. \end{aligned}$$

Let

$$l = \sigma \frac{2t-1}{2t}, \quad s = \frac{2t-1}{t}.$$

These are feasible, since

$$\delta = \frac{1}{s} - \frac{1}{l} = \frac{t}{2t-1} - \frac{2t}{\sigma(2t-1)} = \frac{\sigma-2}{\sigma} \frac{t}{2t-1}$$

is positive and smaller than $\frac{\Theta}{N}$, thanks to the choice of σ . We then obtain

$$\|\phi\|_{L^\sigma(\Omega)} \leq C S^{\frac{1}{2t}} \left[\left(\frac{1-\delta}{\frac{\Theta}{N}-\delta} \right)^{1-\delta} \omega_N^{\frac{N-\Theta}{N}} |\Omega|^{\frac{\Theta}{N}-\delta} \right]^{\frac{2t-1}{2t}} \|F\|_{L^{\frac{2t-1}{2t-1}}(\Omega)}^{\frac{2t-1}{2t}}.$$

By recalling (2.27), we thus obtain

$$\sqrt{\mathcal{T}} \|\phi\|_{L^\sigma(\Omega)} \leq \left(\int_{\Omega} |\nabla u_p|^{m-2} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}$$

with the constant \mathcal{T} given by

$$\mathcal{T} = \frac{1}{C^2 S^{\frac{1}{t}}} \left[\left(\frac{1-\delta}{\frac{\Theta}{N}-\delta} \right)^{1-\delta} \omega_N^{\frac{N-\Theta}{N}} |\Omega|^{\frac{\Theta}{N}-\delta} \right]^{\frac{1-2t}{t}}.$$

Corollary 2.1 *Let $\delta_1 \in (\frac{1}{2}, 1)$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. For every $m \in [1 + \delta_1, 1 + \frac{1}{\delta_1}]$ and $m \leq p \leq m + \delta_0$, let $u_p \in W_0^{1,m}(\Omega)$ be a positive solution of (2.21) and $\|u_p\|_{L^\infty(\Omega)} = 1$. Then there exists an exponent $\theta \in (1, 2)$ and a constant $C = C(N, \delta_0, \delta_1, \lambda, \alpha, \Omega, \hat{\mu}) > 0$ such that*

$$\|\phi\|_{W^{1,\theta}(\Omega)} \leq C \left(\int_{\Omega} |\nabla u_p|^{m-2} |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \quad \text{for every } \phi \in W_0^{1,m}(\Omega), \quad p \in [m, m + \delta_0].$$

Proof We take $1 < \theta < \frac{2m-2}{2m-3}$. Then Hölder's inequality with conjugate exponents

$$\frac{2}{\theta} \quad \text{and} \quad \frac{2}{2-\theta},$$

implies

$$\begin{aligned} \left(\int_{\Omega} |\nabla \phi|^{\theta} dx \right)^{\frac{1}{\theta}} &\leq \left(\int_{\Omega} |\nabla u_p|^{m-2} |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \frac{1}{|\nabla u_p|^{\frac{\theta}{2-\theta}(m-2)}} dx \right)^{\frac{2-\theta}{2\theta}} \\ &\leq \tilde{\mathcal{S}}^{\frac{2-\theta}{2\theta}} \left(\int_{\Omega} |\nabla u_p|^{m-2} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

where we have used that

$$\frac{\theta}{2-\theta}(m-2) < (m-1),$$

which allows us to use estimate (A.14) from Theorem A.5, with $r = \frac{(m-2)\theta}{2-\theta}$. Since Ω is bounded, by Hölder's inequality we have that $W_0^{1,m}(\Omega) \subset W_0^{1,\theta}(\Omega)$ with continuous inclusion. Moreover, by Poincaré inequality,

$$\|\nabla \phi\|_{L^{\theta}(\Omega)} \quad \text{and} \quad \|\phi\|_{W^{1,\theta}(\Omega)}$$

are equivalent norms on $W_0^{1,\theta}(\Omega)$. These facts conclude the proof.

Corollary 2.2 (Uniform compact embedding) *Let $m \in [1 + \delta_1, 1 + \frac{1}{\delta_1}]$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. We take a sequence $\{p_k\}_{k \in \mathbb{N}} \subset [m, m + \delta_0]$ and consider accordingly $u_k \in W_0^{1,m}(\Omega)$ a positive solutions of (2.21) with $p = p_k$ and $\|u_k\|_{L^{\infty}(\Omega)} = 1$. If $\{\phi_k\}_{k \in \mathbb{N}} \subset W_0^{1,m}(\Omega) \cap C^1(\overline{\Omega})$ is a sequence of functions satisfying*

$$\int_{\Omega} |\nabla u_k|^{m-2} |\nabla \phi_k|^2 dx \leq C \quad \text{for every } k \in \mathbb{N},$$

then $\{\phi_k\}_{k \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ and weakly in $W_0^{1,\theta}(\Omega)$, up to a subsequence. Here θ is the same exponent as in Corollary 2.1.

Proof The assumption, in conjunction with Corollary 2.1, entails that $\{\phi_k\}_{k \in \mathbb{N}}$ is a bounded sequence in $W_0^{1,\theta}(\Omega)$. By the classical Rellich-Kondrašov Theorem, we get that this sequence converges weakly in $W^{1,\theta}(\Omega)$ and strongly in $L^{\theta}(\Omega)$, up to a subsequence. Moreover, since $W_0^{1,\theta}(\Omega)$ is also weakly closed, we get that the limit still belongs to $W_0^{1,\theta}(\Omega)$.

In order to get the strong L^2 convergence, we observe that, if we denote by σ_0 the exponent of Theorem 2.4, for every $2 < \sigma < \sigma_0$ and $n, k \in \mathbb{N}$ we have

$$\|\phi_n - \phi_k\|_{L^2(\Omega)} \leq \|\phi_n - \phi_k\|_{L^{\sigma}(\Omega)}^{1-\tau} \|\phi_n - \phi_k\|_{L^{\theta}(\Omega)}^{\tau} \leq C \|\phi_n - \phi_k\|_{L^{\theta}(\Omega)}^{\tau}.$$

This ends the proof of the corollary.

Similarly, we have the following corollary.

Corollary 2.3 (Uniform compact embedding) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. We take a sequence $\{m_k\}_{k \in \mathbb{N}} \subset [2, 2 + \delta_1]$*

and consider accordingly $u_k \in W_0^{1,m_k}(\Omega)$ a positive solutions of (2.21) with $m = m_k$ and $\|u_k\|_{L^\infty(\Omega)} = 1$. If $\{\phi_k\}_{k \in \mathbb{N}} \subset W_0^{1,m_k}(\Omega) \cap C^1(\overline{\Omega})$ is a sequence of functions satisfying

$$\int_{\Omega} |\nabla u_k|^{m_k-2} |\nabla \phi_k|^2 dx \leq C \quad \text{for every } k \in \mathbb{N},$$

then $\{\phi_k\}_{k \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ and weakly in $W_0^{1,\theta}(\Omega)$, up to a subsequence. Here θ is the same exponent as in Corollary 2.1.

3 Eigenvalue Problem

In this section, we treat a weighted linear eigenvalue problem. It is convenient to introduce the notation

$$\mathcal{H}(z) = \frac{1}{m} |z|^m \quad \text{for every } z \in \mathbb{R}^N.$$

Then we observe that

$$\nabla \mathcal{H}(z) = |z|^{m-2} z \quad \text{for every } z \in \mathbb{R}^N$$

and

$$D^2 \mathcal{H}(z) = |z|^{m-2} \text{Id} + (m-2) |z|^{m-4} z \otimes z \quad \text{for every } z \in \mathbb{R}^N. \quad (3.1)$$

In particular, we have the following facts

$$D^2 \mathcal{H}(z) z = (m-1) |z|^{m-2} z, \quad \langle D^2 \mathcal{H}(z) \xi, \xi \rangle \leq (m-1) |z|^{m-2} |\xi|^2 \quad \text{for } z, \xi \in \mathbb{R}^N. \quad (3.2)$$

We have the following elementary inequality.

Lemma 3.1 *Let $1 < m < +\infty$ and $\Omega \subset \mathbb{R}^N$ be an open set. For every $v, w, \phi \in W_{\text{loc}}^{1,1}(\Omega)$, we have*

$$\begin{aligned} & |\langle D^2 \mathcal{H}(\nabla \phi) \nabla v, \nabla v \rangle - \langle D^2 \mathcal{H}(\nabla \phi) \nabla w, \nabla w \rangle| \\ & \leq (m-1) |\nabla \phi|^{m-2} |\nabla v - \nabla w| (|\nabla v| + |\nabla w|), \quad \text{a.e. on } \Omega. \end{aligned}$$

Proof By [4, Lemma A.2], we have

$$|\langle D^2 \mathcal{H}(\nabla \phi) \nabla v, \nabla v \rangle - \langle D^2 \mathcal{H}(\nabla \phi) \nabla w, \nabla w \rangle| \leq |D^2 \mathcal{H}(\nabla \phi) (\nabla v - \nabla w)| (|\nabla v| + |\nabla w|).$$

Since the Hessian matrix is given by (3.1), we get

$$|D^2 \mathcal{H}(\nabla \phi) (\nabla v - \nabla w)| \leq (m-1) |\nabla \phi|^{m-2} |\nabla v - \nabla w|.$$

By using this inequality in the first estimate, we conclude the proof.

Proposition 3.1 *Let $1 < m < +\infty$ and $\Omega \subset \mathbb{R}^N$ be an open connected set with finite volume. Let $U \in W_0^{1,m}(\Omega)$ be the unique positive extremal of*

$$\lambda_m(\Omega) = \inf_{\phi \in W_0^{1,m}(\Omega)} \left\{ \int_{\Omega} |\nabla \phi|^m dx : \int_{\Omega} |\phi|^m dx = 1 \right\}. \quad (3.3)$$

By setting

$$\lambda(\Omega; U) = \inf_{\phi \in C_0^\infty(\Omega)} \left\{ \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla \phi, \nabla \phi \rangle dx : \int_{\Omega} U^{m-2} |\phi|^2 dx = 1 \right\},$$

we have

$$\lambda(\Omega; U) = (m-1)\lambda_m(\Omega).$$

Proof The inequality

$$\lambda(\Omega; U) \leq (m-1)\lambda_m(\Omega) \quad (3.4)$$

is straightforward. Indeed, for every $\varepsilon > 0$, we take a non-negative $U_\varepsilon \in C_0^\infty(\Omega)$ such that

$$\int_{\Omega} |\nabla U_\varepsilon|^m dx \leq \lambda_m(\Omega) + \varepsilon \int_{\Omega} U_\varepsilon^m dx = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla U_\varepsilon - \nabla U|^m dx = 0.$$

Thus in particular we have convergence in $L^m(\Omega)$, as well. Denote

$$\tilde{U}_\varepsilon := \frac{U_\varepsilon}{\left(\int_{\Omega} U^{m-2} U_\varepsilon^2 dx \right)^{\frac{1}{2}}}.$$

It is easy to see that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} U^{m-2} |\tilde{U}_\varepsilon|^2 dx = \int_{\Omega} U^m dx = 1. \quad (3.5)$$

Also

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla U_\varepsilon, \nabla U_\varepsilon \rangle dx = \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla U, \nabla U \rangle dx. \quad (3.6)$$

Indeed, by applying Lemma 3.1, we get

$$\begin{aligned} & \left| \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla U_\varepsilon, \nabla U_\varepsilon \rangle dx - \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla U, \nabla U \rangle dx \right| \\ & \leq (m-1) \int_{\Omega} |\nabla U|^{m-2} |\nabla U_\varepsilon - \nabla U| (|\nabla U_\varepsilon| + |\nabla U|) dx. \end{aligned}$$

Then (3.6) follows by using Hölder's inequality. By (3.2), (3.5) and (3.6), we get

$$\lambda(\Omega; U) \leq (m-1)\lambda_m(\Omega), \quad \text{as } \varepsilon \rightarrow 0.$$

For the converse inequality, we first recall that u satisfies

$$\int_{\Omega} \langle |\nabla U|^{m-2} \nabla U, \nabla \phi \rangle dx = \lambda_m(\Omega) \int_{\Omega} U^{m-1} \phi dx \quad \text{for every } \phi \in W_0^{1,m}(\Omega) \quad (3.7)$$

by minimality. By using (3.2), this can be also written as

$$\int_{\Omega} \langle \Delta^2 \mathcal{H}(\nabla U) \nabla U, \nabla \phi \rangle dx = (m-1)\lambda_m(\Omega) \int_{\Omega} U^{m-1} \phi dx \quad \text{for every } \phi \in W_0^{1,m}(\Omega). \quad (3.8)$$

We take $\varepsilon > 0$ and $\phi_\varepsilon \in C_0^\infty(\Omega)$ such that

$$\int_{\Omega} \langle \Delta^2 \mathcal{H}(\nabla U) \nabla \phi_\varepsilon, \nabla \phi_\varepsilon \rangle dx < \lambda(\Omega; U) + \varepsilon \quad \text{and} \quad \int_{\Omega} U^{m-2} \phi_\varepsilon^2 dx = 1.$$

Then we insert the test function ϕ_ε^2/U in (3.8), so to get

$$(m-1)\lambda_m(\Omega) \int_{\Omega} U^{m-1} \frac{\phi_\varepsilon^2}{U} dx = \int_{\Omega} \left\langle \Delta^2 \mathcal{H}(\nabla U) \nabla U, \nabla \left(\frac{\phi_\varepsilon^2}{U} \right) \right\rangle dx. \quad (3.9)$$

We now use Picone's identity of Lemma A.1 in [4] with the choice $A = D^2 \mathcal{H}(\nabla U)$. This gives

$$\begin{aligned} & \left\langle \Delta^2 \mathcal{H}(\nabla U) \nabla U, \nabla \left(\frac{\phi_\varepsilon^2}{U} \right) \right\rangle \\ &= \langle \Delta^2 \mathcal{H}(\nabla U) \nabla \phi_\varepsilon, \nabla \phi_\varepsilon \rangle - \left\langle \Delta^2 \mathcal{H}(\nabla U) \left(\phi_\varepsilon \frac{\nabla U}{U} - \nabla \phi_\varepsilon \right), \left(\phi_\varepsilon \frac{\nabla U}{U} - \nabla \phi_\varepsilon \right) \right\rangle. \end{aligned}$$

By (3.9), we get

$$\begin{aligned} & (m-1)\lambda_m(\Omega) \int_{\Omega} U^{m-2} \phi_\varepsilon^2 dx \\ &= \int_{\Omega} \langle \Delta^2 \mathcal{H}(\nabla U) \nabla \phi_\varepsilon, \nabla \phi_\varepsilon \rangle dx \\ & \quad - \int_{\Omega} \left\langle \Delta^2 \mathcal{H}(\nabla U) \left(\phi_\varepsilon \frac{\nabla U}{U} - \nabla \phi_\varepsilon \right), \left(\phi_\varepsilon \frac{\nabla U}{U} - \nabla \phi_\varepsilon \right) \right\rangle dx \\ & \leq \lambda(\Omega, U) + \varepsilon. \end{aligned}$$

Thanks to the fact that $D^2 \mathcal{H}(\nabla U)$ is positive semidefinite. By recalling that

$$\int_{\Omega} U^{m-2} \phi_\varepsilon^2 dx = 1$$

and using the arbitrariness of $\varepsilon > 0$, we finally get the desired conclusion.

Definition 3.1 For $m > 1$, we define the weighted Sobolev space

$$X^{1,2}(\Omega; |\nabla U|^{m-2}) := \left\{ \phi \in W_{\text{loc}}^{1,1}(\Omega) \cap L^2(\Omega) : \int_{\Omega} |\nabla U|^{m-2} |\nabla \phi|^2 dx < +\infty \right\}$$

endowed with the natural norm

$$\|\phi\|_{X^{1,2}(\Omega; |\nabla U|^{m-2})} = \|\phi\|_{L^2(\Omega)} + \left(\int_{\Omega} |\nabla U|^{m-2} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}.$$

We set $X_0^{1,2}(\Omega; |\nabla U|^{m-2})$ for the completion of $C_0^\infty(\Omega)$ with respect to this norm.

Lemma 3.2 Let $m > 1$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha < 1$. With the notation above, we have

$$X_0^{1,2}(\Omega; |\nabla U|^{m-2}) \subset W_0^{1,1}(\Omega),$$

with continuous inclusions.

Proof It is sufficient to prove that there exist two constants $C_1 > 0$ such that

$$C_1 \|\phi\|_{W^{1,1}(\Omega)} \leq \|\phi\|_{X^{1,2}(\Omega; |\nabla U|^{m-2})} \quad \text{for every } \phi \in C_0^\infty(\Omega).$$

The estimate follows from Corollary 2.1, with $m = p$.

Remark 3.1 It is easy to verify that $U \in X_0^{1,2}(\Omega; |\nabla U|^{m-2})$.

Proposition 3.2 Let $1 < m < +\infty$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha < 1$. With the notation above, the infimum $\lambda(\Omega; U)$ is uniquely attained on the space $X_0^{1,2}(\Omega; |\nabla U|^{m-2})$ by the function U or $-U$.

Proof We first notice that if $v \in X_0^{1,2}(\Omega; |\nabla U|^{m-2})$ and $\{v_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\Omega)$ is such that

$$\lim_{k \rightarrow \infty} \|v_k - v\|_{X^{1,2}(\Omega; |\nabla U|^{m-2})} = 0,$$

then

$$\lim_{k \rightarrow \infty} \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla v_k, \nabla v_k \rangle dx = \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla v, \nabla v \rangle dx. \quad (3.10)$$

Indeed, by Lemma 3.1, we have

$$|\langle D^2 \mathcal{H}(\nabla U) \nabla v_k, \nabla v_k \rangle - \langle D^2 \mathcal{H}(\nabla U) \nabla v, \nabla v \rangle| \leq (m-1) |\nabla U|^{m-2} |\nabla v_k - \nabla v| (|\nabla v_k| + |\nabla v|).$$

By integrating over Ω and using Hölder's inequality, we have

$$\begin{aligned} & \left| \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla v_k, \nabla v_k \rangle dx - \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla v, \nabla v \rangle dx \right| \\ & \leq C \int_{\Omega} |\nabla U|^{m-2} |\nabla v_k - \nabla v| (|\nabla v_k| + |\nabla v|) dx \\ & \leq C \left(\int_{\Omega} |\nabla U|^{m-2} |\nabla v_k - \nabla v|^2 dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\Omega} |\nabla U|^{m-2} (|\nabla v_k| + |\nabla v|)^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

By observing that the last term converges to 0, we get (3.10). Since $C_0^\infty(\Omega)$ is dense in $X_0^{1,2}(\Omega; |\nabla U|^{m-2})$ by definition, our previous computations show that

$$\lambda(\Omega; U) = \inf_{\phi \in X_0^{1,2}(\Omega; |\nabla U|^{m-2})} \left\{ \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla \phi, \nabla \phi \rangle dx : \int_{\Omega} U^{m-2} \phi^2 dx = 1 \right\}.$$

In order to prove that any minimizer must coincide either with U or $-U$, we assume that there is another minimizer $v \in X_0^{1,2}(\Omega; |\nabla U|^{m-2})$. By definition, there exists a sequence $\{v_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\Omega)$ such that

$$\lim_{k \rightarrow +\infty} \left[\int_{\Omega} |\nabla U|^{m-2} |\nabla v_k - \nabla v|^2 dx + \int_{\Omega} |v_k - v|^2 dx \right] = 0.$$

We recall that u satisfies (3.8). For every $k \in \mathbb{N}$, the choice $\phi = \frac{v_k^2}{U}$ is feasible in (3.8) and it yields

$$\begin{aligned} & \lambda(\Omega; U) \int_{\Omega} U^{m-2} v_k^2 dx \\ & = \int_{\Omega} \left\langle D^2 \mathcal{H}(\nabla U) \nabla U, \nabla \left(\frac{v_k^2}{U} \right) \right\rangle dx \\ & = \int_{\Omega} \langle D^2 \mathcal{H}(\nabla U) \nabla v_k, \nabla v_k \rangle dx \end{aligned}$$

$$- \int_{\Omega} \left\langle D^2 \mathcal{H}(\nabla U) \left(v_k \frac{\nabla U}{U} - \nabla v_k \right), \left(v_k \frac{\nabla U}{U} - \nabla v_k \right) \right\rangle dx, \quad (3.11)$$

where in the second equality we used the general version of Picone identity given by [4, Lemma A.1], with the positive semi-definite matrix $A = D^2 \mathcal{H}(\nabla U)$. We now wish to pass the limit as k goes to ∞ in the previous identity. We notice at first that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} U^{m-2} v_k^2 dx = \int_{\Omega} U^{m-2} v^2 dx = 1.$$

As for the first term on the right-hand side of (3.11), we simply use (3.10). We are left with handling the last term in (3.11). We have that $\{(v_k, \nabla v_k)\}_{k \in \mathbb{N}}$ converges almost everywhere to $(v, \nabla v)$, possibly up to extracting a subsequence. By observing that $\Delta^2 \mathcal{H}(\nabla U)$ is positive semidefinite, an application of Fatou's Lemma yields

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \int_{\Omega} \left\langle \Delta^2 \mathcal{H}(\nabla U) \left(v_k \frac{\nabla U}{U} - \nabla v_k \right), \left(v_k \frac{\nabla U}{U} - \nabla v_k \right) \right\rangle dx \\ & \geq \int_{\Omega} \left\langle \Delta^2 \mathcal{H}(\nabla U) \left(v \frac{\nabla U}{U} - \nabla v \right), \left(v \frac{\nabla U}{U} - \nabla v \right) \right\rangle dx. \end{aligned}$$

Thus, by taking the limit as k goes to ∞ in (3.11), we get

$$\lambda(\Omega; U) + \int_{\Omega} \left\langle \Delta^2 \mathcal{H}(\nabla U) \left(v \frac{\nabla U}{U} - \nabla v \right), \left(v \frac{\nabla U}{U} - \nabla v \right) \right\rangle dx \leq \lambda(\Omega; U).$$

This entails that we must have

$$\left\langle \Delta^2 \mathcal{H}(\nabla U) \left(v \frac{\nabla U}{U} - \nabla v \right), \left(v \frac{\nabla U}{U} - \nabla v \right) \right\rangle = 0, \quad \text{a.e. in } \Omega. \quad (3.12)$$

From the definition of $D^2 \mathcal{H}$, it is clear that $D^2 \mathcal{H}(\nabla U)$ is positive definite whenever ∇U does not vanish, which is true almost everywhere by (A.14). Therefore, from (3.12) we must have

$$v \frac{\nabla U}{U} - \nabla v = 0, \quad \text{a.e. in } \Omega. \quad (3.13)$$

We now observe that $v \in W_0^{1,1}(\Omega)$ thanks to Lemma 3.4. $U \in C^1(\overline{\Omega})$ as in [4, Proposition 3.5] and it has the following property: For every $\Omega' \Subset \Omega$, there exists a constant $C = C(\Omega') > 0$ such that $U \geq \frac{1}{C}$ on Ω' . Thus we have

$$\frac{v}{U} \in W_{\text{loc}}^{1,1}(\Omega),$$

and Leibnitz's rule holds for its distributional gradient. This is given by

$$\nabla \left(\frac{v}{U} \right) = \frac{U \nabla v - v \nabla U}{U^2}, \quad \text{a.e. in } \Omega,$$

and thus it identically vanishes almost everywhere in Ω , by virtue of (3.13). Since Ω is connected, this implies that $\frac{v}{U}$ is constant in Ω . Thus we get that v is proportional to U in Ω . This desired result is now a consequence of the normalization taken.

4 Proof of the Theorems 1.2

Proof of Theorem 1.2 We divide the proof into three parts, for ease of readability.

Part 1 Linearized equation. We argue by contradiction: We suppose that for $\{p_k\}_{k \in \mathbb{N}}$ with $p_k \searrow m$, the problem (1.1) admits (at least) two distinct positive solutions u_k and v_k for $p = p_k$.

Without loss of generality, we can assume $\|u_k\|_{L^\infty(\Omega)} \geq \|v_k\|_{L^\infty(\Omega)}$ for each k . Let $M_k := \|u_k\|_{L^\infty(\Omega)}$. We may choose a subsequence such that $\lim_{k \rightarrow \infty} \frac{\|v_k\|_{L^\infty(\Omega)}}{M_k} = \mu' \in [0, 1]$. Define $\tilde{u}_k = \frac{u_k}{M_k}$, $\tilde{v}_k = \frac{v_k}{M_k}$. Observing that they solve

$$-\Delta_m \tilde{u}_k = \lambda \tilde{u}_k^{m-1} + M_k^{p_k-m} \tilde{u}_k^{p_k-1} \quad \text{and} \quad -\Delta_m \tilde{v}_k = \lambda \tilde{v}_k^{m-1} + M_k^{p_k-m} \tilde{v}_k^{p_k-1} \quad \text{in } \Omega$$

with

$$\tilde{u}_k = \tilde{v}_k = 0 \quad \text{on } \partial\Omega.$$

According to Lemma 2.3, we can choose a subsequence such that \tilde{u}_k converges to \tilde{u} in $C^{1,\beta}$, where \tilde{u} is the first eigenfunction of $-\Delta_m$ in Ω . In a similar way, \tilde{v}_k converges to $\mu' \tilde{u}$ in $C^{1,\beta}$. The equations solved by \tilde{u}_k and \tilde{v}_k can be written as

$$\int_{\Omega} \langle \nabla \mathcal{H}(\nabla \tilde{u}_k), \nabla \phi \rangle dx = \lambda \int_{\Omega} \tilde{u}_k^{m-1} \phi dx + M_k^{p_k-m} \int_{\Omega} \tilde{u}_k^{p_k-1} \phi dx \quad (4.1)$$

and

$$\int_{\Omega} \langle \nabla \mathcal{H}(\nabla \tilde{v}_k), \nabla \phi \rangle dx = \lambda \int_{\Omega} \tilde{v}_k^{m-1} \phi dx + M_k^{p_k-m} \int_{\Omega} \tilde{v}_k^{p_k-1} \phi dx \quad (4.2)$$

for any $\phi \in W_0^{1,m}(\Omega)$.

We now observe that for every $z, w \in \mathbb{R}^N$ we have

$$\begin{aligned} \nabla \mathcal{H}(z) - \nabla \mathcal{H}(w) &= \int_0^1 \frac{d}{dt} (\nabla \mathcal{H}(tz + (1-t)w)) dt \\ &= \left\langle \int_0^1 (D^2 \mathcal{H}(tz + (1-t)w)) dt, z - w \right\rangle. \end{aligned} \quad (4.3)$$

Similarly, for every $a, b \in [0, +\infty)$ we have

$$\begin{aligned} a^p - b^p &= \int_0^1 \frac{d}{dt} (ta + (1-t)b)^p dt \\ &= p \left(\int_0^1 (ta + (1-t)b)^{p-1} dt \right) (a - b). \end{aligned} \quad (4.4)$$

By subtracting the two equations (4.1)–(4.2), using (4.3) with $z = \nabla \tilde{u}_k(x)$, $w = \nabla \tilde{v}_k(x)$ and (4.4), we get

$$\begin{aligned} &\int_{\Omega} \langle A_k(x) \nabla (\tilde{u}_k - \tilde{v}_k), \nabla \phi \rangle dx \\ &= (m-1) \lambda \int_{\Omega} \int_0^1 (t\tilde{u}_k + (1-t)\tilde{v}_k)^{m-2} (\tilde{u}_k - \tilde{v}_k) \phi dt dx \end{aligned}$$

$$+ M_k^{p_k-m}(p_k-1) \int_{\Omega} \int_0^1 (t\tilde{u}_k + (1-t)\tilde{v}_k)^{p_k-2} (\tilde{u}_k - \tilde{v}_k) \phi dt dx, \quad (4.5)$$

where

$$A_k(x) = \int_0^1 D^2 \mathcal{H}(t\nabla \tilde{u}_k + (1-t)\nabla \tilde{v}_k) dt.$$

Set

$$w_k = \frac{\tilde{u}_k - \tilde{v}_k}{\|\tilde{u}_k - \tilde{v}_k\|_{L^2(\Omega)}} \in W_0^{1,m}(\Omega).$$

Then by (4.5) we get that w_k satisfies

$$\begin{aligned} & \int_{\Omega} \langle A_k(x) \nabla w_k, \nabla \phi \rangle dx \\ &= (m-1)\lambda \int_{\Omega} \int_0^1 (t\tilde{u}_k + (1-t)\tilde{v}_k)^{m-2} w_k \phi dt dx \\ &+ M_k^{p_k-m}(p_k-1) \int_{\Omega} \int_0^1 (t\tilde{u}_k + (1-t)\tilde{v}_k)^{p_k-2} w_k \phi dt dx. \end{aligned} \quad (4.6)$$

Let $\phi = w_k$. Then

$$\begin{aligned} & \int_{\Omega} \langle A_k(x) \nabla w_k, \nabla w_k \rangle dx \\ &= (m-1)\lambda \int_{\Omega} \int_0^1 (t\tilde{u}_k + (1-t)\tilde{v}_k)^{m-2} w_k^2 dt dx \\ &+ M_k^{p_k-m}(p_k-1) \int_{\Omega} \int_0^1 (t\tilde{u}_k + (1-t)\tilde{v}_k)^{p_k-2} w_k^2 dt dx. \end{aligned} \quad (4.7)$$

Part 2 Convergence of w_k . Since

$$\|w_k\|_{L^2(\Omega)} = 1, \quad (4.8)$$

one has

$$\int_{\Omega} \langle A_k(x) \nabla w_k, \nabla w_k \rangle dx \leq C.$$

An application of in [4, Lemma A.3] yields

$$\begin{aligned} \langle A_k \xi, \xi \rangle &= \int_0^1 \left\langle D^2 \mathcal{H}(t\nabla \tilde{u}_k + (1-t)\nabla \tilde{v}_k) \xi, \xi \right\rangle dt \\ &\geq \left(\int_0^1 |t\nabla \tilde{u}_k + (1-t)\nabla \tilde{v}_k|^{m-2} dt \right) |\xi|^2 \geq \frac{1}{4^{m-1}} (|\nabla \tilde{u}_k| + |\nabla \tilde{v}_k|)^{m-2} |\xi|^2. \end{aligned} \quad (4.9)$$

Therefore, we obtain

$$\int_{\Omega} (|\nabla \tilde{u}_k| + |\nabla \tilde{v}_k|)^{m-2} |\nabla w_k|^2 dx \leq C \quad \text{for every } k \in \mathbb{N}.$$

By Corollary 2.2, there exist $\theta = \theta(m) \in (1, 2)$ and $w \in L^2(\Omega) \cap W_0^{1,\theta}(\Omega)$ such that $\{w_k\}_{k \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ and weakly in $W^{1,\theta}(\Omega)$ to w , up to a subsequence. In addition, from the $C^1(\overline{\Omega})$ convergence of \tilde{u}_k and \tilde{v}_k , we have

$$A_k \rightarrow \int_0^1 D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt \quad \text{uniformly on } \Omega. \quad (4.10)$$

This is now enough to pass the limit in (4.6)–(4.7). Indeed, the convergence of the right-hand side of (4.6) follows from the uniform convergences of \tilde{u}_k and \tilde{v}_k . As for the left-hand side, we have for every $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} & \left| \int_{\Omega} \langle A_k(x) \nabla w_k, \nabla \phi \rangle dx - \int_{\Omega} \int_0^1 \langle D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) \nabla w, \nabla \phi \rangle dx \right| \\ & \leq \left| \int_{\Omega} \left\langle \left(A_k(x) - \int_0^1 \langle D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt \right) \nabla w_k, \nabla \phi \right\rangle dx \right| \\ & \quad + \left| \int_{\Omega} \left\langle \int_0^1 D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt (\nabla w_k - \nabla w), \nabla \phi \right\rangle dx \right| \\ & \leq \left\| A_k - \int_0^1 \langle D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt \right\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla w_k| dx \\ & \quad + \left| \int_{\Omega} \left\langle \int_0^1 D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt (\nabla w_k - \nabla w), \nabla \phi \right\rangle dx \right| \rightarrow 0. \end{aligned}$$

The last estimate of the above comes from (4.10), $\|w_k\|_{W^{1,\theta}(\Omega)} \leq C$ and $D^2 \mathcal{H}(\nabla \tilde{u}) \in L^\infty(\Omega)$.

Thus w satisfies

$$\begin{aligned} & \int_{\Omega} \left\langle \int_0^1 D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt \nabla w, \nabla \phi \right\rangle dx \\ & = \lambda_m(\Omega)(m-1) \int_{\Omega} \int_0^1 (t \tilde{u} + (1-t) \mu' \tilde{u})^{m-2} w \phi dt dx, \end{aligned}$$

i.e.,

$$\int_{\Omega} \langle D^2 \mathcal{H}(\nabla \tilde{u}) \nabla w, \nabla \phi \rangle dx = \lambda_m(\Omega)(m-1) \int_{\Omega} \tilde{u}^{m-2} w \phi dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

In order to pass the limit in (4.7), we observe that for every $n, k \in \mathbb{N}$,

$$\begin{aligned} & \int_{\Omega} \langle A_k \nabla w_k, \nabla w_k \rangle dx \\ & \geq \int_{\{|\nabla w| \leq n\}} \langle A_k \nabla w_k, \nabla w_k \rangle dx \\ & \geq \int_{\{|\nabla w| \leq n\}} \langle A_k \nabla w, \nabla w \rangle dx + 2 \int_{\{|\nabla w| \leq n\}} \langle A_k \nabla w, \nabla w_k - \nabla w \rangle dx. \end{aligned}$$

Since $A_k \rightarrow D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u})$ uniformly and $w_k \rightarrow w$ weakly in $W^{1,\theta}(\Omega)$. We get

$$\lim_{k \rightarrow +\infty} \int_{\{|\nabla w| \leq n\}} \langle A_k \nabla w, \nabla w_k - \nabla w \rangle dx = 0.$$

This implies that for every $k \in \mathbb{N}$ we have

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \langle A_k \nabla w_k, \nabla w_k \rangle dx$$

$$\begin{aligned}
&\geq \liminf_{k \rightarrow \infty} \int_{\{|\nabla w| \leq n\}} \langle A_k \nabla w, \nabla w \rangle dx \\
&= \int_{\{|\nabla w| \leq n\}} \left\langle \int_0^1 D^2 \mathcal{H}(t \nabla \tilde{u} + (1-t) \mu' \nabla \tilde{u}) dt \nabla w, \nabla w \right\rangle dx \\
&= \int_0^1 (t + (1-t) \mu')^{m-2} dt \int_{\{|\nabla w| \leq n\}} \langle D^2 \mathcal{H}(\nabla \tilde{u}) \nabla w, \nabla w \rangle dx.
\end{aligned}$$

Taking $n \rightarrow \infty$, one gets

$$\int_{\Omega} \langle D^2 \mathcal{H}(\nabla \tilde{u}) \nabla w, \nabla w \rangle dx \leq (m-1) \lambda_m(\Omega) \int_{\Omega} \tilde{u}^{m-2} w^2 dx. \quad (4.11)$$

Observe that in the right-hand side we used the strong convergence in $L^2(\Omega)$ of $\{w_k\}_{k \in \mathbb{N}}$. By recalling that

$$\langle D^2 \mathcal{H}(z) \xi, \xi \rangle \geq |z|^{m-2} |\xi|^2 \quad \text{for every } z, \xi \in \mathbb{R}^N,$$

the estimate (4.11) shows that w also belongs to the weighted Sobolev space $X^{1,2}(\Omega; |\nabla u|^{m-2})$. Note also that the strong convergence of w_k in $L^2(\Omega)$ together with (4.8) implies that $\|w\|_{L^2(\Omega)} = 1$, so that w is non-trivial.

Finally, from the properties above we have

$$w \in X^{1,2}(\Omega; |\nabla \tilde{u}|^{m-2}) \cap W_0^{1,1}(\Omega) = X_0^{1,2}(\Omega; |\nabla \tilde{u}|^{m-2}),$$

Part 3 Conclusion. From the fact that $w \in X_0^{1,2}(\Omega; |\nabla \tilde{u}|^{m-2})$ is nontrivial together with Proposition 3.1, Proposition 3.2 and (4.11), it follows that w must be proportional either to \tilde{u} or to $-\tilde{u}$. In particular, w doesn't change sign.

On the other hand, by Lemma 3.1, we know that $\tilde{u}_k - \tilde{v}_k$ must change sign. Accordingly, if w_k^\pm stand for the positive and negative part of w_k , respectively, we have that each

$$\Omega_k^\pm := \{x \in \Omega : w_k^\pm(x) > 0\}$$

has positive measure. Testing equation (4.6) with w_k^\pm , we obtain by using (4.8),

$$\begin{aligned}
&\int_{\Omega} \langle A_k \nabla w_k^\pm, \nabla w_k^\pm \rangle dx \\
&= (m-1) \lambda \int_{\Omega} \int_0^1 (t \tilde{u}_k + (1-t) \tilde{v}_k)^{m-2} |w_k^\pm|^2 dt dx \\
&\quad + M_k^{p_k-m} (p_k-1) \int_{\Omega} \int_0^1 (t \tilde{u}_k + (1-t) \tilde{v}_k)^{p_k-2} |w_k^\pm|^2 dt dx \\
&\leq C \int_{\Omega} |w_k^\pm|^2 dx.
\end{aligned} \quad (4.12)$$

By Hölder's inequality, Theorem 2.4, (4.9) and (4.12), we have for an exponent $2 < \sigma < \sigma_0$,

$$\begin{aligned}
\int_{\Omega} |w_k^\pm|^2 dx &\leq \left(\int_{\Omega} |w_k^\pm|^\sigma dx \right)^{\frac{2}{\sigma}} |\Omega_k^\pm|^{\frac{\sigma-2}{\sigma}} \\
&\leq \frac{1}{\mathcal{T}} |\Omega_k^\pm|^{\frac{\sigma-2}{\sigma}} \int_{\Omega} |\nabla \tilde{u}_k|^{m-2} |\nabla w_k^\pm|^2 dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{4^{m-1}}{\mathcal{T}} |\Omega_k^\pm|^{\frac{\sigma-2}{\sigma}} \int_{\Omega} \langle A_k \nabla w_k^\pm, \nabla w_k^\pm \rangle dx \\
&\leq C \frac{4^{m-1}}{\mathcal{T}} |\Omega_k^\pm|^{\frac{\sigma-2}{\sigma}} \int_{\Omega} |w_k^\pm|^2 dx.
\end{aligned}$$

This implies

$$|\Omega_k^\pm| \geq \frac{1}{\widetilde{C}} \quad \text{for every } k \in \mathbb{N}, \quad (4.13)$$

for some constant \widetilde{C} , not depending on k . This contradicts the fact that w_k strongly converges in $L^2(\Omega)$ to the constant sign function w . The proof is over.

5 Proof of the Theorems 1.1

Lemma 5.1 *For $2 < p < \frac{2N}{N-2}$, let $u \in W_0^{1,m}(\Omega)$ be a positive solution of (1.1). Let $M = \|u\|_{L^\infty(\Omega)}$. Then there exists a constant $\delta_1 = \delta_1(N, \Omega, \lambda) > 0$ such that $M \leq C$ for all $m \in [2, 2 + \delta_1]$, where $C = C(N, \Omega, \lambda, \delta_1) > 0$.*

Proof We argue by contradiction. Choose $\{m_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow +\infty} m_k = 2$, and let $\{u_k\}_{k \in \mathbb{N}} \in W_0^{1,m}(\Omega)$ be the solution of (1.1) with $m = m_k$ and such that $M_k := \|u_k\|_{L^\infty(\Omega)} \rightarrow +\infty$. Let x_k be the point where the maximum of u_k is achieved. Define

$$\widetilde{u}_k(y) := \frac{u_k(\mu_k y + x_k)}{M_k},$$

then \widetilde{u}_k is a function satisfying $0 \leq \widetilde{u}_k \leq 1$, $\widetilde{u}_k(0) = 1$ and

$$\begin{cases} -\Delta_{m_k} \widetilde{u}_k = \lambda \mu_k^{m_k} \widetilde{u}_k^{m_k-1} + M_k^{p-m_k} \mu_k^{m_k} \widetilde{u}_k^{p-1} := f_k & \text{in } \Omega_k, \\ \widetilde{u}_k = 0 & \text{on } \partial\Omega_k, \end{cases} \quad (5.1)$$

where $\Omega_k = \{y \in \mathbb{R}^N : \mu_k y + x_k \in \Omega\}$. Let $M_k^{p-m_k} \mu_k^{m_k} = 1$, then $\mu_k \rightarrow 0$ (as $k \rightarrow +\infty$). \widetilde{u}_k satisfies

$$\begin{cases} -\Delta_{m_k} \widetilde{u}_k = \lambda \mu_k^{m_k} \widetilde{u}_k^{m_k-1} + \widetilde{u}_k^{p-1} & \text{in } \Omega_k, \\ \widetilde{u}_k = 0 & \text{on } \partial\Omega_k. \end{cases} \quad (5.2)$$

Up to a subsequence, two situations may occur:

$$\text{Either } \text{dist}(x_k, \partial\Omega) \mu_k^{-1} \rightarrow +\infty \quad \text{or} \quad \text{dist}(x_k, \partial\Omega) \mu_k^{-1} \rightarrow d \geq 0.$$

Case 1 $\lim_{k \rightarrow +\infty} \text{dist}(x_k, \partial\Omega) \mu_k^{-1} \rightarrow +\infty$. Then $\Omega_k \rightarrow \mathbb{R}^N$ as $k \rightarrow +\infty$. For any $r > 1$ with $B_r(0) \subset \Omega_k$, f_k satisfies (2.10). By Theorem 2.1, there exists an $\alpha \in (0, 1)$ and a constant $C > 0$ which does not depend on k such that

$$\|\widetilde{u}_k\|_{C^{1,\alpha}(B_r(0))} \leq C.$$

By choosing a subsequence, $\widetilde{u}_k \rightarrow \widetilde{u}$ in $C_{\text{loc}}^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0, \alpha)$. Moreover, $v(0) = 1$, $v \geq 0$ in \mathbb{R}^N , $f_k \rightarrow \widetilde{u}^{p-1}$ in $C_{\text{loc}}^{1,\beta}(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} \nabla \widetilde{u} \nabla \psi = \int_{\mathbb{R}^N} \widetilde{u}^{p-1} \psi \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N),$$

namely, \tilde{u} is a distributional solution of

$$-\Delta \tilde{u} = \tilde{u}^{p-1} \quad \text{in } \mathbb{R}^N. \quad (5.3)$$

Standard elliptic regularity yields that \tilde{u} is classical solution of (5.3) which is positive. But this contradicts the classical Liouville Theorem in the whole space (cf. [12, Theorem 1.2]).

Case 2 $\text{dist}(x^k, \partial\Omega)\mu_k^{-1} \rightarrow d \geq 0$. Then $x^k \rightarrow x^0 \in \partial\Omega$. Then a similar argument as in Lemma 2.2, we choose a subsequence, $v_k \rightarrow v$ in $C_c^{1,\beta}(\mathbb{R}_+^N)$, with $\|v\|_{L^\infty(\Omega)} = 1$. v satisfies

$$\begin{cases} -\Delta v = v^{p-1} & \text{in } \mathbb{R}_+^N, \\ v = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (5.4)$$

By Liouville Theorem in [12, Theorem 1.2], $v \equiv 0$. But this contradicts $\|v\|_{L^\infty} = 1$. The lemma is proved.

Proof of Theorem 1.1 We argue by contradiction. We suppose that for every $m > 2$, the equation (1.1) always admits (at least) two distinct positive solutions. We then take a sequence $\{m_k\}_{k \in \mathbb{N}} \subset (2, 2 + \delta_1]$ such that

$$\lim_{k \rightarrow +\infty} m_k = 2.$$

Correspondingly, for every $k \in \mathbb{N}$ there exist two distinct positive solutions u_k and v_k of (1.1). By Theorem 2.3 and assumptions of Theorem 1.1, $u_k, v_k \rightarrow u \in C^{1,\beta}(\overline{\Omega})$, up to a subsequence. u solves

$$\begin{cases} -\Delta u = \lambda u + u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.5)$$

Claim: $u \not\equiv 0$. By contradiction, assume $\lim_{k \rightarrow +\infty} M_k = 0$ where $M_k := \|u_k\|_{L^\infty(\Omega)}$. Define $\tilde{u}_k = \frac{u_k}{M_k}$. Then a similar argument as in Lemma 2.3. $\tilde{u}_k \rightarrow \tilde{u}$ in $C^{1,\beta}(\Omega)$ where \tilde{u} is the first eigenfunction of $-\Delta$. But this contradicts $\lambda < \lambda_2(\Omega)$.

The equations solved by u_k and v_k can be written as

$$\int_{\Omega} |\nabla u_k|^{m_k-2} \nabla u_k \cdot \nabla \phi \, dx = \lambda \int_{\Omega} u_k^{m_k-1} \phi \, dx + \int_{\Omega} u_k^{p-1} \phi \, dx \quad (5.6)$$

and

$$\int_{\Omega} |\nabla v_k|^{m_k-2} \nabla v_k \cdot \nabla \phi \, dx = \lambda \int_{\Omega} v_k^{m_k-1} \phi \, dx + \int_{\Omega} v_k^{p-1} \phi \, dx \quad (5.7)$$

for any $\phi \in W_0^{1,m_k}(\Omega)$. Let $w_k = \frac{u_k - v_k}{\|u_k - v_k\|_{L^2(\Omega)}}$. By using a similar way in Section 4, we get

$$\begin{aligned} \int_{\Omega} \langle A_k(x) \nabla w_k, \nabla \phi \rangle \, dx &= \lambda(m_k - 1) \int_{\Omega} \int_0^1 (tu_k + (1-t)v_k)^{m_k-2} w_k \phi \, dt \, dx \\ &\quad + (p-1) \int_{\Omega} \int_0^1 (tu_k + (1-t)v_k)^{p-2} w_k \phi \, dt \, dx, \end{aligned} \quad (5.8)$$

where

$$A_k(x) = \int_0^1 D^2 \mathcal{H}_k(t \nabla u_k(x) + (1-t) \nabla v_k(x)) \, dt,$$

$$\mathcal{H}_k(z) = |z|^{m_k-2} \text{Id} + (m_k - 2)|z|^{m_k-4} z \otimes z \quad \text{for } z \in \mathbb{R}^N.$$

Then a similar argument as in the proof of Theorem 1.2, $\{w_k\}_{k \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ and weakly in $W^{1,\theta}(\Omega)$ for some $\theta > 1$. In Remark A.1, $|\{x \in \Omega : |\nabla u(x)| = 0\}| = 0$. Then $\forall \varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$ such that $|E| < \varepsilon_1$ where $E = \{x \in \Omega : |\nabla u| \leq \varepsilon_2\}$.

$$\begin{aligned} & \left| \int_{\Omega} \langle A_k(x) \nabla w_k, \nabla \phi \rangle dx - \int_{\Omega} \langle \nabla w, \nabla \phi \rangle dx \right| \\ & \leq \int_{E^c} |\langle A_k(x) \nabla w_k, \nabla \phi \rangle - \langle \nabla w, \nabla \phi \rangle| dx \\ & \quad + \left| \int_E \langle A_k(x) \nabla w_k, \nabla \phi \rangle dx \right| + \left| \int_E \langle \nabla w, \nabla \phi \rangle dx \right|. \end{aligned}$$

The last two term converges to 0 by the continuity of integral. Thus w satisfies

$$\int_{\Omega} \langle \nabla w, \nabla \phi \rangle dx = \lambda \int_{\Omega} w \phi dx + (p-1) \int_{\Omega} u^{p-2} w \phi dx.$$

This contradicts the nondegeneracy of (1.6).

Appendix A The Linearized Equation

We first observe that $u \in C^{1,\beta}(\Omega)$ and the critical set

$$Z := \{x \in \Omega : |\nabla u| = 0\}$$

is a compact set contained in Ω , thanks to Theorem 2.3. Thus $\Omega \setminus Z$ is an open set. On this set, by classical Elliptic Regularity we can infer that $u \in C^2(\Omega \setminus Z)$.

Remark A.1 (Hessian terms) We seize the opportunity to mention that, since $u \in L^\infty(\Omega)$. The right-hand side of (2.21) is bounded. Then we have

$$|\nabla u|^{m-2} \nabla u \in W_{\text{loc}}^{1,2}(\Omega), \quad (\text{A.1})$$

thanks to (cf. [7, Theorem 2.1]). In addition, as noted in (cf. [8, Remark 2.3]), the weak gradient of $|\nabla u|^{m-2} \nabla u$ coincides with the classical gradient in $\Omega \setminus Z$, while

$$\nabla(|\nabla u|^{m-2} \nabla u) = 0, \quad \text{a.e. on } Z,$$

since by definition Z coincides with the zero level set of $|\nabla u|^{m-2} \nabla u$, thus it is sufficient to use a standard property of Sobolev functions. By further observing that $|Z| = 0$ (by virtue of [8, Theorem 2.3]). We can actually say that the weak gradient of $|\nabla u|^{m-2} \nabla u$ coincides with the classical gradient almost everywhere in Ω .

We consider the following equation

$$\begin{cases} -\Delta_m u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{A.2})$$

where $f \in C^{1,1}(\mathbb{R}^+)$. We assume that u satisfies the following properties.

(u1) There exist $\chi \in (0, 1)$ and $L > 0$ such that

$$\|u\|_{C^{1,\chi}(\Omega)} \leq L.$$

(u2) There exist $\tau > 0$, $\mu_0 > 0$ and $\mu_1 > 0$, such that

$$|\nabla u| \geq \mu_0 \quad \text{in } \Omega_\tau$$

and

$$u \geq \mu_1 \quad \text{in } \overline{\Omega \setminus \Omega_\tau},$$

where $\Omega_\tau = \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \tau\}$.

We then take $\psi \in C_0^\infty(\Omega \setminus Z)$ and test the weak formulation of (A.2) against a partial derivative ψ_{x_i} . We obtain

$$\begin{aligned} & \int_{\Omega} \langle |\nabla u|^{m-2} \nabla u_{x_i}, \nabla \psi \rangle dx + (m-2) \int_{\Omega} |\nabla u|^{m-4} \langle \nabla u, \nabla u_{x_i} \rangle \langle \nabla u, \nabla \psi \rangle dx \\ &= \int_{\Omega} f'(u) u_{x_i} \psi dx \quad \text{for every } \psi \in C_0^\infty(\Omega \setminus Z). \end{aligned} \quad (\text{A.3})$$

Proposition A.1 *Let $m > 1$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. Let $u \in W_0^{1,m}(\Omega)$ be a positive solution of (A.2). Assume u satisfies (u1) and (u2). Let $\beta \in [0, 1)$ and*

$$\begin{cases} \gamma < N - 2, & \text{if } N \geq 3, \\ \gamma \leq 0, & \text{if } N = 2. \end{cases}$$

Then for every $i \in \{1, \dots, N\}$, if we set $Z_i = \{y \in \Omega : u_{x_i}(y) = 0\}$, we have the estimate

$$\sup_{x \in \Omega} \int_{E \setminus Z_i} \frac{|\nabla u(y)|^{m-2} |u_{x_i}(y)|^{-\beta} |\nabla u_{x_i}(y)|^2}{|x - y|^\gamma} dy \leq C_1 \quad (\text{A.4})$$

for some $C_1 = C_1(N, L, \|f\|_{C^{1,1}(\mathbb{R})}, \alpha, \beta, \gamma, \Omega, \text{dist}(E, \partial\Omega))(1 + (m-1)L^{m-\beta}) > 0$.

Proof Without loss of generality, we prove the result for $\gamma \geq 0$. The heuristic idea is to test the linearized equation (A.3) with $u_{x_i} |u_{x_i}|^{-\beta} |x - y|^{-\gamma} \phi^2$, where ϕ is a smooth cut-off function. In order to do this rigorously, we fix $x \in \Omega$ and for $0 < \varepsilon < 1$ use the test function

$$\psi(y) = G_\varepsilon(u_{x_i}(y)) |u_{x_i}(y)|^{-\beta} (|x - y| + \varepsilon)^{-\gamma} \phi^2(y),$$

where $\phi \in C_0^\infty(\Omega)$ is such that

$$\phi \equiv 1 \quad \text{on } E, \quad 0 \leq \phi \leq 1, \quad \|\nabla \phi\|_{L^\infty(E)} \leq \frac{C}{\text{dist}(E, \partial\Omega)},$$

and the odd Lipschitz function G_ε is given by

$$G_\varepsilon(t) = \max\{t - \varepsilon, 0\} \quad \text{for } t \geq 0 \quad \text{and} \quad G_\varepsilon(t) = -G_\varepsilon(-t) \quad \text{for } t \leq 0.$$

The function ψ is a product of a Lipschitz function of u_{x_i} , which vanishes in a neighborhood of the critical set Z , and a smooth function with compact support in Ω . Therefore ψ is an admissible test function for (A.3). This gives

$$\int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x - y| + \varepsilon)^\gamma} \left[G'_\varepsilon(u_{x_i}) - \beta \frac{G_\varepsilon(u_{x_i})}{u_{x_i}} \right] \phi^2 dy$$

$$\begin{aligned}
& + (m-2) \int_{\Omega} \frac{|\nabla u|^{m-4} (\langle \nabla u, \nabla u_{x_i} \rangle)^2 |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} \left[G'_\varepsilon(u_{x_i}) - \beta \frac{G_\varepsilon(u_{x_i})}{u_{x_i}} \right] \phi^2 dy \\
& + 2 \int_{\Omega} \frac{|\nabla u|^{m-2} \langle \nabla u_{x_i}, \nabla \phi \rangle |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} \phi G_\varepsilon(u_{x_i}) dy \\
& + 2(m-2) \int_{\Omega} \frac{|\nabla u|^{m-4} \langle \nabla u, \nabla u_{x_i} \rangle \langle \nabla u, \nabla \phi \rangle |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} \phi G_\varepsilon(u_{x_i}) dy \\
& + \int_{\Omega} |\nabla u|^{m-2} \langle \nabla u_{x_i}, \nabla (|x-y|+\varepsilon)^{-\gamma} \rangle G_\varepsilon(u_{x_i}) |u_{x_i}|^{-\beta} \phi^2 dy \\
& + (m-2) \int_{\Omega} |\nabla u|^{m-4} \langle \nabla u, \nabla u_{x_i} \rangle \langle \nabla u, \nabla (|x-y|+\varepsilon)^{-\gamma} \rangle G_\varepsilon(u_{x_i}) |u_{x_i}|^{-\beta} \phi^2 dy \\
& = \int_{\Omega} \frac{f'(u) G_\varepsilon(u_{x_i}) |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} \phi^2 dy.
\end{aligned}$$

Note that

$$G'_\varepsilon(t) - \beta \frac{G_\varepsilon(t)}{t} \geq 0 \quad \text{for every } t \in \mathbb{R},$$

therefore, by dropping the second term on the left-hand side and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} \left[G'_\varepsilon(u_{x_i}) - \beta \frac{G_\varepsilon(u_{x_i})}{u_{x_i}} \right] \phi^2 dy \\
& \leq 2(m-1) \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}| |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} G_\varepsilon(u_{x_i}) \phi |\nabla \phi| dy \\
& \quad + \gamma(m-1) \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}| |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^{\gamma+1}} G_\varepsilon(u_{x_i}) \phi^2 dy \\
& \quad + \int_{\Omega} \frac{f'(u) G_\varepsilon(u_{x_i}) |u_{x_i}|^{-\beta}}{(|x-y|+\varepsilon)^\gamma} \phi^2 dy \\
& := I_1 + I_2 + I_3.
\end{aligned} \tag{A.5}$$

By using Theorem 2.3, the properties of the cut-off function ϕ , the last term can be estimated as

$$I_3 \leq C \int_{\Omega} \frac{1}{|x-y|^\gamma} dy$$

for a constant C depending on N , $\|f\|_{C^{1,1}(\mathbb{R})}$, L , α , β and Ω , only. In turn, the last integral is easily estimated as follows

$$\begin{aligned}
\int_{\Omega} \frac{1}{|x-y|^\gamma} dy & \leq \int_{\{y \in \mathbb{R}^N : |y-x| \leq \text{diam}(\Omega)\}} \frac{1}{|y-x|^\gamma} dy \\
& = N\omega_N \int_0^{\text{diam}(\Omega)} \rho^{N-1-\gamma} d\rho = \frac{N\omega_N}{N-\gamma} (\text{diam}(\Omega))^{N-\gamma}.
\end{aligned} \tag{A.6}$$

In the last integral we used that $\gamma < N-2$.

As for the terms I_1 and I_2 , we first observe that by using

$$(|x-y|+\varepsilon)^\gamma \geq \frac{(|x-y|+\varepsilon)^{\gamma+1}}{\text{diam}(\Omega)+1} \quad \text{for every } x, y \in \Omega, \tag{A.7}$$

we have

$$I_1 + I_2 \leq (2 + |\gamma|)(m-1) \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}| |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma+1}} G_{\varepsilon}(u_{x_i}) \phi [\phi + |\nabla \phi|] dy.$$

Thus, from (A.5), we have obtained

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma}} \left[G'_{\varepsilon}(u_{x_i}) - \beta \frac{G_{\varepsilon}(u_{x_i})}{u_{x_i}} \right] \phi^2 dy \\ & \leq C + (2 + |\gamma|)(m-1) \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}| |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma+1}} G_{\varepsilon}(u_{x_i}) \phi [\phi + |\nabla \phi|] dy. \end{aligned} \quad (A.8)$$

In order to estimate the last integral, we used Young's inequality. For every $\delta > 0$, we have

$$\begin{aligned} & (2 + |\gamma|)(m-1) \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}| |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma+1}} G_{\varepsilon}(u_{x_i}) \phi [\phi + |\nabla \phi|] dy \\ & \leq \frac{\delta}{2} \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma}} \frac{G_{\varepsilon}(u_{x_i})}{u_{x_i}} \phi^2 dy \\ & \quad + \frac{((2 + |\gamma|)(m-1))^2}{2\delta} \int_{\Omega} \frac{|\nabla u|^{m-2} |u_{x_i}|^{1-\beta}}{(|x-y| + \varepsilon)^{\gamma+2}} G_{\varepsilon}(u_{x_i}) [\phi + |\nabla \phi|]^2 dy. \end{aligned} \quad (A.9)$$

We make the choice $\delta = (2 + |\gamma|)(m-1)$. Then observe that

$$(1 - \beta) \frac{G_{\varepsilon}}{t} \leq G'_{\varepsilon}(t) - \beta \frac{G_{\varepsilon}(t)}{t} \quad \text{for every } |t| \neq \varepsilon.$$

Then the first term in the right-hand side of (A.9) can now be estimated by

$$\frac{(2 + |\gamma|)(m-1)}{2} \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma}} \left[G'_{\varepsilon}(u_{x_i}) - \beta \frac{G_{\varepsilon}(u_{x_i})}{u_{x_i}} \right] \phi^2 dy,$$

which can be absorbed into the right-hand side of (A.8). Thus, up to now, we obtained

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma}} \left[G'_{\varepsilon}(u_{x_i}) - \beta \frac{G_{\varepsilon}(u_{x_i})}{u_{x_i}} \right] \phi^2 dy \\ & \leq C + (2 + |\gamma|)(m-1) \int_{\Omega} \frac{|\nabla u|^{m-2} |u_{x_i}|^{1-\beta}}{(|x-y| + \varepsilon)^{\gamma+2}} G_{\varepsilon}(u_{x_i}) [\phi + |\nabla \phi|]^2 dy \end{aligned} \quad (A.10)$$

possibly for a different constant, independent of $x \in \Omega$, and $\varepsilon \in (0, 1)$. Using that $|G_{\varepsilon}(t)| \leq |t|$ and the properties of ϕ , the last integral of (A.10) can be estimated by

$$\int_{\Omega} \frac{|\nabla u|^{m-2} |u_{x_i}|^{2-\beta}}{(|x-y| + \varepsilon)^{\gamma+2}} [\phi + |\nabla \phi|]^2 dy \leq CL^{m-\beta} \int_{\Omega} \frac{1}{|x-y|^{\gamma+2}} dy,$$

and the last integral is uniformly (in $x \in \Omega$) bounded by a constant depending only on N and $\text{diam}(\Omega)$. From (A.10) and using that $\phi \equiv 1$ on E , we thus obtain

$$\int_E \frac{|\nabla u|^{m-2} |\nabla u_{x_i}|^2 |u_{x_i}|^{-\beta}}{(|x-y| + \varepsilon)^{\gamma}} \left[G'_{\varepsilon}(u_{x_i}) - \beta \frac{G_{\varepsilon}(u_{x_i})}{u_{x_i}} \right] \phi^2 dy \leq C(1 + (m-1)L^{m-\beta}) \quad (A.11)$$

for some $C = C(N, L, \|f\|_{C^{1,1}(\mathbb{R})}, \gamma, \alpha, \beta, \Omega, \text{dist}(E, \partial\Omega)) > 0$. We have

$$\lim_{\varepsilon \rightarrow 0} \left[G'_{\varepsilon} - \beta \frac{G_{\varepsilon}(\tau)}{\tau} \right] = 1 - \beta \quad \text{for } \tau \neq 0. \quad (A.12)$$

By using Fatou's Lemma together with (A.12), we may take the limit as ε goes to 0 in (A.11) and obtain

$$\int_{E \setminus Z_i} \frac{|\nabla u|^{m-2} |u_{x_i}|^{-\beta} |\nabla u_{x_i}|^2}{|x - y|^\gamma} dy \leq C.$$

The previous bound holds uniformly with respect to $x \in \Omega$, thus we get the desired conclusion.

Corollary A.1 *Under the assumption of Proposition A.1, for every $\beta \in (-\infty, 1)$, we have*

$$\sup_{x \in \Omega} \int_E \frac{|\nabla u(y)|^{m-2-\beta} |D^2 u(y)|^2}{|x - y|^\gamma} dy \leq C_1 (1 + (m-1)L^{m-\beta}).$$

Here we still denote by C_1 the same constant as in Proposition A.1.

Proof From (A.4), by using that $|u_{x_i}|^{-\beta} \geq |\nabla u|^{-\beta}$, we immediately get

$$\sup_{x \in \Omega} \int_{E \setminus Z_i} \frac{|\nabla u(y)|^{m-2-\beta} |\nabla u_{x_i}(y)|^2}{|x - y|^\gamma} dy \leq C_1 (1 + (m-1)L^{m-\beta}).$$

We then observe that $u_{x_i} \in C^1(\Omega \setminus Z)$ and thus it belongs to $W_{\text{loc}}^{1,1}(\Omega \setminus Z)$. By appealing to [21, Theorem 6.19], we have

$$\nabla u_{x_i} = 0, \text{ a.e. in } Z_i \cap (\Omega \setminus Z).$$

By using this fact and summing over $i = 1, \dots, N$, we get the claimed inequality, by recalling that $|Z| = 0$ (see Remark A.1).

The case $\beta < 0$ can be reduced to the case $\beta = 0$: It is sufficient to use that $\|\nabla u\|_{L^\infty} \leq L \leq +\infty$, thus we get

$$\sup_{x \in \Omega} \int_E \frac{|\nabla u(y)|^{m-2-\beta} |D^2 u(y)|^2}{|x - y|^\gamma} dy \leq L^{-\beta} \sup_{x \in \Omega} \int_E \frac{|\nabla u(y)|^{m-2} |D^2 u(y)|^2}{|x - y|^\gamma} dy.$$

Proposition A.2 *Let $m > 1$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. Let $\lambda > 0$ and $f(t) - \lambda t^{m-1} \geq 0$ for all t . Let $u \in W_0^{1,m}(\Omega)$ be a positive solution of (A.2). Assume u satisfies (u1). Let*

$$\begin{cases} \gamma < N - 2, & \text{if } N \geq 3, \\ \gamma \leq 0, & \text{if } N = 2. \end{cases}$$

Then for every $K \Subset E \Subset \Omega$ and every $b < 1$, we have

$$\sup_{x \in \Omega} \int_K \frac{1}{|\nabla u|^{(m-1)b} |x - y|^\gamma} dy \leq C_2 \left(\frac{1}{\inf_E u} \right)^{m-1} \left(L^{(m-1)(1-b)} + \frac{(m-1)^3 L^{m-b}}{\left(\inf_E u \right)^{m-1}} \right),$$

where $C_2 = C_2(N, L, \alpha, b, \gamma, \lambda, \text{dist}(K, \partial E), \text{dist}(E, \partial \Omega)) > 0$.

Proof Without loss of generality, we prove the result for $\gamma \geq 0$. The heuristic idea is to test equation (A.2) with $|\nabla u|^{-(m-1)b} |x - y|^{-\gamma} \phi$, where $\phi \in C_0^\infty(E)$ is a cut-off function such that

$$\phi \equiv 1 \text{ in } K, \quad 0 \leq \phi \leq 1, \quad \|\phi\|_{L^\infty(E)} \leq \frac{C}{\text{dist}(K, \partial E)}.$$

To make this precise, we fix $x \in \Omega$ and use for every $\varepsilon > 0$ the test function

$$\psi(y) = (|\nabla u|^{m-1} + \varepsilon)^{-b} (|x - y| + \varepsilon)^{-\gamma} \phi(y). \quad (\text{A.13})$$

This is a product of a Lipschitz function of $|\nabla u|^{m-1}$ and a smooth function with compact support in Ω . In light of (A.3), we have that this function belongs to $W_0^{1,2}(\Omega)$. If we now use that $u \in L^\infty(\Omega)$, we see that in the weak formulation of (A.2) we can in particular admit test functions $\psi \in W_0^{1,2}(\Omega)$. Therefore the test function in (A.13) is feasible.

This gives

$$\begin{aligned} & \lambda \int_{\Omega} \frac{u^{m-1}}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi}{(|x - y| + \varepsilon)^\gamma} dy + \int_{\Omega} \frac{f(u) - \lambda u^{m-1}}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi}{(|x - y| + \varepsilon)^\gamma} dy \\ &= \int_{\Omega} \frac{\langle |\nabla u|^{m-2} \nabla u, \nabla \phi \rangle}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{1}{(|x - y| + \varepsilon)^\gamma} dy \\ & \quad - \beta(m-1) \int_{\Omega} \frac{|\nabla u|^{2m-5}}{(|\nabla u|^{m-1} + \varepsilon)^{b+1}} \langle D^2 u \nabla u, \nabla u \rangle \frac{\phi}{(|x - y| + \varepsilon)^\gamma} dy \\ & \quad + \int_{\Omega} \frac{|\nabla u|^{m-2}}{(|\nabla u|^{m-1} + \varepsilon)^b} \langle \nabla u, \nabla |x - y| + \varepsilon|^{-\gamma} \rangle \phi dy. \end{aligned}$$

Dropping the second term in the left-hand side and using that

$$u \geq \inf_E u > 0,$$

we obtain

$$\begin{aligned} & (\inf_E u)^{m-1} \int_{\Omega} \frac{1}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi}{(|x - y| + \varepsilon)^\gamma} dy \\ & \leq C \int_{\Omega} \frac{|\nabla u|^{m-1} |\nabla \phi|}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{1}{(|x - y| + \varepsilon)^\gamma} dy \\ & \quad + Cb(m-1) \int_{\Omega} \frac{|\nabla u|^{2m-5}}{(|\nabla u|^{m-1} + \varepsilon)^{b+1}} \langle D^2 u \nabla u, \nabla u \rangle \frac{\phi}{(|x - y| + \varepsilon)^\gamma} dy \\ & \quad + C\gamma \int_{\Omega} \frac{|\nabla u|^{m-1}}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi}{(|x - y| + \varepsilon)^{\gamma+1}} dy \\ & := J_1 + J_2 + J_3, \end{aligned} \quad (\text{A.14})$$

where $C = C(N, \Omega, \lambda) > 0$. We observe that, by using (A.7), we have

$$J_1 + J_3 \leq C \int_{\Omega} \frac{|\nabla u|^{m-1}}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi + |\nabla \phi|}{(|x - y| + \varepsilon)^{\gamma+1}} dy$$

for a constant C depending on $\text{diam}(\Omega)$ and γ . We can then go by observing that $|\nabla u|^{m-1} \leq |\nabla u|^{m-1} + \varepsilon$. This gives

$$J_1 + J_3 \leq CL^{(m-1)(1-b)} \int_{\Omega} \frac{1}{(|x - y| + \varepsilon)^{\gamma+1}} dy$$

for a constant $C = C(N, \alpha, b, \gamma, \Omega, \text{dist}(K, \partial E), \lambda) > 0$. Then we can estimate the last integral as in (A.6). For J_2 we have

$$J_2 \leq C(m-1) \int_{\Omega} \frac{|\nabla u|^{2m-3}}{(|\nabla u|^{m-1} + \varepsilon)^{b+1}} \frac{|D^2 u|}{(|x - y| + \varepsilon)^\gamma} \phi dy.$$

In the last integral above, we use Young's inequality as follows

$$\begin{aligned}
& C(m-1) \int_{\Omega} \frac{|\nabla u|^{2m-3}}{(|\nabla u|^{m-1} + \varepsilon)^{b+1}} \frac{|D^2 u|}{(|x-y| + \varepsilon)^{\gamma}} \phi dy \\
& \leq \frac{(\inf_E u)^{m-1}}{2} \int_{\Omega} \frac{1}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi}{(|x-y| + \varepsilon)^{\gamma}} dy \\
& \quad + \frac{(C(m-1))^2}{2(\inf_E u)^{m-1}} \int_{\Omega} \frac{|\nabla u|^{4m-6}}{(|\nabla u|^{m-1} + \varepsilon)^{b+2}} \frac{|D^2 u|^2 \phi}{(|x-y| + \varepsilon)^{\gamma}} dy \\
& \leq \frac{(\inf_E u)^{m-1}}{2} \int_{\Omega} \frac{1}{(|\nabla u|^{m-1} + \varepsilon)^b} \frac{\phi}{(|x-y| + \varepsilon)^{\gamma}} dy \\
& \quad + \frac{(C(m-1))^2}{2(\inf_E u)^{m-1}} \int_{\Omega} |\nabla u|^{(2-b)(m-2)-b} \frac{|D^2 u|^2 \phi}{(|x-y| + \varepsilon)^{\gamma}} dy.
\end{aligned}$$

The last integral is uniformly bounded, thanks to Corollary A.1. We then obtain from (A.14)

$$\begin{aligned}
& \left(\inf_E u \right)^{m-1} \int_{\Omega} \frac{1}{(|\nabla u|^{m-1} + \varepsilon)^{\beta}} \frac{\phi}{(|x-y| + \varepsilon)^{\gamma}} dy \\
& \leq C \left(L^{(m-1)(1-b)} + \frac{(m-1)^3 L^{m-b}}{\left(\inf_E u \right)^{m-1}} \right) \\
& \quad + \frac{\left(\inf_E u \right)^{m-1}}{2} \int_{\Omega} \frac{1}{(|\nabla u|^{m-1} + \varepsilon)^{\beta}} \frac{\phi}{(|x-y| + \varepsilon)^{\gamma}} dy,
\end{aligned}$$

where the constant $C = C(N, \alpha, b, \gamma, \lambda, \Omega, \text{dist}(E, \cdot, \partial E)) > 0$. The term on the right-hand side can now be absorbed in the left-hand side. Since $\phi = 1$ on K , this implies the desired result upon letting ε go to 0 and using Fatou's Lemma.

Theorem A.1 *Let $m > 1$ and $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with boundary of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. Let $\lambda > 0$ and let $f(t) - \lambda t^{m-1} \geq 0$ for all t . Let $u \in W_0^{1,m}(\Omega)$ be a positive solution of (A.2). Assume u satisfies (u1) and (u2). Let*

$$\begin{cases} \gamma < N - 2, & \text{if } N \geq 3, \\ \gamma \leq 0, & \text{if } N = 2. \end{cases}$$

Then every $r < m - 1$, there exists $\mathcal{S} = \mathcal{S}(\alpha, N, \Omega, L, \mu_0, \mu_1, r, \gamma, \lambda, m) > 0$ such that

$$\sup_{x \in \Omega} \int_{\Omega} \frac{1}{|\nabla u_p(y)|^r |y - x|^{\gamma}} dy \leq \mathcal{S}.$$

In particular, we also have

$$\sup_{\Omega} \int_{\Omega} \frac{1}{|\nabla u_p(y)|^r} dy \leq \tilde{\mathcal{S}} \tag{A.15}$$

for some $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(\alpha, N, \Omega, L, \mu_0, \mu_1, r, \gamma, \lambda, m) > 0$.

Proof We only treat the case $N \geq 3$ in detail. This is (cf. [8, Theorem 2.3]): As claimed, we just want to pay attention to the dependence of the constant \mathcal{S} on the data. We have

$$\int_{\Omega_{\tau}} \frac{1}{|\nabla u_p(y)|^r |y - x|^{\gamma}} dy \leq \frac{1}{\mu_0^r} \int_{\Omega_{\tau}} \frac{1}{|y - x|^{\gamma}} dy \leq \frac{N \omega_N}{(N - \gamma) \mu_0^r} (\text{diam}(\Omega))^{N-\gamma}. \tag{A.16}$$

Thus we have a uniform estimate, at least when integrating in fixed neighborhood of the boundary. In order to prove a uniform estimate on

$$\int_{\Omega \setminus \Omega_\tau} \frac{1}{|\nabla u_p(y)|^r |y-x|^\gamma} dy \quad \text{for every } x \in \Omega,$$

we apply Proposition A.2 with

$$E = \Omega \setminus \Omega_{\tau/2}, \quad K = \Omega \setminus \Omega_\tau, \quad b = \frac{r}{m-1},$$

and with the constant μ_1 provided by Theorem 2.3. This yields

$$\int_{\Omega \setminus \Omega_\tau} \frac{1}{|\nabla u_p(y)|^r |y-x|^\gamma} dy \leq C_2 \left(\frac{1}{\inf_E u} \right)^{m-1} \left(L^{(m-1)(1-b)} + \frac{(m-1)^3 L^{m-b}}{\left(\inf_E u \right)^{m-1}} \right),$$

as desired.

Finally, the estimate (A.15) is an easy consequence of the previous one, it is sufficient to take $\gamma = 0$.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

- [1] Allegretto, W. and Huang, Y. X., A Picone's identity for the p -Laplacian and applications, *Nonlinear Analysis: Theory, Methods & Applications*, **32**(7), 1998, 819–830.
- [2] Angeles, F. and Saldaña, A., Small order limit of fractional Dirichlet sublinear-type problems, *Fract. Calc. and Appl. Anal.*, **26**(4), 2023, 1594–1631.
- [3] Brasco, L. and Franzina, G., An overview on constrained critical points of Dirichlet integrals, *Rend. Semin. Mat. Univ. Politec. Torino.*, **78**(2), 2020, 7–50.
- [4] Brasco, L. and Lindgren, E., Uniqueness of extremals for some sharp Poincaré-Sobolev constants, *Tran. Amer. Math. Soc.*, **376**(5), 2023, 3541–3584.
- [5] Brezis, H. and Oswald, L., Remarks on sublinear elliptic equations, *Nonlinear Analysis: Theory, Methods & Applications*, **10**(1), 1986, 55–64.
- [6] Cheng, T. Z., Huang G. G. and Xu, X. H., Uniqueness of nontrivial solutions for degenerate Monge-Ampère equations, *SIAM Journal On Mathematical Analysis*, **56**(1), 2024, 234–253.
- [7] Cianchi, A. and G. Maz'ya V., Second-order two-sided estimates in nonlinear elliptic problems, *Archive for Rational Mechanics and Analysis*, **229**, 2018, 569–599.
- [8] Damascelli, L. and Sciunzi B., Regularity, monotonicity and symmetry of positive solutions of m -Laplace equations, *Journal of Differential Equations*, **206**(2), 2004, 483–515.
- [9] Dieb, A., Ianni, I. and Saldaña, A., Uniqueness and nondegeneracy for Dirichlet fractional problems in bounded domains via asymptotic methods, *Nonlinear Anal.*, **236**, 2023, 113354, 21 pp.
- [10] Fan, X., Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, *Journal of Differential Equations*, **235**(2), 2007, 397–417.
- [11] García Azorero, J. P. and Alonso I. P., Existence and nonuniqueness for the p -Laplacian, *Communications in Partial Differential Equations*, **12**(12), 1987, 126–202.

- [12] Gidas, B. and Spruck J, A priori bounds for positive solutions of nonlinear elliptic equations, *Communications in Partial Differential Equations*, **6**(8), 1981, 883–901.
- [13] Gilbarg, D. and Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, New York, 1977.
- [14] Guedda, M. and Véron L., Quasilinear elliptic equations involving critical Sobolev exponents, *Nonlinear Analysis: Theory Methods & Applications*, **13**(8), 1989, 879–902.
- [15] Han, Q., *Nonlinear Elliptic Equations of the Second Order*, American Mathematical Soc, Dover., 2016.
- [16] Henrot, A., *Extremum Problems for Eigenvalues of Elliptic Operators*, Frtont. Math., Birkhäuser-Verlag, Basel, 2006.
- [17] Huang, G., Uniqueness of least energy solutions for Monge-Ampère functional, *Calculus of Variations and Partial Differential Equations*, **58**, 2019, 1–20.
- [18] Idogawa, T. and Ôtani, M., The first eigenvalues of some abstract elliptic operators, *Funkcialaj Ekvacioj*, **38**(1), 1995, 1–9.
- [19] Kawohl, B., Symmetry results for functions yielding best constants in Sobolev-type inequalities, *Discrete & Continuous Dynamical Systems*, **6**(3), 2000, 683–690.
- [20] Kwong, M. K. and Li, Y., Uniqueness of radial solutions of semilinear elliptic equations, *Transactions of the American Mathematical Society*, **333**(1), 1992, 339–363.
- [21] Lieb, E. H. and Loss, M., *Analysis*, American Mathematical Soc., Providence, RI, 2001.
- [22] Lin, C., Uniqueness of least energy solutions to a semilinear elliptic equation in \mathbb{R}^2 , *Manuscripta Mathematica*, **84**, 1994, 13–19.
- [23] Lions, P. L., Two remarks on Monge-Ampère equations, *Annali di matematica pura ed applicata*, **142**, 1985, 263–275.
- [24] Long, W., Yan, S. and Yang, J., A critical elliptic problem involving fractional Laplacian operator in domains with shrinking holes, *J. Diff. Eq.*, **267**(7), 2019, 4117–4147.
- [25] Mikayelyan, H. and Shahgholian, H., Hopf’s lemma for a class of singular/degenerate PDE’s, *Ann. Acad. Sci. Fenn.*, **40**(3), 2015, 475–484.
- [26] Tso, K., On a real Monge–Ampère functional, *Inventiones Mathematicae*, **101**(2), 1990, 425–448.