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Divisible Properties for Asymptotically Tracially Approximation of C*-algebras

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Abstract The authors show that m-almost divisibility and weak (m,n)-divisibility of C*-algebras in a class $\mathcal{P}$ are preserved to the simple unital C*-algebras which are asymptotically tracially in $\mathcal{P}$.

Keywords C*-algebras, Asymptotically tracially approximation, Cuntz semigroup
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1 Introduction

The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF algebras in [1]. Since then, many classes of C*-algebras have been classified by the Elliott invariant. A major next step was the classification of simple AH algebras without dimension growth (in the real rank zero case see [2], and in the general case see [3]). A crucial intermediate step was Lin's axiomatization of Elliott-Gong's decomposition theorem for simple AH algebras of real rank zero (classified by Elliott-Gong in [2]) and Gong's decomposition theorem (see [4]) for simple AH algebras (classified by Elliott-Gong-Li in [3]). Heavily inspired by Gong's work in [4], Lin introduced the concepts of TAF and TAI (see [5–6]). Instead of assuming inductive limit structure, Lin started with a certain abstract (tracial) approximation property. This led eventually to the classification of simple separable amenable stably finite C*-algebras with finite nuclear dimension in the UCT class (see [7–10]).

In the classification of simple separable nuclear C*-algebras, it is necessary to invoke some regularity property of the C*-algebras. There are three regularity properties of particular interest: $\mathcal{Z}$-stability, finite nuclear dimension and certain comparison property of positive elements. Winter and Toms have conjectured that these three properties are equivalent for all separable, simple, nuclear C*-algebras.

In order to be easier to verify a C*-algebra being $\mathcal{Z}$-stable, as well as Hirshberg and Oroviz introduced tracial $\mathcal{Z}$-stability in [11], they showed that a unital simple separable nuclear C*-algebra A is $\mathcal{Z}$-stable if and only if A is tracially $\mathcal{Z}$-stable in [11].

Inspired by the work of Elliott, Gong, Lin and Niu in [12–14], and the work of Hirshberg and Oroviz's tracial $\mathcal{Z}$-stability, in order to search a tracial version of Toms-Winter conjecture, Fu and Lin introduced asymptotically tracially approximation of C*-algebras and also the concept of tracial nuclear dimensional in [15]. They showed that a unital separable simple C*-algebra

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A has tracial nuclear dimensional no more than k if and only if A is asymptotically tracially in $\mathcal{F}_k$, where $\mathcal{F}_k$ are C*-algebras with nuclear dimension at most k.

In [15], Fu and Lin showed that the class of stably finite C*-algebras, quasidiagonal C*-algebras, purely infinite simple C*-algebras, and the properties almost unperforated; almost unperforated of Cuntz semigroup are preserved to the simple unital C*-algebras which are asymptotically tracially in the same class.

In [16], Fan and Fang showed that the class of certain comparison properties C*-algebras are preserved to the simple unital C*-algebras which are asymptotically tracially in the same class.

In this paper, we show the following two results.

• Let $\mathcal{P}$ be a class of unital m-almost divisible C*-algebras (introduced by Robert and Tikuisis in [17]). If a unital separable stably finite simple C*-algebra A is asymptotically tracially in $\mathcal{P}$, then A is m-almost divisible.

• Let $\mathcal{P}$ be a class of unital weakly (m, n)-divisible C*-algebras (introduced by Robert and Rørdam in [18]). If a unital separable stably finite simple C*-algebra A is asymptotically tracially in $\mathcal{P}$, then A is weakly (m, n)-divisible.

2 Definitions and Preliminaries

Let A be a C*-algebra. Given two positive elements $a, b \in A$, we call that a is Cuntz subequivalent to b and write $a \precsim b$, if there exist $(s_n)_{n=1}^{\infty}$ in A, such that

$$\lim_{n\to\infty} \|s_n b s_n^* - a\| = 0.$$

We call that a and b are Cuntz equivalent (written as $a \sim b$), if $a \precsim b$ and $b \precsim a$. We write $\langle a\rangle$ for the equivalence class of a. Cuntz equivalent for positive elements of C*-algebra was first introduced by Cuntz in [19].

Given a C*-algebra A, we denote $\mathrm{M}_\infty(A)_+ = \bigcup\limits_{n\in\mathbb{N}} \mathrm{M}_n(A)_+$, and for $a \in \mathrm{M}_n(A)_+$ and $b \in \mathrm{M}_m(A)_+$, denote $a \oplus b := \mathrm{diag}(a, b) \in \mathrm{M}_{n+m}(A)_+$.

Given $a, b \in \mathrm{M}_\infty(A)_+$, then there exist integers n, m, such that $a \in \mathrm{M}_n(A)_+$ and $b \in \mathrm{M}_m(A)_+$. We call a is Cuntz subequivalent to b and write $a \precsim b$ if $a \oplus 0_{\max((m-n),0)} \precsim b \oplus 0_{\max((n-m),0)}$ as elements in $\mathrm{M}_{\max(n,m)}(A)_+$.

The object $\mathrm{Cu}(A) := \mathrm{M}_\infty(A\otimes\mathcal{K})_+/\sim$ will be called the Cuntz semigroup of A (see [20–22]). $\mathrm{Cu}(A)$ becomes an ordered semigroup when equipped with the addition operation

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$$

and the order relation

$$\langle a\rangle \le \langle b\rangle \Leftrightarrow a \precsim b.$$

Given a C*-algebra A, a positive element a in A is called purely positive, if a is not Cuntz equivalent to a projection. Let A be a unital stably finite C*-algebra. For any $a \in A_+$, then either a is a purely positive element or a is equivalent to a projection. Given a positive element a in A and $\varepsilon > 0$, we denote by $(a-\varepsilon)_+$ the element in A via the functional calculus to the function $f(t) = \max(0, t-\varepsilon)$, $t \in \sigma(a)$. It is easy to see that $((a-\varepsilon_1)_+ - \varepsilon_2)_+ = (a-(\varepsilon_1+\varepsilon_2))_+$ for any $\varepsilon_1, \varepsilon_2 > 0$.

The property of m-almost divisible was introduced by Robert and Tikuisis in [17].

Definition 2.1 (see [17]) *Given integer* $m \in \mathbb{N}$*, we say that a* C^**-algebra* A *is* m*-almost divisible, if for each positive element* $a \in M_\infty(A \otimes \mathcal{K})$*, any* $k \in \mathbb{N}$ *and any* $\varepsilon > 0$*, there exists a positive element* $b \in M_\infty(A \otimes \mathcal{K})$*, such that* $k\langle b\rangle \leq \langle a\rangle$ *and* $\langle (a-\varepsilon)_+\rangle \leq (k+1)(m+1)\langle b\rangle$*.*

The property of weakly (m,n)-divisible was introduced by Robert and Rørdam in [18].

Definition 2.2 (see [18]) *Given two integers* $m, n \geq 1$*, we say that a* C^**-algebra* A *is weakly* (m,n)*-divisible, if for every* u *in* $\mathrm{Cu}(A)$*, any* $\varepsilon > 0$*, there exist elements* $x_1, x_2, \cdots, x_n \in \mathrm{Cu}(A)$*, such that* $mx_j \leq u$ *for all* $j = 1, 2, \cdots, n$ *and* $(u-\varepsilon)_+ \leq x_1 + x_2 + \cdots + x_n$*.*

Definition 2.3 *Given two* C^**-algebras* A *and* B*, let* $\varphi : A \to B$ *be a map, let* $\mathcal{G} \subset A$*, and* $\varepsilon > 0$*. The map* φ *is called* $\mathcal{G}$*-*ε*-multiplicative, or called* ε*-multiplicative on* $\mathcal{G}$*, if for any* $a, b \in \mathcal{G}$*,* $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon$*. If, in addition, for any* $a \in \mathcal{G}$*,* $|\|\varphi(a)\| - \|a\|| < \varepsilon$*, then we say* φ *is a* $\mathcal{G}$*-*ε*-approximate embedding.*

Let A and B be two C^*-algebras. We say that a map $\phi : A \to B$ is a c.p.c map when ϕ is a completely positive contraction linear map. We say that a linear map $\psi : A \to B$ is an order zero map which means preserving orthogonality, i.e., $\psi(e)\psi(f) = 0$ for all $e, f \in M_n$ with $ef = 0$.

Fu and Lin introduced the asymptotically tracially approximation of C^*-algebras in [15].

Definition 2.4 (see [15, Definition 3.1]) *Let* $\mathcal{P}$ *be a class of* C^**-algebra. We say that a unital* C^**-algebra* A *is asymptotically tracially in* $\mathcal{P}$*, if for any finite subset* $\mathcal{F} \subseteq A$*, any* $\varepsilon > 0$ *and any non-zero positive element* a*, there is a* C^**-algebra* $B \in \mathcal{P}$ *and* c.p.c *maps* $\alpha : A \to B$*,* $\beta_n : B \to A$ *and* $\gamma_n : A \to A$ *such that*

(1) $\|x - \gamma_n(x) - \beta_n(\alpha(x))\| < \varepsilon$*, for all* $x \in \mathcal{F}$*, and for all* $n \in \mathbb{N}$*,*

(2) α *is an* $\mathcal{F}$*-*ε *approximate embedding,*

(3) $\lim\limits_{n\to\infty} \|\beta_n(xy) - \beta_n(x)\beta_n(y)\| = 0$ *and* $\lim\limits_{n\to\infty} \|\beta_n(x)\| = \|x\|$ *for all* $x, y \in B$*,*

(4) $\gamma_n(1_A) \precsim a$ *for all* $n \in \mathbb{N}$*.*

The following theorem is in [15, Proposition 3.8].

Theorem 2.1 (see [15]) *Let* $\mathcal{P}$ *be a class of* C^**-algebras. Let* A *be a simple unital* C^**-algebra which is asymptotically tracially in* $\mathcal{P}$*. Then the following conditions hold: For any finite subset* $\mathcal{F} \subseteq A$*, any* $\varepsilon > 0$ *and any non-zero positive element* a*, there is a* C^**-algebra* B *in* $\mathcal{P}$ *and* c.p.c *maps* $\alpha : A \to B$*,* $\beta_n : B \to A$ *and* $\gamma_n : A \to A \cap \beta_n(B)^\perp$ *such that*

(1) *the map* α *is a unital completely positive linear map,* $\beta_n(1_B)$ *and* $\gamma_n(1_A)$ *are projections and* $\beta_n(1_B) + \gamma_n(1_A) = 1_A$*, for all* $n \in \mathbb{N}$*,*

(2) $\|x - \gamma_n(x) - \beta_n(\alpha(x))\| < \varepsilon$*, for all* $x \in \mathcal{F}$*, and for all* $n \in \mathbb{N}$*,*

(3) α *is an* $\mathcal{F}$*-*ε*-approximate embedding,*

(4) $\lim\limits_{n\to\infty} \|\beta_n(xy) - \beta_n(x)\beta_n(y)\| = 0$ *and* $\lim\limits_{n\to\infty} \|\beta_n(x)\| = \|x\|$ *for all* $x, y \in B$*,*

(5) $\gamma_n(1_A) \precsim a$*, for all* $n \in \mathbb{N}$*.*

Lemma 2.1 (see [15]) *If the class* $\mathcal{P}$ *is closed under tensoring with matrix algebras and under passing to unital hereditary* C^**-subalgebras, then the class which is asymptotically tracially in* $\mathcal{P}$ *is closed under tensoring with matrix algebras and under passing to unital hereditary* C^**-subalgebras.*

The following lemma is obvious, and we omit the proof.

Lemma 2.2 *The* m*-almost divisible (or weakly* (m,n)*-divisible) is preserved under tensoring with matrix algebras and under passing to unital hereditary* C^**-subalgebras.*

3 The Main Results

Theorem 3.1 *Let $\mathcal{P}$ be a class of unital m-almost divisible* C*-*algebras. If a unital separable stably finite simple* C*-*algebra A is asymptotically tracially in $\mathcal{P}$, then A is m-almost divisible.*

Proof We must show that there exists $b \in \mathrm{M}_\infty(A)_+$ such that $kb \precsim a$ and $(a-\varepsilon)_+ \precsim (k+1)(m+1)b$, for any given $a \in \mathrm{M}_\infty(A)_+$ (in fact we must assume that $a \in \mathrm{M}_\infty(A\otimes\mathcal{K})_+$, since there exist $a' \in \mathrm{M}_\infty(A)_+$ such that $\|a-a'\|$ sufficiently small, we can replace a with a') any given $\varepsilon > 0$ and any given $k \in \mathbb{N}$. We may assume that $\|a\| = 1$.

By Lemmas 2.1–2.2, we may assume that $a \in A_+$.

With $F = \{a\}$, any $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$, since A is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra B in $\mathcal{P}$, and c.p.c maps $\alpha : A \to B$, $\beta_n : B \to A$ and $\gamma_n : A \to A \cap \beta_n(B)^\perp$ such that

(1) the map α is a unital completely positive linear map, $\beta_n(1_B)$ and $\gamma_n(1_A)$ are all projections, and $\beta_n(1_B)+\gamma_n(1_A) = 1_A$, for any $n \in \mathbb{N}$,

(2) $\|x-\gamma_n(x)-\beta_n(\alpha(x))\| < \varepsilon'$, for any $x \in F$, and for any $n \in \mathbb{N}$,

(3) α is an F-ε'-approximate embedding,

(4) $\lim\limits_{n\to\infty}\|\beta_n(xy)-\beta_n(x)\beta_n(y)\| = 0$ and $\lim\limits_{n\to\infty}\|\beta_n(x)\| = \|x\|$ for all $x, y \in B$.

Since B is m-almost divisible, and given $\alpha(a) \in B$, given $\varepsilon' > 0$, given $k \in \mathbb{N}$, we may assume that there exists $b_1 \in B$ such that

$$kb_1 \precsim \alpha(a)$$

and

$$(\alpha(a)-\varepsilon')_+ \precsim (k+1)(m+1)b_1.$$

Since A is a stably finite C*-algebra, we divide the proof into two cases.

Case 1 we assume that $(\alpha(a)-\varepsilon')_+$ is Cuntz equivalent to a projection.

Case 1.1 If $(\alpha(a)-\varepsilon')_+$ is not Cuntz equivalent to $(k+1)(m+1)b_1$.

By [23, Theorem 2.1 (2)], we may assume that there exist non-zero positive element $c \in B$ such that $(\alpha(a)-\varepsilon')_+ + c \precsim (k+1)(m+1)b_1$.

Since $kb_1 \precsim \alpha(a)$, for any $\overline{\varepsilon} > 0$, there exists $v \in M_k(B)$, such that

$$\|v^*\mathrm{diag}(\alpha(a), 0\otimes 1_{k-1})v - b_1 \otimes 1_k\| < \overline{\varepsilon}.$$

We assume that $\|v\| \le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer N_1 such that for any $n > N_1$, we have

$$\|\beta_n \otimes \mathrm{id}_{M_k}(v^*)\mathrm{diag}(\beta_n\alpha(a), 0\otimes 1_{k-1})\beta_n \otimes \mathrm{id}_{M_k}(v) - \beta_n(b_1)\otimes 1_k\| < \varepsilon'.$$

Therefore we have

$$k(\beta_n(b_1)-4\varepsilon')_+ \precsim (\beta_n\alpha(a)-2\varepsilon')_+.$$

Since $(\alpha(a)-\varepsilon')_+ + c \precsim (k+1)(m+1)b_1$, there exists $w \in M_{(k+1)(m+1)}(B)$ such that

$$\|w^*(b_1 \otimes 1_{(k+1))m+1)}w - \mathrm{diag}((\alpha(a)-\varepsilon')_+ + c, 0\otimes 1_{(k+1)(m+1)-1)})\| < \overline{\varepsilon}.$$

We assume that $\|w\| \le N(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer N_2 such that for any $N_2 < n$, we have

$$\|\beta_n \otimes \mathrm{id}_{M_{(k+1)(m+1)}}(w^*)\beta_n(b_1)\otimes 1_{(k+1)(m+1)}\beta_n \otimes \mathrm{id}_{M_{(k+1)(m+1)}}(w)$$

$$-\operatorname{diag}(\beta_n\alpha((a)-\varepsilon')_+ + c, 0\otimes 1_{(k+1)(m+1)-1})\| < \varepsilon'.$$

Therefore we have

$$(\beta_n\alpha(a)-8\varepsilon')_+ + \beta_n(c) \precsim (k+1)(m+1)(\beta_n(b_1)-4\varepsilon')_+.$$

For sufficiently large $n > \max\{N_1, N_2\}$, with $G = \{\gamma_n(a)\}$, any $\varepsilon'' > 0$, with $\varepsilon'' < \varepsilon'$, let $E = \gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha' : E \to D$, $\beta'_n : D \to E$ and $\gamma'_n : E \to E \cap \beta'_n(D)^\perp$ such that

(1)′ the map α' is a unital completely positive linear map, $\beta'_n(1_D)$ and $\gamma'_n(1_E)$ are all projections, $\beta'_n(1_D) + \gamma'_n(1_E) = 1_E$, for any $n \in \mathbb{N}$,

(2)′ $\|x - \gamma'_n(x) - \beta'_n(\alpha'(x))\| < \varepsilon''$ for any $x \in G$ and for any $n \in \mathbb{N}$,

(3)′ α' is a G-ε''-approximate embedding,

(4)′ $\lim\limits_{n\to\infty}\|\beta'_n(xy) - \beta'_n(x)\beta_n(y)\| = 0$ and $\lim\limits_{n\to\infty}\|\beta'_n(x)\| = \|x\|$ for all $x, y \in D$,

(5)′ $\gamma'_n\gamma_n(1_A) \precsim \gamma_n(1_A)\beta_n(c)\gamma_n(1_A) \precsim_A \beta_n(c)$ for all $n \in \mathbb{N}$.

Since D is m-almost divisible, and $(\alpha'\gamma_n(a)-3\varepsilon')_+ \in D$, there exists $b_2 \in D_+$ such that

$$kb_2 \precsim (\alpha'\gamma_n(a)-3\varepsilon')_+$$

and

$$(\alpha'\gamma_n(a)-4\varepsilon')_+ \precsim (k+1)(m+1)b_2.$$

With the same argument, as above, we can get

$$k(\beta'_n(b_2)-4\varepsilon')_+ \precsim (\beta'_n\alpha'\gamma_n(a)-2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a)-8\varepsilon')_+ \precsim (k+1)(m+1)(\beta'_n(b_2)-2\varepsilon')_+.$$

Therefore, we have

$$\begin{aligned}
&k((\beta_n(b_1)-4\varepsilon')_+ \oplus (\beta'_n(b_2)-4\varepsilon')_+) \sim k((\beta_n(b_1)-4\varepsilon')_+ + (\beta'_n(b_2)-4\varepsilon')_+)\\
&\precsim (\beta_n\alpha(a)-2\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a)-2\varepsilon')_+\\
&\precsim (\beta_n\alpha(a)-\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a)-\varepsilon')_+ + (\gamma'_n\gamma_n(a)-3\varepsilon')_+\\
&\precsim a,
\end{aligned}$$

and we also have

$$\begin{aligned}
&(a-\varepsilon)_+\\
&\precsim (\beta_n\alpha(a)-8\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a)-8\varepsilon')_+ \oplus (\gamma'_n\gamma_n(a)-4\varepsilon')_+\\
&\precsim (\beta_n\alpha(a)-8\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a)-8\varepsilon')_+ \oplus \gamma'_n\gamma_n(1_E)\\
&\precsim (\beta_n\alpha(a)-8\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a)-8\varepsilon)_+ \oplus \beta_n(c)\\
&\precsim (k+1)(m+1)(\beta_n(b_1)-4\varepsilon')_+ \oplus (k+1)(m+1)(\beta'_n(b_1)-4\varepsilon')_+.
\end{aligned}$$

Case 1.2 If $(\alpha(a)-\varepsilon')_+$ is Cuntz equivalent to $(k+1)(m+1)b_1$.

Since $k(b_1 \oplus b_1) \precsim \alpha(a)$, for any $\overline{\varepsilon} > 0$, there exists $v \in M_{2k}(B)$, such that

$$\|v^*\operatorname{diag}(\alpha(a), 0\otimes 1_{2k-1})v - b_1 \otimes 1_{2k}\| < \overline{\varepsilon}.$$

We assume that $\|v\| \le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer N_1 such that for any $n > N_1$, we have

$$\|\beta_n \otimes \mathrm{id}_{M_{2k}}(v^*)\mathrm{diag}(\beta_n\alpha(a), 0 \otimes 1_{k-1}\beta_n \otimes \mathrm{id}_{M_{2k}}(v) - \beta_n(b_1) \otimes 1_{2k}\| < \varepsilon'.$$

Therefore, we have

$$k(\beta_n(b_1) \oplus \beta_n(b_1) - 4\varepsilon')_+ \precsim (k+1)(m+1)(\beta_n\alpha(a) - 2\varepsilon')_+.$$

With the same argument, we have

$$(\beta_n\alpha(a) - 8\varepsilon')_+ \precsim (k+1)(m+1)(\beta_n\alpha(a) - 4\varepsilon')_+$$

and

$$(\beta_n\alpha(a) - 6\varepsilon')_+ + \beta_n(c) \precsim (\beta_n(b_1) \oplus \beta_n(b_1) - 2\varepsilon')_+.$$

For sufficiently large $n > \max\{N_1, N_2\}$, with $G = \{\gamma_n(a)\}$, any $\varepsilon'' > 0$ with $\varepsilon'' < \varepsilon'$ sufficiently small, let $E = \gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha' : E \to D$, $\beta'_n : D \to E$ and $\gamma'_n : E \to E \cap \beta'_n(D)^\perp$ such that

(1)$'$ the map α' is a unital completely positive linear map, $\beta'_n(1_D)$ and $\gamma'_n(1_E)$ are projections, $\beta'_n(1_D) + \gamma'_n(1_E) = 1_E$, for all $n \in \mathbb{N}$,

(2)$'$ $\|x - \gamma'_n(x) - \beta'_n(\alpha'(x))\| < \varepsilon''$, for all $x \in G$, and for all $n \in \mathbb{N}$,

(3)$'$ α' is a G-ε''-approximate embedding,

(4)$'$ $\lim\limits_{n\to\infty} \|\beta'_n(xy) - \beta'_n(x)\beta'_n(y)\| = 0$ and $\lim\limits_{n\to\infty} \|\beta'_n(x)\| = \|x\|$ for all $x, y \in D$,

(5)$'$ $\gamma'_n\gamma_n(1_A) \precsim \gamma_n(1_A)\beta_n(b_1)\gamma_n(1_A) \precsim_A \beta_n(b_1)$ for any $n \in \mathbb{N}$.

Since D is m-almost divisible and $(\beta'_n\alpha'\gamma_n(a) - \varepsilon')_+ \in B$, there exists $b_2 \in D_+$ such that $kb_2 \precsim (\beta'_n\alpha'\gamma_n(a) - \varepsilon')_+$ and $(\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+ \precsim (k+1)(m+1)b_2$.

With the same argument as above we have

$$k(\beta'_n(b_2) - 4\varepsilon')_+ \precsim (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a) - 8\varepsilon')_+ \precsim (k+1)(m+1)(\beta'_n(b_1) - 4\varepsilon')_+.$$

Therefore we have

$$\begin{aligned}
&k((\beta_n(b_1 \oplus b_1) - 4\varepsilon')_+ \oplus (\beta'_n(b_2) - 4\varepsilon')_+) \\
\sim\ &k((\beta_n(b_1 \oplus b_1) - 4\varepsilon')_+ + (\beta'_n(b_2) - 4\varepsilon')_+) \\
\precsim\ &(\beta_n\alpha(a) - 2\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+ \\
\precsim\ &(\beta_n\alpha(a) - \varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - \varepsilon')_+ + (\gamma'_n\gamma_n(a) - 3\varepsilon')_+ \\
\precsim\ &a,
\end{aligned}$$

and we also have

$$\begin{aligned}
&(a - \varepsilon)_+ \\
\precsim\ &(\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 8\varepsilon')_+ \oplus (\gamma'_n\gamma_n(a) - 4\varepsilon')_+ \\
\precsim\ &(\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 8\varepsilon')_+ \oplus \gamma'_n\gamma_n(1_E) \\
\precsim\ &(\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 8\varepsilon')_+ \oplus \beta_n(b_1)
\end{aligned}$$

$$\precsim (k+1)(m+1)(\beta_n(b_1)-4\varepsilon')_+ \oplus (k+1)(m+1)(\beta_n'(b_1)-4\varepsilon')_+$$
$$\precsim (k+1)(m+1)(\beta_n(b_1\oplus b_1)-4\varepsilon')_+ \oplus (k+1)(m+1)(\beta_n'(b_1)-4\varepsilon')_+.$$

Case 2 we assume that $(\alpha(a)-\varepsilon')_+$ is not Cuntz equivalent to a projection.

By [23, Theorem 2.1(4)], there is a non-zero positive element c such that $(\alpha(a)-2\varepsilon')_+ + c \precsim (\alpha(a)-\varepsilon')_+$.

Since $(\alpha(a)-\varepsilon')_+ + c \precsim (k+1)(m+1)b_1$, for any $\overline{\varepsilon}>0$, there exists $w\in M_{(k+1)(m+1)}(B)$ such that

$$\|w^*b_1\otimes 1_{(k+1)(m+1)}w - \mathrm{diag}(\alpha(a)-\varepsilon')_+ + c, 0\otimes 1_{(k+1)(m+1)-1}\| < \overline{\varepsilon}.$$

We assume that $\|w\|\le N(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\|\beta_n(w^*)\beta_n(b_1)\otimes 1_{(k+1)(m+1)}\beta_n(w) - \mathrm{diag}(\beta_n\alpha((a-\varepsilon')_+)+\beta_n(c)), 0\otimes 1_{(k+1)(m+1)}\| < \overline{\varepsilon}.$$

Therefore we have

$$(\beta_n\alpha(a)-6\varepsilon')_+ + \beta_n(c) \precsim (\beta_n(b_1)-4\varepsilon')_+.$$

Since $kb_1 \precsim \alpha(a)$, for any $\overline{\varepsilon}>0$, there exists $v\in M_k(B)$ such that

$$\|v^*\mathrm{diag}(\alpha(a), 0\otimes 1_{k-1})v - b_1\otimes 1_k\| < \overline{\varepsilon}.$$

We assume that $\|v\|\le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer N_1 such that for any $n>N_1$, we have

$$\|\beta_n\otimes \mathrm{id}_{M_k}(v^*)\mathrm{diag}(\beta_n\alpha(a), 0\otimes 1_{k-1})\beta_n\otimes \mathrm{id}_{M_k}(v) - \beta_n(b_1)\otimes 1_k\| < \varepsilon'.$$

Therefore we have

$$k(\beta_n(b_1)-4\varepsilon')_+ \precsim (\beta_n\alpha(a)-2\varepsilon')_+$$

and

$$(\beta_n\alpha(a)-8\varepsilon')_+ \precsim (k+1)(m+1)(\beta_n(b_1)-4\varepsilon')_+.$$

For sufficiently large $n>\max\{N_1,N_2\}$, with $G=\{\gamma_n(a)\}$, any $\varepsilon''>0$ with $\varepsilon''<\varepsilon'$, let $E=\gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha': E\to D$, $\beta_n': D\to E$ and $\gamma_n': E\to E\cap\beta_n'(D)^\perp$ such that

(1)′ the map α' is a unital completely positive linear map, $\beta_n'(1_D)$ and $\gamma_n'(1_E)$ are all projections, $\beta_n'(1_D)+\gamma_n'(1_E)=1_E$, for all $n\in\mathbb{N}$,

(2)′ $\|x-\gamma_n'(x)-\beta_n'(\alpha'(x))\|<\varepsilon''$, for any $x\in G$, and for any $n\in\mathbb{N}$,

(3)′ α' is a G-ε''-approximate embedding,

(4)′ $\lim_{n\to\infty}\|\beta_n'(xy)-\beta_n'(x)\beta_n'(y)\|=0$ and $\lim_{n\to\infty}\|\beta_n'(x)\|=\|x\|$, for all $x,y\in D$,

(5)′ $\gamma_n'\gamma_n(1_A)\precsim \gamma_n(1_A)\beta_n(c)\gamma_n(1_A)\precsim_A \beta_n(c)$, for all $n\in\mathbb{N}$.

Since D is m-almost divisible and $(\beta_n'\alpha'\gamma_n(a)-\varepsilon')_+\in B$, there exists $b_2\in D_+$ such that

$$kb_2 \precsim (\beta_n'\alpha'\gamma_n(a)-\varepsilon')_+$$

and

$$(\beta_n'\alpha'\gamma_n(a)-2\varepsilon')_+ \precsim (k+1)(m+1)b_2.$$

With the same argument, as above, we have

$$k(\beta_n'(b_2)-4\varepsilon')_+ \precsim (\beta_n'\alpha'\gamma_n(a)-2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a)-8\varepsilon')_+ \precsim (k+1)(m+1)(\beta'_n(b_1)-4\varepsilon')_+.$$

Therefore we have

$$\begin{aligned}&k((\beta_n(b_1)-4\varepsilon')_+\oplus(\beta'_n(b_2)-4\varepsilon')_+)\sim k((\beta_n(b_1)-4\varepsilon')_+ + (\beta'_n(b_2)-4\varepsilon')_+)\\ &\precsim (\beta_n\alpha(a)-2\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-2\varepsilon')_+\\ &\precsim (\beta_n\alpha(a)-\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-\varepsilon')_+ + (\gamma'_n\gamma_n(a)-3\varepsilon')_+\\ &\precsim a,\end{aligned}$$

and we also have

$$\begin{aligned}&(a-\varepsilon)_+\\ &\precsim (\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-8\varepsilon')_+\oplus(\gamma'_n\gamma_n(a)-4\varepsilon')_+\\ &\precsim (\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-8\varepsilon')_+\oplus\gamma'_n\gamma_n(1_E)\\ &\precsim (\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-8\varepsilon)_+\oplus\beta_n(c)\\ &\precsim (k+1)(m+1)(\beta_n(b_1)-4\varepsilon')_+\oplus(k+1)(m+1)(\beta'_n(b_2)-4\varepsilon')_+.\end{aligned}$$

Theorem 3.2 *Let $\mathcal{P}$ be a class of unital weakly (m,n)-divisible* C*-*algebras. If a unital separable stably finite simple* C*-*algebra A is asymptotically tracially in $\mathcal{P}$, then A is weakly (m,n)-divisible.*

Proof We must show that for any given $a\in \mathrm{M}_\infty(A)_+$ (as Theorem 3.1, we may assume that $a\in \mathrm{M}_\infty(A)_+$), any given $\varepsilon>0$, there are $x_1,x_2,\cdots,x_n\in \mathrm{M}_\infty(A)_+$ such that $x_j\oplus x_j\oplus\cdots\oplus x_j\precsim a$, for any $1\le j\le n$, where x_j repeat m times, and $(a-\varepsilon)_+\precsim \bigoplus_{i=1}^{n} x_i$.

By Lemmas 2.1–2.2, we may assume $a\in A_+$ and $\|a\|\le 1$.

With $F=\{a\}$, any $\varepsilon'>0$ with $\varepsilon'<\varepsilon$, since A is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra B in $\mathcal{P}$, and c.p.c maps $\alpha: A\to B$, $\beta_n: B\to A$ and $\gamma_n: A\to A\cap\beta_n(B)^\perp$ such that

(1) the map α is a unital completely positive linear map, $\beta_n(1_B)$ and $\gamma_n(1_A)$ are all projections, and $\beta_n(1_B)+\gamma_n(1_A)=1_A$, for any $n\in\mathbb{N}$,

(2) $\|x-\gamma_n(x)-\beta_n(\alpha(x))\|<\varepsilon'$, for any $x\in F$, and for any $n\in\mathbb{N}$,

(3) α is an F-ε'-approximate embedding,

(4) $\lim\limits_{n\to\infty}\|\beta_n(xy)-\beta_n(x)\beta_n(y)\|=0$ and $\lim\limits_{n\to\infty}\|\beta_n(x)\|=\|x\|$, for all $x,y\in B$.

Since B is weakly (m,n)-divisible, there exist $x'_1,x'_2,\cdots,x'_n\in \mathrm{M}_\infty(B)_+$ such that

$$x'_j\oplus x'_j\oplus\cdots\oplus x'_j\precsim\alpha(a),$$

where x'_j repeat m times and

$$(\alpha(a)-\varepsilon')_+\precsim\bigoplus_{i=1}^{n}x'_i.$$

Since A is a stably finite C*-algebra, we divide the proof into two cases.

Case 1 we assume that $(\alpha(a)-\varepsilon')_+$ is Cuntz equivalent to a projection.

Case 1.1 If $(\alpha(a)-\varepsilon')_+$ is Cuntz equivalent to $\bigoplus_{i=1}^{n}x'_i$.

Case 1.1.1 If $x'_1,x'_2,\cdots,x'_n\in \mathrm{M}_\infty(B)_+$ are all Cuntz equivalent to projections and $(\alpha(a)-\varepsilon')_+\bigoplus_{i=1}^{n}x'_i$, then there exist some j and a nonzero projection r such that $(x'_j\oplus r)\oplus(x'_j\oplus r)\oplus\cdots\oplus(x'_j\oplus r)\precsim\alpha(a)$, where $x'_j\oplus r$ repeat m times.

Since $(x'_j\oplus r)\oplus(x'_j\oplus r)\oplus\cdots\oplus(x'_j\oplus r)\precsim\alpha(a)$, for any $\overline{\varepsilon}>0$, there exists $v\in M_\infty(B)$ such that

$$\|v^*\mathrm{diag}(\alpha(a),0\otimes 1_{m-1})v-(x'_j\oplus r)\oplus(x'_j\oplus r)\oplus\cdots\oplus(x'_j\oplus r)\|<\overline{\varepsilon}.$$

We assume that $\|v\|\le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\|\beta_n(v^*)\mathrm{diag}(\beta_n\alpha(a),0\otimes 1_{m-1})\beta_n(v)-\beta_n((x'_j\oplus r)\oplus(x'_j\oplus r)\oplus\cdots\oplus(x'_j\oplus r))\|<\varepsilon'.$$

Therefore we have

$$(\beta_n(x'_j\oplus r)-3\varepsilon')+\oplus(\beta_n(x'_j\oplus r)-3\varepsilon')_+\oplus\cdots\oplus(\beta_n(x'_j\oplus r)-3\varepsilon')_+\precsim(\beta_n\alpha(a)-2\varepsilon')_+.$$

Since $(\alpha(a)-2\varepsilon')_+\precsim\bigoplus_{i=1}^n x'_i$, with the same argument as above, we have

$$(\beta_n\alpha(a)-6\varepsilon')_+\precsim(\beta_n(x'_i)-3\varepsilon')_+\oplus(\beta_n(x'_i)-3\varepsilon')_+\oplus\cdots\oplus(\beta_n(x'_i)-3\varepsilon')_+.$$

With $G=\{\gamma_n(a)\}$, any $\varepsilon''>0$ with $\varepsilon''<\varepsilon'$, let $E=\gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$, and c.p.c maps $\alpha':E\to D$, $\beta'_n:D\to E$ and $\gamma'_n:E\to E\cap\beta'_n(D)^\perp$ such that

$(1)'$ the map α' is a unital completely positive linear map, $\beta'_n(1_D)$ and $\gamma'_n(1_E)$ are all projections, $\beta'_n(1_D)+\gamma'_n(1_E)=1_E$, for any $n\in\mathbb{N}$,

$(2)'$ $\|x-\gamma'_n(x)-\beta'_n(\alpha'(x))\|<\varepsilon''$, for any $x\in G$, and for any $n\in\mathbb{N}$,

$(3)'$ α' is a G-ε''-approximate embedding,

$(4)'$ $\lim\limits_{n\to\infty}\|\beta'_n(xy)-\beta'_n(x)\beta'_n(y)\|=0$ and $\lim\limits_{n\to\infty}\|\beta'_n(x)\|=\|x\|$, for all $x,y\in D$,

$(5)'$ $\gamma'_n\gamma_n(1_A)\precsim\gamma_n(1_A)\beta_n(r)\gamma_n(1_A)\precsim_A\beta_n(r)$, for all $n\in\mathbb{N}$.

Since D is weakly (m,n)-divisible, there exist $x''_1,x''_2,\cdots,x''_n\in\mathrm{M}_\infty(D)_+$ such that

$$x''_j\oplus x''_j\oplus\cdots\oplus x''_j\precsim(\alpha'\gamma_n(a)-\varepsilon')_+,$$

where x''_j repeat m times and

$$(\alpha'\gamma_n(a)-2\varepsilon')_+\precsim\bigoplus_{i=1}^n x''_i.$$

With the same argument as above, we have

$$(\beta'_n(x''_j)-3\varepsilon')_+\oplus(\beta'_n(x''_j)-3\varepsilon')_+\oplus\cdots\oplus(\beta'_n(x''_j)-3\varepsilon')_+\precsim(\beta'_n\alpha'\gamma_n(a)-2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a)-6\varepsilon')_+\precsim\bigoplus_{i=1}^n(\beta'_n(x''_i)-3\varepsilon')_+.$$

Therefore we have

$$\begin{aligned}&(\beta_n(x'_j\oplus r)-3\varepsilon')_+\oplus(\beta'_n(x''_j)-3\varepsilon')_+\oplus(\beta_n(x'_j\oplus r)-3\varepsilon')_+\oplus(\beta'_n(x''_j)-3\varepsilon')_+\oplus\cdots\\&\quad\oplus(\beta_n(x'_j\oplus r)-3\varepsilon')_+\oplus(\beta'_n(x''_j)-3\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-2\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-2\varepsilon')_+\\&\precsim a,\end{aligned}$$

where $(\beta_n(x'_j \oplus r) - 3\varepsilon')_+ \oplus (\beta'_n(x''_j) - 3\varepsilon')_+$ repeat m times,

$$\begin{aligned}&(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon)_+ \oplus \cdots\\ &\oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+\\ \precsim\ &(\beta_n\alpha(a) - 2\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+\\ \precsim\ &a\end{aligned}$$

for all $i \neq j$ and $1 \leq i \leq n$, and $(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+$ repeat m times.

We also have

$$\begin{aligned}&(a-\varepsilon)_+\\ \precsim\ &(\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(a) - 4\varepsilon')_+\\ \precsim\ &(\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(1_A) - \varepsilon')_+\\ \precsim\ &(\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus \beta_n(r)\\ \precsim\ &\bigoplus_{i=1, i\neq j}^{n} ((\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+)\\ &\oplus (\beta_n(x'_j \oplus r) - 3\varepsilon)_+ \oplus (\beta'_n(x''_j) - 3\varepsilon')_+.\end{aligned}$$

Case 1.1.2 If $x'_1, x'_2, \cdots, x'_k \in \mathrm{M}_\infty(B)_+$ are all projections and $(\alpha(a) - \varepsilon')_+ < \bigoplus_{i=1}^{k} x'_i$. By [23, Theorem 2.1], then there exists a nonzero projection s such that $(\alpha(a) - \varepsilon')_+ \oplus s \precsim \bigoplus_{i=1}^{n} x'_i$.

Since $(\alpha(a) - \varepsilon')_+ \oplus s \precsim \bigoplus_{i=1}^{n} x'_i$ and $(\alpha(a) - \varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x'_i$, for any $\overline{\varepsilon} > 0$, there exist $v, w \in M_\infty(B)$ such that

$$\Big\| w^* \mathrm{diag}((\alpha(a) - \varepsilon')_+, 0 \otimes 1_{n-2}) w - \bigoplus_{i=1}^{n} x'_i \Big\| < \overline{\varepsilon}$$

and

$$\Big\| v^* \mathrm{diag}((\alpha(a) - \varepsilon')_+, s, 0 \otimes 1_{n-2}) v - \bigoplus_{i=1}^{n} x'_i \Big\| < \overline{\varepsilon}.$$

We assume that $\|v\|, \|w\| \leq M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\Big\| \beta_n(w^*) \mathrm{diag}((\beta_n\alpha(a) - \varepsilon')_+, 0 \otimes 1_{n-2}) \beta_n(w) - \beta_n\Big(\bigoplus_{i=1}^{n} x'_i\Big) \Big\| < \varepsilon'$$

and

$$\Big\| \beta_n(v^*) \mathrm{diag}((\beta_n\alpha(a) - \varepsilon')_+, \beta_n(s), 0 \otimes 1_{k-2}) \beta_n(v) - \beta_n\Big(\bigoplus_{i=1}^{n} x'_i\Big) \Big\| < \varepsilon'.$$

Therefore, with the same argument as Case 1.1.1, we have

$$\bigoplus_{i=1}^{n} (\beta_n(x'_i) - 3\varepsilon')_+ \precsim (\beta_n\alpha(a) - 2\varepsilon')_+$$

and

$$(\beta_n\alpha(a) - 6\varepsilon')_+ + \beta_n(s) \precsim \bigoplus_{i=1}^{n} (\beta_n(x'_i) - 3\varepsilon')_+.$$

With $G = \{\gamma_n(a)\}$, given $\varepsilon'' > 0$ with $\varepsilon'' < \varepsilon'$, let $E = \gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha' : E \to D$, $\beta'_n : D \to E$ and $\gamma'_n : E \to E \cap \beta'_n(D)^{\perp}$ such that

$(1)'$ the map α' is a unital completely positive linear map, $\beta'_n(1_D)$ and $\gamma'_n(1_E)$ are all projections, $\beta'_n(1_D) + \gamma'_n(1_E) = 1_E$, for any $n \in \mathbb{N}$,

$(2)'$ $\|x - \gamma'_n(x) - \beta'_n(\alpha'(x))\| < \varepsilon''$, for any $x \in G$, and for all $n \in \mathbb{N}$,

$(3)'$ α' is a G-ε''-approximate embedding,

$(4)'$ $\lim_{n\to\infty} \|\beta'_n(xy) - \beta'_n(x)\beta'_n(y)\| = 0$ and $\lim_{n\to\infty} \|\beta'_n(x)\| = \|x\|$, for any $x, y \in D$,

$(5)'$ $\gamma'_n\gamma_n(1_A) \precsim \gamma_n(1_A)\beta_n(s)\gamma_n(1_A) \precsim_A \beta_n(s)$, for any $n \in \mathbb{N}$.

Since D is weakly (m, n)-divisible, there exist $x''_1, x''_2, \cdots, x''_n \in \mathrm{M}_\infty(D)_+$ such that

$$\beta'_n(x''_j) \oplus \beta'_n(x''_j) \oplus \cdots \oplus \beta'_n(x''_j) \precsim (\beta'_n\alpha'\gamma_n(a) - \varepsilon')_+,$$

where x''_j repeat m times and

$$(\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} \beta'_n(x''_i).$$

With the same argument as above, we have

$$(\beta'_n(x''_j) - 3\varepsilon')_+ \oplus (\beta'_n(x''_j) - 3\varepsilon')_+ \oplus \cdots \oplus (\beta'_n(x''_j) - 3\varepsilon')_+ \precsim (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} (\beta'_n(x''_i) - 3\varepsilon')_+.$$

We have

$$\begin{aligned} &(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus \cdots \\ &\quad \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \\ &\precsim (\beta_n\alpha(a) - 2\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+ \\ &\precsim a \end{aligned}$$

for $1 \le i \le n$, where $(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+$ repeat m times.

We also have

$$\begin{aligned} &(a - \varepsilon)_+ \\ &\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(a) - 4\varepsilon')_+ \\ &\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(1_A) - \varepsilon')_+ \\ &\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus \beta_n(s) \\ &\precsim \bigoplus_{i=1}^{n} (\beta_n(x'_i) - 3\varepsilon')_+ \bigoplus_{i=1}^{n} (\beta'_n(x''_i) - 3\varepsilon)_+. \end{aligned}$$

Case 1.1.3 we assume that there is a purely positive element x'_1. Since $(\alpha(a) - \varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x'_i$, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $(\alpha(a) - 2\varepsilon')_+ \precsim (x'_1 - \delta)_+ \bigoplus_{i=2}^{n} x'_i$.

By [23, Theorem 2.1(4)], there exists a nonzero positive element d such that $(x'_1 - \delta)_+ + d \precsim x'_1$.

Since $(\alpha(a)-2\varepsilon')_+ + d \precsim \bigoplus_{i=1}^{n} x_i'$ and $(\alpha(a)-\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x_i'$, for any $\overline{\varepsilon} > 0$, there exist $v, w \in M_\infty(B)$ such that

$$\Big\|w^*\mathrm{diag}((\alpha(a)-\varepsilon')_+, 0\otimes 1_{n-1})w - \bigoplus_{i=1}^{n} x_i'\Big\| < \overline{\varepsilon}$$

and

$$\Big\|v^*\mathrm{diag}((\alpha(a)-2\varepsilon')_+, d, 0\otimes 1_{k-2})v - \bigoplus_{i=1}^{n} x_i'\Big\| < \overline{\varepsilon}.$$

We assume that $\|v\|, \|w\| \leq M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\Big\|\beta_n(w^*)\mathrm{diag}((\beta_n\alpha(a)-\varepsilon')_+, 0\otimes 1_{n-1})\beta_n(w) - \beta_n\Big(\bigoplus_{i=1}^{n} x_i'\Big)\Big\| < \varepsilon'$$

and

$$\Big\|\beta_n(v^*)\mathrm{diag}((\beta_n\alpha(a)-\varepsilon')_+, \beta_n(d), 0\otimes 1_{n-2})\beta_n(v) - \beta_n\Big(\bigoplus_{i=1}^{n} x_i'\Big)\Big\| < \varepsilon'.$$

Therefore, with the same argument as Case 1.1.1, we have

$$\bigoplus_{i=1}^{n}(\beta_n(x_i')-3\varepsilon')_+ \precsim (\beta_n\alpha(a)-2\varepsilon')_+$$

and

$$(\beta_n\alpha(a)-6\varepsilon')_+ + \beta_n(d) \precsim \bigoplus_{i=1}^{n}(\beta_n(x_i')-3\varepsilon')_+.$$

With $G = \{\gamma_n(a)\}$, any $\varepsilon'' > 0$ with $\varepsilon'' < \varepsilon'$, let $E = \gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$, and c.p.c maps $\alpha': E \to D$, $\beta_n': D \to E$ and $\gamma_n': E \to E \cap \beta_n'(D)^\perp$ such that

$(1)'$ the map α' is a unital completely positive linear map, $\beta_n'(1_D)$ and $\gamma_n'(1_E)$ are all projections, $\beta_n'(1_D) + \gamma_n'(1_E) = 1_E$, for any $n \in \mathbb{N}$,

$(2)'$ $\|x - \gamma_n'(x) - \beta_n'(\alpha'(x))\| < \varepsilon''$ for any $x \in G$, and for any $n \in \mathbb{N}$,

$(3)'$ α' is a G-ε''-approximate embedding,

$(4)'$ $\lim_{n\to\infty} \|\beta_n'(xy) - \beta_n'(x)\beta_n'(y)\| = 0$ and $\lim_{n\to\infty} \|\beta_n'(x)\| = \|x\|$ for all $x, y \in D$,

$(5)'$ $\gamma_n'\gamma_n(1_A) \precsim \gamma_n(1_A)\beta_n(d)\gamma_n(1_A) \precsim_A \beta_n(d)$ for all $n \in \mathbb{N}$.

Since D is weakly (m,n)-divisible, there exist $x_1'', x_2'', \cdots, x_n'' \in \mathrm{M}_\infty(D)_+$ such that

$$x_j'' \oplus x_j'' \oplus \cdots \oplus x_j'' \precsim (\gamma_n'\gamma_n(a)-2\varepsilon')_+,$$

where x_j'' repeat m times and

$$(\gamma_n'\gamma_n(a)-3\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x_i''.$$

With the same argument as above, we have

$$(\beta_n'(x_j'')-3\varepsilon')_+ \oplus (\beta_n'(x_j'')-3\varepsilon')_+ \oplus \cdots \oplus (\beta_n'(x_j'')-3\varepsilon')_+ \precsim (\beta_n'\alpha'\gamma_n(a)-2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a)-6\varepsilon')_+\precsim\Big(\beta'_n\Big(\bigoplus_{i=1}^{n}x''_i\Big)-3\varepsilon'\Big)_+.$$

We have

$$\begin{aligned}&(\beta_n(x'_i)-3\varepsilon')_+\oplus(\beta'_n(x''_i)-3\varepsilon')_+\oplus(\beta_n(x'_i)-3\varepsilon')_+\oplus(\beta'_n(x''_i)-3\varepsilon')_+\oplus\cdots\\&\quad\oplus(\beta_n(x'_i)-3\varepsilon')_+\oplus(\beta'_n(x''_i)-3\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-2\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-2\varepsilon')_+\\&\precsim a\end{aligned}$$

for $1\le i\le n$, where $(\beta_n(x'_i)-3\varepsilon')_+\oplus(\beta'_n(x''_i)-3\varepsilon')_+$ repeat m times.

We also have

$$\begin{aligned}&(a-\varepsilon)_+\\&\precsim(\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-6\varepsilon')_+\oplus(\gamma'_n\gamma_n(a)-4\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-6\varepsilon')_+\oplus(\gamma'_n\gamma_n(1_A)-\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta'_n\alpha'\gamma_n(a)-6\varepsilon')_+\oplus\beta_n(d)\\&\precsim\bigoplus_{i=1}^{n}(\beta_n(x'_i)-3\varepsilon')_+\bigoplus_{i=1}^{n}(\beta'_n(x''_i)-3\varepsilon')_+.\end{aligned}$$

Case 1.1.4 We assume that there exists a nonzero projection s such that $(\alpha(a)-2\varepsilon')_++s\precsim(\alpha(a)-\varepsilon')_+$.

Since $(\alpha(a)-2\varepsilon')_++s\precsim\bigoplus_{i=1}^{n}x'_i$ and $(\alpha(a)-\varepsilon')_+\precsim\bigoplus_{i=1}^{n}x'_i$, for any $\overline{\varepsilon}>0$, there exist $v,w\in M_\infty(B)$ such that

$$\Big\|w^*\mathrm{diag}((\alpha(a)-\varepsilon')_+,0\otimes 1_{n-1})w-\bigoplus_{i=1}^{n}x'_i\Big\|<\overline{\varepsilon}$$

and

$$\Big\|v^*\mathrm{diag}((\alpha(a)-2\varepsilon')_+,s,0\otimes 1_{k-2})v-\bigoplus_{i=1}^{n}x'_i\Big\|<\overline{\varepsilon}.$$

We assume that $\|v\|,\|w\|\le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\Big\|\beta_n(w^*)\mathrm{diag}((\beta_n\alpha(a)-\varepsilon')_+,0\otimes 1_{n-1})\beta_n(w)-\beta_n\Big(\bigoplus_{i=1}^{n}x'_i\Big)\Big\|<\varepsilon'$$

and

$$\Big\|\beta_n(v^*)\mathrm{diag}((\beta_n\alpha(a)-\varepsilon')_+,\beta_n(s),0\otimes 1_{n-2})\beta_n(v)-\beta_n\Big(\bigoplus_{i=1}^{n}x'_i\Big)\Big\|<\varepsilon'.$$

Therefore, with the same argument as Case 1.1.1, we have

$$\bigoplus_{i=1}^{n}(\beta_n(x'_i)-3\varepsilon')_+\precsim(\beta_n\alpha(a)-2\varepsilon')_+$$

and

$$(\beta_n\alpha(a)-6\varepsilon')_++\beta_n(s)\precsim\bigoplus_{i=1}^{n}(\beta_n(x'_i)-3\varepsilon')_+.$$

With $G = \{\gamma_n(a)\}$, any $\varepsilon'' > 0$ with $\varepsilon'' < \varepsilon'$, let $E = \gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha' : E \to D$, $\beta'_n : D \to E$ and $\gamma'_n : E \to E \cap \beta'_n(D)^{\perp}$ such that

$(1)'$ the map α' is a unital completely positive linear map, $\beta'_n(1_D)$ and $\gamma'_n(1_E)$ are all projections, $\beta'_n(1_D) + \gamma'_n(1_E) = 1_E$, for any $n \in \mathbb{N}$,

$(2)'$ $\|x - \gamma'_n(x) - \beta'_n(\alpha'(x))\| < \varepsilon''$, for any $x \in G$, and for any $n \in \mathbb{N}$,

$(3)'$ α' is a G-ε''-approximate embedding,

$(4)'$ $\lim_{n\to\infty} \|\beta'_n(xy) - \beta'_n(x)\beta'_n(y)\| = 0$ and $\lim_{n\to\infty} \|\beta'_n(x)\| = \|x\|$ for all $x, y \in D$,

$(5)'$ $\gamma'_n\gamma_n(1_A) \precsim \gamma_n(1_A)\beta_n(s)\gamma_n(1_A) \precsim_A \beta_n(s)$ for all $n \in \mathbb{N}$.

Since D is weakly (m, n)-divisible, there exist $x''_1, x''_2, \cdots, x''_n \in \mathrm{M}_\infty(D)_+$ such that

$$x''_j \oplus x''_j \oplus \cdots \oplus x''_j \precsim (\gamma'_n\gamma_n(a) - 2\varepsilon')_+,$$

where x''_j repeat m times and

$$(\gamma'_n\gamma_n(a) - 3\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x''_i.$$

With the same argument as above, we have

$$(\beta'_n(x''_j) - 3\varepsilon')_+ \oplus (\beta'_n(x''_j) - 3\varepsilon')_+ \oplus \cdots \oplus (\beta'_n(x''_j) - 3\varepsilon')_+ \precsim (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} (\beta'_n(x''_i - 3\varepsilon'))_+.$$

We have

$$\begin{aligned} &(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus \cdots \\ &\quad \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \\ &\precsim (\beta_n\alpha(a) - 2\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+ \\ &\precsim a \end{aligned}$$

for all $1 \le i \le n$, where $(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+$ repeat m times.

We also have

$$\begin{aligned} &(a - \varepsilon)_+ \\ &\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(a) - 4\varepsilon')_+ \\ &\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(1_A) - \varepsilon')_+ \\ &\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus \beta_n(s) \\ &\precsim \bigoplus_{i=1}^{n} (\beta_n(x'_i) - 3\varepsilon')_+ \bigoplus_{i=1}^{n} (\beta'_n(x''_i) - 3\varepsilon')_+. \end{aligned}$$

Case 1.2 If $(\alpha(a) - \varepsilon')_+$ is not Cuntz equivalent to $\bigoplus_{i=1}^{n} x'_i$.

By [23, Theorem 2.1(2)], we may assume that there exists a non-zero $c \in B_+$ such that $(\alpha(a) - \varepsilon')_+ + c \precsim \bigoplus_{i=1}^{n} x'_i$.

Since $(\alpha(a)-\varepsilon')_+ + c \precsim \bigoplus_{i=1}^{n} x_i'$ and $(\alpha(a)-\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x_i'$, for any $\overline{\varepsilon} > 0$, there exist $v, w \in M_\infty(B)$ such that

$$\Big\|w^*\mathrm{diag}((\alpha(a)-\varepsilon')_+, 0\otimes 1_{n-1})w - \bigoplus_{i=1}^{n} x_i'\Big\| < \overline{\varepsilon}$$

and

$$\Big\|v^*\mathrm{diag}((\alpha(a)-\varepsilon')_+, c, 0\otimes 1_{k-2})v - \bigoplus_{i=1}^{n} x_i'\Big\| < \overline{\varepsilon}.$$

We assume that $\|v\|, \|w\| \le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\Big\|\beta_n(w^*)\mathrm{diag}((\beta_n\alpha(a)-\varepsilon')_+, 0\otimes 1_{n-1})\beta_n(w) - \beta_n\Big(\bigoplus_{i=1}^{n} x_i'\Big)\Big\| < \varepsilon'$$

and

$$\Big\|\beta_n(v^*)\mathrm{diag}((\beta_n\alpha(a)-\varepsilon')_+, \beta_n(c), 0\otimes 1_{n-2})\beta_n(v) - \beta_n\Big(\bigoplus_{i=1}^{n} x_i'\Big)\Big\| < \varepsilon'.$$

Therefore, with the same argument as Case 1.1.1, we have

$$\bigoplus_{i=1}^{n}(\beta_n(x_i') - 3\varepsilon')_+ \precsim (\beta_n\alpha(a) - 2\varepsilon')_+$$

and

$$(\beta_n\alpha(a) - 6\varepsilon')_+ + \beta_n(c) \precsim \bigoplus_{i=1}^{n}(\beta_n(x_i') - 3\varepsilon')_+.$$

For sufficiently large $n > \max\{N_1, N_2\}$, with $G = \{\gamma_n(a)\}$, any $\varepsilon'' > 0$, with $\varepsilon'' < \varepsilon$, let $E = \gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C^*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha' : E \to D$, $\beta_n' : D \to E$ and $\gamma_n' : E \to E \cap \beta_n'(D)^\perp$ such that

(1)′ the map α' is a unital completely positive linear map, $\beta_n'(1_D)$ and $\gamma_n'(1_E)$ are all projections, $\beta_n'(1_D) + \gamma_n'(1_E) = 1_E$, for any $n \in \mathbb{N}$,

(2)′ $\|x - \gamma_n'(x) - \beta_n'(\alpha'(x))\| < \varepsilon''$, for any $x \in G$, and for any $n \in \mathbb{N}$,

(3)′ α' is a G-ε''-approximate embedding,

(4)′ $\lim_{n\to\infty} \|\beta_n'(xy) - \beta_n'(x)\beta_n'(y)\| = 0$ and $\lim_{n\to\infty} \|\beta_n'(x)\| = \|x\|$ for all $x, y \in D$,

(5)′ $\gamma_n'\gamma(1_E) \precsim \gamma_n(1_A)\beta_n(c)\gamma_n(1_A) \precsim \beta_n(c)$ for all $n \in \mathbb{N}$.

Since D is weakly (m, n)-divisible, there exist $x_1'', x_2'', \cdots, x_n'' \in \mathrm{M}_\infty(D)_+$ such that

$$x_j'' \oplus x_j'' \oplus \cdots \oplus x_j'' \precsim (\gamma_n'\gamma_n(a) - 2\varepsilon')_+,$$

where x_j'' repeat m times and

$$(\gamma_n'\gamma_n(a) - 3\varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x_i''.$$

With the same argument, as above, we have

$$(\beta_n'(x_j'') - 3\varepsilon')_+ \oplus (\beta_n'(x_j'') - 3\varepsilon')_+ \oplus \cdots \oplus (\beta_n'(x_j'') - 3\varepsilon')_+ \precsim (\beta_n'\alpha'\gamma_n(a) - 2\varepsilon')_+$$

and

$$(\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \precsim \bigoplus_{i=1}^{n}(\beta'_n(x''_i) - 3\varepsilon')_+.$$

We have

$$\begin{aligned}
&(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \oplus \cdots \\
&\quad \oplus (\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+ \\
&\precsim (\beta_n\alpha(a) - 2\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 2\varepsilon')_+ \\
&\precsim a
\end{aligned}$$

for $1 \le i \le n$, where $(\beta_n(x'_i) - 3\varepsilon')_+ \oplus (\beta'_n(x''_i) - 3\varepsilon')_+$ repeat m times.

We also have

$$\begin{aligned}
&(a - \varepsilon)_+ \\
&\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(a) - 4\varepsilon')_+ \\
&\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus (\gamma'_n\gamma_n(1_A) - \varepsilon')_+ \\
&\precsim (\beta_n\alpha(a) - 6\varepsilon')_+ \oplus (\beta'_n\alpha'\gamma_n(a) - 6\varepsilon')_+ \oplus \beta_n(c) \\
&\precsim \bigoplus_{i=1}^{n}(\beta_n(x'_i) - 3\varepsilon')_+ \bigoplus_{i=1}^{n}(\beta'_n(x''_i) - 3\varepsilon')_+.
\end{aligned}$$

Case 2 If $(\alpha(a) - \varepsilon')_+$ is not Cuntz equivalent to a projection.

By [23, Theorem 2.1(4)], there is a non-zero positive element d such that $(\alpha(a) - 2\varepsilon')_+ + d \precsim (\alpha(a) - \varepsilon')_+$.

Since $(\alpha(a) - 2\varepsilon')_+ + d \precsim \bigoplus_{i=1}^{n} x'_i$ and $(\alpha(a) - \varepsilon')_+ \precsim \bigoplus_{i=1}^{n} x'_i$, for any $\overline{\varepsilon} > 0$, there exist $v, w \in M_\infty(B)$ such that

$$\Big\| w^*\mathrm{diag}((\alpha(a) - \varepsilon')_+, 0 \otimes 1_{n-1})w - \bigoplus_{i=1}^{n} x'_i \Big\| < \overline{\varepsilon}$$

and

$$\Big\| v^*\mathrm{diag}((\alpha(a) - 2\varepsilon')_+, d, 0 \otimes 1_{k-2})v - \bigoplus_{i=1}^{n} x'_i \Big\| < \overline{\varepsilon}.$$

We assume that $\|v\|, \|w\| \le M(\overline{\varepsilon})$, by (4), there exists a sufficiently large integer n such that

$$\Big\| \beta_n(w^*)\mathrm{diag}((\beta_n\alpha(a) - \varepsilon')_+, 0 \otimes 1_{n-1})\beta_n(w) - \beta_n\Big(\bigoplus_{i=1}^{n} x'_i\Big) \Big\| < \varepsilon'$$

and

$$\Big\| \beta_n(v^*)\mathrm{diag}((\beta_n\alpha(a) - \varepsilon')_+, \beta_n(d), 0 \otimes 1_{n-2})\beta_n(v) - \beta_n\Big(\bigoplus_{i=1}^{n} x'_i\Big) \Big\| < \varepsilon'.$$

Therefore, with the same argument as Case 1.1.1, we have

$$\bigoplus_{i=1}^{n}(\beta_n(x'_i) - 3\varepsilon')_+ \precsim (\beta_n\alpha(a) - 2\varepsilon')_+$$

and

$$(\beta_n\alpha(a)-6\varepsilon')_+ + \beta_n(d) \precsim \bigoplus_{i=1}^{n}(\beta_n(x_i')-3\varepsilon')_+.$$

With $G=\{\gamma_n(a)\}$, any $\varepsilon''>0$, let $E=\gamma_n(1_A)A\gamma_n(1_A)$. By Lemma 2.2, E is asymptotically tracially in $\mathcal{P}$, by Theorem 2.1, there is a C*-algebra D in $\mathcal{P}$ and c.p.c maps $\alpha': E\to D$, $\beta_n': D\to E$ and $\gamma_n': E\to E\cap\beta_n'(D)^{\perp}$ such that

(1)′ the map α' is a unital completely positive linear map, $\beta_n'(1_D)$ and $\gamma_n'(1_E)$ are all projections, $\beta_n'(1_D)+\gamma_n'(1_E)=1_E$, for any $n\in\mathbb{N}$,

(2)′ $\|x-\gamma_n'(x)-\beta_n'(\alpha'(x))\|<\varepsilon''$ for any $x\in G$, and for any $n\in\mathbb{N}$,

(3)′ α' is a G-ε''-approximate embedding,

(4)′ $\lim\limits_{n\to\infty}\|\beta_n'(xy)-\beta_n'(x)\beta_n'(y)\|=0$ and $\lim\limits_{n\to\infty}\|\beta_n'(x)\|=\|x\|$ for all $x,y\in D$,

(5)′ $\gamma_n'(\gamma(1_A))\gamma_n(1_A)\beta_n(d)\gamma_n(1_A)\precsim\beta_n(d)$ for all $n\in\mathbb{N}$.

Since D is weakly (m,n)-divisible, there exist $x_1'',x_2'',\cdots,x_n''\in \mathrm{M}_\infty(D)_+$ such that

$$x_j''\oplus x_j''\oplus\cdots\oplus x_j''\precsim(\gamma_n'\gamma_n(a)-2\varepsilon')_+,$$

where x_j'' repeat m times and

$$(\gamma_n'\gamma_n(a)-3\varepsilon')_+\precsim\bigoplus_{i=1}^{n}x_i''.$$

With the same argument as above, we have

$$(\beta_n'(x_j'')-3\varepsilon')_+\oplus(\beta_n'(x_j'')-3\varepsilon')_+\oplus\cdots\oplus(\beta_n'(x_j'')-3\varepsilon')_+\precsim(\beta_n'\alpha'\gamma_n(a)-2\varepsilon')_+$$

and

$$(\beta_n'\alpha'\gamma_n(a)-6\varepsilon')_+\precsim\bigoplus_{i=1}^{n}(\beta_n'(x_i'')-3\varepsilon')_+.$$

We have

$$\begin{aligned}&(\beta_n(x_i')-3\varepsilon')_+\oplus(\beta_n'(x_i'')-3\varepsilon')_+\oplus(\beta_n(x_i')-3\varepsilon')_+\oplus(\beta_n'(x_i'')-3\varepsilon')_+\oplus\cdots\\&\quad\oplus(\beta_n(x_i')-3\varepsilon')_+\oplus(\beta_n'(x_i'')-3\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-2\varepsilon')_+\oplus(\beta_n'\alpha'\gamma_n(a)-2\varepsilon')_+\\&\precsim a\end{aligned}$$

for $1\le i\le n$, where $(\beta_n(x_i')-3\varepsilon')_+\oplus(\beta_n'(x_i'')-3\varepsilon')_+$ repeat m times.

We also have

$$\begin{aligned}&(a-\varepsilon)_+\\&\precsim(\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta_n'\alpha'\gamma_n(a)-6\varepsilon')_+\oplus(\gamma_n'\gamma_n(a)-4\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta_n'\alpha'\gamma_n(a)-6\varepsilon')_+\oplus(\gamma_n'\gamma_n(1_A)-\varepsilon')_+\\&\precsim(\beta_n\alpha(a)-6\varepsilon')_+\oplus(\beta_n'\alpha'\gamma_n(a)-6\varepsilon')_+\oplus\beta_n(d)\\&\precsim\bigoplus_{i=1}^{n}(\beta_n(x_i')-3\varepsilon')_+\bigoplus_{i=1}^{n}(\beta_n'(x_i'')-3\varepsilon')_+.\end{aligned}$$

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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