Chin. Ann. Math. Ser. B
46(3), 2025, 443–448
DOI: 10.1007/s11401-025-0024-1

Chinese Annals of Mathematics, Series B

A Note on Translating Solitons to Lagrangian Mean Curvature Flows*

Hongbing QIU[1] Chenyu ZHU[1]

Abstract The authors prove a rigidity result of Lagrangian translating solitons in $\mathbb{R}^{2n}$, which extends the result of [Han, X. and Sun, J., Translating solitons to symplectic mean curvature flows, *Ann. Global Anal. Geom.*, **38**(2), 2010, 161–169] to higher dimension.

Keywords Rigidity, Lagrangian translating solitons, Lagrangian angle, Mean curvature
2020 MR Subject Classification 53C24, 53E10

1 Introduction

Let J, ω be the standard complex structure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and the standard Kähler form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, respectively. The closed complex-valued n-form is given by

$$\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

where $z_j = x_j + \mathrm{i}y_j$ are complex coordinates of $\mathbb{C}^n$.

A smooth n-dimensional submanifold Σ in $\mathbb{C}^n$ is said to be Lagrangian if $\omega|_\Sigma = 0$. The induced volume form $\mathrm{d}\mu_\Sigma$ on a Lagrangian submanifold Σ is related to Ω by

$$\Omega|_\Sigma = \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\mu_\Sigma = \cos\theta \mathrm{d}\mu_\Sigma + \mathrm{i}\sin\theta \mathrm{d}\mu_\Sigma,$$

where θ is some multi-valued function called the Lagrangian angle and is well-defined up to an additive constant $2k\pi, k \in \mathbb{Z}$. Nevertheless, $\cos\theta$ and $\sin\theta$ are single valued functions on Σ. If $\cos\theta \geq \delta$ for some positive constant δ, then Σ is said to be almost calibrated. The relation between the Lagrangian angle and the mean curvature is given by (see [13, 16])

$$H = J\nabla\theta. \tag{1.1}$$

Recall that Σ^n is said to be a translating soliton in $\mathbb{R}^{2n}$ if it satisfies

$$H = V_0^N, \tag{1.2}$$

where V_0 is a fixed vector in $\mathbb{R}^{2n}$ with unit length and V_0^N denotes the orthogonal projection of V_0 onto the normal bundle of Σ^n.

Manuscript received May 6, 2022. Revised March 3, 2023.
[1]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail: hbqiu@whu.edu.cn 2019300001093@whu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 12471050) and Hubei Provincial Natural Science Foundation of China (No. 2024AFB746).

Huisken showed, in the seminal paper [8], that the closed convex hypersurfaces in Euclidean space $\mathbb{R}^{m+1}(m > 1)$ contracts to a single point under the mean curvature flow in finite time and the normalized flow (area is fixed) converges to a sphere of the same area in infinite time. As time evolves, the mean curvature flow may develop singularities which can be classified as Type I and Type II according to the blow up rate of the second fundamental form with respect to time t. And Huisken [9] proved that after appropriate rescaling near the Type I singularity the hypersurfaces converge to a self-similar solution of the mean curvature flow.

Due to the fact that the Lagrangian condition is preserved under the mean curvature flow (see [17]), the Lagrangian mean curvature flow has attracted special attention. Smoczyk [18] proved that there do not exist any compact Type I singularities with trivial Maslov class. Afterward, Chen-Li [3] and Wang [23] showed independently that there is no finite time Type I singularity along the almost calibrated Lagrangian MCF. Later, this result was extended to the complete zero-Maslov class case by Neves [12]. Therefore it is of great interest to understand the geometric and analytic nature of Type II singularities in the Lagrangian mean curvature flow with zero-Maslov class.

One of the most important examples of Type II singularities is the translating soliton. There are plenty of works on the subject of Lagrangian translating solitons, see [1–2, 4–6, 11, 16, 19–22] and the references therein. Joyce-Lee-Tsui [10] constructed many translating solitons for Lagrangian mean curvature flow with oscillation of the Lagrangian angle arbitrarily small, these translating solitons are important in studying the regularity of Lagrangian mean curvature flow. In [7], Han-Sun verified that any complete almost calibrated Lagrangian translating soliton with nonnegative sectional curvature in $\mathbb{R}^4$ must be an affine plane. Afterward, Neves-Tian [13] showed that the Lagrangian translating soliton in $\mathbb{R}^4$ with L^2-bound on the mean curvature has to be an affine plane, so does the static and almost calibrated Lagrangian translating soliton in $\mathbb{R}^4$. The dimension 4 of the ambient space is necessary, which was explained in their paper [13]. So it natural to ask that whether we can generalize the result of [7] to higher dimension.

In this paper, by using the strategy of Xin's work on translating solitons in [24] (see also [15]), we prove the following rigidity result of Lagrangian translating solitons in $\mathbb{R}^{2n}$.

Theorem 1.1 *Let $X : \Sigma^n \to \mathbb{R}^{2n}$ be an n-dimensional complete proper almost calibrated Lagrangian translating soliton with nonnegative scalar curvature. Then Σ has to be an affine n-plane.*

Remark 1.1 Han-Sun [7] proved that any complete almost calibrated Lagrangian translating soliton in $\mathbb{R}^4$ with nonnegative sectional curvature must be flat (see also [14]). Hence the above Theorem 1.1 extends their results to higher dimension.

Remark 1.2 By the example of "grim reaper" $(x, y, -\ln\cos x, 0), |x| < \frac{\pi}{2}, y \in \mathbb{R}$, we easily know that $\cos\theta \geq \delta > 0$ is necessary. Moreover, Joyce-Lee-Tsui [10] showed that for any $0 < \delta < 1$, there exist nontrivial Lagrangian translating solitons which satisfy $\cos\theta \geq \delta$, where θ is the Lagrangian angle. Thus we can not conclude that Σ is flat without the condition on the scalar curvature in Theorem 1.1.

2 Basic Notations

Let $\overline{\nabla}, \nabla$ be the Levi-Civita connection on the ambient space $\mathbb{R}^{2n}$ and the submanifold Σ^n, respectively. If there is no confusion, we also denote the normal connection on the normal bundle $N\Sigma$ by ∇.

The second fundamental form B of Σ^n in $\mathbb{R}^{2n}$ is defined by $B(U,W) := (\overline{\nabla}_U W)^N$ for $U, W \in \Gamma(T\Sigma)$. We use the notation $(\cdot)^T$ and $(\cdot)^N$ for the orthogonal projections into the tangent bundle $T\Sigma$ and the normal bundle $N\Sigma$, respectively. For $\nu \in \Gamma(N\Sigma)$ we define the shape operator $A^\nu : T\Sigma \to T\Sigma$ by $A^\nu(U) := -(\overline{\nabla}_U \nu)^T$. Taking the trace of B gives the mean curvature vector H of Σ in $\mathbb{R}^{2n}$ and

$$H := \mathrm{trace}(B) = \sum_{i=1}^{n} B(e_i, e_i),$$

where $\{e_i\}$ is a local orthonormal frame field of Σ.

3 Proof of Theorem 1.1

Let $V := V_0^T$ and $\Delta_V := \Delta + \langle V, \nabla \cdot \rangle$.

Proof Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field on Σ such that $\nabla e_i = 0$ at the considered point. By (1.1), we get $e_i(\theta) = \langle H, Je_i \rangle$.

It follows that

$$\Delta\theta = \sum_{i=1}^{n} e_i \langle H, Je_i \rangle = \sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} H, Je_i \rangle + \sum_{i=1}^{n} \langle H, \overline{\nabla}_{e_i}(Je_i) \rangle. \tag{3.1}$$

Since Σ is Lagrangian, we derive

$$\sum_{i=1}^{n} \langle H, \overline{\nabla}_{e_i}(Je_i) \rangle = \langle H, JH \rangle = 0. \tag{3.2}$$

The translating soliton equation (1.2) and (1.1) imply that

$$\begin{aligned}\sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} H, Je_i \rangle &= \sum_{i=1}^{n} \langle \overline{\nabla}_{e_i}(V_0 - V), Je_i \rangle = -\sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} V, Je_i \rangle \\ &= -\sum_{i=1}^{n} e_i \langle V, Je_i \rangle + \sum_{i=1}^{n} \langle V, \overline{\nabla}_{e_i}(Je_i) \rangle \\ &= \sum_{i=1}^{n} \langle V, J\overline{\nabla}_{e_i} e_i \rangle = \langle V, JH \rangle = -\langle V, \nabla\theta \rangle. \end{aligned} \tag{3.3}$$

Substituting (3.2)–(3.3) into (3.1), we obtain

$$\Delta\theta = -\langle V, \nabla\theta \rangle. \tag{3.4}$$

By (3.4) and (1.1), we have

$$\Delta \cos\theta = -\cos\theta |\nabla\theta|^2 - \sin\theta \Delta\theta = -\cos\theta |\nabla\theta|^2 + \sin\theta \langle V, \nabla\theta \rangle = -\cos\theta |H|^2 - \langle V, \nabla \cos\theta \rangle.$$

Namely

$$\Delta_V \cos\theta = -\cos\theta |H|^2. \tag{3.5}$$

From the translating soliton equation (1.2), we derive

$$\nabla_{e_j} H = \left(\overline{\nabla}_{e_j}\left(V_0 - \sum_{k=1}^n \langle V_0, e_k\rangle e_k\right)\right)^N = -\sum_{k=1}^n \langle V_0, e_k\rangle B_{jk}$$

and

$$\nabla_{e_i}\nabla_{e_j} H = -\sum_{k=1}^n \langle V_0, e_k\rangle \nabla_{e_i} B_{jk} - \sum_{k=1}^n \langle H, B_{ik}\rangle B_{jk},$$

where $B_{jk} = B(e_j, e_k)$. Hence using the Codazzi equation, we obtain

$$\begin{aligned}
\Delta_V |H|^2 =& \Delta |H|^2 + \langle V, \nabla |H|^2\rangle = 2\sum_{i=1}^n \langle \nabla_{e_i}\nabla_{e_i} H, H\rangle + 2|\nabla H|^2 + \langle V, \nabla |H|^2\rangle \\
=& -2\sum_{i,k=1}^n \langle H, B_{ik}\rangle^2 - 2\langle \nabla_{V_0^T} H, H\rangle + 2|\nabla H|^2 + \langle V, \nabla |H|^2\rangle \\
=& -2\sum_{i,k=1}^n \langle H, B_{ik}\rangle^2 - \nabla_V |H|^2 + 2|\nabla H|^2 + \langle V, \nabla |H|^2\rangle \\
=& -2\sum_{i,k=1}^n \langle H, B_{ik}\rangle^2 + 2|\nabla H|^2.
\end{aligned}$$

It follows that

$$\Delta_V |H|^2 \geq 2|\nabla H|^2 - 2|B|^2|H|^2. \tag{3.6}$$

By the Gauss equation, we get $R(e_i, e_j, e_i, e_j) = \langle B(e_i, e_i), B(e_j, e_j)\rangle - \langle B(e_i, e_j), B(e_j, e_i)\rangle$, where R is the curvature operator of Σ. Thus the scalar curvature S of Σ satisfies

$$S = \sum_{i,j} R(e_i, e_j, e_i, e_j) = |H|^2 - |B|^2. \tag{3.7}$$

For any $X = (x_1, x_2, \cdots, x_{2n}) \in \mathbb{R}^{2n}$, let $r = |X|$, we have

$$\begin{aligned}
\nabla r^2 =& 2X^T, \quad |\nabla r| \leq 1 \\
\Delta r^2 =& 2n + 2\langle H, X\rangle \leq 2n + 2r.
\end{aligned} \tag{3.8}$$

Let $\psi := 1 - \cos\theta$. By the assumption, $\cos\theta \geq \delta$ for some positive constant δ. So we can choose a constant b, such that $1 - \delta < b < 1$. Let $B_a(o)$ be the closed ball centered at the origin o with radius a in $\mathbb{R}^{2n}$ and $D_a(o) = \Sigma^n \cap B_a(o)$. Define $f : D_a(o) \to \mathbb{R}$ by $f = \frac{(a^2-r^2)^2|H|^2}{(b-\psi)^2}$.

Since $f|_{\partial D_a(o)} = 0$, f achieves an absolute maximum in the interior of $D_a(o)$, say $f \leq f(q)$, for some q inside $D_a(o)$. We may also assume $|H|(q) \neq 0$. Then from $\nabla f(q) = 0, \Delta_V f(q) \leq 0$, we obtain the following at the point q

$$-\frac{2\nabla r^2}{a^2 - r^2} + \frac{\nabla |H|^2}{|H|^2} + \frac{2\nabla\psi}{b - \psi} = 0, \tag{3.9}$$

$$-\frac{2\Delta_V r^2}{a^2 - r^2} - \frac{2\left|\nabla r^2\right|^2}{(a^2 - r^2)^2} + \frac{\Delta_V |H|^2}{|H|^2} - \frac{\left|\nabla |H|^2\right|^2}{|H|^4} + \frac{2\Delta_V\psi}{b - \psi} + \frac{2\left|\nabla\psi\right|^2}{(b - \psi)^2} \leq 0. \tag{3.10}$$

Direct computation gives us

$$|\nabla|H|^2|^2 = |2\langle \nabla H, H\rangle|^2 \leq 4|\nabla H|^2|H|^2, \tag{3.11}$$

$$|\nabla\psi|^2 = |\nabla \cos\theta|^2 \leq |\nabla\theta|^2 = |H|^2. \tag{3.12}$$

It follows from (3.6) and (3.11) that

$$\frac{\Delta_V |H|^2}{|H|^2} \geq \frac{|\nabla|H|^2|^2}{2|H|^4} - 2|B|^2. \tag{3.13}$$

From (3.9), we obtain

$$\frac{|\nabla|H|^2|^2}{|H|^4} \leq \frac{4|\nabla r^2|^2}{(a^2-r^2)^2} + \frac{8|\nabla r^2||\nabla\psi|}{(a^2-r^2)(b-\psi)} + \frac{4|\nabla\psi|^2}{(b-\psi)^2}. \tag{3.14}$$

By (3.5), we get

$$\Delta_V \psi = \cos\theta |H|^2. \tag{3.15}$$

Substituting (3.7)–(3.8) and (3.12)–(3.15) into (3.10), we have

$$\Big(\frac{\cos\theta}{b-\psi} - 1\Big)|H|^2 - \frac{4r}{(a^2-r^2)(b-\psi)}|H| - \frac{2n+4r}{a^2-r^2} - \frac{8r^2}{(a^2-r^2)^2} + S \leq 0.$$

By the assumption that the scalar curvature $S \geq 0$, we have

$$\Big(\frac{\cos\theta}{b-\psi} - 1\Big)|H|^2 - \frac{4r}{(a^2-r^2)(b-\psi)}|H| - \frac{2n+4r}{a^2-r^2} - \frac{8r^2}{(a^2-r^2)^2} \leq 0.$$

It is easy to see that there is a constant $C > 0$ such that $\frac{\cos\theta}{b-\psi} - 1 > C$. Therefore, at the point q,

$$|H|^2 \leq \max\Big\{\frac{64r^2}{C^2(a^2-r^2)^2(b-\psi)^2}, \frac{4(2n+4r)}{C(a^2-r^2)} + \frac{32r^2}{C(a^2-r^2)^2}\Big\} \tag{3.16}$$

and

$$f(q) \leq \max\Big\{\frac{64a^2}{C^2(b-(1-\delta))^4}, \frac{4(2n+4a)a^2}{C(b-(1-\delta))^2} + \frac{32a^2}{C(b-(1-\delta))^2}\Big\}.$$

Then for any point $x \in D_{\frac{a}{2}}(o)$, we have

$$\begin{aligned} |H|^2(x) \leq& \frac{(b-\psi)^2}{(a^2-r^2)^2} f(q) \\ \leq& \frac{16b^2}{9a^4} \max\Big\{\frac{64a^2}{C^2(b-(1-\delta))^4}, \frac{4(2n+4a)a^2}{C(b-(1-\delta))^2} + \frac{32a^2}{C(b-(1-\delta))^2}\Big\}. \end{aligned} \tag{3.17}$$

Hence we may fix x and let $a \to \infty$ in (3.17), we then derive that $H \equiv 0$. Then by (3.7) and the assumption that the scalar curvature $S \geq 0$, we have $B \equiv 0$. Namely, Σ is an affine n-plane.

Acknowledgement The authors thank Dr. J. Sun for helpful discussions.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Castro, I. and Lerma, A. M., Translating solitons for Lagrangian mean curvature flow in complex Euclidean plane, *Internat. J. Math.*, **23**(10), 2012, 16 pp.

[2] Castro, I. and Lerma, A. M., Lagrangian translators under mean curvature flow, 2015, https://idus.us.es/server/api/core/bitstreams/03eabbdf-a90c-4d3c-bd0e-477bc8eff1de/content.

[3] Chen, J. and Li, J., Singularity of mean curvature flow of Lagrangian submanifolds, *Invent. Math.,* **156**(1), 2004, 25–51.

[4] Chen, Q. and Qiu, H., Rigidity of self-shrinkers and translating solitons of mean curvature flows, *Adv. Math.,* **294**, 2016, 517–531.

[5] Cooper, A. A., Singularities of the Lagrangian mean curvature flow, 2015, arXiv:1505.01856.

[6] Han, X. and Li, J., Translating solitons to symplectic and Lagrangian mean curvature flows, *Internat. J. Math.,* **20**(4), 2009, 443–458.

[7] Han, X. and Sun, J., Translating solitons to symplectic mean curvature flows, *Ann. Global Anal. Geom.,* **38**(2), 2010, 161–169.

[8] Huisken, G., Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.*, **20**(1), 1984, 237–266.

[9] Huisken, G., Asymptotic behavior for singularities of the mean curvature flow, *J. Differential Geom.*, **31**(1), 1990, 285–299.

[10] Joyce, D., Lee, Y.-I. and Tsui, M.-P., Self-similar solutions and translating solitons for Lagrangian mean curvature flow, *J. Differential Geom.,* **84**(1), 2010, 127–161.

[11] Li, X., Liu, Y. and Qiao, R., A uniqueness theorem of complete Lagrangian translator in $\mathbb{C}^2$, *Manuscripta Math.*, **164**(1–2), 2021, 251–265.

[12] Neves, A., Singularities of Lagrangian mean curvature flow: Zero-Maslov class case, *Invent. Math.*, **168**(3), 2007, 449–484.

[13] Neves, A. and Tian, G., Translating solutions to Lagrangian mean curvature flow, *Trans. Amer. Math. Soc.*, **365**(11), 2013, 5655–5680.

[14] Qiu, H. B., Rigidity of symplectic translating solitons, *J. Geom. Anal.*, **32**(11), 2022, 19 pp.

[15] Qiu, H. B., A Bernstein type result of translating solitons, *Calc. Var. Partial Differential Equations*, **61**(6), 2022, 9 pp.

[16] Schoen, R. and Wolfson, J., Minimizing area among Lagrangian surfaces: The mapping problem, *J. Differential Geom.*, **58**, 2001, 1–86.

[17] Smoczyk, K., A canonical way to deform a Lagrangian submanifold, 1996, arXiv:dg-ga/9605005.

[18] Smoczyk, K., The Lagrangian mean curvature flow, (Der Lagrangesche mittlere Krmmungsfluss), Leipzig, Univ. Leipzig (Habil.-Schr.), 2000, https://www.researchgate.net/publication/268305622.

[19] Smoczyk, K., Local non-collapsing of volume for the Lagrangian mean curvature flow, *Calc. Var. Partial Differential Equations*, **58**(1), 2019, 14 pp.

[20] Sun, J., A gap theorem for translating solitons to Lagrangian mean curvature flow, *Differential Geom. Appl.,* **31**(5), 2013, 568–576.

[21] Sun, J., Mean curvature decay in symplectic and Lagrangian translating solitons, *Geom. Dedicata*, **172**, 2014, 207–215.

[22] Sun, J., Rigidity results on Lagrangian and symplectic translating solitons, *Commun. Math. Stat.*, **3**(1), 2015, 63–68.

[23] Wang, M.-T., Mean curvature flow of surfaces in Einstein four-manifolds, *J. Differential Geom.*, **57**(2), 2001, 301–338.

[24] Xin, Y. L., Translating solitons of the mean curvature flow, *Calc. Var. Partial Differential Equations*, **54**, 2015, 1995–2016.