Chin. Ann. Math. Ser. B
46(5), 2025, 733–758
DOI: 10.1007/s11401-025-0039-7

Chinese Annals of Mathematics, Series B

The Jordan Algebra of Complex Symmetric Operators*

Cun WANG[1] Sen ZHU[2]

Abstract For a conjugation C on a separable, complex Hilbert space $\mathcal{H}$, the set $\mathcal{S}_C$ of C-symmetric operators on $\mathcal{H}$ forms a weakly closed, selfadjoint, Jordan operator algebra. In this paper, the authors study $\mathcal{S}_C$ in comparison with the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, and obtain $\mathcal{S}_C$-analogues of some classical results concerning $\mathcal{B}(\mathcal{H})$. The authors determine the Jordan ideals of $\mathcal{S}_C$ and their dual spaces. Jordan automorphisms of $\mathcal{S}_C$ are classified. The authors determine the spectra of Jordan multiplication operators on $\mathcal{S}_C$ and their different parts. It is proved that those Jordan invertible ones constitute a dense, path connected subset of $\mathcal{S}_C$.

Keywords Complex symmetric operator, Jordan operator algebra, Cartan factor, Jordan ideal, Automorphism
2020 MR Subject Classification 47B99, 46L70, 17C65, 46K70

1 Introduction

The aim of this paper is to study a class of Jordan operator algebras consisting of complex symmetric operators. Before proceeding let us first recall a few definitions.

Throughout this paper, $\mathcal{H}$ ($\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$, $\cdots$, etc.) will always denote a separate, complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$. The Banach space of all bounded linear operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$ will be denoted by $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$. For convenience, we shall write $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H},\mathcal{H})$.

Definition 1.1 *A map $C:\mathcal{H}\to\mathcal{H}$ is called a conjugation if*

(i) *C is anti-linear, i.e., $C(\alpha x+y)=\overline{\alpha}Cx+Cy$ for $x,y\in\mathcal{H}$ and $\alpha\in\mathbb{C}$,*

(ii) *C is invertible with $C^{-1}=C$,*

(iii) *$\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathcal{H}$.*

Definition 1.2 *An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be complex symmetric (c.s. for short) if $CTC=T^*$ for some conjugation C on $\mathcal{H}$; in this case, T is called C-symmetric.*

The term "symmetric" stems from the fact that an operator T is complex symmetric if and only if there is an orthonormal basis $\{e_i\}$ of $\mathcal{H}$ such that $\langle Te_i,e_j\rangle=\langle Te_j,e_i\rangle$ for all i,j, that is, T can be written as a (complex) symmetric matrix relative to $\{e_i\}$. Following Garcia

Manuscript received December 24, 2021. Revised March 4, 2023.
[1]Department of Mathematics, School of Science, Jimei University, Xiamen 361021, Fujian, China.
E-mail: wangcun18@mails.jlu.edu.cn
[2]Corresponding author. Department of Mathematics, Jilin University, Changchun 130012, China.
E-mail: zhusen@jlu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 12401149, 12171195) and the National Key Research and Development Program of China (No. 2020YFA0714101).

and Wogen [18], we let $\mathcal{S}(\mathcal{H})$ denote the collection of c.s. operators on $\mathcal{H}$. Many important examples of c.s. operators have been identified, such as normal operators, binormal operators, Hankel operators, truncated Toeplitz operators, the Volterra integral operator, certain partial isometries, weighted shifts, composition operators and many integral operators (see [16–18, 40–41]).

The study of abstract c.s. operators, initiated in [16–17], is relatively new, although it has classical roots in the work on automorphic functions (see [23]), projective geometry (see [24]), quadratic forms (see [35]), symplectic geometry (see [37]), function theory (see [38]) and extension theory of differential operators (see [19]). People's current interest in c.s. operators is chiefly inspired by many interesting results of Garcia, Putinar and Wogen [16–18] concerning the structure of c.s. operators as well as their connections to concrete operators (see [12–14]) and applications to Mathematical physics (see [15, 21, 30]). In particular, the study of c.s. operators is closely related to that of truncated Toeplitz operators, which was initiated in Sarason's seminal paper [33]. In their recent paper [3], Bercovici and Timotin showed that truncated Toeplitz operators can be characterized by a collection of complex symmetries.

Recently there has been a growing interest in the algebraic aspect of c.s. operators. In [20], certain connections between c.s. operators and anti-automorphisms of singly generated C^*-algebras are established. Furthermore, a general answer to the norm closure problem for c.s. operators was provided in [42] which relies on an intensive analysis of singly generated C^*-algebras. In [36, 44], c.s. generator problem for operator algebras has been studied. In a recent paper [4], Blecher and Wang studied involutive operator algebras and obtained a characterization of operator algebras with linear involutions in terms of c.s. operators.

Since $\mathcal{S}(\mathcal{H})$ is not closed under addition or multiplication, we may pay attention to a typical linear subspace of $\mathcal{B}(\mathcal{H})$ contained in $\mathcal{S}(\mathcal{H})$. Given a conjugation C on $\mathcal{H}$, we denote by $\mathcal{S}_C$ the set of all C-symmetric operators on $\mathcal{H}$. Clearly, $\mathcal{S}_C$ is a subspace of $\mathcal{B}(\mathcal{H})$ closed in the weak operator topology. Note that $\mathcal{S}(\mathcal{H}) = \bigcup\limits_C \mathcal{S}_C$, where the union is taken over all conjugations on $\mathcal{H}$. Later we shall show that if C_1, C_2 are conjugations on $\mathcal{H}$, then there exists a unitary operator U on $\mathcal{H}$ such that $U\mathcal{S}_{C_1}U^* = \mathcal{S}_{C_2}$ (see Corollary 2.1). So, up to unitary equivalence, $\mathcal{S}_C$ contains all c.s. operators on $\mathcal{H}$. Throughout the following, we always assume that C is a conjugation on $\mathcal{H}$, unless stated otherwise. Also, we always assume that $\dim \mathcal{H} = \infty$, since almost all results in finite dimensional case follow readily from that in infinite dimensional case.

Some interesting results concerning $\mathcal{S}_C$ as a subspace of $\mathcal{B}(\mathcal{H})$ have been obtained. For example, Garcia [13, Theorem 1] proved that each contraction in $\mathcal{S}_C$ is the mean of two unitary ones in $\mathcal{S}_C$, whose proof relies on a refined polar decomposition for c.s. operators obtained by Garcia and Putinar [17]. This result is a C-symmetric analogue of a classical result in $\mathcal{B}(\mathcal{H})$. In [8], Danciger, Garcia and Putinar established for $\mathcal{S}_C$ a natural analogue of Courant's minimax principle to estimate singular values of compact ones in $\mathcal{S}_C$. It was shown in [27] that $\mathcal{S}_C$ is transitive and 2-hyperreflexive.

The aim of this paper is to study $\mathcal{S}_C$ in the Jordan setting. This is inspired by the observation that $\mathcal{S}_C$ is closed under the Jordan product $\circ$, defined by

$$A \circ B = \frac{1}{2}(AB + BA), \quad \forall A, B \in \mathcal{B}(\mathcal{H}).$$

That is, $\mathcal{S}_C$ forms a Jordan operator algebra. By a Jordan operator algebra we mean a norm-closed subspace of $\mathcal{B}(\mathcal{H})$ closed under the Jordan product $\circ$. Thus $\mathcal{S}_C$ is a Jordan subalgebra of $\mathcal{B}(\mathcal{H})$.

Jordan algebras arose from the search for a new algebraic setting for quantum mechanics (see [25]), and turned out to have illuminating connections with many areas of mathematics. Any associative algebra gives rise to a Jordan algebra under the Jordan product $\circ$. Note that $A \in \mathcal{S}_C$ implies $A^* \in \mathcal{S}_C$. Hence $\mathcal{S}_C$ is also selfadjoint. Selfadjoint Jordan operator algebras are known as JC*-algebras (see [29]).

We remark that $\mathcal{S}_C$ lies naturally in many more general contexts. In fact, $\mathcal{S}_C$ has been studied under the name of Hermitian type Cartan factor for many years. There are six types of Cartan factors, namely rectangular type, Hermitian type, symplectic type, triple spin factors and two finite-dimensional exceptional Cartan factors. Cartan factors arose in Cartan's classification of finite dimensional bounded symmetric domains (see [5, 26] or [6, Theorem 2.5.9]) and play an important role in the proof of the Gelfand-Naimark theorem for JB*-triples (see [11]). Any JB*-triple is triple isomorphic to a closed subtriple of an l_∞-sum of Cartan factors (see [6, Theorem 3.3.19]).

This paper focuses on the special Jordan operator algebra $\mathcal{S}_C$, and it aims to present some applications of recent advances in the theory of c.s. operators to the Jordan structure of $\mathcal{S}_C$. We shall prove some results concerning the algebraic aspect of the space $\mathcal{S}_C$, including Jordan ideals, Schatten p-classes, automorphisms, invertibles, multiplication operators. We obtain $\mathcal{S}_C$-analogues of some classical results concerning $\mathcal{B}(\mathcal{H})$, showing that $\mathcal{S}_C$ is far more similar to the associative operator algebra $\mathcal{B}(\mathcal{H})$ than was suspected.

In Section 2, we study the Jordan ideal structure of $\mathcal{S}_C$. A linear subspace $\mathcal{J}$ of $\mathcal{S}_C$ is called a Jordan ideal if $A \circ X \in \mathcal{J}$ for every $A \in \mathcal{S}_C$ and $X \in \mathcal{J}$. Note that Jordan ideals of $\mathcal{S}_C$ coincide with its triple ideals as a Jordan triple system (see [6, p. 38]). We shall show that Jordan ideals of $\mathcal{S}_C$ arise by intersection from ideals of $\mathcal{B}(\mathcal{H})$ and hence are selfadjoint (see Theorem 2.1). By an (associative) ideal of $\mathcal{B}(\mathcal{H})$, we mean a two-sided ideal (not necessarily norm closed) of $\mathcal{B}(\mathcal{H})$ under the usual multiplication of operators. Note that Jordan ideals of $\mathcal{B}(\mathcal{H})$ coincide with its ideals (see [10]). So our result shows that $\mathcal{S}_C$ and $\mathcal{B}(\mathcal{H})$ have the same Jordan ideal structure. Moreover, we observe that $\mathcal{S}_C$ is universal in the sense that any Jordan subalgebra of $\mathcal{B}(\mathcal{H})$ is Jordan isomorphic to a Jordan subalgebra of $\mathcal{S}_C$ (see Corollary 2.3). This inspires us to examine whether some classical facts about $\mathcal{B}(\mathcal{H})$ still hold or have analogues in $\mathcal{S}_C$.

In Subsection 2.2, we study those norm ideals of $\mathcal{S}_C$ induced by Schatten p-classes. The Schatten p-class of compact operators on $\mathcal{H}$ is denoted by $\mathcal{B}_p(\mathcal{H})$, $1 \le p < \infty$. In addition, we write $\mathcal{B}_0(\mathcal{H})$ for the collection of compact operators on $\mathcal{H}$. Denote $\mathcal{S}_{C,p} = \mathcal{S}_C \cap \mathcal{B}_p(\mathcal{H})$. We establish in Subsection 2.2 the dual relations among $\mathcal{S}_C$ and its Jordan ideals $\mathcal{S}_{C,p}$ ($p \in \{0\} \cup [1, \infty)$) (see Proposition 2.2). As an application, we classify Jordan automorphisms of $\mathcal{S}_C$ and show that Jordan automorphisms of $\mathcal{S}_C$ are implemented by those unitary operators on $\mathcal{H}$ commuting with C (see Theorem 2.2).

In Section 3, we concentrate on Jordan multiplication operators on $\mathcal{S}_C$. For $T \in \mathcal{S}_C$, we define $J_T : \mathcal{S}_C \to \mathcal{S}_C$ as $J_T(X) = T \circ X$ for $X \in \mathcal{S}_C$. We call J_T the Jordan multiplication operator with symbol T. Note that J_T is called left multiplication by T in [6], and is called the Jordan translation of T in [39]. These operators play a basic role in the study of Jordan algebras. In particular, quadratic product, trilinear product as well as the centre can be defined in terms of them. We shall determine the spectra of Jordan multiplication operators J_T and their restrictions to some Jordan ideals of $\mathcal{S}_C$ (see Theorem 3.1 and Corollary 3.3). One shall see that their proofs rely on the dual relations among $\mathcal{S}_C$ and its Jordan ideals $\mathcal{S}_{C,p}$ ($p \in [1,\infty]$).

In Section 4, we discuss Jordan invertible elements in $\mathcal{S}_C$. For $T \in \mathcal{S}_C$, we define the quadratic operator $Q_T : \mathcal{S}_C \to \mathcal{S}_C$ by $Q_T(X) = TXT$ for $X \in \mathcal{S}_C$. An element T of $\mathcal{S}_C$ is called Jordan invertible if Q_T is invertible as a bounded linear operator on $\mathcal{S}_C$ with $Q_T^{-1} = Q_A$ for some $A \in \mathcal{S}_C$ (see [6, p. 107]). One can check that an element $T \in \mathcal{S}_C$ is Jordan invertible if and only if T is bijective or, equivalently, T is an invertible bounded operator on $\mathcal{H}$. It is proved that those invertible ones in $\mathcal{S}_C$ constitute a dense, path connected subset of $\mathcal{S}_C$. This is an analogue of a result of Apostol, Fialkow, Herrero and Voiculescu concerning invertible approximation in $\mathcal{B}(\mathcal{H})$ (see [1, Proposition 10.1]).

Results obtained in this paper suggest a rich structure theory of $\mathcal{S}_C$, provide interesting contrasts between $\mathcal{S}_C$ and $\mathcal{B}(\mathcal{H})$, and also show that the structure of $\mathcal{S}_C$ deserves further study. Concerning our methodology, most techniques used in this paper are developed based on some recent results concerning the structure of c.s. operators. Also some of our results employ tools from noncommutative approximation of Hilbert space operators, as represented in the two-volume monographs of Apostol-Fialkow-Herrero-Voiculescu [1, 22].

2 Jordan Ideals of $\mathcal{S}_C$

This section focuses on Jordan ideals of $\mathcal{S}_C$. It is proved that each proper Jordan ideal of $\mathcal{S}_C$ is the intersection of $\mathcal{S}_C$ with some ideal of $\mathcal{B}(\mathcal{H})$ and hence consists of some compact operators on $\mathcal{H}$. We shall extend some results concerning compact operators in $\mathcal{B}(\mathcal{H})$ to $\mathcal{S}_C$.

2.1 Characterization of Jordan ideals

The main result of this subsection is the following theorem, which shows that each Jordan ideal of $\mathcal{S}_C$ is induced by an ideal of $\mathcal{B}(\mathcal{H})$.

Theorem 2.1 *A subset $\mathcal{J}$ of $\mathcal{S}_C$ is a Jordan ideal of $\mathcal{S}_C$ if and only if $\mathcal{J} = \mathcal{I} \cap \mathcal{S}_C$ for some ideal $\mathcal{I}$ of $\mathcal{B}(\mathcal{H})$.*

To prove Theorem 2.1, we first make some preparation.

Let C be a conjugation on $\mathcal{H}$. For $X \in \mathcal{B}(\mathcal{H})$, we denote $X^t = CX^*C$. Define $D : \mathcal{H}^{(2)} \to \mathcal{H}^{(2)}$ as

$$D : (x_1, x_2) \mapsto (Cx_2, Cx_1), \quad \forall (x_1, x_2) \in \mathcal{H}^{(2)}.$$

Then D can be written as

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

One can check that D is a conjugation on $\mathcal{H}^{(2)}$.

Lemma 2.1 *Let C be a conjugation on $\mathcal{H}$ and*

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}.$$

Assume that $T \in \mathcal{B}(\mathcal{H}^{(2)})$ and

$$T = \begin{bmatrix} A & E \\ F & B \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Then

(i) *$T \in \mathcal{S}_D$ if and only if $B = A^t$, $E = E^t$ and $F = F^t$;*

(ii) *if $\mathcal{J}$ is a Jordan ideal of $\mathcal{S}_D$ and $T \in \mathcal{J}$, then $\mathcal{J}$ contains the following operators on $\mathcal{H}^{(2)}$:*

$$\begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}, \quad \begin{bmatrix} 0 & A+B \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ A+B & 0 \end{bmatrix}, \quad \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}.$$

Proof (i) By the definition, $T \in \mathcal{S}_D$ if and only if $DT = T^*D$. The result follows from a direct matrix calculation.

(ii) Define $Y_1, Y_2 \in \mathcal{B}(\mathcal{H}^{(2)})$ as

$$Y_1 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix},$$

where I is the identity operator on $\mathcal{H}$. Then, by (i), $Y_1, Y_2 \in \mathcal{S}_D$.

It is easy to check that

$$[Y_1 \circ (Y_1 \circ T)] \circ Y_2 = \frac{1}{4} \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad \{[Y_1 \circ (Y_1 \circ T)] \circ Y_2\} \circ Y_2 = \frac{1}{4} \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}$$

and

$$Y_1 \circ [(T \circ Y_2) \circ Y_2] = \frac{1}{4} \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad Y_1 \circ \{Y_1 \circ [(T \circ Y_2) \circ Y_2]\} = \frac{1}{4} \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}, \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} \in \mathcal{J}.$$

It follows immediately that $A \oplus B \in \mathcal{J}$,

$$Y_1 \circ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ A+B & 0 \end{bmatrix} \in \mathcal{J}, \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \circ Y_2 = \frac{1}{2} \begin{bmatrix} 0 & A+B \\ 0 & 0 \end{bmatrix} \in \mathcal{J},$$

$$Y_1 \circ \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \circ Y_2 \right) = \frac{1}{4} \begin{bmatrix} A+B & 0 \\ 0 & A+B \end{bmatrix} \in \mathcal{J}.$$

Therefore we conclude that $B \oplus A \in \mathcal{J}$.

Lemma 2.2 *If C_1, C_2 are two conjugations on $\mathcal{H}$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $U^*C_1U = C_2$.*

Proof Since C_1, C_2 are conjugations on $\mathcal{H}$, by [15, Lemma 2.11], there exist two orthonormal bases $\{e_n : n \geq 1\}$ and $\{f_n : n \geq 1\}$ such that $C_1 e_n = e_n$ and $C_2 f_n = f_n$ for all n. Define a unitary operator U on $\mathcal{H}$ as $Uf_n = e_n$, $n \geq 1$. Then it is easy to check that $U^*C_1U = C_2$.

Corollary 2.1 *If C_1, C_2 are two conjugations on $\mathcal{H}$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{S}_{C_2} = U^*\mathcal{S}_{C_1}U$.*

The preceding result shows that $\mathcal{S}(\mathcal{H})$ is the union of all Jordan operator algebras unitarily equivalent to $\mathcal{S}_C$.

Corollary 2.2 *The set of all conjugations on $\mathcal{H}$ is path connected (as a subset of the Banach space of bounded anti-linear operators on $\mathcal{H}$).*

Proof Fix two conjugations C_1 and C_2 on $\mathcal{H}$. By Lemma 2.2, we can find unitary $U \in \mathcal{B}(\mathcal{H})$ such that $U^*C_1U = C_2$. It is well known that the set $\mathcal{U}(\mathcal{H})$ of all unitary operators on $\mathcal{H}$ is path connected. Then there exists continuous $V : [0,1] \to \mathcal{U}(\mathcal{H})$ such that $V(0) = I$ and $V(1) = U$. Thus $\{V(\lambda)^*C_1V(\lambda); \lambda \in [0,1]\}$ is a path of conjugations from C_1 to C_2.

Proof of Theorem 2.1 The sufficiency is obvious.

"$\Rightarrow$". Set

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}.$$

Since $\dim \mathcal{H} = \dim \mathcal{H}^{(2)} = \infty$, in view of Corollary 2.1, there exists a unitary $U : \mathcal{H} \to \mathcal{H}^{(2)}$ such that $U\mathcal{S}_CU^* = \mathcal{S}_D$. Thus it suffices to prove the conclusion for $\mathcal{S}_D$.

Assume that $\mathcal{J}$ is a Jordan ideal of $\mathcal{S}_D$. Set

$$\mathcal{J}_0 = \left\{A \in \mathcal{B}(\mathcal{H}) : \exists A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H}) \text{ with } \begin{bmatrix} A & A_1 \\ A_2 & A_3 \end{bmatrix} \in \mathcal{J}\right\}.$$

Claim 1 $\mathcal{J}_0$ is a Jordan ideal of $\mathcal{B}(\mathcal{H})$.

It is obvious that $\mathcal{J}_0$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. Now assume that $A \in \mathcal{J}_0$ and $B \in \mathcal{B}(\mathcal{H})$. So, by Lemma 2.1(i), there exists an element $T \in \mathcal{J}$ with the form

$$T = \begin{bmatrix} A & E \\ F & A^t \end{bmatrix},$$

where $E = E^t$ and $F = F^t$. By Lemma 2.1(ii), we have

$$\begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix} \in \mathcal{J}. \tag{2.1}$$

Note that

$$\begin{bmatrix} B & 0 \\ 0 & B^t \end{bmatrix} \in \mathcal{S}_D.$$

We have

$$\begin{bmatrix} A \circ B & 0 \\ 0 & A^t \circ B^t \end{bmatrix} \in \mathcal{J}.$$

Thus $A \circ B \in \mathcal{J}_0$. This proves Claim 1.

Moreover, in view of (2.1), it follows from Lemma 2.1(ii) that $A^t \in \mathcal{J}_0$. Thus $\mathcal{J}_0$ is invariant under the map $X \mapsto X^t$.

By [10, Theorem 3], $\mathcal{J}_0$ is an ideal of $\mathcal{B}(\mathcal{H})$. It follows immediately that $M_2(\mathcal{J}_0)$ is an ideal of $\mathcal{B}(\mathcal{H}^{(2)})$, where

$$M_2(\mathcal{J}_0) = \left\{ \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} : X, Y, Z, W \in \mathcal{J}_0\right\}.$$

Thus it remains to check that $\mathcal{J} = M_2(\mathcal{J}_0) \cap \mathcal{S}_D$.

Claim 2 $M_2(\mathcal{J}_0) \cap \mathcal{S}_D = \mathcal{E}$, where

$$\mathcal{E} = \left\{ \begin{bmatrix} A & E+E^t \\ F+F^t & A^t \end{bmatrix} : A, E, F \in \mathcal{J}_0 \right\}.$$

Note that an operator X lies in $\mathcal{J}_0$ if and only if $X^t \in \mathcal{J}_0$. Then, by Lemma 2.1(i), the inclusion $\mathcal{E} \subset M_2(\mathcal{J}_0) \cap \mathcal{S}_D$ is obvious. Conversely, if $A, B, E, F \in \mathcal{J}_0$ and

$$\begin{bmatrix} A & E \\ F & B \end{bmatrix} \in \mathcal{S}_D.$$

Then by Lemma 2.1(i), $B = A^t$, $E = E^t$ and $F = F^t$. It follows that $\mathcal{E} = M_2(\mathcal{J}_0) \cap \mathcal{S}_D$.

Now it remains to check that $\mathcal{J} = \mathcal{E}$.

"$\mathcal{J} \subset \mathcal{E}$". Choose an element $T \in \mathcal{J}$ and assume that

$$T = \begin{bmatrix} A & E \\ F & A^t \end{bmatrix},$$

where $E = E^t$ and $F = F^t$. Thus $A \in \mathcal{J}_0$, and by Lemma 2.1(ii), we have

$$\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \in \mathcal{J}, \quad \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} \in \mathcal{J}.$$

This implies that $E, F \in \mathcal{J}_0$. So

$$T = \begin{bmatrix} A & E \\ F & A^t \end{bmatrix} = \begin{bmatrix} A & \frac{E}{2} + \frac{E^t}{2} \\ \frac{F}{2} + \frac{F^t}{2} & A^t \end{bmatrix} \in \mathcal{E}.$$

"$\mathcal{J} \supset \mathcal{E}$". Choose $A, E, F \in \mathcal{J}_0$. We shall prove that

$$T = \begin{bmatrix} A & E+E^t \\ F+F^t & A^t \end{bmatrix} \in \mathcal{J}.$$

If $X \in \mathcal{J}_0$, then one can see from the proof of Claim 1 that $X \oplus X^t \in \mathcal{J}$. So we have

$$\begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix}, \begin{bmatrix} E & 0 \\ 0 & E^t \end{bmatrix}, \begin{bmatrix} F & 0 \\ 0 & F^t \end{bmatrix} \in \mathcal{J}.$$

By Lemma 2.1(ii), we have $T \in \mathcal{J}$. This ends the proof.

Corollary 2.3 *Let C be a conjugation on $\mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$ is Jordan isomorphic to a Jordan subalgebra of $\mathcal{S}_C$.*

Proof Set

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}.$$

Thus D is a conjugation on $\mathcal{H}^{(2)}$. By Corollary 2.1, $\mathcal{S}_C$ and $\mathcal{S}_D$ are unitarily equivalent. Hence it suffices to prove that $\mathcal{B}(\mathcal{H})$ is Jordan isomorphic to a Jordan subalgebra of $\mathcal{S}_D$. Define

$$\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{S}_D, \quad X \mapsto X \oplus X^t.$$

It is easy to see that ϕ is a linear isometry satisfying $\phi(X^*) = \phi(X)^*$ and $\phi(X \circ Y) = \phi(X) \circ \phi(Y)$ for $X, Y \in \mathcal{B}(\mathcal{H})$. Thus ϕ induces a Jordan isomorphism between $\mathcal{B}(\mathcal{H})$ and $\phi(\mathcal{B}(\mathcal{H}))$.

We denote by $\mathcal{F}(\mathcal{H})$ the set of all finite-rank operators on $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ the set of all compact operators on $\mathcal{H}$.

Corollary 2.4 *If $\mathcal{J}$ is a nontrivial Jordan ideal of $\mathcal{S}_C$, then*

$$[\mathcal{S}_C \cap \mathcal{F}(\mathcal{H})] \subset \mathcal{J} \subset [\mathcal{S}_C \cap \mathcal{K}(\mathcal{H})].$$

Proof Note that each nontrivial ideal $\mathcal{I}$ of $\mathcal{B}(\mathcal{H})$ satisfies $\mathcal{F}(\mathcal{H}) \subset \mathcal{I} \subset \mathcal{K}(\mathcal{H})$. The desired result follows readily from Theorem 2.1.

Corollary 2.5 *$\mathcal{S}_C \cap \mathcal{K}(\mathcal{H})$ is the unique nontrivial norm-closed Jordan ideal of $\mathcal{S}_C$.*

Remark 2.1 The result of Theorem 2.1 still holds in the case that $\dim \mathcal{H} < \infty$. In fact, if $\dim \mathcal{H} < \infty$, then one can prove that neither $\mathcal{B}(\mathcal{H})$ nor $\mathcal{S}_C$ has a nontrivial Jordan ideal. This shows that $\mathcal{S}_C$ and $\mathcal{B}(\mathcal{H})$ have the same Jordan ideal structure for separable complex Hilbert space $\mathcal{H}$.

2.2 Schatten p-classes

The Schatten p-class of compact operators on $\mathcal{H}$ is denoted by $\mathcal{B}_p(\mathcal{H})$, $1 \le p < \infty$. It is well known that $\mathcal{B}_p(\mathcal{H})$ is a Banach space under p-norm $\|\cdot\|_p$ and $\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{B}_p(\mathcal{H})$, where $\mathcal{F}(\mathcal{H})$ denotes the collection of finite-rank operators in $\mathcal{B}(\mathcal{H})$. Moreover, $\mathcal{B}_p(\mathcal{H})$ is the dual of $\mathcal{B}_q(\mathcal{H})$ for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and the duality is achieved by the map $(A, B) \mapsto \mathrm{tr}(AB)$. The reader is referred to [31] or [34] for more details. The aim of this subsection is to prove analogues of these facts in $\mathcal{S}_C$.

For convenience, we denote $\mathcal{B}_0(\mathcal{H}) = \mathcal{K}(\mathcal{H})$. For $p \in \{0\} \cup [1, \infty)$, we denote $\mathcal{S}_{C,p} = \mathcal{S}_C \cap \mathcal{B}_p(\mathcal{H})$. Still $\mathcal{S}_{C,p}$ is a Banach space under the norm $\|\cdot\|_p$. Note that $\|\cdot\|_0 = \|\cdot\|$.

Proposition 2.1 *Let $p \in \{0\} \cup [1, \infty)$. Then, given $T \in \mathcal{S}_{C,p}$ and $\varepsilon > 0$, there exists $F \in \mathcal{S}_C \cap \mathcal{F}(\mathcal{H})$ such that $\|T - F\|_p < \varepsilon$.*

Proof Choose an operator $T \in \mathcal{S}_{C,p}$. Then $T \in \mathcal{B}_p(\mathcal{H})$ and there exist finite-rank operators $\{F_n\} \subset \mathcal{B}(\mathcal{H})$ such that $\|T - F_n\|_p \to 0$. For each $n \ge 1$, note that $\frac{1}{2}(F_n + CF_n^*C)$ is of finite rank lying in $\mathcal{S}_C$ and

$$\begin{aligned}\Big\|T - \frac{1}{2}(F_n + CF_n^*C)\Big\|_p &= \Big\|\frac{1}{2}(T + CT^*C) - \frac{1}{2}(F_n + CF_n^*C)\Big\|_p \\ &\le \Big\|\frac{1}{2}(T - F_n)\Big\|_p + \Big\|\frac{1}{2}C(T - F_n)^*C\Big\|_p \\ &= \|T - F_n\|_p.\end{aligned}$$

Thus T can be approximated in p-norm $\|\cdot\|_p$ by finite-rank operators in $\mathcal{S}_C$.

It was found in [43] that $\mathcal{S}_C$ has a topological complement. In fact, the set of skew-symmetric operators relative to C,

$$\mathcal{O}_C := \{X \in \mathcal{B}(\mathcal{H}) : CXC = -X^*\}$$

is a topological complement of $\mathcal{S}_C$; that is, $\mathcal{S}_C + \mathcal{O}_C = \mathcal{B}(\mathcal{H})$ and $\mathcal{S}_C \cap \mathcal{O}_C = \{0\}$. $\mathcal{O}_C$ is just the symplectic type Cartan factor (see [6]). It was shown in [43] that $\mathcal{O}_C$ is Roberts orthogonal to $\mathcal{S}_C$. Recall that two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be Roberts orthogonal, if $\|A - \lambda B\| = \|A + \lambda B\|$ for all complex numbers λ.

Lemma 2.3 $\mathcal{S}_C + \mathcal{K}(\mathcal{H})$ *is a proper, norm-closed Jordan subalgebra of* $\mathcal{B}(\mathcal{H})$.

Proof Clearly, $\mathcal{S}_C + \mathcal{K}(\mathcal{H})$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ and closed under the Jordan product. Also each Fredholm operator in $\mathcal{S}_C + \mathcal{K}(\mathcal{H})$ has an index 0. Hence it suffices to prove that $\mathcal{S}_C + \mathcal{K}(\mathcal{H})$ is norm-closed.

Assume that $\{A_n + K_n\}$ is a Cauchy sequence, where $\{A_n\} \subset \mathcal{S}_C$ and $\{K_n\} \subset \mathcal{K}(\mathcal{H})$. Set $K_n^+ = \frac{1}{2}(K_n + CK_n^*C)$ and $K_n^- = \frac{1}{2}(K_n - CK_n^*C)$. Then $K_n^+ \in \mathcal{S}_C$ and $K_n^- \in \mathcal{O}_C \cap \mathcal{K}(\mathcal{H})$. By [43, Theorem 2.1], we have $\|E\| \leq \|E + F\|$ for any $E \in \mathcal{O}_C$ and any $F \in \mathcal{S}_C$. Then

$$\|K_n^- - K_m^-\| \leq \|(K_n^- - K_m^-) + (K_n^+ - K_m^+ + A_n - A_m)\| \leq \|(A_n + K_n) - (A_m + K_m)\|.$$

Thus $\{K_n^-\}$ is a Cauchy sequence and converges to a compact operator K^-. It follows that $\{A_n + K_n^+\}$ is a Cauchy sequence in $\mathcal{S}_C$ and converges to an operator $A \in \mathcal{S}_C$. Therefore we conclude that $A_n + K_n \to A + K^- \in \mathcal{S}_C + \mathcal{K}(\mathcal{H})$.

The aim of the rest of this subsection is to prove the following result which exhibits the dual relation among $\mathcal{S}_C$ and $\mathcal{S}_{C,p}$ ($p \in \{0\} \cup [1, \infty)$).

Proposition 2.2 (i) $(\mathcal{S}_{C,1}, \|\cdot\|_1)$ *is isometrically isomorphic to the dual of* $(\mathcal{S}_{C,0}, \|\cdot\|)$.

(ii) $\mathcal{S}_C$ *is isometrically isomorphic to the dual of* $(\mathcal{S}_{C,1}, \|\cdot\|_1)$.

(iii) $(\mathcal{S}_{C,q}, \|\cdot\|_q)$ *is isometrically isomorphic to the dual of* $(\mathcal{S}_{C,p}, \|\cdot\|_p)$, *where* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

In order to prove the preceding proposition, we need to make some preparation.

Given a complex matrix A, we denote by A^{T} the transpose of A.

Lemma 2.4 *Let* n *be a positive integer and* $M_n(\mathbb{C})$ *be the collection of* $n \times n$ *complex matrices. If* $A, B \in M_n(\mathbb{C})$ *with* $A = A^{\mathrm{T}}$ *and* $B = -B^{\mathrm{T}}$, *then* $\mathrm{tr}(AB) = 0$, *where* $\mathrm{tr}(\cdot)$ *is the trace function.*

Proof Note that $\mathrm{tr}(AB) = \mathrm{tr}(AB)^{\mathrm{T}} = \mathrm{tr}(B^{\mathrm{T}}A^{\mathrm{T}}) = -\mathrm{tr}(BA) = -\mathrm{tr}(AB)$. So $\mathrm{tr}(AB) = 0$.

Corollary 2.6 *Let* C *be a conjugation on* $\mathcal{H}$. *Assume that* $A \in \mathcal{S}_C$ *and* $B \in \mathcal{O}_C$. *If* (i) $A \in \mathcal{B}_1(\mathcal{H})$ *or* (ii)$A \in \mathcal{B}_p(\mathcal{H})$ *and* $B \in \mathcal{B}_q(\mathcal{H})$, *where* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then* $\mathrm{tr}(AB) = 0$.

Proof We just give the proof in the case (i). The proof for the case (ii) is similar.

Since C is a conjugation on $\mathcal{H}$, by [15, Lemma 2.11], there exists an orthonormal basis $\{e_n\}$ such that $Ce_n = e_n$ for all n. For each $n \geq 1$, denote by P_n the projection of $\mathcal{H}$ onto $\vee\{e_i : 1 \leq i \leq n\}$.

Note that $P_n \to I$ in the strong operator topology. It follows that $\lim\limits_n \|P_nAP_n - A\|_1 = 0$ and furthermore

$$\|P_nAP_nB - AB\|_1 \leq \|P_nAP_n - A\|_1 \cdot \|B\| \to 0$$

as $n \to \infty$. Thus $\operatorname{tr}(AB) = \lim_n \operatorname{tr}(P_nAP_nB)$. It suffices to prove that $\operatorname{tr}(P_nAP_nB) = 0$ for all n.

For each n, assume that

$$A = \begin{bmatrix} A_n & * \\ * & * \end{bmatrix} \begin{matrix} \operatorname{ran} P_n \\ \operatorname{ran}(I - P_n) \end{matrix}, \quad B = \begin{bmatrix} B_n & * \\ * & * \end{bmatrix} \begin{matrix} \operatorname{ran} P_n \\ \operatorname{ran}(I - P_n) \end{matrix}.$$

It follows that $\operatorname{tr}(P_nAP_nB) = \operatorname{tr}(A_nB_n)$. For $1 \le i, j \le n$, note that

$$\begin{aligned} \langle A_n e_i, e_j\rangle &= \langle Ae_i, e_j\rangle = \langle Ae_i, Ce_j\rangle \\ &= \langle e_i, A^*Ce_j\rangle = \langle e_i, CAe_j\rangle \\ &= \langle Ae_j, Ce_i\rangle = \langle Ae_j, e_i\rangle \\ &= \langle A_n e_j, e_i\rangle \end{aligned}$$

and similarly that $\langle B_n e_i, e_j\rangle = -\langle B_n e_j, e_i\rangle$. Thus, relative to $\{e_1, e_2, \cdots, e_n\}$, A_n admits a symmetric matrix representation and B_n admits a skew symmetric matrix representation (that is, $R = -R^{\mathrm{T}}$). By Lemma 2.4, $\operatorname{tr}(P_nAP_nB) = \operatorname{tr}(A_nB_n) = 0$. Therefore we conclude that $\operatorname{tr}(AB) = 0$.

Given a Banach space $\mathcal{X}$, we let $\mathcal{X}'$ denote its dual.

Proof of Proposition 2.2 (i) For $K \in \mathcal{S}_{C,1}$, define $\phi_K : \mathcal{S}_{C,0} \to \mathbb{C}$ as

$$\phi_K(X) = \operatorname{tr}(XK), \quad \forall X \in \mathcal{S}_{C,0}.$$

Clearly, $|\phi_K(X)| = |\operatorname{tr}(XK)| \le \|K\|_1 \cdot \|X\|$. Thus $\phi_K \in (\mathcal{S}_{C,0})'$.

It suffices to prove that the map $\Phi : K \mapsto \phi_K$ is an isometric isomorphism of $\mathcal{S}_{C,1}$ onto $(\mathcal{S}_{C,0})'$. Clearly, Φ is linear. It remains to check that Φ is isometric and surjective.

Step 1 Φ is isometric.

Fix a $K \in \mathcal{S}_{C,1}$. ϕ_K can be extended to the linear functional $\widetilde{\phi}_K$ on $\mathcal{K}(\mathcal{H})$ defined by

$$\widetilde{\phi}_K(X) = \operatorname{tr}(XK), \quad \forall X \in \mathcal{K}(\mathcal{H}).$$

Then, by [7, Theorem 19.1], $\|\phi_K\| \le \|\widetilde{\phi}_K\| = \|K\|_1$.

For any $Y \in \mathcal{K}(\mathcal{H})$, denote $Y_1 = \frac{1}{2}(Y + CY^*C)$ and $Y_2 = \frac{1}{2}(Y - CY^*C)$. Note that $\|Y_1\| \le \|Y\|$, $Y_1 \in \mathcal{S}_{C,0}$ and $Y_2 \in \mathcal{O}_C \cap \mathcal{K}(\mathcal{H})$. By Corollary 2.6,

$$\begin{aligned} |\widetilde{\phi}_K(Y)| &= |\operatorname{tr}(KY_1 + KY_2)| = |\operatorname{tr}(KY_1)| \\ &= |\phi_K(Y_1)| \le \|\phi_K\| \cdot \|Y_1\| \\ &\le \|\phi_K\| \cdot \|Y\|. \end{aligned}$$

Since $Y \in \mathcal{K}(\mathcal{H})$ was arbitrary operator in $\mathcal{K}(\mathcal{H})$, we deduce that $\|\widetilde{\phi}_K\| \le \|\phi_K\|$. Furthermore, we obtain $\|\phi_K\| = \|\widetilde{\phi}_K\| = \|K\|_1$. This shows that Φ is isometric.

Step 2 Φ is surjective.

Assume that l is a bounded linear functional on $\mathcal{S}_{C,0}$. Since $\mathcal{S}_{C,0}$ is a closed subspace of $\mathcal{K}(\mathcal{H})$, l admits an extension $\widetilde{l}$ to $\mathcal{K}(\mathcal{H})$. Then, by [7, Theorem 19.1], there exists an operator

$K \in \mathcal{B}_1(\mathcal{H})$ such that $\widetilde{l}(X) = \mathrm{tr}(XK)$ for $X \in \mathcal{K}(\mathcal{H})$. Denote $K_1 = \frac{1}{2}(K + CK^*C)$ and $K_2 = \frac{1}{2}(K - CK^*C)$. Then $K_1 \in \mathcal{S}_{C,1}$ and $K_2 \in \mathcal{O}_C \cap \mathcal{B}_1(\mathcal{H})$. For each $X \in \mathcal{S}_{C,0}$, we have

$$l(X) = \widetilde{l}(X) = \mathrm{tr}(XK) = \mathrm{tr}(XK_1) + \mathrm{tr}(XK_2) = \mathrm{tr}(XK_1) = \phi_{K_1}(X),$$

which means that $l = \phi_{K_1}$. Thus we conclude that Φ is surjective.

(ii) For $K \in \mathcal{S}_C$, define $\psi_K : \mathcal{S}_{C,1} \to \mathbb{C}$ as

$$\psi_K(X) = \mathrm{tr}(XK), \quad X \in \mathcal{S}_{C,1}.$$

Clearly, $|\psi_K(X)| = |\mathrm{tr}(XK)| \le \|K\| \cdot \|X\|_1$. Thus $\psi_K \in (\mathcal{S}_{C,1})'$.

It suffices to prove that the map $\Psi : K \mapsto \psi_K$ is an isometric isomorphism of $\mathcal{S}_C$ onto $(\mathcal{S}_{C,1})'$. Clearly, Ψ is linear. It remains to check that Ψ is isometric and surjective.

Step 1 Ψ is isometric.

Fix a $K \in \mathcal{S}_C$. Obviously, ψ_K can be extended to the linear functional $\widetilde{\psi}_K$ on $\mathcal{B}_1(\mathcal{H})$ defined by

$$\widetilde{\psi}_K(X) = \mathrm{tr}(XK), \quad X \in \mathcal{B}_1(\mathcal{H}).$$

Then, by [7, Theorem 19.2], $\|\psi_K\| \le \|\widetilde{\psi}_K\| = \|K\|$.

For any $Y \in \mathcal{B}_1(\mathcal{H})$, denote $Y_1 = \frac{1}{2}(Y + CY^*C)$ and $Y_2 = \frac{1}{2}(Y - CY^*C)$. Note that $\|Y_1\|_1 \le \|Y\|_1$, $Y_1 \in \mathcal{S}_{C,1}$ and $Y_2 \in \mathcal{O}_C \cap \mathcal{B}_1(\mathcal{H})$. By Corollary 2.6,

$$\begin{aligned} |\widetilde{\psi}_K(Y)| &= |\mathrm{tr}(KY_1 + KY_2)| = |\mathrm{tr}(KY_1)| \\ &= |\psi_K(Y_1)| \le \|\psi_K\| \cdot \|Y_1\|_1 \\ &\le \|\psi_K\| \cdot \|Y\|_1. \end{aligned}$$

Since Y was arbitrary operator in $\mathcal{B}_1(\mathcal{H})$, we deduce that $\|\widetilde{\psi}_K\| \le \|\psi_K\|$. Furthermore, we obtain $\|\psi_K\| = \|\widetilde{\psi}_K\| = \|K\|$. This shows that Ψ is isometric.

Step 2 Ψ is surjective.

Assume that l is a bounded linear functional on $\mathcal{S}_{C,1}$. Since $\mathcal{S}_{C,1}$ is a closed subspace of $\mathcal{B}_1(\mathcal{H})$, l admits an extension $\widetilde{l}$ to $\mathcal{B}_1(\mathcal{H})$. Then, by [7, Theorem 19.2], there exists an operator $K \in \mathcal{B}(\mathcal{H})$ such that $\widetilde{l}(X) = \mathrm{tr}(XK)$ for $X \in \mathcal{B}_1(\mathcal{H})$. Denote $K_1 = \frac{1}{2}(K + CK^*C)$ and $K_2 = \frac{1}{2}(K - CK^*C)$. Then $K_1 \in \mathcal{S}_C$ and $K_2 \in \mathcal{O}_C$. For each $X \in \mathcal{S}_{C,1}$, we have

$$l(X) = \widetilde{l}(X) = \mathrm{tr}(XK) = \mathrm{tr}(XK_1) + \mathrm{tr}(XK_2) = \mathrm{tr}(XK_1) = \psi_{K_1}(X),$$

which means that $l = \psi_{K_1}$. Thus we conclude that Ψ is surjective.

(iii) The proof follows similar lines as those of (i) and (ii), and is omitted.

2.3 C^*-algebras contained in $\mathcal{S}_C$

This subsection aims to characterize the C^*-algebras contained in $\mathcal{S}_C$ and the C^*-algebras generated by $\mathcal{S}_C$. This helps to develop operator theory in $\mathcal{S}_C$.

For $T \in \mathcal{B}(\mathcal{H})$, we denote by $J^*(T)$ the Jordan operator algebra generated by T, T^* and the identity operator I, and denote by $W^*(T)$ the von Neumann algebra generated by T.

The first result shows that $\mathcal{S}_C$ contains no noncommutative C^*-algebra.

Proposition 2.3 *Let C be a conjugation on $\mathcal{H}$ and $T \in \mathcal{S}_C$. Then the following are equivalent*:

(i) $C^*(T) \subset \mathcal{S}_C$;

(ii) $W^*(T) \subset \mathcal{S}_C$;

(iii) $|T| \in \mathcal{S}_C$;

(iv) T *is normal*;

(v) $C^*(T) = J^*(T)$.

Proof Since $\mathcal{S}_C$ is closed in the weak operator topology, the implication (i)⇒(ii) is obvious.

(ii)⇒(iii). This is obvious.

(iii)⇒(iv). Since $|T| \in \mathcal{S}_C$, we have $C|T|C = |T|$ and hence $C|T|^2C = |T|^2 = T^*T$. On the other hand,

$$C|T|^2C = CT^*TC = (CT^*C)(CTC) = TT^*.$$

Thus $T^*T = TT^*$.

(iv)⇒(v). Since $C^*(T)$ is an associative algebra and hence a Jordan algebra containing T, T^*, it follows that $J^*(T) \subset C^*(T)$. Note that T is normal. Thus $C^*(T)$ is the closed linear span of $\{T^mT^{*n} : m, n \geq 0\}$. Noting that

$$T = T \circ I, \quad T^2 = T \circ T, \quad T^3 = (T \circ T) \circ T, \quad \cdots,$$

we have $T^m, T^{*n} \in J^*(T)$. Also, one can check

$$T^mT^{*n} = T^m \circ T^{*n}$$

since $T^mT^{*n} = T^{*n}T^m$. This shows that $C^*(T) \subset J^*(T)$. We conclude that $C^*(T) = J^*(T)$.

(v)⇒(i). Note that $T, T^* \in \mathcal{S}_C$. Thus $J^*(T)$ is a Jordan subalgebra of $\mathcal{S}_C$. It follows that $C^*(T) \subset \mathcal{S}_C$.

Remark 2.2 The Jordan product $\circ$ on $\mathcal{S}_C$ is not associative. In fact, choose a non-normal $T \in \mathcal{S}_C$. Thus $T^* \in \mathcal{S}_C$. By Proposition 2.3, $|T| \notin \mathcal{S}_C$ and hence $|T|^2 = T^*T \notin \mathcal{S}_C$.

By Proposition 2.3, a good functional calculus is permitted in $\mathcal{S}_C$. This will have many useful corollaries.

Corollary 2.7 *Let C be a conjugation on $\mathcal{H}$ and $\mathcal{A}$ be a JC*-subalgebra of $\mathcal{S}_C$. If $T \in \mathcal{A}$, then $C^*(T) \subset \mathcal{A}$ if and only if T is normal.*

Corollary 2.8 *If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with $\mathcal{A} \subset \mathcal{S}_C$, then $\mathcal{A} \subset W^*(N)$ for some normal operator $N \in \mathcal{S}_C$.*

Corollary 2.9 *The selfadjoint elements of $\mathcal{S}_C$ with finite spectra are norm dense in the selfadjoint elements of $\mathcal{S}_C$.*

Since $\mathcal{S}_C \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{S}_C$ is not an associative algebra, it is natural to determine the associative algebra generated by $\mathcal{S}_C$.

Proposition 2.4 *$\mathcal{B}(\mathcal{H})$ is the C^*-algebra generated by $\mathcal{S}_C$.*

Proof Set
$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}.$$
Thus D is a conjugation on $\mathcal{H}^{(2)}$. It suffices to prove that each operator T on $\mathcal{H}^{(2)}$ lies in the C^*-algebra $C^*(\mathcal{S}_D)$ generated by $\mathcal{S}_D$.

Note that
$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathcal{S}_D.$$
It follows immediately that $M_2(\mathbb{C}I) \subset C^*(\mathcal{S}_D)$. For any $X \in \mathcal{B}(\mathcal{H})$, $X \oplus CX^*C \in \mathcal{S}_D$. One can check that
$$\begin{bmatrix} X & 0 \\ 0 & CX^*C \end{bmatrix} \circ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \in C^*(\mathcal{S}_D)$$
and
$$\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \in C^*(\mathcal{S}_D).$$
Since $C^*(\mathcal{S}_D)$ is selfadjoint, it follows that
$$\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} \in C^*(\mathcal{S}_D)$$
and
$$\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix} \in C^*(\mathcal{S}_D).$$
Therefore we conclude that $\mathcal{B}(\mathcal{H}^{(2)}) = C^*(\mathcal{S}_D)$.

2.4 Jordan automorphisms

A map $\varphi : \mathcal{S}_C \to \mathcal{S}_C$ is called a Jordan automorphism of $\mathcal{S}_C$ if φ is linear, bijective and
$$\varphi(X^*) = \varphi(X)^*, \quad \varphi(X \circ Y) = \varphi(X) \circ \varphi(Y)$$
for all $X, Y \in \mathcal{S}_C$.

The main result of this subsection is the following theorem which determines Jordan automorphisms of $\mathcal{S}_C$.

Theorem 2.2 *A map $\varphi : \mathcal{S}_C \to \mathcal{S}_C$ is a Jordan automorphism of $\mathcal{S}_C$ if and only if there exists a unitary operator $V \in \mathcal{B}(\mathcal{H})$ with $CV = VC$ such that $\varphi(X) = VXV^*$ for all $X \in \mathcal{S}_C$.*

To give the proof of Theorem 2.2, we make some preparation.

Lemma 2.5 *Let C and D be conjugations on $\mathcal{H}$. Then the following are equivalent*:
(i) $\mathcal{S}_C = \mathcal{S}_D$;
(ii) $\mathcal{S}_C \subset \mathcal{S}_D$;
(iii) $\mathcal{S}_C \supset \mathcal{S}_D$;
(iv) $C = \alpha D$ *for some unimodular number* $\alpha \in \mathbb{C}$.

Proof It suffices to prove (ii)$\Rightarrow$(iv).

"(ii)$\Rightarrow$(iv)". Assume that $\{e_i\}_{i\geq 1}$ is an orthonormal basis (ONB for short) of $\mathcal{H}$ such that $Ce_i = e_i$ for all i. Easy to see $e_i \otimes e_i \in \mathcal{S}_C$ for $i \geq 1$. Thus $e_i \otimes e_i \in \mathcal{S}_D$. That is,

$D(e_i \otimes e_i)D = e_i \otimes e_i$. Since $D(e_i \otimes e_i)D = (De_i) \otimes (De_i)$, there exists a unimodular number $\alpha_i \in \mathbb{C}$ such that $De_i = \alpha_i e_i$.

Now it remains to show that $\alpha_i = \alpha_1$ for all $i \geq 2$. Assume that $i \geq 2$. Note that $X = e_1 \otimes e_i + e_i \otimes e_1 \in \mathcal{S}_C$. Thus $DXD = X^* = X$. In particular, $DXDe_i = Xe_i$. Since $Xe_i = e_1$ and $DXDe_i = DX(\alpha_i e_i) = \alpha_1 \overline{\alpha_i} e_1$, it follows that $e_1 = \alpha_1 \overline{\alpha_i} e_1$, that is, $\alpha_1 = \alpha_i$.

Lemma 2.6 *Let C be a conjugation on $\mathcal{H}$ and U be a unitary operator on $\mathcal{H}$. For $X \in \mathcal{B}(\mathcal{H})$, $\psi(X) = UXU^*$. Then $\psi(\mathcal{S}_C) = \mathcal{S}_C$ if and only if C commutes with αU for some unimodular number $\alpha \in \mathbb{C}$.*

Proof The sufficiency is obvious. We need only prove the necessity.

"⇒". Since U is unitary, it is easy to check that $\psi(\mathcal{S}_C) = \mathcal{S}_{UCU^*}$. So $\mathcal{S}_C = \mathcal{S}_{UCU^*}$. By Lemma 2.5, there exists a unimodular number $\beta \in \mathbb{C}$ such that $UCU^* = \beta C$. Thus $UC = \beta CU$. Assume that $\alpha \in \mathbb{C}$ satisfies $\alpha^2 = \overline{\beta}$. Then

$$(\alpha U)C = \alpha(UC) = (\alpha\beta)CU = \overline{\alpha}CU = C(\alpha U).$$

This ends the proof.

If C is a conjugation on $\mathcal{H}$, then we denote by $\mathcal{F}_C$ the collection of all finite-rank operators in $\mathcal{S}_C$.

Lemma 2.7 *Let φ be a Jordan automorphism of $\mathcal{S}_C$. Then*

(i) $\varphi(\mathcal{F}_C) = \mathcal{F}_C$;

(ii) $\varphi(X^2) = \varphi(X)^2$ *for all* $X \in \mathcal{S}_C$;

(iii) *if* $X \in \mathcal{S}_C$ *is positive, then so is* $\varphi(X)$;

(iv) *if* $X \in \mathcal{S}_C$ *is a projection of rank one, then so is* $\varphi(X)$;

(v) *if* $X \in \mathcal{S}_C$ *is selfadjoint, then* $\|\varphi(X)\| = \|X\|$.

Proof (i) By Corollary 2.4, $\mathcal{F}_C$ is the smallest nonzero Jordan ideal of $\mathcal{S}_C$. It follows immediately that $\varphi(\mathcal{F}_C) = \mathcal{F}_C$.

(ii) For $X \in \mathcal{B}(\mathcal{H})$, note that $X \circ X = X^2$. Then the result follows readily.

(iii) Since X is positive, it follows that $X = Y^2$ for some positive $Y \in \mathcal{B}(\mathcal{H})$. In view of Proposition 2.3, it can be required that $Y \in \mathcal{S}_C$. Then $\varphi(X) = \varphi(Y^2) = \varphi(Y)^2$. Note that $\varphi(Y) = \varphi(Y^*) = \varphi(Y)^*$. Thus $\varphi(X)$ is positive.

(iv) From (ii) and (iii), one can see that $\varphi(X)$ is a projection. It remains to check that $\operatorname{rank}\varphi(X) = 1$. Note that φ maps positive operators to positive operators. Then φ maps minimal projections to minimal projections. So the desired result follows readily.

(v) Denote by $C^*(X)$ the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by X and the identity I. Then $C^*(X)$ is commutative and $C^*(X) \subset \mathcal{S}_C$. One can check that $\varphi|_{C^*(X)}$ is a faithful representation of $C^*(X)$, and hence is isometric.

Proposition 2.5 *Let φ be a Jordan automorphism of $\mathcal{S}_C$ and $\{e_i\}_{i\geq 1}$ be an* ONB *of $\mathcal{H}$ such that $Ce_i = e_i$ for all i. Then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\varphi(X) = UXU^*$ for all $X \in \Theta$, where*

$$\Theta = \{X \in \mathcal{S}_C : \exists\, n \geq 1 \text{ such that } \langle Xe_i, e_j\rangle = 0 \text{ whenever } i + j \geq n\}.$$

Proof For $i \geq 1$, set $P_i = e_i \otimes e_i$ and $Q_i = \varphi(P_i)$. Clearly, P_i is a projection of rank one. In view of Lemma 2.7, Q_i is a projection of rank one. Then there exists a unit vector f_i such that $Q_i = f_i \otimes f_i$. Using Lemma 2.7 again, one can see that $\{f_i\}_{i\geq 1}$ is an ONB of $\mathcal{H}$.

Claim There exist unimodular numbers $\{\alpha_i\}_{i\geq 1}$ such that $\varphi(e_i \otimes e_j + e_j \otimes e_i) = (\alpha_i f_i) \otimes (\alpha_j f_j) + (\alpha_j f_j) \otimes (\alpha_i f_i)$.

For $i, j \geq 1$ with $i < j$, denote $E_{i,j} = e_i \otimes e_j + e_j \otimes e_i$. Then $E_{i,j}$ lies in $\mathcal{S}_C$ and is selfadjoint. Denote $F_{i,j} = \varphi(E_{i,j})$. One can see that $F_{i,j}$ is selfadjoint and, by Lemma 2.7(v), $\|F_{i,j}\| = 1$.

Now fix $i, j \geq 1$ with $i < j$. Note that $Q_k \circ F_{i,j} = \varphi(P_k \circ E_{i,j}) = 0$ whenever $k \notin \{i, j\}$. So

$$F_{i,j} = a_{i,j} f_i \otimes f_i + b_{i,j} f_j \otimes f_i + \overline{b_{i,j}} f_i \otimes f_j + c_{i,j} f_j \otimes f_j$$

for some $a_{i,j}, b_{i,j}, c_{i,j} \in \mathbb{C}$. Thus

$$Q_i \circ F_{i,j} = a_{i,j} f_i \otimes f_i + \frac{b_{i,j}}{2} f_j \otimes f_i + \frac{\overline{b_{i,j}}}{2} f_i \otimes f_j.$$

On the other hand, note that

$$Q_i \circ F_{i,j} = \varphi(P_i \circ E_{i,j}) = \frac{1}{2}\varphi(E_{i,j}) = \frac{1}{2}F_{i,j}.$$

It follows that $a_{i,j} = 0 = c_{i,j}$ and $|b_{i,j}| = 1$. Hence

$$F_{i,j} = b_{i,j} f_j \otimes f_i + \overline{b_{i,j}} f_i \otimes f_j.$$

Set $\alpha_1 = 1$ and $\alpha_i = \prod_{k=1}^{i-1} b_{k,k+1}$ for $i \geq 2$. Set $g_i = \alpha_i f_i$ for $i \geq 1$ and $G_{i,j} = g_i \otimes g_j + g_j \otimes g_i$ for $i > j \geq 1$.

For $i \geq 1$,

$$\begin{aligned} G_{i,i+1} &= g_i \otimes g_{i+1} + g_{i+1} \otimes g_i \\ &= (\alpha_i f_i) \otimes (\alpha_{i+1} f_{i+1}) + (\alpha_{i+1} f_{i+1}) \otimes (\alpha_i f_i) \\ &= \overline{b_{i,i+1}}(f_i \otimes f_{i+1}) + b_{i,i+1}(f_{i+1} \otimes f_i) = \varphi(E_{i,i+1}). \end{aligned}$$

That is

$$\varphi(E_{i,i+1}) = G_{i,i+1}. \tag{2.2}$$

For $i \geq 1$ and $k \geq 1$, we check that

$$E_{i,i+k} \circ E_{i+k,i+k+1} = \frac{1}{2} E_{i,i+k+1} \quad \text{and} \quad G_{i,i+k} \circ G_{i+k,i+k+1} = \frac{1}{2} G_{i,i+k+1}.$$

In view of (2.2), we have $\varphi(E_{i,i+k}) = G_{i,i+k}$. This proves claim.

Clearly, $\{g_i\}_{i\geq 1}$ is an ONB of $\mathcal{H}$. Define a unitary operator $U \in \mathcal{B}(\mathcal{H})$ as $Ue_i = g_i$ for $i \geq 1$. Then, for $i, k \geq 1$,

$$\varphi(P_i) = Q_i = f_i \otimes f_i = g_i \otimes g_i = (Ue_i) \otimes (Ue_i) = UP_iU^*$$

and

$$\varphi(E_{i,i+k}) = G_{i,i+k} = g_i \otimes g_{i+k} + g_{i+k} \otimes g_i = UE_{i,i+k}U^*.$$

Clearly, Θ equals the linear span of $\{P_i, E_{i,j} : j > i \geq 1\}$. So $\varphi(X) = UXU^*$ for all $X \in \Theta$.

Corollary 2.10 *If φ is a Jordan automorphism of $\mathcal{S}_C$, then $\|\varphi(X)\| = \|X\|$ for $X \in \mathcal{F}_C$.*

Proof Choose an $X \in \mathcal{F}_C$. Set $M = \overline{\operatorname{ran} X + \operatorname{ran} X^*}$. Then M reduces both C and X. Assume that $n = \dim M$. Then we can choose an ONB $\{e_i\}_{i\geq 1}$ of $\mathcal{H}$ such that $M = \vee\{e_i : 1 \leq i \leq n\}$ and $Ce_i = e_i$ for all $i \geq 1$.

By Proposition 2.5, there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\varphi(X) = UXU^*$ for all $X \in \Theta$, where

$$\Theta = \{Y \in \mathcal{S}_C : \exists\, n \geq 1 \text{ such that } \langle Ye_i, e_j\rangle = 0 \text{ whenever } i + j \geq n\}.$$

Clearly, $X \in \Theta$. Thus $\|\varphi(X)\| = \|X\|$. This completes the proof.

Proof of Theorem 2.2 First we choose an ONB $\{e_i\}_{i\geq 1}$ of $\mathcal{H}$ such that $Ce_i = e_i$ for all $i \geq 1$.

By Proposition 2.5, there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\varphi(X) = UXU^*$ for all $X \in \Theta$, where

$$\Theta = \{Y \in \mathcal{S}_C : \exists\, n \geq 1 \text{ such that } \langle Ye_i, e_j\rangle = 0 \text{ whenever } i + j \geq n\}.$$

Claim $\varphi(X) = UXU^*$ for $X \in \mathcal{F}_C$.

Arbitrarily choose an $X \in \mathcal{F}_C$. For each $n \geq 1$, denote by R_n the projection of $\mathcal{H}$ onto $\vee\{e_i : 1 \leq i \leq n\}$. Then $R_nXR_n \in \Theta$ and $R_nXR_n \to X$ in norm. Since $\varphi|_{\mathcal{F}_C}$ is isometric, it follows that

$$\varphi(X) = \lim_n \varphi(R_nXR_n) = \lim_n U(R_nXR_n)U^* = UXU^*.$$

This proves the claim.

For $i \geq 1$, denote $g_i = Ue_i$ and $Q_i = g_i \otimes g_i$. Then $\{g_i\}$ is an ONB of $\mathcal{H}$ and $\varphi(P_i) = UP_iU^* = U(e_i \otimes e_i)U^* = Q_i$.

Now we shall prove that $\varphi(Y) = UYU^*$ for $Y \in \mathcal{S}_C$.

Fix an operator $Y \in \mathcal{S}_C$. For each $i \geq 1$, since $Y \circ P_i \in \mathcal{F}_C$, it follows that

$$\begin{aligned}\varphi(Y) \circ Q_i &= \varphi(Y) \circ \varphi(P_i) = \varphi(Y \circ P_i)\\ &= U(Y \circ P_i)U^* = (UYU^*) \circ (UP_iU^*) = (UYU^*) \circ Q_i.\end{aligned}$$

Then $\langle \varphi(Y)g_i, g_j\rangle = \langle UYU^*g_i, g_j\rangle$ for all i, j with $i \neq j$. On the other hand, for each $i \geq 1$, note that

$$\begin{aligned}Q_i\varphi(Y)Q_i &= 2Q_i \circ (Q_i \circ \varphi(Y)) - Q_i^2 \circ \varphi(Y)\\ &= 2\varphi(P_i) \circ (\varphi(P_i) \circ \varphi(Y)) - \varphi(P_i^2) \circ \varphi(Y)\\ &= \varphi(2P_i \circ (P_i \circ Y) - P_i^2 \circ Y)\\ &= U(2P_i \circ (P_i \circ Y) - P_i^2 \circ Y)U^*\\ &= U(P_iYP_i)U^* = Q_i(UYU^*)Q_i,\end{aligned}$$

which implies that $\langle \varphi(Y)g_i, g_i\rangle = \langle UYU^*g_i, g_i\rangle$. Therefore we conclude that $\varphi(Y) = UYU^*$.

By Lemma 2.6, there exists a unimodular number $\alpha \in \mathbb{C}$ such that $(\alpha U)C = C(\alpha U)$. Set $V = \alpha U$. Then V is unitary and one can see that $\varphi(Y) = VYV^*$ for all $Y \in \mathcal{S}_C$.

3 Jordan Multiplication Operators

The Jordan product $\circ$ naturally induces a class of multiplication operators on $\mathcal{S}_C$. For $T \in \mathcal{S}_C$, define $J_T \in \mathcal{B}(\mathcal{S}_C)$ as $J_T : X \mapsto T \circ X$. Thus J_T is a bounded linear operator on $\mathcal{S}_C$.

For $T \in \mathcal{S}_C$, J_T is closely related to the Rosenblum operator induced by T. For $A, B \in \mathcal{B}(\mathcal{H})$, the Rosenblum operator $\tau_{A,B}$ on $\mathcal{B}(\mathcal{H})$ is defined by

$$\tau_{A,B}(X) = AX - XB, \quad \forall X \in \mathcal{B}(\mathcal{H}).$$

Thus J_T is the restriction of the Rosenblum operator $\frac{1}{2}\tau_{T,-T}$ to $\mathcal{S}_C$. Rosenblum operators, which arose in the study of operator equations, were first systematically studied by Rosenblum in [32].

We wish to determine the spectrum of J_T and its different parts for $T \in \mathcal{S}_C$, since the spectrum of a Rosenblum operator has been clearly described (see [22, Chapter 4] or [28, 32]).

Let A be a bounded linear operator acting on some Banach space. Denote by $\ker A$ and $\operatorname{ran} A$ the kernel of A and the range of A, respectively. We let $\sigma_p(A)$, $\sigma_\pi(A)$ and $\sigma_\delta(A)$ denote respectively the point spectrum of A, the approximate point spectrum of A and the approximate defect spectrum of A. Thus

$$\sigma_\pi(A) = \{z \in \mathbb{C} : A - z \text{ is not bounded below}\}$$

and

$$\sigma_\delta(A) = \{z \in \mathbb{C} : A - z \text{ is not surjective}\}.$$

We let $\sigma_l(A)$ and $\sigma_r(A)$ denote respectively the left spectrum of A and the right spectrum of A. That is

$$\sigma_l(A) = \{z \in \mathbb{C} : A - z \text{ does not have a left inverse}\}$$

and

$$\sigma_r(A) = \{z \in \mathbb{C} : A - z \text{ does not have a right inverse}\}.$$

The main result of this section is the following theorem.

Theorem 3.1 *Let $T \in \mathcal{S}_C$. Then*
(i) $\|J_T\| = \|T\|$;
(ii) $\sigma(J_T) = \sigma_r(J_T) = \sigma_\delta(J_T) = \sigma_l(J_T) = \sigma_\pi(J_T) = \frac{1}{2}[\sigma(T) + \sigma(T)]$.

Remark 3.1 Let $T \in \mathcal{S}_C$. Then $C(T-z)C = (T-z)^*$ and $\dim\ker(T-z)^* = \dim\ker(T-z)$ for all $z \in \mathbb{C}$. Thus a Fredholm C-symmetric operator has zero index. In addition, it is easy to check that

$$\sigma(T) = \sigma_r(T) = \sigma_\delta(T) = \sigma_l(T) = \sigma_\pi(T).$$

This explains to certain extent why the preceding result holds. In general, one can not say anything more about the spectrum of a complex symmetric operator even for the finite dimensional case.

To give the proof of Theorem 3.1, we first make some preparation.

For the reader's convenience, we write down some elementary facts.

Lemma 3.1 *Let $e, f \in \mathcal{H}$ and $X = e \otimes f$. If $A \in \mathcal{B}(\mathcal{H})$ and C is a conjugation on $\mathcal{H}$, then*
(i) $AX = (Ae) \otimes f$,
(ii) $XA = e \otimes (A^* f)$,
(iii) $CXC = (Ce) \otimes (Cf)$.

Lemma 3.2 *Let $e, f \in \mathcal{H}$ with $\|e\| = \|f\| = 1$. Set $X = e \otimes f + (Cf) \otimes (Ce)$. Then $X \in \mathcal{S}_C$ and $1 \le \|X\| \le \|X\|_p \le 2$ for all $p \in [1, \infty)$.*

Proof It is easy to check that $CXC = X^*$ and $\|X\|_p \le 2$.
On the other hand, compute to see

$$\begin{aligned}\|X\| &\ge |\langle Xf, e\rangle| = |1 + \langle f, Ce\rangle \cdot \langle Cf, e\rangle| \\ &= |1 + \langle f, Ce\rangle \cdot \langle Ce, f\rangle| = 1 + |\langle Ce, f\rangle|^2.\end{aligned}$$

It follows that $\|X\| \ge 1$, which completes the proof.

Lemma 3.3 *Let $A \in \mathcal{S}_C$. Then*
(i) $\sigma_\pi(A) = \sigma_\delta(A) = \sigma(A)$;
(ii) $\sigma(J_A) \subset \frac{1}{2}[\sigma(A) + \sigma(A)]$.

Proof (i) For any $\lambda \in \mathbb{C}$, note that $C(A - \lambda)C = (A - \lambda)^*$. Then $A - \lambda$ is bounded below if and only if so is $(A - \lambda)^*$, which equals that $A - \lambda$ is surjective. Hence the result follows readily.

(ii) Note that $A \circ X \in \mathcal{O}_C$ for all $X \in \mathcal{O}_C$. Thus $\mathcal{O}_C$ and $\mathcal{S}_C$ are both invariant under $\tau_{A,-A}$. Thus, by [22, Corollary 3.20],

$$\sigma(J_A) = \sigma\Big(\frac{1}{2}\tau_{A,-A}|_{\mathcal{S}_C}\Big) \subset \sigma\Big(\frac{1}{2}\tau_{A,-A}\Big) = \frac{1}{2}[\sigma(A) + \sigma(A)].$$

By [22, Theorem 3.19 & Corollary 3.20], the following corollary is clear.

Corollary 3.1 *If $A \in \mathcal{S}_C$, then*

$$\sigma_\pi\Big(\frac{1}{2}\tau_{A,-A}\Big) = \sigma_\delta\Big(\frac{1}{2}\tau_{A,-A}\Big) = \sigma\Big(\frac{1}{2}\tau_{A,-A}\Big) = \frac{1}{2}[\sigma(A) + \sigma(A)].$$

Note that each $\mathcal{S}_{C,p}$ is invariant under J_T for $p \in \{0\} \cup [1, \infty)$. Recall that $\mathcal{S}_{C,p} = \mathcal{S}_C \cap \mathcal{B}_p(\mathcal{H})$. Denote $J_{T,p} = J_T|_{\mathcal{S}_{C,p}}$. We view $J_{T,p}$ as a linear operator on $(\mathcal{S}_{C,p}, \|\cdot\|_p)$. By [31, Theorem 2.3.10], $J_{T,p}$ is bounded.

Lemma 3.4 *If $T \in \mathcal{S}_C$, then $\sigma_\pi(J_{T,p}) \subset \frac{1}{2}[\sigma(T) + \sigma(T)]$ for all $p \in \{0\} \cup [1, \infty)$.*

Proof Assume that $z \in \sigma_\pi(J_{T,p})$. Then there exist $\{X_n\} \in \mathcal{S}_{C,p}$ with $\|X_n\|_p = 1$ for all n such that $\|J_{T,p}X_n - zX_n\|_p \to 0$. That is, $\big\|\frac{1}{2}\tau_{T,-T}(X_n) - zX_n\big\|_p \to 0$. Thus $\frac{1}{2}\tau_{T,-T}\big|_{\mathcal{B}_p(\mathcal{H})} - z$ is not bounded below. By [22, Theorem 3.54], we deduce that $\frac{1}{2}\tau_{T,-T} - z$ is not bounded below. In view of Corollary 3.1, we have $z \in \frac{1}{2}[\sigma(T) + \sigma(T)]$.

Given a Banach space $\mathcal{X}$, we let $\mathcal{X}'$ denote its dual. If $T : \mathcal{X} \to \mathcal{X}$ is a bounded linear operator, we denote by T' the adjoint of T acting on $\mathcal{X}'$.

Lemma 3.5 *If $T \in \mathcal{S}_C$, then $\sigma_\pi(J_T) = \sigma_\delta(J_{T,1})$ and $\sigma_\delta(J_T) = \sigma_\pi(J_{T,1})$.*

Proof It suffices to prove that J_T is similar to the adjoint $J'_{T,1}$ of $J_{T,1}$.

Denote by ϕ the isometrical isomorphism of $\mathcal{S}_C$ onto $(\mathcal{S}_{C,1})'$ defined by $\phi(K) = \phi_K$, where

$$\phi_K(X) = \mathrm{tr}(XK), \quad \forall X \in \mathcal{S}_{C,1}.$$

Then it suffices to check that $\phi J_T = J'_{T,1}\phi$.

Fix $Z \in \mathcal{S}_C$. Denote $J_1 = [\phi J_T](Z)$ and $J_2 = [J'_{T,1}\phi](Z)$. Then $J_i \in \mathcal{S}'_{C,1}$, $i = 1, 2$. It suffices to prove that $J_1 = J_2$. Since

$$[\phi J_T](Z) = \phi[J_T(Z)] = \phi_{T\circ Z}, \quad [J'_{T,1}\phi](Z) = J'_{T,1}[\phi(Z)] = J'_{T,1}(\phi_Z)$$

for any $X \in \mathcal{S}_{C,1}$, we have

$$\begin{aligned} J_1(X) &= \phi_{T\circ Z}(X) = \mathrm{tr}[(T\circ Z)X] \\ &= \frac{1}{2}\mathrm{tr}(TZX + ZTX) = \frac{1}{2}\mathrm{tr}(TXZ + XTZ) \\ &= \mathrm{tr}[(T\circ X)Z] = \phi_Z(T\circ X) \\ &= \phi_Z[J_{T,1}(X)] = J_2(X). \end{aligned}$$

This shows that $J_1 = J_2$.

Using dual relations among $\mathcal{S}_{C,p} (p \in \{0\} \cup [1, \infty))$ (see Proposition 2.2), one can prove as in Lemma 3.5 the following corollary.

Corollary 3.2 *Let* $T \in \mathcal{S}_C$ *and* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*
(i) $\sigma_\pi(J_{T,0}) = \sigma_\delta(J_{T,1})$ *and* $\sigma_\delta(J_{T,0}) = \sigma_\pi(J_{T,1})$.
(ii) $\sigma_\pi(J_{T,p}) = \sigma_\delta(J_{T,q})$ *and* $\sigma_\delta(J_{T,p}) = \sigma_\pi(J_{T,q})$.

Now we are going to prove Theorem 3.1.

Proof of Theorem 3.1 (i) Clearly, $\|T\circ X\| \le \|T\|\cdot\|X\|$ for all $X \in \mathcal{S}_C$. Thus $\|J_T\| \le \|T\|$. Noting that $I \in \mathcal{S}_C$, we have $\|T\| = \|J_T(I)\| \le \|J_T\|$. Hence $\|J_T\| = \|T\|$.

(ii) Fix $p \in \{0\} \cup [1, \infty)$. We first prove a key claim.

Claim $\frac{1}{2}[\sigma(T) + \sigma(T)] \subset [\sigma_\pi(J_T) \cap \sigma_\pi(J_{T,p})]$.

Let $\lambda, \mu \in \sigma(T)$. Then $\lambda \in \sigma_\pi(T)$ and, by Lemma 3.3(i), $\overline{\mu} \in \sigma_\pi(T^*)$. We can choose unit vectors $\{e_n, f_n : n \ge 1\}$ such that $(T - \lambda)e_n \to 0$ and $(T - \mu)^* f_n \to 0$ as $n \to \infty$.

For $n \ge 1$, set $X_n = e_n \otimes f_n + (Cf_n) \otimes (Ce_n)$. By Lemma 3.2, $X_n \in \mathcal{S}_C$ and $1 \le \|X_n\| \le \|X_n\|_p \le 2$. By Lemma 3.1,

$$\begin{aligned} 2J_T(X_n) = TX_n + X_nT &= (Te_n) \otimes f_n + (TCf_n) \otimes (Ce_n) \\ &\quad + e_n \otimes (T^* f_n) + (Cf_n) \otimes (T^* Ce_n) \\ &= (Te_n) \otimes f_n + (CT^* f_n) \otimes (Ce_n) \\ &\quad + e_n \otimes (T^* f_n) + (Cf_n) \otimes (CTe_n). \end{aligned}$$

Note that

$$\lambda X_n = (\lambda e_n) \otimes f_n + (Cf_n) \otimes [C(\lambda e_n)]$$

and

$$\mu X_n = e_n \otimes (\overline{\mu} f_n) + [C(\overline{\mu} f_n)] \otimes (Ce_n).$$

Then, as $n \to \infty$,

$$\begin{aligned} 2J_T(X_n) - (\lambda+\mu)X_n &= [(Te_n)\otimes f_n + (Cf_n)\otimes (CTe_n) - \lambda X_n] \\ &\quad + [(CT^* f_n)\otimes (Ce_n) + e_n \otimes (T^* f_n) - \mu X_n] \\ &= [(T-\lambda)e_n]\otimes f_n + (Cf_n)\otimes [C(T-\lambda)e_n] \\ &\quad + [C(T^*-\overline{\mu})f_n]\otimes (Ce_n) + e_n \otimes [(T^*-\overline{\mu})f_n] \xrightarrow{\|\cdot\|_p} 0. \end{aligned}$$

This shows that $\frac{1}{2}(\lambda+\mu) \in \sigma_\pi(J_{T,p}) \cap \sigma_\pi(J_T)$.

Since $\lambda, \mu \in \sigma(T)$ can be chosen arbitrarily, we deduce that $\frac{1}{2}(\sigma(T)+\sigma(T)) \subset \sigma_\pi(J_{T,p}) \cap \sigma_\pi(J_T)$. This proves the claim.

In view of Lemma 3.3(ii) and Lemma 3.4, we conclude that

$$\sigma(J_T) = \sigma_\pi(J_T) = \sigma_\pi(J_{T,p}) = \frac{1}{2}[\sigma(T)+\sigma(T)].$$

It follows immediately from Lemma 3.5 and Corollary 3.2 that

$$\sigma_\delta(J_T) = \sigma_\delta(J_{T,p}) = \frac{1}{2}[\sigma(T)+\sigma(T)].$$

For any bounded linear operator A on a Banach space, note that

$$\sigma_\delta(A) \subset \sigma_r(A) \subset \sigma(A), \quad \sigma_\pi(A) \subset \sigma_l(A) \subset \sigma(A).$$

Hence we conclude the proof.

Corollary 3.3 *Let $T \in \mathcal{S}_C$ and $p \in \{0\} \cup [1,\infty)$. Then*

$$\sigma(J_{T,p}) = \sigma_\delta(J_{T,p}) = \sigma_\pi(J_{T,p}) = \frac{1}{2}[\sigma(T)+\sigma(T)].$$

Remark 3.2 By Theorem 3.1, Corollaries 3.1 and 3.3, if $T \in \mathcal{S}_C$, then the spectra, the approximate point spectra and the approximate defect spectra of J_T and $J_{T,p}$ ($p \in \{0\} \cup [1,\infty)$) coincide with that of $\frac{1}{2}\tau_{T,-T}$, all equaling $\frac{1}{2}[\sigma(T)+\sigma(T)]$.

Corollary 3.4 *Let $T \in \mathcal{S}_C$ and $p \in \{0\} \cup [1,\infty)$. Then the following are equivalent*:

(i) *For any $Y \in \mathcal{B}(\mathcal{H})$, the operator equation $TX + XT = Y$ has at least one solution in $\mathcal{B}(\mathcal{H})$;*

(ii) *for any $Y \in \mathcal{B}(\mathcal{H})$, the operator equation $TX + XT = Y$ has exactly one solution in $\mathcal{B}(\mathcal{H})$;*

(iii) *for any $Y \in \mathcal{S}_C$, the operator equation $TX + XT = Y$ has at least one solution in $\mathcal{S}_C$;*

(iv) *for any $Y \in \mathcal{S}_C$, the operator equation $TX + XT = Y$ has exactly one solution in $\mathcal{S}_C$;*

(v) *for any $Y \in \mathcal{S}_{C,p}$, the operator equation $TX + XT = Y$ has at least one solution in $\mathcal{S}_{C,p}$;*

(vi) *for any $Y \in \mathcal{S}_{C,p}$, the operator equation $TX + XT = Y$ has exactly one solution in $\mathcal{S}_{C,p}$;*

(vii) $0 \notin \sigma(T) + \sigma(T)$.

Proposition 3.1 *If $T \in \mathcal{S}_C$, then $\frac{1}{2}[\sigma_p(T) + \sigma_p(T)] \subset \sigma_p(J_T)$.*

Proof Now choose $\lambda, \mu \in \sigma_p(T)$. It suffices to prove that $\frac{1}{2}(\lambda + \mu) \in \sigma_p(J_T)$.

Since $\lambda \in \sigma_p(T)$, we can find a unit vector $e \in \mathcal{H}$ such that $Te = \lambda e$. On the other hand, note that $C(T-\mu)^*C = T - \mu$. Since $\mu \in \sigma_p(T)$, it follows that $\overline{\mu} \in \sigma_p(T^*)$ and we can find a unit vector $f \in \mathcal{H}$ such that $T^*f = \overline{\mu} f$.

Set $X = e \otimes f + (Cf) \otimes (Ce)$. Then $X \in \mathcal{S}_C$ and, by Lemma 3.2, $X \neq 0$. Moreover, we have

$$\begin{aligned} 2J_T(X) &= TX + XT \\ &= (Te) \otimes f + (TCf) \otimes (Ce) + e \otimes (T^*f) + (Cf) \otimes (T^*Ce) \\ &= (\lambda e) \otimes f + (CT^*f) \otimes (Ce) + e \otimes (\overline{\mu} f) + (Cf) \otimes (CTe) \\ &= \lambda(e \otimes f) + (C\overline{\mu} f) \otimes (Ce) + \mu(e \otimes f) + (Cf) \otimes (C\lambda e) \\ &= \lambda(e \otimes f) + \mu[(Cf) \otimes (Ce)] + \mu(e \otimes f) + \lambda[(Cf) \otimes (Ce)] \\ &= (\lambda + \mu)X. \end{aligned}$$

Hence $\frac{1}{2}(\lambda + \mu) \in \sigma_p(J_T)$.

By Theorem 3.1 and Proposition 3.1, the invertibility of an operator $T \in \mathcal{S}_C$ in general does not imply the invertibility or even the injectivity of J_T.

Example 3.1 Let C be a conjugation on $\mathcal{H}$ and

$$D = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Then one can check that

$$\mathcal{S}_D = \left\{ \begin{bmatrix} A & E^* \\ CEC & B \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix} : A, B \in \mathcal{S}_C, E \in \mathcal{B}(\mathcal{H}) \right\}.$$

Define

$$T = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Clearly, $T \in \mathcal{S}_D$, $\sigma(T) = \{1, -1\}$ and, by Theorem 3.1, $\sigma(J_T) = \{0, 1, -1\}$. Moreover, one can check that $\sigma_p(J_T) = \{0, 1, -1\}$,

$$\ker J_T = \left\{ \begin{bmatrix} 0 & E^* \\ CEC & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix} : E \in \mathcal{B}(\mathcal{H}) \right\},$$

$$\ker(J_T - 1) = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix} : A \in \mathcal{S}_C \right\}$$

and

$$\ker(J_T + 1) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix} : B \in \mathcal{S}_C \right\}.$$

4 Invertible Operators in $\mathcal{S}_C$

This section focuses on some topics concerning Jordan invertible elements of $\mathcal{S}_C$.

We remark that an element $T \in \mathcal{S}_C$ is Jordan invertible if and only if T is an invertible operator. In fact, if $T \in \mathcal{S}_C$ is an invertible operator, then it is easy to see that $T^{-1} \in \mathcal{S}_C$. One can check that Q_T is invertible with $Q_T^{-1} = Q_{T^{-1}}$. On the other hand, if Q_T is invertible, then there exists $X \in \mathcal{S}_C$ such that $Q_T(X) = TXT = I$. So T is invertible.

4.1 Connectedness of the invertibles

It is well known that the collection of invertible operators in $\mathcal{B}(\mathcal{H})$ is path connected. The following result is its analogue in $\mathcal{S}_C$.

Theorem 4.1 *The set of invertible operators in $\mathcal{S}_C$ is path connected.*

Proof Choose an invertible operator $T \in \mathcal{S}_C$ and assume that $T = U|T|$ is its polar decomposition. So U is unitary and $|T|$ is invertible. By [17, Theorem 2], $U \in \mathcal{S}_C$ and there exists a conjugation J on $\mathcal{H}$ such that $U = CJ$ and $J|T| = |T|J$.

Claim There is an arc $\{T_\lambda : \lambda \in [0,1]\}$ in $\mathcal{S}_C$ connecting T to U.

Denote $m = \min\sigma(|T|)$ and $M = \max\sigma(|T|)$. For $\lambda \in [0,1]$, define $f_\lambda \in C[m,M]$ as $f_\lambda(t) = \lambda + (1-\lambda)t$. Set $T_\lambda = Uf_\lambda(|T|)$, $\lambda \in [0,1]$. Note that $f_\lambda(|T|)$ is continuous with respect to λ and the path $\{T_\lambda : \lambda \in [0,1]\}$ connects T to U. It suffices to prove that $T_\lambda \in \mathcal{S}_C$ for each $\lambda \in [0,1]$.

Since $J|T| = |T|J$, it follows from Proposition 2.3 that $f_\lambda(|T|)J = Jf_\lambda(|T|)$. Noting that $U^* = U^{-1} = JC$, we obtain

$$CT_\lambda C = CUf_\lambda(|T|)C = Jf_\lambda(|T|)C = f_\lambda(|T|)JC = f_\lambda(|T|)U^* = T_\lambda^*.$$

That is, $T_\lambda \in \mathcal{S}_C$. This proves the claim.

Now it remains to prove that there is an arc $\{U_\lambda : \lambda \in [0,1]\}$ in $\mathcal{S}_C$ connecting the identity operator to U. Since U is unitary, by [9, Proposition 5.29], the set of unitary operators in $W^*(U)$ is path connected. So there is an arc $\{U_\lambda : \lambda \in [0,1]\}$ in $W^*(U)$ connecting the identity operator to U. On the other hand, since $U \in \mathcal{S}_C$, by Proposition 2.3, $W^*(U) \subset \mathcal{S}_C$. This shows that $U_\lambda \in \mathcal{S}_C$. Thus we complete the proof.

The following result follows directly from the proof of Theorem 4.1.

Corollary 4.1 *The set of unitary operators in $\mathcal{S}_C$ is path connected.*

Proposition 4.1 *The set of Fredholm operators in $\mathcal{S}_C$ is path connected.*

Proof Assume that $T \in \mathcal{S}_C$ is Fredholm. It suffices to find a path of Fredholm operators in $\mathcal{S}_C$ connecting T to I. In view of Theorem 4.1, we may directly assume that T is not invertible. Thus 0 is an isolated point of $\sigma(|T|)$ and $0 < \dim\ker T < \infty$.

By [17, Theorem 2], we assume that $T = CJ|T|$, where J is a partial conjugation J acting on $\mathcal{H}$ and supported on $\overline{\operatorname{ran}|T|}$ such that $J|T| = |T|J$. Then we may assume

$$|T| = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \ker|T| \\ (\ker|T|)^\perp \end{matrix}, \quad J = \begin{bmatrix} 0 & 0 \\ 0 & J_0 \end{bmatrix} \begin{matrix} \ker|T| \\ (\ker|T|)^\perp \end{matrix},$$

where J_0 is a conjugation on $(\ker|T|)^\perp$. Clearly, A is invertible.

Choose a conjugation J_1 on $\ker|T|$. Set

$$\widetilde{J} = \begin{bmatrix} J_1 & 0 \\ 0 & J_0 \end{bmatrix} \begin{matrix} \ker|T| \\ (\ker|T|)^\perp \end{matrix}, \quad P_\lambda = \begin{bmatrix} \lambda I_1 & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \ker|T| \\ (\ker|T|)^\perp \end{matrix}, \quad \lambda \in [0,1],$$

where I_1 is the identity operator on $\ker|T|$.

For $\lambda \in [0,1]$, define $T_\lambda = C\widetilde{J}P_\lambda$. Clearly, $T_0 = T$, T_λ is invertible for $\lambda \in (0,1]$ and

$$CT_\lambda C = \widetilde{J}P_\lambda C = P_\lambda \widetilde{J}C = T_\lambda^*.$$

Thus $\{T_\lambda : \lambda \in [0,1]\}$ is a path of invertible operators in $\mathcal{S}_C$ connecting T to T_1 (which is invertible). In view of Theorem 4.1, one can see the conclusion.

In the proof of Proposition 4.1, one can see that $T_\lambda - T \in \mathcal{K}(\mathcal{H})$, which implies that $T_\lambda^{-1}T - I \in \mathcal{K}(\mathcal{H})$ and $TT_\lambda^{-1} - I \in \mathcal{K}(\mathcal{H})$ for $\lambda \in (0,1]$. Thus the following corollary is clear.

Corollary 4.2 *Let $T \in \mathcal{S}_C$. Then the following are equivalent*:

(i) *T is a Fredholm operator*;

(ii) *given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is invertible and $T + K \in \mathcal{S}_C$*;

(iii) *there exists invertible $A \in \mathcal{S}_C$ such that $TA - I, AT - I \in \mathcal{K}(\mathcal{H})$.*

Remark 4.1 Note that $I \in \mathcal{S}_C$ for any conjugation C on $\mathcal{H}$. Then, in view of Theorem 4.1 and Proposition 4.1, we conclude that the set of invertible operators in $\mathcal{S}(\mathcal{H})$, the set of unitary operators in $\mathcal{S}(\mathcal{H})$ and the set of Fredholm operators in $\mathcal{S}(\mathcal{H})$ are all path connected.

4.2 Invertible approximation

This subsection focuses on invertible approximation in $\mathcal{S}_C$, that is, describing which operators can be approximated in norm by invertible operators in $\mathcal{S}_C$.

Denote by $\mathcal{G}(\mathcal{H})$ the collection of invertible operators in $\mathcal{B}(\mathcal{H})$. By [1, Proposition 10.1], an operator T lies in the norm closure of $\mathcal{G}(\mathcal{H})$ if and only if either T is not a semi-Fredholm operator or T is a semi-Fredholm operator with $\operatorname{ind} T = 0$. Recall that an operator R is called a semi-Fredholm operator if $\operatorname{ran} R$ is closed and either $\dim\ker R$ or $\dim\ker R^*$ is finite; in this case, $\operatorname{ind} R = \dim\ker R - \dim\ker R^*$ is called the index of R. If, in addition, $-\infty < \operatorname{ind} R < \infty$, then T is called a Fredholm operator.

Let $T \in \mathcal{S}_C$. Then $CTC = T^*$ and $\dim\ker T = \dim\ker T^*$. This shows that if T is a semi-Fredholm operator, then $\operatorname{ind} T = 0$. In view of the invertible approximation in $\mathcal{B}(\mathcal{H})$, it is natural to conjecture that every operator in $\mathcal{S}_C$ is a norm limit of invertible operators in $\mathcal{S}_C$. This is indeed the case.

Proposition 4.2 *If C is a conjugation on $\mathcal{H}$, then $\mathcal{S}_C \cap \mathcal{G}(\mathcal{H})$ is norm dense in $\mathcal{S}_C$.*

Proof Choose an operator $T \in \mathcal{S}_C$. By [17, Theorem 2], there exists a partial conjugation J supported on $\overline{\operatorname{ran}|T|}$ such that $T = CJ|T|$ and $J|T| = |T|J$. Then

$$|T| = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \ker|T| \\ (\ker|T|)^\perp \end{matrix}, \quad J = \begin{bmatrix} 0 & 0 \\ 0 & J_1 \end{bmatrix} \begin{matrix} \ker|T| \\ (\ker|T|)^\perp \end{matrix},$$

where A is positive, J_1 is a conjugation on $\overline{\operatorname{ran}|T|} = (\ker|T|)^\perp$ and $J_1A = AJ_1$.

Fix an $\varepsilon > 0$. Assume that $E(\cdot)$ is the projection-valued spectral measure associated with A. Set $P = E\big([0, \frac{\varepsilon}{2}]\big)$. From Proposition 2.3 one can see $J_1P = PJ_1$.

Hence

$$|T| = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{matrix} \ker|T| \\ \operatorname{ran} P \\ (\ker|T|)^{\perp} \ominus \operatorname{ran} P \end{matrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J_{1,1} & 0 \\ 0 & 0 & J_{1,2} \end{bmatrix} \begin{matrix} \ker|T| \\ \operatorname{ran} P \\ (\ker\ |T|)^{\perp} \ominus \operatorname{ran} P \end{matrix}.$$

Choose a conjugation J_0 on $\ker|T|$ and set

$$Q = \begin{bmatrix} \frac{\varepsilon I_0}{2} & 0 & 0 \\ 0 & \frac{\varepsilon I_1}{2} & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{matrix} \ker|T| \\ \operatorname{ran} P \\ (\ker|T|)^{\perp} \ominus \operatorname{ran} P \end{matrix}, \quad \widetilde{J} = \begin{bmatrix} J_0 & 0 & 0 \\ 0 & J_{1,1} & 0 \\ 0 & 0 & J_{1,2} \end{bmatrix} \begin{matrix} \ker|T| \\ \operatorname{ran} P \\ (\ker|T|)^{\perp} \ominus \operatorname{ran} P \end{matrix},$$

where I_0 is the identity operator on $\ker|T|$ and I_1 is the identity operator on ran$P$. Then Q is positive, invertible, and $\widetilde{J}$ is a conjugation on $\mathcal{H}$ commuting with Q. Set $T_\varepsilon = C\widetilde{J}Q$. Then T_ε is invertible, $T_\varepsilon \in \mathcal{S}_C$ and

$$\begin{aligned} \|T_\varepsilon - T\| &\le \|T_\varepsilon - C\widetilde{J}|T|\| + \|C\widetilde{J}|T| - CJ|T|\| \\ &= \|T_\varepsilon - C\widetilde{J}|T|\| \\ &= \|Q - |T|\| \le \frac{\varepsilon}{2}. \end{aligned}$$

Since ε can be chosen arbitrarily small, we conclude that T is the norm limit of invertible ones in $\mathcal{S}_C$.

By a classical approximation result of Apostol and Morrel (see [2] or [22, Theorem 6.15]), if an operator T is biquasitriangular (that is, $\operatorname{ind}(T - z) = 0$ whenever defined), then T can be approximated in norm by operators with finite spectra. Note that each operator in $\mathcal{S}_C$ is biquasitriangular. Thus it is natural to ask the following question.

Question 4.1 Are those elements with finite spectra norm dense in $\mathcal{S}_C$?

Acknowledgement The authors thank the referee for helpful comments and suggestions which greatly improve the readability of this paper.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Apostol, C., Fialkow, L. A., Herrero, D. A. and Voiculescu, D., Approximation of Hilbert Space Operators, **2**, Pitman (Advanced Publishing Program), Boston, 1984.

[2] Apostol, C. and Morrel, B. B., On uniform approximation of operators by simple models, *Indiana Univ. Math. J.*, **26**, 1977, 427–442.

[3] Bercovici, H. and Timotin, D., Truncated Toeplitz operators and complex symmetries, *Proc. Amer. Math. Soc.*, **146**, 2018, 261–266.

[4] Blecher, D. P. and Wang, Z., Involutive operator algebras, *Positivity*, **24**, 2020, 13–53.

[5] Cartan, É., Sur les domaines bornés homogènes de l'espace de n variables complexes, *Abh. Math. Semin. Univ. Hamburg*, **11**, 1935, 116–162.

[6] Chu, C-H., Jordan structures in geometry and analysis, Cambridge Tracts in Mathematics, **190**, Cambridge University Press, Cambridge, 2012.

[7] Conway, J. B., A Course in Operator Theory, American Mathematical Society, Providence, 2000.

[8] Danciger, J., Garcia, S. R. and Putinar, M., Variational principles for symmetric bilinear forms, *Math. Nachr.*, **281**, 2008, 786–802.

[9] Douglas, R. G., Banach Algebra Techniques in Operator Theory, 2nd ed., Springer-Verlag, New York, 1998.

[10] Fong, C. K., Miers, C. R. and Sourour, A. R., Lie and Jordan ideals of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **84**, 1982, 516–520.

[11] Friedman, Y. and Russo, B., The Gelfand-Naimark theorem for JB*-triples, *Duke Math. J.*, **53**, 1986, 139–148.

[12] Garcia, S. R., Conjugation and Clark operators, Recent Advances in Operator-Related Function Theory, Contemp. Math., **393**, Amer. Math. Soc., Providence, 2006, 67–111.

[13] Garcia, S. R., Means of unitaries, conjugations, and the Friedrichs operator, *J. Math. Anal. Appl.*, **335**, 2007, 941–947.

[14] Garcia, S. R., The norm and modulus of a Foguel operator, *Indiana Univ. Math. J.*, **58**, 2009, 2305–2315.

[15] Garcia, S. R., Prodan, E. and Putinar, M., Mathematical and physical aspects of complex symmetric operators, *J. Phys. A*, **47**, 2014, 353001.

[16] Garcia, S. R. and Putinar, M., Complex symmetric operators and applications, *Trans. Amer. Math. Soc.*, **358**, 2006, 1285–1315.

[17] Garcia, S. R. and Putinar, M., Complex symmetric operators and applications, II, *Trans. Amer. Math. Soc.*, **359**, 2007, 3913–3931.

[18] Garcia, S. R. and Wogen, W. R., Complex symmetric partial isometries, *J. Funct. Anal.*, **257**, 2009, 1251–1260.

[19] Glazman, I. M., An analogue of the extension theory of Hermitian operators and a non-symmetric one-dimensional boundary problem on a half-axis, *Dokl. Akad. Nauk SSSR* (*N.S.*), **115**, 1957, 214–216.

[20] Guo, K. Y., Ji, Y. Q. and Zhu, S., A C^*-algebra approach to complex symmetric operators, *Trans. Amer. Math. Soc.*, **367**, 2015, 6903–6942.

[21] Hai, P. V. and Putinar, M., Complex symmetric evolution equations, *Anal. Math. Phys.*, **10**, 2020.

[22] Herrero, D. A., Approximation of Hilbert Space Operators, **1**, 2nd ed., Longman Scientific & Technical, Harlow, 1989.

[23] Hua, L.-K., On the theory of automorphic functions of a matrix level I, Geometrical basis, *Amer. J. Math.*, **66**, 1944, 470–488.

[24] Jacobson, N., Normal semi-linear transformations, *Amer. J. Math.*, **61**, 1939, 45–58.

[25] Jordan, P., von Neumann, J. and Wigner, E., On an algebraic generalization of the quantum mechanical formalism, *Ann. of Math.*, **35**, 1934, 29–64.

[26] Kaup, W., A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.*, **183**, 1983, 503–529.

[27] Kliś-Garlicka, K. and Ptak, M., C-symmetric operators and reflexivity, *Oper. Matrices*, **9**, 2015, 225–232.

[28] Lumer, G. and Rosenblum, M., Linear operator equations, *Proc. Amer. Math. Soc.*, **10**, 1959, 32–41.

[29] Maitland Wright, J. D., Jordan C^*-algebras, *Michigan Math. J.*, **24**, 1977, 291–302.

[30] Prodan, E., Garcia, S. R. and Putinar, M., Norm estimates of complex symmetric operators applied to quantum systems, *J. Phys. A*, **39**, 2006, 389–400.

[31] Ringrose, J. R., Compact Non-self-adjoint Operators, Van Nostrand Reinhold Co., London, 1971.

[32] Rosenblum, M., On the operator equation $BX - XA = Q$, *Duke Math. J.*, **23**, 1956, 263–269.

[33] Sarason, D., Algebraic properties of truncated Toeplitz operators, *Oper. Matrices*, **1**, 2007, 491–526.

[34] Schatten, R., Norm Ideals of Completely Continuous Operators, 2nd ed., Springer-Verlag, Berlin, New York, 1970.

[35] Schur, I., Ein Satz ueber quadratische Formen mit komplexen Koeffizienten, *Amer. J. Math.*, **67**, 1945, 472–480.

[36] Shen, J. and Zhu, S., Complex symmetric generators for operator algebras, *J. Operator Theory*, **77**, 2017, 421–454.

[37] Siegel, C. L., Symplectic geometry, *Amer. J. Math.*, **65**, 1943, 1–86.

[38] Takagi, T., On an algebraic problem related to an analytic theorem of Carathéodory and Fejér and on an allied theorem of Landau, *Japan J. Math.*, **1**, 1925, 83–93.

[39] Topping, D. M., Jordan algebras of self-adjoint operators, *Mem. Amer. Math. Soc.*, **53**, 1965, 48 pp.

[40] Waleed Noor, S., On an example of a complex symmetric composition operator on $H^2(\mathbb{D})$, *J. Funct. Anal.*, **269**, 2015, 1899–1901.

[41] Wang, M. and Yao, X., Complex symmetry of weighted composition operators in several variables, *Internat. J. Math.*, **27**, 2016, 1650017.

[42] Zhu, S., Approximation of complex symmetric operators, *Math. Ann.*, **364**, 2016, 373–399.

[43] Zhu, S., Complex symmetric operators, skew symmetric operators and reflexivity, *Oper. Matrices*, **11**, 2017, 941–951.

[44] Zhu, S. and Zhao, J., Complex symmetric generators of C^*-algebras, *J. Math. Anal. Appl.*, **456**, 2017, 796–822.