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Conformal Perturbations of Twisted Dirac Operators and Noncommutative Residue*

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Abstract In this paper, the authors obtain two kinds of Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of signature operators by a vector bundle with a non-unitary connection on six-dimensional manifolds with (respectively without) boundary.

Keywords Conformal perturbations of twisted Dirac operators, Conformal perturbations of twisted signature operators, Noncommutative residue, Non-unitary connection
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1 Introduction

The noncommutative residue found in [1–2] plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler [3] in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact n-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [2] using the theory of zeta functions of elliptic pseudodifferential operators. In [4], Connes used the noncommutative residue to derive a conformal four-dimensional Polyakov action analogy. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in [5]. In [6], Kastler gave a brute-force proof of this theorem. In [7], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [8].

In [9], Ponge defined lower dimensional volumes of Riemannian manifolds by the Wodzicki residue. Fedosov et al. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [10]. In [11], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In [12], Wang generalized the Kastler-Kalau-Walze type theorem to the cases of three-, four-dimensional spin manifolds with boundary and proved a Kastler-Kalau-Walze type theorem.

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In [12–13, 15–17], Wang and his coauthors computed the lower dimensional volumes for five-, six-, seven-dimensional spin manifolds with boundary and also got some Kastler-Kalau-Walze type theorems. In [18], authors computed $\widetilde{\mathrm{Wres}}[(\pi^+D^{-2})\circ(\pi^+D^{-n+2})]$ for any dimensional manifolds with boundary, and proved a general Kastler-Kalau-Walze type theorem.

In [19], Wang and Wang proved two kinds of Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of signature operators by a vector bundle with a non-unitary connection on four-dimensional manifolds with (respectively without) boundary.

The motivation of this paper is to establish two Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of signature operators with non-unitary connections on six-dimensional manifolds with boundary. We know that the leading symbol of conformal perturbations of twisted Dirac operators is not $\sqrt{-1}c(\xi)$. This is the reason that we study the residue of conformal perturbations of twisted Dirac operators.

This paper is organized as follows: In Section 2, we recall some basic facts and formulas about Boutet de Monvel's calculus. In Section 3, we give a Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators on six-dimensional manifolds with boundary. In Sections 4–5, we recall the definition of conformal perturbations of signature operators and compute their symbols, and we give a Kastler-Kalau-Walze type theorems for conformal perturbations of signature operators on six-dimensional manifolds with boundary. The main results are Theorems 3.1 and 5.1 in this paper.

2 Boutet de Monvel's Calculus and Noncommutative Residue

In this section, we shall recall some basic facts and formulas about Boutet de Monvel's calculus. Let

$$F: L^2(\mathbf{R}_t)\to L^2(\mathbf{R}_v),\quad F(u)(v)=\int_{\mathbf{R}}\mathrm{e}^{-\mathrm{i}vt}u(t)\mathrm{d}t$$

denote the Fourier transformation and $\varphi(\overline{\mathbf{R}^+})=r^+\varphi(\mathbf{R})$ (similarly define $\varphi(\overline{\mathbf{R}^-})$), where $\varphi(\mathbf{R})$ denotes the Schwartz space and

$$r^+: C^\infty(\mathbf{R})\to C^\infty(\overline{\mathbf{R}^+}),\quad f\to f|\overline{\mathbf{R}^+},\quad \overline{\mathbf{R}^+}=\{x\geq 0; x\in\mathbf{R}\}. \tag{2.1}$$

We define $H^+=F(\varphi(\overline{\mathbf{R}^+}))$; $H_0^-=F(\varphi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h\in H^+$ (resp. H_0^-) if and only if $h\in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (resp. upper) complex half-plane $\{\operatorname{Im}\xi<0\}$ (resp. $\{\operatorname{Im}\xi>0\}$) such that for all nonnegative integer l,

$$\frac{\mathrm{d}^l h}{\mathrm{d}\xi^l}(\xi)\sim\sum_{k=1}^{\infty}\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\Big(\frac{c_k}{\xi^k}\Big) \tag{2.2}$$

as $|\xi|\to+\infty$, $\operatorname{Im}\xi\leq 0$ (resp. $\operatorname{Im}\xi\geq 0$).

Let H' be the space of all polynomials and $H^-=H_0^-\oplus H'$; $H=H^+\oplus H^-$. Denote by π^+ (resp. π^-) the projection on H^+ (resp. H^-). For calculations, we take $H=\widetilde{H}=\{$rational functions having no poles on the real axis$\}$ ($\widetilde{H}$ is a dense set in the topology of H). Then on $\widetilde{H}$,

$$\pi^+h(\xi_0)=\frac{1}{2\pi\mathrm{i}}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{h(\xi)}{\xi_0+\mathrm{i}u-\xi}\mathrm{d}\xi, \tag{2.3}$$

where Γ^+ is a Jordan close curve included $\mathrm{Im}\,\xi > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on $\widetilde{H}$,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi)\mathrm{d}\xi. \tag{2.4}$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(\mathbb{R})$, $\pi' h = \frac{1}{2\pi} \int_{\mathbb{R}} h(v)\mathrm{d}v$ and for $h \in H^+ \cap L^1(\mathbb{R})$, $\pi' h = 0$.

Denote by $\mathcal{B}$ Boutet de Monvel's algebra. For a detailed introduction to Boutet de Monvel's algebra see Boutet de Monvel [21], Grubb [22], Rempel-Schulze [23] or Schrohe-Schulze [24]. In the following we will give a review of some basic fact we need.

An operator of order $m \in \mathbf{Z}$ and type d is a matrix

$$A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(\partial X, F_1) \end{matrix} \to \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2) \end{matrix},$$

where X is a manifold with boundary ∂X and E_1, E_2 (resp. F_1, F_2) are vector bundles over X (resp. ∂X). Here, $P : C_0^\infty(\Omega, \overline{E_1}) \to C^\infty(\Omega, \overline{E_2})$ is a classical pseudodifferential operator of order m on Ω, where Ω is an open neighborhood of X and $\overline{E_i}|X = E_i$ $(i = 1, 2)$. Then P has an extension: $\mathcal{E}'(\Omega, \overline{E_1}) \to \mathcal{D}'(\Omega, \overline{E_2})$, where $\mathcal{E}'(\Omega, \overline{E_1})$ and $\mathcal{D}'(\Omega, \overline{E_2})$ are the dual space of $C^\infty(\Omega, \overline{E_1})$ and $C_0^\infty(\Omega, \overline{E_2})$, respectively. Let $e^+ : C^\infty(X, E_1) \to \mathcal{E}'(\Omega, \overline{E_1})$ denote the extension by zero from X to Ω and $r^+ : \mathcal{D}'(\Omega, \overline{E_2}) \to \mathcal{D}'(\Omega, E_2)$ denote the restriction from Ω to X, then define

$$\pi^+ P = r^+ P e^+ : C^\infty(X, E_1) \to \mathcal{D}'(\Omega, E_2).$$

In addition, P is supposed to have the transmission property; this means that, for all j, k, α, the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies

$$\partial^k_{x_n} \partial^\alpha_{\xi'} p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|} \partial^k_{x_n} \partial^\alpha_{\xi'} p_j(x', 0, 0, -1),$$

then $\pi^+ P$ maps $C^\infty(X, E_1)$ into $C^\infty(X, E_2)$ by [14, Section 2.1].

Let G, T be respectively the singular Green operator and the trace operator of order m and type d. K is a potential operator and S is a classical pseudodifferential operator of order m along the boundary (for detailed definition, see [11]). Denote by $B^{m,d}$ the collection of all operators of order m and type d, and $\mathcal{B}$ is the union over all m and d.

Recall $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^{m,d} \times B^{m',d'} \to B^{m+m', \max\{m'+d, d'\}}$. Write

$$\widetilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \widetilde{S} \end{pmatrix} \in B^{m,d}, \quad \widetilde{A}' = \begin{pmatrix} \pi^+ P' + G' & K' \\ T' & \widetilde{S}' \end{pmatrix} \in B^{m',d'}.$$

The composition $\widetilde{A}\widetilde{A}'$ is obtained by multiplication of the matrices (for more details see [14]). For example $\pi^+ P \circ G'$ and $G \circ G'$ are singular Green operators of type d' and

$$\pi^+ P \circ \pi^+ P' = \pi^+(PP') + L(P, P').$$

Here PP' is the usual composition of pseudodifferential operators and $L(P, P')$ called leftover term is a singular Green operator of type $m' + d$. For our case, P, P' are classical pseudodifferential operators, in other words $\pi^+ P \in \mathcal{B}^\infty$ and $\pi^+ P' \in \mathcal{B}^\infty$.

In the following, write $\pi^+D^{-1}=\begin{pmatrix}\pi^+D^{-1} & 0\\ 0 & 0\end{pmatrix}$. Let M be a compact manifold with boundary ∂M. We assume that the metric g^M on M has the following form near the boundary

$$g^M=\frac{1}{h(x_n)}g^{\partial M}+\mathrm{d}x_n^2, \tag{2.5}$$

where $g^{\partial M}$ is the metric on ∂M. Let $U\subset M$ be a collar neighborhood of ∂M which is diffeomorphic to $\partial M\times[0,1)$. By the definition of $h(x_n)\in C^\infty([0,1))$ and $h(x_n)>0$, there exists $\widetilde{h}\in C^\infty((-\varepsilon,1))$ such that $\widetilde{h}|_{[0,1)}=h$ and $\widetilde{h}>0$ for some sufficiently small $\varepsilon>0$. Then there exists a metric $\widehat{g}$ on $\widehat{M}=M\bigcup_{\partial M}\partial M\times(-\varepsilon,0]$ which has the form on $U\bigcup_{\partial M}\partial M\times(-\varepsilon,0]$,

$$\widehat{g}=\frac{1}{\widetilde{h}(x_n)}g^{\partial M}+\mathrm{d}x_n^2, \tag{2.6}$$

such that $\widehat{g}|_M=g^M$. We fix a metric $\widehat{g}$ on the $\widehat{M}$ such that $\widehat{g}|_M=g^M$.

Consider the $(n-1)$-form

$$\sigma(\xi)=\sum_{j=1}^n(-1)^{j+1}\xi_j\mathrm{d}\xi_1\wedge\cdots\wedge\widehat{\mathrm{d}\xi_j}\wedge\cdots\wedge\mathrm{d}\xi_n,$$

where the hat indicates that the corresponding factor has been omitted.

Restricted $\sigma(\xi)$ to the $(n-1)$-dimensional unit sphere $|\xi|=1$, $\sigma(\xi)$ gives the volume form on $|\xi|=1$. Denote by $|\xi'|=1$ and $\sigma(\xi')$ the $(n-2)$-dimensional unit sphere and the corresponding $(n-2)$-form.

Denote by $\mathcal{B}^\infty$ the algebra of all operators in Boutet de Monvel's calculus (with integral order) and by $\mathcal{B}^{-\infty}$ the ideal of all smoothing operators in $\mathcal{B}^\infty$. We assume that $E_1=E_2=E$, $F_1=F_2=F$. Denote by $b(x',\xi',\xi_n,\eta_n)$ the symbol of the singular Green operator G. Then

$$\mathrm{tr}(b)=\frac{1}{2\pi}\int_{\Gamma^+}b(x',\xi',\xi_n,\eta_n)\mathrm{d}\xi_n=\overline{b}(x',\xi')$$

is the symbol on ∂X and $\overline{b}_{1-n}$ can be represented by b_{-n}. Now we recall the main theorem in [10].

Theorem 2.1 (Fedosov-Golse-Leichtnam-Schrohe) *Let X and ∂X be connected,* $\dim X=n\geq 3$, $A=\begin{pmatrix}\pi^+P+G & K\\ T & S\end{pmatrix}\in\mathcal{B}$, *and denote by p, b and s the local symbols of P, G and S, respectively. Define*

$$\begin{aligned}\widetilde{\mathrm{Wres}}(A)=&\int_X\int_{|\xi|=\mathbf{1}}\mathrm{tr}_E[p_{-n}(x,\xi)]\sigma(\xi)\mathrm{d}x\\&+2\pi\int_{\partial X}\int_{|\xi'|=\mathbf{1}}\{\mathrm{tr}_E[(\mathrm{tr}\,b_{-n})(x',\xi')]\\&+\mathrm{tr}_F[s_{1-n}(x',\xi')]\}\sigma(\xi')\,\mathrm{d}x',\end{aligned} \tag{2.7}$$

then

(a) $\widetilde{\mathrm{Wres}}([A,B])=0$ *for any $A,B\in\mathcal{B}$;*

(b) *it is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.*

3 Conformal Perturbations of Twisted Dirac Operator and Noncommutative Residue

In this section we consider an n-dimensional oriented Riemannian manifold (M, g^M) equipped with a fixed spin structure. Let $S(TM)$ be the spinor bundle and F be an additional smooth vector bundle equipped with a non-unitary connection $\widetilde{\nabla}^F$. Let $S_1, S_2 \in \Gamma(F)$, g^F be a metric on F. We define the dual connection $\widetilde{\nabla}^{F,*}$ by

$$g^F(\widetilde{\nabla}^F_X S_1, S_2) + g^F(S_1, \widetilde{\nabla}^{F,*}_X S_2) = X(g^F(S_1, S_2))$$

for $X \in \Gamma(TM)$ and define

$$\nabla^F = \frac{\widetilde{\nabla}^F + \widetilde{\nabla}^{F,*}}{2}, \quad A = \frac{\widetilde{\nabla}^F - \widetilde{\nabla}^{F,*}}{2}, \tag{3.1}$$

then ∇^F is a metric connection and A is an endomorphism of F with a 1-form coefficient. We consider the tensor product vector bundle $S(TM) \otimes F$, which becomes a Clifford module via the definition

$$c(a) = c(a) \otimes \mathrm{id}_F, \quad a \in TM \tag{3.2}$$

and which we equip with the compound connection

$$\widetilde{\nabla}^{S(TM)\otimes F} = \nabla^{S(TM)} \otimes \mathrm{id}_F + \mathrm{id}_{S(TM)} \otimes \widetilde{\nabla}^F. \tag{3.3}$$

Let

$$\nabla^{S(TM)\otimes F} = \nabla^{S(TM)} \otimes \mathrm{id}_F + \mathrm{id}_{S(TM)} \otimes \nabla^F, \tag{3.4}$$

then the spinor connection $\widetilde{\nabla}^{S(TM)\otimes F}$ induced by $\nabla^{S(TM)\otimes F}$ is locally given by

$$\widetilde{\nabla}^{S(TM)\otimes F} = \nabla^{S(TM)} \otimes \mathrm{id}_F + \mathrm{id}_{S(TM)} \otimes \nabla^F + \mathrm{id}_{S(TM)} \otimes A. \tag{3.5}$$

Let $\{e_i\}(1 \le i, j \le n)$ (resp. $\{\partial_i\}$) be the orthonormal frames (resp. natural frames) on TM. Set

$$D_F = \sum_{i,j=1}^n g^{ij} c(\partial_i) \nabla^{S(TM)\otimes F}_{\partial_j} = \sum_{j=1}^n c(e_j) \nabla^{S(TM)\otimes F}_{e_j}, \tag{3.6}$$

where $\nabla^{S(TM)\otimes F}_{\partial_j} = \partial_j + \sigma^s_j + \sigma^F_j$ and $\sigma^s_j = \frac{1}{4} \sum_{j,k=1}^n \langle \nabla^{TM}_{\partial_i} e_j, e_k \rangle c(e_j) c(e_k)$, σ^F_j is the connection matrix of ∇^F.

Then the twisted Dirac operators $\widetilde{D}_F$ and $\widetilde{D}^*_F$ associated with the connections $\widetilde{\nabla}^F$ and $\widetilde{\nabla}^{F,*}$ satisfy that for $\psi \otimes \chi \in S(TM) \otimes F$, we have

$$\widetilde{D}_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) + c(A)(\psi \otimes \chi), \tag{3.7}$$

$$\widetilde{D}^*_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) - c(A^*)(\psi \otimes \chi), \tag{3.8}$$

where $c(A) = \sum_{i=1}^n c(e_i) \otimes A(e_i)$ and $c(A^*) = \sum_{i=1}^n c(e_i) \otimes A^*(e_i)$, $A^*(e_i)$ denotes the adjoint of $A(e_i)$.

Then, we obtain

$$\widetilde{D}_F=\sum_{j=1}^{n}c(e_j)\nabla^{S(TM)\otimes F}_{e_j}+c(A), \tag{3.9}$$

$$\widetilde{D}_F^*=\sum_{j=1}^{n}c(e_j)\nabla^{S(TM)\otimes F}_{e_j}-c(A^*). \tag{3.10}$$

Let ∇^{TM} denote the Levi-Civita connection about g^M. In the local coordinates $\{x_i; 1\leq i\leq n\}$ and the fixed orthonormal frame $\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^{TM}(\widetilde{e_1},\cdots,\widetilde{e_n})=(\widetilde{e_1},\cdots,\widetilde{e_n})(\omega_{s,t}). \tag{3.11}$$

Let $c(\widetilde{e_i})$ denote the Clifford action, $g^{ij}=g(\mathrm{d}x_i,\mathrm{d}x_j)$, $\nabla^{TM}_{\partial_i}\partial_j=\sum\limits_k\Gamma^k_{ij}\partial_k$, $\Gamma^k=g^{ij}\Gamma^k_{ij}$ and the cotangent vector $\xi=\sum\xi_j\mathrm{d}x_j$ and $\xi^j=g^{ij}\xi_i$. By [13, Lemma 1] and [12, Lemma 2.1], for any fixed point $x_0\in\partial M$, choosing the normal coordinates U of x_0 in ∂M (not in M). Denote by $\sigma_l(P)$ the l-order symbol of an operator P. By the composition formula and [12, (2.2.11)], we obtain the following lemma (see [19, Lemma 2.6]).

Lemma 3.1 *Let $\widetilde{D}_F^*,\widetilde{D}_F$ be the twisted Dirac operators on $\Gamma(S(TM)\otimes F)$, then*

$$\sigma_{-1}(\widetilde{D}_F^*)^{-1}=\sigma_{-1}(\widetilde{D}_F^{-1})=\frac{\sqrt{-1}c(\xi)}{|\xi|^2}, \tag{3.12}$$

$$\sigma_{-2}(\widetilde{D}_F^*)^{-1}=\frac{c(\xi)\sigma_0(\widetilde{D}_F^*)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}[c(\xi)]|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)], \tag{3.13}$$

$$\sigma_{-2}(\widetilde{D}_F^{-1})=\frac{c(\xi)\sigma_0(\widetilde{D}_F)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}[c(\xi)]|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)], \tag{3.14}$$

where

$$\sigma_0(\widetilde{D}_F^*)=-\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_l)c(e_l)c(e_s)c(e_t)+\sum_{j=1}^{n}c(e_j)(\sigma_j^F-A^*(e_j)), \tag{3.15}$$

$$\sigma_0(\widetilde{D}_F)=-\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_l)c(e_l)c(e_s)c(e_t)+\sum_{j=1}^{n}c(e_j)(\sigma_j^F+A(e_j)). \tag{3.16}$$

For convenience, let $\lambda=\sum\limits_{j=1}^{n}c(e_j)(\sigma_j^F-A^*(e_j))$, $\mu=\sum\limits_{j=1}^{n}c(e_j)(\sigma_j^F+A(e_j))$. Let M be a six-dimensional compact spin manifolds with the boundary ∂M. In the following, we will compute the more general case $\widetilde{\mathrm{Wres}}[\pi^+(f\widetilde{D}_F^{-1})\circ\pi^+(f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1})]$ for nonzero smooth functions $f,\ f^{-1}$. An application of [14, (3.5)–(3.6)] shows that

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(f\widetilde{D}_F^{-1})\circ\pi^+(f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1})]\\&=\int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)\otimes F}[\sigma_{-n}((\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1})^{-2})]\sigma(\xi)\mathrm{d}x+\int_{\partial M}\Phi,\end{aligned} \tag{3.17}$$

where

$$\Phi=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-\mathrm{i})^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!}\mathrm{trace}_{S(TM)\otimes F}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r(f\widetilde{D}_F^{-1})(x',0,\xi',\xi_n)$$

$$\times \partial_{x_n}^{\alpha}\partial_{\xi_n}^{j+1}\partial_{x_n}^{k}\sigma_l(f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1})(x',0,\xi',\xi_n)]\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x', \tag{3.18}$$

and the sum is taken over $r-k+|\alpha|+\ell-j-1=-n=-6, r\leq -1, \ell\leq -3$.

Note that

$$\begin{aligned}
&f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1}\\
&=(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)^{-1}\\
&=(\widetilde{D}_F^*f\cdot\widetilde{D}_F\widetilde{D}_F^*f^{-1}\cdot f-\widetilde{D}_F^*f\cdot\widetilde{D}_F\cdot[\widetilde{D}_F^*,f^{-1}]\cdot f)^{-1}\\
&=(\widetilde{D}_F^*f\cdot\widetilde{D}_F\widetilde{D}_F^*-\widetilde{D}_F^*f\cdot\widetilde{D}_Fc(\mathrm{d}f^{-1})f)^{-1}\\
&=(f\cdot\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*+[\widetilde{D}_F^*,f]\widetilde{D}_F\widetilde{D}_F^*-\widetilde{D}_F^*f\cdot\widetilde{D}_Fc(\mathrm{d}f^{-1})f)^{-1}\\
&=(f\cdot\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*+c(\mathrm{d}f)\widetilde{D}_F\widetilde{D}_F^*-\widetilde{D}_F^*f\cdot\widetilde{D}_Fc(\mathrm{d}f^{-1})f)^{-1}\\
&=(f\cdot\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*+c(\mathrm{d}f)\widetilde{D}_F\widetilde{D}_F^*-\widetilde{D}_F^*\widetilde{D}_Ff\cdot c(\mathrm{d}f^{-1})\cdot f+\widetilde{D}_F^*\cdot c(\mathrm{d}f)c(\mathrm{d}f^{-1})f)^{-1}.
\end{aligned} \tag{3.19}$$

In order to get the symbol of operators $\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f$. We first give the specification of $\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*$, $\widetilde{D}_F^*\widetilde{D}_F$ and $\widetilde{D}_F\widetilde{D}_F^*$. By (3.9)–(3.10), we have

$$\begin{aligned}
&\widetilde{D}_F\widetilde{D}_F^*\\
&=D_F^2-D_Fc(A^*)+c(A)D_F-c(A)c(A^*)\\
&=-g^{ij}\partial_i\partial_j-2\sigma^j_{S(TM)\otimes F}\partial_j+\Gamma^k\partial_k+\sum_{j=1}^{n}[c(A)c(e_j)-c(e_j)c(A^*)]e_j-\sum_{j=1}^{n}c(e_j)\sigma_j^{S(TM)\otimes F}\\
&\quad\times c(A^*)-g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F})+\sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F}-\Gamma^k_{ij}\sigma^k_{S(TM)\otimes F}]+\frac{1}{4}s+\frac{1}{2}\\
&\quad\times\sum_{i\neq j}R^F(e_i,e_j)c(e_i)c(e_j)+\sum_{j=1}^{n}[c(A)c(e_j)]\sigma_j^{S(TM)\otimes F}-\sum_{j=1}^{n}c(e_j)e_j(c(A^*))\\
&\quad-c(A)c(A^*)
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\widetilde{D}_F^*\widetilde{D}_F\\
&=D_F^2-c(A^*)D_F+D_Fc(A)-c(A)c(A^*)\\
&=-g^{ij}\partial_i\partial_j-2\sigma^j_{S(TM)\otimes F}\partial_j+\Gamma^k\partial_k+\sum_{j=1}^{n}[c(e_j)c(A)-c(A^*)c(e_j)]e_j+\sum_{j=1}^{n}c(e_j)\sigma_j^{S(TM)\otimes F}\\
&\quad\times c(A)-g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F})+\sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F}-\Gamma^k_{ij}\sigma^k_{S(TM)\otimes F}]+\frac{1}{4}s+\frac{1}{2}\\
&\quad\times\sum_{i\neq j}R^F(e_i,e_j)c(e_i)c(e_j)-\sum_{j=1}^{n}[c(A^*)c(e_j)]\sigma_j^{S(TM)\otimes F}+\sum_{j=1}^{n}c(e_j)e_j(c(A))\\
&\quad-c(A^*)c(A).
\end{aligned} \tag{3.21}$$

Combining (3.10) and (3.20), we obtain

$$\begin{aligned}
&\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*\\
&=\sum_{i,j,l=1}^{n}\sum_{r=1}^{n}c(e_r)\langle e_r,\mathrm{d}x_l\rangle(-g^{ij}\partial_l\partial_i\partial_j)+\sum_{r,l=1}^{n}c(e_r)\langle e_r,\mathrm{d}x_l\rangle\Big\{-\sum_{i,j=1}^{n}(\partial_lg^{ij})\partial_i\partial_j-\sum_{i,j,k=1}^{n}g^{ij}
\end{aligned}$$

$$\times (4\sigma_i^{S(TM)\otimes F}\partial_j - 2\Gamma_{ij}^k\partial_k)\partial_l\Big\} + \sum_{r,l=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle\Big\{ - 2\sum_{i,j=1}^n (\partial_l g^{ij})\sigma_i^{S(TM)\otimes F}\partial_j + \sum_{i,j,k=1}^n g^{ij}$$
$$\times (\partial_l\Gamma_{ij}^k)\partial_k - 2\sum_{i,j=1}^n g^{ij}(\partial_l\sigma_i^{S(TM)\otimes F})\partial_j + \sum_{i,j,k=1}^n (\partial_l g^{ij})\Gamma_{ij}^k\partial_k + \sum_{j,k=1}^n [\partial_l(c(A)c(e_j) - c(e_j)$$
$$\times c(A^*))]\langle e_j, \mathrm{d}x^k\rangle\partial_k + \sum_{j,k=1}^n (c(A)c(e_j) - c(e_j)c(A^*))[\partial_l\langle e_j, \mathrm{d}x^k\rangle]\partial_k\Big\} + \sum_{r,l=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle$$
$$\times \partial_l\Big\{ - \sum_{i,j,k=1}^n g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F}) + \sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F} - \Gamma_{ij}^k\sigma^k_{S(TM)\otimes F}] + \frac{1}{4}s + \sum_{j=1}^n [c(A)$$
$$\times c(e_j)]\sigma_j^{S(TM)\otimes F} - \sum_{j=1}^n c(e_j)e_j(c(A^*)) - \sum_{j=1}^n c(e_j)\sigma_j^{S(TM)\otimes F}c(A^*) - c(A)c(A^*) + \frac{1}{2}$$
$$\times \sum_{i\neq j} R^F(e_i, e_j)c(e_i)c(e_j)\Big\} + \sigma_0(\widetilde{D}_F^*)\sum_{j,i=1}^n (-g^{ij}\partial_i\partial_j) + \sum_{r,l=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle\Big\{2\sum_{j,k=1}^n [c(A)$$
$$\times c(e_j) - c(e_j)c(A^*)]\langle e_i, \mathrm{d}x_k\rangle\Big\}\partial_l\partial_k + \sigma_0(\widetilde{D}_F^*)\Big\{ - 2\sigma^j_{S(TM)\otimes F}\partial_j + \Gamma^k\partial_k + \sum_{j=1}^n [c(A)c(e_j)$$
$$- c(e_j)c(A^*)]e_j - \sum_{j=1}^n c(e_j)e_j(c(A^*)) - g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F}) + \sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F}$$
$$- \Gamma_{ij}^k\sigma^k_{S(TM)\otimes F}] + \frac{1}{4}s - c(A)c(A^*) + \sum_{j=1}^n [c(A)c(e_j)]\sigma_j^{S(TM)\otimes F} - \sum_{j=1}^n c(e_j)\sigma_j^{S(TM)\otimes F}$$
$$\times c(A^*) + \frac{1}{2}\sum_{i\neq j} R^F(e_i, e_j)c(e_i)c(e_j)\Big\}. \tag{3.22}$$

Thus, using (3.19)–(3.22), we get the specification of $\widetilde{D}_F^* f \cdot \widetilde{D}_F f^{-1} \cdot \widetilde{D}_F^* f$.

$$\widetilde{D}_F^* f \cdot \widetilde{D}_F f^{-1} \cdot \widetilde{D}_F^* f$$
$$= f \cdot \widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^* + c(\mathrm{d}f)\widetilde{D}_F\widetilde{D}_F^* - \widetilde{D}_F^*\widetilde{D}_F f \cdot c(\mathrm{d}f^{-1}) \cdot f + \widetilde{D}_F^* \cdot c(\mathrm{d}f)c(\mathrm{d}f^{-1})f$$
$$= f \cdot \Big\{ \sum_{i,j,l=1}^n \sum_{r=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle(-g^{ij}\partial_l\partial_i\partial_j) + \sum_{r,l=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle\Big\{ - \sum_{i,j=1}^n (\partial_l g^{ij})\partial_i\partial_j$$
$$- \sum_{i,j,k=1}^n g^{ij}(4\sigma_i^{S(TM)\otimes F}\partial_j - 2\Gamma_{ij}^k\partial_k)\partial_l\Big\} + \sum_{r,l=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle\Big\{ - 2\sum_{i,j=1}^n (\partial_l g^{ij})\sigma_i^{S(TM)\otimes F}\partial_j$$
$$+ \sum_{i,j,k=1}^n g^{ij}(\partial_l\Gamma_{ij}^k)\partial_k - 2\sum_{i,j=1}^n g^{ij}(\partial_l\sigma_i^{S(TM)\otimes F})\partial_j + \sum_{i,j,k=1}^n (\partial_l g^{ij})\Gamma_{ij}^k\partial_k + \sum_{j,k=1}^n [\partial_l(c(A)$$
$$\times c(e_j) - c(e_j)c(A^*))]\langle e_j, \mathrm{d}x^k\rangle\partial_k + \sum_{j,k=1}^n (c(A)c(e_j) - c(e_j)c(A^*))[\partial_l\langle e_j, \mathrm{d}x^k\rangle]\partial_k\Big\}$$
$$+ \sum_{r,l=1}^n c(e_r)\langle e_r, \mathrm{d}x_l\rangle\partial_l\Big\{ - \sum_{i,j,k=1}^n g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F}) + \sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F} - \Gamma_{ij}^k\sigma^k_{S(TM)\otimes F}]$$
$$+ \frac{1}{4}s + \sum_{j=1}^n [c(A)c(e_j)]\sigma_j^{S(TM)\otimes F} - \sum_{j=1}^n c(e_j)e_j(c(A^*)) - \sum_{j=1}^n c(e_j)\sigma_j^{S(TM)\otimes F}c(A^*) - c(A)c(A^*)$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i\neq j}R^F(e_i,e_j)c(e_i)c(e_j)\Big\}+\sigma_0(\widetilde{D}_F^*)\sum_{j,i=1}^{n}(-g^{ij}\partial_i\partial_j)+\sum_{r,l=1}^{n}c(e_r)\langle e_r,\mathrm{d}x_l\rangle\Big\{2\sum_{j,k=1}^{n}[c(A)c(e_j)\\
&-c(e_j)c(A^*)]\langle e_i,\mathrm{d}x_k\rangle\Big\}\partial_l\partial_k+\sigma_0(\widetilde{D}_F^*)\Big\{-2\sigma^j_{S(TM)\otimes F}\partial_j+\Gamma^k\partial_k+\sum_{j=1}^{n}[c(A)c(e_j)-c(e_j)\\
&\times c(A^*)]e_j-\sum_{j=1}^{n}c(e_j)e_j(c(A^*))-g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F})+\sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F}-\Gamma^k_{ij}\sigma^k_{S(TM)\otimes F}]\\
&+\frac{1}{4}s-c(A)c(A^*)+\sum_{j=1}^{n}[c(A)c(e_j)]\sigma_j^{S(TM)\otimes F}-\sum_{j=1}^{n}c(e_j)\sigma_j^{S(TM)\otimes F}c(A^*)\\
&+\frac{1}{2}\sum_{i\neq j}R^F(e_i,e_j)c(e_i)c(e_j)\Big\}\Big\}+c(\mathrm{d}f)\cdot\Big\{-g^{ij}\partial_i\partial_j-2\sigma^j_{S(TM)\otimes F}\partial_j+\Gamma^k\partial_k\\
&+\sum_{j=1}^{n}[c(A)c(e_j)-c(e_j)c(A^*)]e_j-\sum_{j=1}^{n}c(e_j)\sigma_j^{S(TM)\otimes F}\times c(A^*)-g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F})\\
&+\sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F}-\Gamma^k_{ij}\sigma^k_{S(TM)\otimes F}]+\frac{1}{4}s+\frac{1}{2}\sum_{i\neq j}R^F(e_i,e_j)\times c(e_i)c(e_j)\\
&+\sum_{j=1}^{n}[c(A)c(e_j)]\sigma_j^{S(TM)\otimes F}-\sum_{j=1}^{n}c(e_j)e_j(c(A^*))-c(A)c(A^*)\Big\}-\Big\{-g^{ij}\partial_i\times\partial_j\\
&-2\sigma^j_{S(TM)\otimes F}\partial_j+\Gamma^k\partial_k+\sum_{j=1}^{n}[c(e_j)c(A)-c(A^*)c(e_j)]e_j+\sum_{j=1}^{n}c(e_j)\sigma_j^{S(TM)\otimes F}c(A)\\
&-g^{ij}[(\partial_i\sigma^j_{S(TM)\otimes F})+\sigma^i_{S(TM)\otimes F}\sigma^j_{S(TM)\otimes F}-\Gamma^k_{ij}\sigma^k_{S(TM)\otimes F}]+\frac{1}{4}s+\frac{1}{2}\sum_{i\neq j}R^F(e_i,e_j)c(e_i)\\
&\times c(e_j)-\sum_{j=1}^{n}[c(A^*)c(e_j)]\sigma_j^{S(TM)\otimes F}+\sum_{j=1}^{n}c(e_j)e_j(c(A))-c(A^*)c(A)\Big\}f\cdot c(\mathrm{d}f^{-1})\cdot f\\
&+\Big\{\sum_{i,j=1}^{n}g^{ij}c(\partial_i)(\partial_j+\sigma_j^{S(TM)\otimes F})-c(A^*)\Big\}\cdot c(\mathrm{d}f)c(\mathrm{d}f^{-1})f. \qquad (3.23)
\end{aligned}
$$

Let $\partial^j=g^{ij}\partial_i,\sigma^i=g^{ij}\sigma_j$. By the above formulas, we obtain the following lemma.

Lemma 3.2 *Let $\widetilde{D}_F^*,\widetilde{D}_F$ be the twisted Dirac operators on $\Gamma(S(TM)\otimes F)$,*

$$\sigma_3(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)=f\sigma_3(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)=\sqrt{-1}c(\xi)|\xi|^2f, \qquad (3.24)$$

$$\sigma_2(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)=f\sigma_2(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)+2c(\mathrm{d}f)|\xi|^2, \qquad (3.25)$$

where $\sigma_2(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)=c(\xi)(4\sigma^k-2\Gamma^k)\xi_k-\frac{1}{4}|\xi|^2h'(0)c(\mathrm{d}x_n)+\lambda|\xi|^2-2c(\xi)c(A)c(\xi)-2|\xi|^2c(A^*)$.

For convenience, we write that $\sigma_2(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)=G+\lambda|\xi|^2-2c(\xi)c(A)c(\xi)-2|\xi|^2c(A^*)$. In order to get the symbol of operators $\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f$. We first give the following formulas

$$
\begin{aligned}
D_x^\alpha=(-\sqrt{-1})^{|\alpha|}\partial_x^\alpha,\quad \sigma(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)&=p_3+p_2+p_1+p_0,\\
\sigma((\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)^{-1})&=\sum_{j=3}^{\infty}q_{-j}. \qquad (3.26)
\end{aligned}
$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1&=\sigma[(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)\circ(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)^{-1}]\\&=(p_3+p_2+p_1+p_0)(q_{-3}+q_{-4}+q_{-5}+\cdots)\\&\quad+\sum_j(\partial_{\xi_j}p_3+\partial_{\xi_j}p_2+\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-3}+D_{x_j}q_{-4}+D_{x_j}q_{-5}+\cdots)\\&=p_3q_{-3}+\Big(p_3q_{-4}+p_2q_{-3}+\sum_j\partial_{\xi_j}p_3D_{x_j}q_{-3}\Big)+\cdots.\end{aligned}\tag{3.27}$$

Then

$$q_{-3}=p_3^{-1},\quad q_{-4}=-p_3^{-1}\Big[p_2p_3^{-1}+\sum_j\partial_{\xi_j}p_3D_{x_j}(p_3^{-1})\Big].\tag{3.28}$$

By [12, Lemma 2.1] and (3.24)–(3.25), we obtain the following lemma.

Lemma 3.3 *Let $\widetilde{D}_F^*,\widetilde{D}_F$ be the twisted Dirac operators on $\Gamma(S(TM)\otimes F)$, then*

$$\sigma_{-3}(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)^{-1}=f^{-1}\sigma_{-3}(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1}=\frac{\sqrt{-1}c(\xi)}{f|\xi|^4},\tag{3.29}$$

$$\begin{aligned}\sigma_{-4}(\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1}\cdot\widetilde{D}_F^*f)^{-1}&=f^{-1}\sigma_{-4}(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1}+\frac{2c(\xi)c(\mathrm{d}f)c(\xi)}{f^2|\xi|^6}\\&\quad+\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8},\end{aligned}\tag{3.30}$$

where

$$\begin{aligned}&\sigma_{-4}(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1}\\&=\frac{c(\xi)\sigma_2(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)c(\xi)}{|\xi|^8}+\frac{c(\xi)}{|\xi|^{10}}\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)][\partial_{x_j}[c(\xi)]|\xi|^2-2c(\xi)\partial_{x_j}(|\xi|^2)]\\&=\frac{c(\xi)Gc(\xi)}{|\xi|^8}+\frac{c(\xi)\lambda c(\xi)}{|\xi|^6}-\frac{2c(A)}{|\xi|^4}-\frac{2c(\xi)c(A^*)c(\xi)}{|\xi|^6}+\frac{c(\xi)}{|\xi|^{10}}\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]\\&\quad\times[\partial_{x_j}[c(\xi)]|\xi|^2-2c(\xi)\partial_{x_j}(|\xi|^2)].\end{aligned}\tag{3.31}$$

Locally we can use [19, Theorem 2.5] to compute the interior term of (3.17), then

$$\begin{aligned}&\int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)\otimes F}[\sigma_{-n}((\widetilde{D}_F^*f\cdot\widetilde{D}_Ff^{-1})^{-2})]\sigma(\xi)\mathrm{d}x\\&=8\pi^3\int_M\Big\{\mathrm{trace}\Big[-\frac{s}{12}+c(A^*)c(A)-\frac14\sum_i[c(A^*)c(e_i)-c(e_i)c(A)]^2\\&\quad-\frac12\sum_j\nabla_{e_j}^F(c(A^*))c(e_j)-\frac12\sum_jc(e_j)\nabla_{e_j}^F(c(A))\Big]-2f^{-1}\Delta(f)\\&\quad+4f^{-1}\mathrm{trace}[A(\mathrm{grad}_Mf)]-f^2[|\mathrm{grad}_M(f)|^2+2\Delta(f)]\Big\}\mathrm{d\,vol}_M.\end{aligned}\tag{3.32}$$

So we only need to compute $\int_{\partial M}\Phi$.

From formula (3.18) for the definition of Φ, now we can compute Φ. Since the sum is taken over $r+\ell-k-j-|\alpha|-1=-6$, $r\leq -1$, $\ell\leq -3$, we have $\int_{\partial M}\Phi$ is the sum of the following five cases:

Case (a) (I) $r=-1$, $l=-3$, $j=k=0$, $|\alpha|=1$.

By (3.18), we get

$$\begin{aligned}
&\text{case (a) (I)}\\
&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(f\widetilde{D}_F^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-3}(f^{-1}(\widetilde{D}_F^{*})^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^{*})^{-1})]\\
&\quad(x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}(f\sigma_{-1}(\widetilde{D}_F^{-1}))\times\partial^{\alpha}_{x'}\partial_{\xi_n}(f^{-1}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})](x_0)\\
&\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&\quad-f\sum_{j<n}\partial_j(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\\
&\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned}\tag{3.33}$$

By [12, Lemma 2.2] and (3.29), for $i<n$, we have

$$\begin{aligned}
&\partial_{x_i}\sigma_{-3}((\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})(x_0)=\partial_{x_i}\left[\frac{\sqrt{-1}c(\xi)}{|\xi|^4}\right](x_0)\\
&=\sqrt{-1}\partial_{x_i}[c(\xi)]|\xi|^{-4}(x_0)-2\sqrt{-1}c(\xi)\partial_{x_i}[|\xi|^2]|\xi|^{-6}(x_0)=0.
\end{aligned}\tag{3.34}$$

Thus we have

$$\begin{aligned}
&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\\
&\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'=0.
\end{aligned}\tag{3.35}$$

By (3.12) and direct calculations, for $i<n$, we obtain

$$\begin{aligned}
&\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})(x_0)|_{|\xi'|=1}=\partial_{\xi_i}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})(x_0)|_{|\xi'|=1}\\
&=\frac{c(\mathrm{d}x_i)}{2(\xi_n-\sqrt{-1})}-\frac{\xi_i(\xi_n-2\sqrt{-1})c(\xi')+\xi_ic(\mathrm{d}x_n)}{2(\xi_n-\sqrt{-1})^2},
\end{aligned}\tag{3.36}$$

and we get

$$\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}=\frac{\sqrt{-1}c(\mathrm{d}x_n)}{|\xi|^4}-\frac{4\sqrt{-1}[\xi_nc(\xi')+\xi_n^2c(\mathrm{d}x_n)]}{|\xi|^6}.\tag{3.37}$$

Then for $i<n$, we have

$$\begin{aligned}
&\mathrm{trace}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\\
&=-\xi_i\mathrm{trace}\Big[\frac{c(\mathrm{d}x_n)^2}{2(\xi_n-\sqrt{-1})^2}\Big]-4\sqrt{-1}\xi_n\xi_i\mathrm{trace}\Big[\frac{c(\mathrm{d}x_i)^2}{2(\xi_n-\sqrt{-1})|\xi|^6}\Big]+4\sqrt{-1}\xi_n\xi_i(\xi_n-2\sqrt{-1})
\end{aligned}$$

$$\times\,\mathrm{trace}\Big[\frac{c(\xi')^2}{2(\xi_n-\sqrt{-1})^2|\xi|^6}\Big]+4\sqrt{-1}\xi_n^2\xi_i\mathrm{trace}\Big[\frac{c(\mathrm{d}x_n)^2}{2(\xi_n-\sqrt{-1})^2|\xi|^6}\Big].\tag{3.38}$$

We note that $i<n$, $\int_{|\xi'|=1}\xi_i\sigma(\xi')=0$, so

$$-f\sum_{j<n}\partial_j(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)$$
$$\times\,\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'=0.\tag{3.39}$$

Then we have case (a) (I) $=0$.

Case (a) (II) $r=-1,l=-3,|\alpha|=k=0,j=1$.

By (3.18), we have

$$\begin{aligned}&\text{case (a) (II)}\\ &=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(f\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}(f^{-1}(\widetilde{D}_F^{*})^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^{*})^{-1})](x_0)\\ &\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\ &=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\ &\quad-\frac{1}{2}f^{-1}\partial_{x_n}(f)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\\ &\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.\end{aligned}\tag{3.40}$$

By [12, (2.2.23)] and (3.12), we have

$$\begin{aligned}\pi_{\xi_n}^{+}\partial_{x_n}\sigma_{-1}(\widetilde{D}_F^{-1})(x_0)|_{|\xi'|=1}&=\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-\sqrt{-1})}+\sqrt{-1}h'(0)\Big[\frac{\sqrt{-1}c(\xi')}{4(\xi_n-\sqrt{-1})}\\ &\quad+\frac{c(\xi')+\sqrt{-1}c(\mathrm{d}x_n)}{4(\xi_n-\sqrt{-1})^2}\Big].\end{aligned}\tag{3.41}$$

By (3.29) and direct calculations, we have

$$\partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})=\frac{-4\sqrt{-1}\xi_nc(\xi')+\sqrt{-1}(1-3\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\tag{3.42}$$

and

$$\partial_{\xi_n}^2\sigma_{-3}((\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})=\sqrt{-1}\Big[\frac{(20\xi_n^2-4)c(\xi')+12(\xi_n^3-\xi_n)c(\mathrm{d}x_n)}{(1+\xi_n^2)^4}\Big].\tag{3.43}$$

Since $n=6$, $\mathrm{trace}_{S(TM)\otimes F}[-\mathrm{id}]=-8\dim F$. By the relation of the Clifford action and $\mathrm{trace}\,PQ=\mathrm{trace}\,QP$, we have

$$\begin{aligned}&\mathrm{trace}[c(\xi')c(\mathrm{d}x_n)]=0,\quad\mathrm{trace}[c(\mathrm{d}x_n)^2]=-8\dim F,\quad\mathrm{trace}[c(\xi')^2](x_0)|_{|\xi'|=1}=-8\dim F,\\ &\mathrm{trace}[\partial_{x_n}[c(\xi')]c(\mathrm{d}x_n)]=0,\quad\mathrm{trace}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1}=-4h'(0)\dim F.\end{aligned}\tag{3.44}$$

By (3.41)–(3.44), we get

$$\mathrm{trace}[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}((\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})](x_0)$$

$$= h'(0)\dim F\frac{-8-24\xi_n\sqrt{-1}+40\xi_n^2+24\sqrt{-1}\xi_n^3}{(\xi_n-\sqrt{-1})^6(\xi_n+\sqrt{-1})^4}. \tag{3.45}$$

Then we obtain

$$-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -\frac{15}{16}\pi h'(0)\Omega_4\dim F\mathrm{d}x'. \tag{3.46}$$

On the other hand, by calculations, we have

$$\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi')+\sqrt{-1}c(\mathrm{d}x_n)}{2(\xi_n-\sqrt{-1})}. \tag{3.47}$$

By (3.42), (3.44) and (3.47), we get

$$\mathrm{trace}[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}((\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})](x_0)$$
$$= -16\dim F\frac{5\xi_n^2\sqrt{-1}-\sqrt{-1}-3\xi_n^3+3\xi_n}{(\xi_n-\sqrt{-1})^5(\xi_n+\sqrt{-1})^4}. \tag{3.48}$$

Then we obtain

$$-\frac{1}{2}f^{-1}\partial_{x_n}(f)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}^2\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= \frac{5\sqrt{-1}+44}{4}\pi f^{-1}\partial_{x_n}(f)\cdot\dim F\Omega_4\mathrm{d}x', \tag{3.49}$$

where Ω_4 is the canonical volume of S_4.

Combining (3.40), (3.46) and (3.49), we obtain

$$\text{case (a) (II)} = -\frac{15}{16}\pi h'(0)\Omega_4\dim F\mathrm{d}x' + \frac{5\sqrt{-1}+44}{4}\pi f^{-1}\partial_{x_n}(f)\cdot\Omega_4\dim F\mathrm{d}x'. \tag{3.50}$$

Case (a) (III) $r=-1$, $l=-3$, $|\alpha|=j=0$, $k=1$.

By (3.18), we have

$$\begin{aligned}
&\text{case (a) (III)}\\
&= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}(f\widetilde{D}_F^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(f^{-1}(\widetilde{D}_F^{*})^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^{*})^{-1})]\\
&\quad\times(x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi_{\xi_n}^{+}(\sigma_{-1}(\widetilde{D}_F^{-1}))\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&\quad-\frac{1}{2}f\partial_{x_n}(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)\\
&\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned} \tag{3.51}$$

By [12, (2.2.29)], we have

$$\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi')+\sqrt{-1}c(\mathrm{d}x_n)}{2(\xi_n-\sqrt{-1})^2}. \tag{3.52}$$

By (3.29) and direct calculations, we have

$$\begin{aligned}&\partial_{\xi_n}\partial_{x_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})\\&=-\frac{4\sqrt{-1}\xi_n\partial_{x_n}c(\xi')(x_0)}{(1+\xi_n^2)^3}+\frac{12\sqrt{-1}h'(0)\xi_nc(\xi')}{(1+\xi_n^2)^4}-\frac{\sqrt{-1}(2-10\xi_n^2)h'(0)c(\mathrm{d}x_n)}{(1+\xi_n^2)^4}.\end{aligned}\tag{3.53}$$

Combining (3.44) and (3.52)–(3.53), we have

$$\begin{aligned}&\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})](x_0)|_{|\xi'|=1}\\&=h'(0)\dim F\frac{8\sqrt{-1}-32\xi_n-8\sqrt{-1}\xi_n^2}{(\xi_n-\sqrt{-1})^5(\xi+\mathrm{i})^4}\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}&\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})](x_0)|_{|\xi'|=1}\\&=-4\dim F\frac{4\sqrt{-1}\xi_n+1-3\xi_n^2}{(\xi_n-\sqrt{-1})^5(\xi_n+\sqrt{-1})^3}.\end{aligned}\tag{3.55}$$

Then

$$\begin{aligned}&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}(\sigma_{-1}(\widetilde{D}_F^{-1}))\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=\frac{25}{16}\pi h'(0)\Omega_4\dim F\mathrm{d}x'\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}&-\frac{1}{2}f\partial_{x_n}(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1}](x_0)\\&\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=\frac{\pi\sqrt{-1}}{16}\cdot f\cdot\partial_{x_n}(f^{-1})\Omega_4\dim F\mathrm{d}x',\end{aligned}\tag{3.57}$$

where Ω_4 is the canonical volume of S_4.

Then

$$\text{case (a) (III)}=\Big[\frac{25}{16}\pi h'(0)+\frac{\pi\sqrt{-1}}{16}\cdot f\cdot\partial_{x_n}(f^{-1})\Big]\Omega_4\dim F\mathrm{d}x'.\tag{3.58}$$

Case (b) $r=-1$, $l=-4$, $|\alpha|=j=k=0$.

By (3.18), we have

$$\begin{aligned}&\text{case (b)}\\&=-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-1}(f\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-4}(f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1})](x_0)\\&\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(f\widetilde{D}_F^{-1})\times\partial_{\xi_n}\Big(f^{-1}\sigma_{-4}(\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1}+\frac{2c(\xi)c(\mathrm{d}f)c(\xi)}{f^2|\xi|^6}\\&\quad+\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1}) \times \partial_{\xi_n}(\sigma_{-4}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&\quad - 2\mathrm{i}f^{-1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}\Big[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1}) \times \partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big)\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&\quad - f\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}\Big[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1}) \times \partial_{\xi_n}\Big(\frac{ic(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\Big] \\
&\quad \times (x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned} \tag{3.59}
$$

In the normal coordinate, $g^{ij}(x_0) = \delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0) = 0$, if $j < n$; $\partial_{x_j}(g^{\alpha\beta})(x_0) = h'(0)\delta_\beta^\alpha$, if $j = n$. So by [12, Lemma A.2], we have $\Gamma^n(x_0) = \frac{5}{2}h'(0)$ and $\Gamma^k(x_0) = 0$ for $k < n$. By the definition of δ^k and [12, Lemma 2.3], we have $\delta^n(x_0) = 0$ and $\delta^k = \frac{1}{4}h'(0)c(\widetilde{e}_k)c(\widetilde{e}_n)$ for $k < n$. By (3.30), we obtain

$$
\begin{aligned}
&\sigma_{-4}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1} \\
&= \frac{-17 - 9\xi_n^2}{4(1+\xi_n^2)^4}h'(0)c(\xi')c(\mathrm{d}x_n)c(\xi') + \frac{33\xi_n + 17\xi_n^3}{2(1+\xi_n^2)^4}h'(0)c(\xi') + \frac{49\xi_n^2 + 25\xi_n^4}{2(1+\xi_n^2)^4}h'(0)c(\mathrm{d}x_n) \\
&\quad + \frac{1}{(1+\xi_n^2)^3}c(\xi')c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')](x_0) - \frac{3\xi_n}{(1+\xi_n^2)^3}\partial_{x_n}[c(\xi')](x_0) - \frac{2\xi_n}{(1+\xi_n^2)^3}h'(0)\xi_n c(\xi')(x_0) \\
&\quad + \frac{1-\xi_n^2}{(1+\xi_n^2)^3}h'(0)c(\mathrm{d}x_n)(x_0) + \frac{c(\xi)\lambda c(\xi)}{|\xi|^6} - \frac{2c(\xi)c(A^*)c(\xi)}{|\xi|^6} - \frac{2c(A)}{|\xi|^4}.
\end{aligned} \tag{3.60}
$$

Then

$$
\begin{aligned}
&\partial_{\xi_n}(\sigma_{-4}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})(x_0) \\
&= \frac{59\xi_n + 27\xi_n^3}{2(1+\xi_n^2)^5}h'(0)c(\xi')c(\mathrm{d}x_n)c(\xi') + \frac{33 - 180\xi_n^2 - 85\xi_n^4}{2(1+\xi_n^2)^5}h'(0)c(\xi') + \frac{49\xi_n - 97\xi_n^3 - 50\xi_n^5}{2(1+\xi_n^2)^5} \\
&\quad \times h'(0)c(\mathrm{d}x_n) - \frac{6\xi_n}{(1+\xi_n^2)^4}c(\xi')c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')](x_0) - \frac{3 - 15\xi_n^2}{(1+\xi_n^2)^4}\partial_{x_n}[c(\xi')](x_0) + \frac{4\xi_n^3 - 8\xi_n}{(1+\xi_n^2)^4} \\
&\quad \times h'(0)c(\mathrm{d}x_n) + \frac{2 - 10\xi_n^2}{(1+\xi_n^2)^4}h'(0)c(\xi') + \frac{c(\mathrm{d}x_n)\lambda c(\xi') + c(\xi')\lambda c(\mathrm{d}x_n) + 2\xi_n c(\mathrm{d}x_n)\lambda c(\mathrm{d}x_n)}{(1+\xi_n^2)^3} \\
&\quad - \frac{6\xi_n c(\xi)\lambda c(\xi)}{(1+\xi_n^2)^4} + \frac{c(\mathrm{d}x_n)c(A^*)c(\xi') + c(\xi')c(A^*)c(\mathrm{d}x_n) + 2\xi_n c(\mathrm{d}x_n)c(A^*)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3} \\
&\quad - \frac{6\xi_n c(\xi)c(A^*)c(\xi)}{(1+\xi_n^2)^4} - \frac{2\xi_n c(A)}{(1+\xi_n^2)^3}.
\end{aligned} \tag{3.61}
$$

By (3.47) and (3.61), we obtain

$$
\begin{aligned}
&\mathrm{trace}[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1}) \times \partial_{\xi_n}\sigma_{-4}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1}](x_0)|_{|\xi'|=1} \\
&= h'(0)\dim F\frac{4\mathrm{i}(-17 - 42\mathrm{i}\xi_n + 50\xi_n^2 - 16\mathrm{i}\xi_n^3 + 29\xi_n^4)}{(\xi_n - \mathrm{i})^5(\xi + \mathrm{i})^5} \\
&\quad + \frac{(4\xi_n\mathrm{i} + 2)\mathrm{i}}{2(\xi_n + \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\xi')\lambda] + \frac{4\xi_n\mathrm{i} + 2}{2(\xi_n + \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\mathrm{d}x_n)\lambda] \\
&\quad + \frac{(4\xi_n\mathrm{i} + 2)\mathrm{i}}{2(\xi_n + \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\xi')c(A^*)] + \frac{4\xi_n\mathrm{i} + 2}{2(\xi_n + \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\mathrm{d}x_n)c(A^*)] \\
&\quad + \frac{-2\xi_n}{2(\xi_n - \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\xi')c(A)] + \frac{-2\xi_n\mathrm{i}}{2(\xi_n - \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\mathrm{d}x_n)c(A)].
\end{aligned} \tag{3.62}
$$

By the relation of the Clifford action and trace $QP = \mathrm{trace}\, PQ$, we have the following equalities

$$\begin{aligned}\mathrm{trace}[c(\mathrm{d}x_n)\lambda] &= \mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_{j=1}^{n} c(e_j)(\sigma_j^F - A^*(e_j))\Big]\\ &= \mathrm{trace}[-\mathrm{id}\otimes(\sigma_n^F - A^*(e_n))],\end{aligned} \tag{3.63}$$

$$\begin{aligned}\mathrm{trace}[c(\xi')\lambda] &= \mathrm{trace}\Big[c(\xi')\sum_{j=1}^{n} c(e_j)(\sigma_j^F - A^*(e_j))\Big]\\ &= \mathrm{trace}\Big[-\sum_{j=1}^{n-1}\xi_j(\sigma_j^F - A^*(e_j))\Big],\end{aligned} \tag{3.64}$$

$$\begin{aligned}\mathrm{trace}[c(\mathrm{d}x_n)c(A^*)] &= \mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_{j=1}^{n} c(e_j)\otimes A^*(e_j)\Big]\\ &= \mathrm{trace}[-\mathrm{id}\otimes A^*(e_n)],\end{aligned} \tag{3.65}$$

$$\mathrm{trace}[c(\mathrm{d}x_n)c(A)] = \mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_{j=1}^{n} c(e_j)\otimes A(e_j)\Big] = \mathrm{trace}[-\mathrm{id}\otimes A(e_n)], \tag{3.66}$$

$$\mathrm{trace}[c(\xi')c(A^*)] = \mathrm{trace}\Big[c(\xi')\sum_{j=1}^{n} c(e_j)\otimes A^*(e_j)\Big] = \mathrm{trace}\Big[-\sum_{j=1}^{n-1}\xi_j A^*(e_j)\Big], \tag{3.67}$$

$$\mathrm{trace}[c(\xi')c(A)] = \mathrm{trace}\Big[c(\xi')\sum_{j=1}^{n} c(e_j)\otimes A(e_j)\Big] = \mathrm{trace}\Big[-\sum_{j=1}^{n-1}\xi_j A(e_j)\Big]. \tag{3.68}$$

We note that $i < n$, $\int_{|\xi'|=1}\xi_i\sigma(\xi') = 0$, so $\mathrm{trace}[c(\xi')c(A^*)]$ has no contribution for computing case (b).

By (3.24), we have

$$\begin{aligned}&-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}(\sigma_{-4}(\widetilde{D}_F^{*}\widetilde{D}_F\widetilde{D}_F^{*})^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\ &= \Big\{-\frac{129}{16}h'(0) + \frac{3}{2}\mathrm{trace}[\sigma_n^F - A^*(e_n)] - 3\mathrm{trace}[A^*(e_n)] - \mathrm{trace}[A(e_n)]\Big\}\pi\\ &\quad\times\dim F\Omega_4\mathrm{d}x'.\end{aligned} \tag{3.69}$$

Since

$$\begin{aligned}\partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big) &= \frac{c(\mathrm{d}x_n)c(\mathrm{d}f)c(\xi') + c(\xi')c(\mathrm{d}f)c(\mathrm{d}x_n) + 2\xi_n c(\mathrm{d}x_n)c(\mathrm{d}f)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\\ &\quad - \frac{6\xi_n c(\xi)c(\mathrm{d}f)c(\xi)}{(1+\xi_n^2)^4}\end{aligned}$$

and

$$\begin{aligned}&\partial_{\xi_n}\Big(\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\\ &= \mathrm{i}\Big\{c(\mathrm{d}x_n)\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi') + c(\xi')\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]\\ &\quad\times D_{x_j}(f^{-1})c(\mathrm{d}x_n) + 2\xi_n c(\mathrm{d}x_n)\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\mathrm{d}x_n)\Big\}(1+\xi_n^2)^{-4}\end{aligned}$$

$$-\mathrm{i}\Big\{8\xi_n c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi)\Big\}(1+\xi_n^2)^{-5}, \tag{3.70}$$

we have

$$\begin{aligned}&\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big)\Big](x_0)\\&=\frac{(4\xi_n\mathrm{i}+2)\mathrm{i}}{2(\xi_n+\mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\xi')c(\mathrm{d}f)]+\frac{4\xi_n\mathrm{i}+2}{2(\xi_n+\mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\mathrm{d}x_n)c(\mathrm{d}f)]\end{aligned}$$

and

$$\begin{aligned}&\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\Big](x_0)\\&=\frac{(3\xi_n-\mathrm{i})\mathrm{i}}{(\xi_n+\mathrm{i})(1+\xi_n^2)^4}\mathrm{trace}\Big[c(\xi')\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]D_{x_j}(f^{-1})\Big]\\&\quad+\frac{3\xi_n-\mathrm{i}}{(\xi_n+\mathrm{i})(1+\xi_n^2)^4}\mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]D_{x_j}(f^{-1})\Big].\end{aligned} \tag{3.71}$$

By the relation of the Clifford action and $\mathrm{trace}\,QP=\mathrm{trace}\,PQ$, we have the following equalities

$$\mathrm{trace}[c(\mathrm{d}x_n)c(\mathrm{d}f)]=-g(\mathrm{d}x_n,\mathrm{d}f)$$

and

$$\begin{aligned}&\mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]D_{x_j}(f^{-1})\Big]\\&=\mathrm{trace}(-\mathrm{id})|\xi|^2(-\mathrm{i}\partial_{x_n}(f)f^{-1})+2\sum_j\xi_j\xi_n\mathrm{trace}(-\mathrm{id})(-\mathrm{i}\partial_{x_j}(f)f^{-1})\\&=-8\mathrm{dim}\,F|\xi|^2(-\mathrm{i}\partial_{x_n}(f)f^{-1})+2\sum_j\xi_j\xi_n\mathrm{trace}(-\mathrm{id})(-\mathrm{i}\partial_{x_j}(f)f^{-1}).\end{aligned}$$

We note that $i<n$, $\int_{|\xi'|=1}\xi_i\sigma(\xi')=0$, so $\mathrm{trace}[c(\xi')c(\mathrm{d}f)]$, $\mathrm{trace}\big[c(\xi')\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]$ $D_{x_j}(f^{-1})\big]$ and $2\mathrm{i}\sum_j\xi_j\xi_n\partial_{x_j}(f)f^{-1}\mathrm{trace}[-\mathrm{id}]$ have no contribution for computing case (b). Then we obtain

$$\begin{aligned}&-2\mathrm{i}f^{-1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big)\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=\frac{3}{8f}\pi g(\mathrm{d}x_n,\mathrm{d}f)\Omega_4\mathrm{d}x'\end{aligned} \tag{3.72}$$

and

$$\begin{aligned}&-f\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\Big]\\&\times(x_0)\mathrm{d}\xi_n\sigma(\xi')dx'\\&=-\frac{15\mathrm{i}}{2}\partial_{x_n}(f)\pi\mathrm{dim}\,F\Omega_4\mathrm{d}x'.\end{aligned} \tag{3.73}$$

Thus we have

$$\begin{aligned}
&\text{case (b)}\\
&=\Big\{-\frac{129}{16}h'(0)+\frac{3}{2}\mathrm{trace}[\sigma_n^F-A^*(e_n)]-3\mathrm{trace}[A^*(e_n)]-\mathrm{trace}[A(e_n)]\Big\}\pi\mathrm{dim}\,F\Omega_4\mathrm{d}x'\\
&\quad+\frac{3}{8f}\pi g[\mathrm{d}x_n,\mathrm{d}f]\Omega_4\mathrm{d}x'-\frac{15\mathrm{i}}{2}\partial_{x_n}(f)\pi\mathrm{dim}\,F\Omega_4\mathrm{d}x'.
\end{aligned}\tag{3.74}$$

Case (c) $r=-2$, $l=-3$, $|\alpha|=j=k=0$.
By (3.18), we have

$$\begin{aligned}
&\text{case (c)}\\
&=-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi_{\xi_n}^+\sigma_{-2}(f\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1})](x_0)\\
&\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi_{\xi_n}^+\sigma_{-2}(\widetilde{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned}\tag{3.75}$$

By (3.14), we have

$$\begin{aligned}
\pi_{\xi_n}^+\sigma_{-2}(\widetilde{D}_F^{-1})&=\pi_{\xi_n}^+\Big(\frac{c(\xi)\sigma_0(\widetilde{D}_F)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}[c(\xi)]|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)]\Big)\\
&:=T_1-T_2+\pi_{\xi_n}^+\Big(\frac{c(\xi)\mu c(\xi)}{|\xi|^4}\Big),
\end{aligned}\tag{3.76}$$

where

$$\begin{aligned}
T_1&=-\frac{1}{4(\xi_n-\mathrm{i})^2}[(2+\mathrm{i}\xi_n)c(\xi')\sigma_0(\widetilde{D}_F)c(\xi')+\mathrm{i}\xi_nc(\mathrm{d}x_n)\sigma_0(\widetilde{D}_F)c(\mathrm{d}x_n)+(2+\mathrm{i}\xi_n)c(\xi')\\
&\quad\times c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')]+\mathrm{i}c(\mathrm{d}x_n)\sigma_0(\widetilde{D}_F)c(\xi')+\mathrm{i}c(\xi')\sigma_0(\widetilde{D}_F)c(\mathrm{d}x_n)-\mathrm{i}\partial_{x_n}[c(\xi')]]\\
&=\frac{1}{4(\xi_n-\mathrm{i})^2}\Big[\frac{5}{2}h'(0)c(\mathrm{d}x_n)-\frac{5\mathrm{i}}{2}h'(0)c(\xi')-(2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{\xi_n}[c(\xi')]\\
&\quad+\mathrm{i}\partial_{\xi_n}[c(\xi')]\Big],
\end{aligned}\tag{3.77}$$

$$T_2=\frac{h'(0)}{2}\Big[\frac{c(\mathrm{d}x_n)}{4\mathrm{i}(\xi_n-\mathrm{i})}+\frac{c(\mathrm{d}x_n)-\mathrm{i}c(\xi')}{8(\xi_n-\mathrm{i})^2}+\frac{3\xi_n-7\mathrm{i}}{8(\xi_n-\mathrm{i})^3}(\mathrm{i}c(\xi')-c(\mathrm{d}x_n))\Big].\tag{3.78}$$

On the other hand,

$$\begin{aligned}
&\pi_{\xi_n}^+\Big(\frac{c(\xi)\mu c(\xi)}{|\xi|^4}\Big)(x_0)\Big|_{|\xi'|=1}\\
&=\frac{(-\mathrm{i}\xi_n-2)c(\xi')\mu c(\xi')-\mathrm{i}[c(\mathrm{d}x_n)\mu c(\xi')+c(\xi')\mu c(\mathrm{d}x_n)]-\mathrm{i}\xi_nc(\mathrm{d}x_n)\mu c(\mathrm{d}x_n)}{4(\xi_n-\mathrm{i})^2}.
\end{aligned}\tag{3.79}$$

By (3.42), (3.44) and (3.76), we have

$$\begin{aligned}
&\mathrm{tr}[T_1\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})]|_{|\xi'|=1}\\
&=\mathrm{tr}\Big\{\frac{1}{4(\xi_n-\mathrm{i})^2}\Big[\frac{5}{2}h'(0)c(\mathrm{d}x_n)-\frac{5\mathrm{i}}{2}h'(0)c(\xi')-(2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{\xi_n}c(\xi')+\mathrm{i}\partial_{\xi_n}c(\xi')\Big]
\end{aligned}$$

$$\times \frac{-4\mathrm{i}\xi_n c(\xi') + (\mathrm{i} - 3\mathrm{i}\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\Big\}$$
$$= h'(0)\dim F \frac{3 + 12\mathrm{i}\xi_n + 3\xi_n^2}{(\xi_n - \mathrm{i})^4(\xi_n + \mathrm{i})^3}. \tag{3.80}$$

Similarly, we have

$$\mathrm{trace}[T_2 \times \partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})]|_{|\xi'|=1}$$
$$= \mathrm{trace}\Big\{\frac{h'(0)}{2}\Big[\frac{c(\mathrm{d}x_n)}{4\mathrm{i}(\xi_n - \mathrm{i})} + \frac{c(\mathrm{d}x_n) - \mathrm{i}c(\xi')}{8(\xi_n - \mathrm{i})^2} + \frac{3\xi_n - 7\mathrm{i}}{8(\xi_n - \mathrm{i})^3}(\mathrm{i}c(\xi') - c(\mathrm{d}x_n))\Big]$$
$$\times \frac{-4\mathrm{i}\xi_n c(\xi') + (\mathrm{i} - 3\mathrm{i}\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\Big\}$$
$$= h'(0)\dim F \frac{4\mathrm{i} - 11\xi_n - 6\mathrm{i}\xi_n^2 + 3\xi_n^3}{(\xi_n - \mathrm{i})^5(\xi_n + \mathrm{i})^3}. \tag{3.81}$$

By (3.79)–(3.80), we obtain

$$-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}\Big[(T_1 - T_2) \times \partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -\mathrm{i}\dim F h'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \frac{-7\mathrm{i} + 26\xi_n + 15\mathrm{i}\xi_n^2}{(\xi_n - \mathrm{i})^5(\xi_n + \mathrm{i})^3}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -\mathrm{i}\dim F h'(0)\frac{2\pi\mathrm{i}}{4!}\Big[\frac{-7\mathrm{i} + 26\xi_n + 15\mathrm{i}\xi_n^2}{(\xi_n + \mathrm{i})^3}\Big]^{(5)}\Big|_{\xi_n = \mathrm{i}}\Omega_4\mathrm{d}x'$$
$$= \frac{55}{16}\dim F\pi h'(0)\Omega_4\mathrm{d}x'. \tag{3.82}$$

By (3.55)–(3.56), we have

$$\mathrm{trace}\Big[\pi^+_{\xi_n}\Big(\frac{c(\xi)\mu c(\xi)}{|\xi|^4}\Big) \times \partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})\Big](x_0)$$
$$= \frac{(3\xi_n - \mathrm{i})\mathrm{i}}{2(\xi_n - \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\mathrm{d}x_n)\mu] + \frac{3\xi_n - \mathrm{i}}{2(\xi_n - \mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\xi')\mu]. \tag{3.83}$$

By the relation of the Clifford action and $\mathrm{trace}\, PQ = \mathrm{trace}\, QP$, we have the equalities

$$\mathrm{trace}[c(\mathrm{d}x_n)\mu] = \mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_{j=1}^n c(e_j)(\sigma_j^F + A(e_j))\Big]$$
$$= \mathrm{trace}[-\mathrm{id}\otimes(\sigma_n^F + A(e_n))], \tag{3.84}$$
$$\mathrm{trace}[c(\xi')\mu] = \mathrm{trace}\Big[c(\xi')\sum_{j=1}^n c(e_j)(\sigma_j^F + A(e_j))\Big]$$
$$= \mathrm{trace}\Big[-\sum_{j=1}^{n-1}\xi_j(\sigma_j^F + A(e_j))\Big]. \tag{3.85}$$

We note that $i < n$, $\int_{|\xi'|=1}\xi_i\sigma(\xi') = 0$, so $\mathrm{trace}[c(\xi')\mu]$ has no contribution for computing case (c).

Then, we obtain

$$-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}\Big[\pi^+_{\xi_n}\Big(\frac{c(\xi)\mu c(\xi)}{|\xi|^4}\Big) \times \partial_{\xi_n}\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$

$$= -\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{(3\xi_n-\mathrm{i})\mathrm{i}}{2(\xi_n-\mathrm{i})(1+\xi_n^2)^3}\mathrm{trace}[c(\mathrm{d}x_n)\mu]\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -2\pi\dim F\mathrm{trace}[\sigma_n^F+A(e_n)]\Omega_4\mathrm{d}x'. \tag{3.86}$$

Then

$$\text{case (c)} = \frac{55}{16}\dim F\pi h'(0)\Omega_4\mathrm{d}x' - 2\pi\dim Fh'(0)\mathrm{trace}[\sigma_n^F+A(e_n)]\Omega_4\mathrm{d}x'. \tag{3.87}$$

Now Φ is the sum of the case (a), case (b) and case (c), then

$$\Phi = \Big[-4h'(0)-\mathrm{trace}(A(e_n))-3\mathrm{trace}(A^*(e_n))+\frac{3}{2}\mathrm{trace}(\sigma_n^F-A^*(e_n))$$
$$-2\mathrm{trace}(\sigma_n^F+A(e_n))+\Big(\frac{19\mathrm{i}}{16}+11\Big)\cdot f^{-1}\cdot\partial_{x_n}(f)\Big]\pi\dim F\Omega_4\mathrm{d}x'$$
$$+\frac{3}{8f}\pi g(\mathrm{d}x_n,\mathrm{d}f)\Omega_4\mathrm{d}x' - \frac{15\mathrm{i}}{2}\partial_{x_n}(f)\pi\dim F\Omega_4\mathrm{d}x'. \tag{3.88}$$

By [12, (4.2)], we have

$$K = \sum_{1\le i,j\le n-1}K_{i,j}g_{\partial M}^{i,j}, \quad K_{i,j} = -\Gamma_{i,j}^n,$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. For $n=6$, we have

$$K(x_0) = \sum_{1\le i,j\le n-1}K_{i,j}(x_0)g_{\partial M}^{i,j}(x_0) = \sum_{i=1}^{5}K_{i,i}(x_0) = -\frac{5}{2}h'(0). \tag{3.89}$$

Hence we conclude the following theorem.

Theorem 3.1 *Let M be a six-dimensional compact spin manifolds with the boundary ∂M. Then*

$$\widetilde{\mathrm{Wres}}[\pi^+(f\widetilde{D}_F^{-1})\circ\pi^+(f^{-1}(\widetilde{D}_F^*)^{-1}\cdot f\widetilde{D}_F^{-1}\cdot f^{-1}(\widetilde{D}_F^*)^{-1})]$$
$$= 8\pi^3\int_M\Big\{\mathrm{trace}\Big[-\frac{s}{12}+c(A^*)c(A)-\frac{1}{4}\sum_i[c(A^*)c(e_i)$$
$$-c(e_i)c(A)]^2-\frac{1}{2}\sum_j\nabla_{e_j}^F(c(A^*))$$
$$\times c(e_j)-\frac{1}{2}\sum_j c(e_j)\nabla_{e_j}^F(c(A))\Big]-\frac{2\Delta(f)}{f}+\frac{4\mathrm{trace}[A(\mathrm{grad}_M f)]}{f}-f^2[|\mathrm{grad}_M(f)|^2$$
$$+2\Delta(f)]\Big\}\mathrm{d\,vol}_M+\int_{\partial M}\Big\{\Big[\frac{3}{2}\mathrm{trace}(\sigma_n^F-A^*(e_n))-4h'(0)-\mathrm{trace}(A(e_n))-3$$
$$\times\mathrm{trace}(A^*(e_n))-2\mathrm{trace}(\sigma_n^F+A(e_n))+\Big(\frac{19\mathrm{i}}{16f}+\frac{11}{f}-\frac{15\mathrm{i}}{2}\Big)\partial_{x_n}(f)\Big]\pi\dim F\Omega_4$$
$$+\frac{3\pi g(\mathrm{d}x_n,\mathrm{d}f)}{8f}\Omega_4\Big\}\mathrm{d\,vol}_M, \tag{3.90}$$

where s is the scalar curvature.

4 Twisted Signature Operator and Its Symbol

Let us recall the definition of twisted signature operators. We consider an n-dimensional oriented Riemannian manifold (M, g^M). Let F be a real vector bundle over M. Let g^F be a Euclidean metric on F. Let

$$\wedge^*(T^*M) = \bigoplus_{i=0}^{n} \wedge^i(T^*M) \tag{4.1}$$

be the real exterior algebra bundle of T^*M. Let

$$\Omega^*(M, F) = \bigoplus_{i=0}^{n} \Omega^i(M, F) = \bigoplus_{i=0}^{n} C^\infty(M, \wedge^i(T^*M) \otimes F) \tag{4.2}$$

be the set of smooth sections of $\wedge^*(T^*M) \otimes F$. Let $*$ be the Hodge star operator of g^{TM}. It extends on $\wedge^*(T^*M) \otimes F$ by acting on F as identity. Then $\Omega^*(M, F)$ inherits the following standardly induced inner product

$$\langle \zeta, \eta \rangle = \int_M \langle \zeta \wedge *\eta \rangle_F, \quad \zeta, \eta \in \Omega^*(M, F). \tag{4.3}$$

Let $\widehat{\nabla}^F$ be the non-Euclidean connection on F. Let d^F be the obvious extension of ∇^F on $\Omega^*(M, F)$. Let $\delta^F = d^{F*}$ be the formal adjoint operator of d^F with respect to the inner product. Let $\widehat{D}^F$ be the differential operator acting on $\Omega^*(M, F)$ defined by

$$\widehat{D}^F = d^F + \delta^F. \tag{4.4}$$

Let

$$\omega(F, g^F) = \widehat{\nabla}^{F,*} - \widehat{\nabla}^F, \quad \nabla^{F,e} = \nabla^F + \frac{1}{2}\omega(F, g^F). \tag{4.5}$$

Then $\nabla^{F,e}$ is a Euclidean connection on (F, g^F).

Let $\nabla^{\wedge^*(T^*M)}$ be the Euclidean connection on $\wedge^*(T^*M)$ induced canonically by the Levi-Civita connection ∇^{TM} of g^{TM}. Let ∇^e be the Euclidean connection on $\wedge^*(T^*M) \otimes F$ obtained from the tensor product of $\nabla^{\wedge^*(T^*M)}$ and $\nabla^{F,e}$. Let $\{e_1, \cdots, e_n\}$ be an oriented (local) orthonormal basis of TM. The following result was proved by [20, Proposition 4.12].

The following identity holds

$$d^F + \delta^F = \sum_{i=1}^{n} c(e_i)\nabla^e_{e_i} - \frac{1}{2}\sum_{i=1}^{n} \widehat{c}(e_i)\omega(F, g^F)(e_i). \tag{4.6}$$

Let $D^e_F = \sum\limits_{j=1}^{n} c(e_j)\nabla^e_{e_j}$ and $\omega(F, g^F)$ be any element in $\Omega(M, \operatorname{End} F)$, then we define the generalized twisted signature operators $\widehat{D}_F$, $\widehat{D}^*_F$ as follows.

For sections $\psi \otimes \chi \in \wedge^*(T^*M) \otimes F$,

$$\widehat{D}_F(\psi \otimes \chi) = D^e_F(\psi \otimes \chi) - \frac{1}{2}\sum_{i=1}^{n} \widehat{c}(e_i)\omega(F, g^F)(e_i)(\psi \otimes \chi), \tag{4.7}$$

$$\widehat{D}^*_F(\psi \otimes \chi) = D^e_F(\psi \otimes \chi) - \frac{1}{2}\sum_{i=1}^{n} \widehat{c}(e_i)\omega^*(F, g^F)(e_i)(\psi \otimes \chi). \tag{4.8}$$

Here $\omega^*(F,g^F)(e_i)$ denotes the adjoint of $\omega(F,g^F)(e_i)$.

In the local coordinates $\{x_i; 1\leq i\leq n\}$ and the fixed orthonormal frame $\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\widetilde{\nabla}(\widetilde{e_1},\cdots,\widetilde{e_n})=(\widetilde{e_1},\cdots,\widetilde{e_n})(\omega_{s,t}). \tag{4.9}$$

Let M be a six-dimensional compact oriented Riemannian manifold with boundary ∂M. We define that $\widehat{D}_F: C^\infty(M,\wedge^*(T^*M)\otimes F)\to C^\infty(M,\wedge^*(T^*M)\otimes F)$ is the generalized twisted signature operator. Take the coordinates and the orthonormal frame as in Section 3. Let $\varepsilon(\widetilde{e_j*})$, $\iota(\widetilde{e_j*})$ be the exterior and interior multiplications, respectively. Write

$$c(\widetilde{e_j})=\varepsilon(\widetilde{e_j*})-\iota(\widetilde{e_j*}),\quad \widehat{c}(\widetilde{e_j})=\varepsilon(\widetilde{e_j*})+\iota(\widetilde{e_j*}). \tag{4.10}$$

We will compute $\mathrm{tr}_{\wedge^*(T^*M)\otimes F}$ in the frame $\{e^*_{i_1}\wedge\cdots\wedge e^*_{i_k}\mid 1\leq i_1<\cdots<i_k\leq 6\}$. By [12, (3.2) and (4.8)], we have

$$\begin{aligned}\widehat{D}_F&=\sum_{i=1}^n c(e_i)\nabla^e_{e_i}-\frac{1}{2}\sum_{i=1}^n\widehat{c}(e_i)\omega(F,g^F)(e_i)\\&=\sum_{i=1}^n c(e_i)(\nabla^{\wedge^*(T^*M)}_{e_i}\otimes \mathrm{id}_F+\mathrm{id}_{\wedge^*(T^*M)}\otimes\nabla^{F,e}_{e_i})-\frac{1}{2}\sum_{i=1}^n\widehat{c}(e_i)\omega(F,g^F)(e_i)\\&=\sum_{i=1}^n c(\widetilde{e_i})\Big[\widetilde{e_i}+\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\widehat{c}(\widetilde{e_s})\widehat{c}(\widetilde{e_t})-c(\widetilde{e_s})c(\widetilde{e_t})]\otimes\mathrm{id}_F+\mathrm{id}_{\wedge^*(T^*M)}\otimes\sigma_i^{F,e}\Big]\\&\quad-\frac{1}{2}\sum_{i=1}^n\widehat{c}(e_i)\omega(F,g^F)(e_i).\end{aligned} \tag{4.11}$$

Similarly, we have

$$\begin{aligned}\widehat{D}_F^*&=\sum_{i=1}^n c(\widetilde{e_i})\Big[\widetilde{e_i}+\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\widehat{c}(\widetilde{e_s})\widehat{c}(\widetilde{e_t})-c(\widetilde{e_s})c(\widetilde{e_t})]\otimes\mathrm{id}_F+\mathrm{id}_{\wedge^*(T^*M)}\otimes\sigma_i^{F,e}\Big]\\&\quad-\frac{1}{2}\sum_{i=1}^n\widehat{c}(e_i)\omega^*(F,g^F)(e_i).\end{aligned} \tag{4.12}$$

For convenience, let $\widehat{c}(\omega)=\sum_i\widehat{c}(e_i)\omega(F,g^F)(e_i)$ and $\widehat{c}(\omega^*)=\sum_i\widehat{c}(e_i)\omega^*(F,g^F)(e_i)$, by the composition formula and [12, (2.2.11)], we obtain the following lemma (see [19]).

Lemma 4.1 *Let $\widehat{D}_F^*,\widehat{D}_F$ be the twisted signature operators on $\Gamma(\wedge^*(T^*M)\otimes F)$, then*

$$\sigma_1(\widehat{D}_F)=\sigma_1(\widehat{D}_F^*)=\sqrt{-1}c(\xi), \tag{4.13}$$

$$\begin{aligned}\sigma_0(\widehat{D}_F)&=\sum_{i=1}^n c(\widetilde{e_i})\Big[\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\widehat{c}(\widetilde{e_s})\widehat{c}(\widetilde{e_t})-c(\widetilde{e_s})c(\widetilde{e_t})]\otimes\mathrm{id}_F+\mathrm{id}_{\wedge^*(T^*M)}\otimes\sigma_i^{F,e}\Big]\\&\quad-\frac{\widehat{c}(\omega)}{2},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\sigma_0(\widehat{D}_F^*)&=\sum_{i=1}^n c(\widetilde{e_i})\Big[\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\widehat{c}(\widetilde{e_s})\widehat{c}(\widetilde{e_t})-c(\widetilde{e_s})c(\widetilde{e_t})]\otimes\mathrm{id}_F+\mathrm{id}_{\wedge^*(T^*M)}\otimes\sigma_i^{F,e}\Big]\\&\quad-\frac{\widehat{c}(\omega^*)}{2}.\end{aligned} \tag{4.15}$$

By the composition formula of pseudodifferential operators in [12, Section 2.2.1], we have the following lemma.

Lemma 4.2 *The symbol of the twisted signature operators $\widehat{D}_F^*, \widehat{D}_F$ are as follows:*

$$\sigma_{-1}(\widehat{D}_F^{-1}) = \sigma_{-1}((\widehat{D}_F^*)^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}, \tag{4.16}$$

$$\sigma_{-2}(\widehat{D}_F^{-1}) = \frac{c(\xi)\sigma_0(\widehat{D}_F)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)], \tag{4.17}$$

$$\sigma_{-2}((\widehat{D}_F^*)^{-1}) = \frac{c(\xi)\sigma_0(\widehat{D}_F^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)]. \tag{4.18}$$

Since Ψ is a global form on ∂M, for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Psi(x_0)$ in the coordinates $\widetilde{U} = U \times [0,1)$ and the metric $\frac{1}{h(x_n)}g^{\partial M} + \mathrm{d}x_n^2$. The dual metric of $g^{\partial M}$ on $\widetilde{U}$ is $\frac{1}{\tilde{h}(x_n)}g^{\partial M} + \mathrm{d}x_n^2$. Write $g_{ij}^M = g^M\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big)$, $g^{ij}_M = g^M(\mathrm{d}x_i, \mathrm{d}x_j)$, then

$$[g_{ij}^M] = \begin{bmatrix} \frac{1}{h(x_n)}[g_{ij}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}, \quad [g^{ij}_M] = \begin{bmatrix} h(x_n)[g^{ij}_{\partial M}] & 0 \\ 0 & 1 \end{bmatrix} \tag{4.19}$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, \quad 1 \le i,j \le n-1, \quad g_{ij}^M(x_0) = \delta_{ij}. \tag{4.20}$$

Let $\{e_1, \cdots, e_{n-1}\}$ be an orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $e_i = \frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{e_1} = \sqrt{h(x_n)}e_1, \cdots, \widetilde{e_{n-1}} = \sqrt{h(x_n)}e_{n-1}, \widetilde{e_n} = \mathrm{d}x_n\}$ is the orthonormal frame field in $\widetilde{U}$ about g^M. Locally $\wedge^*(T^*M)|\widetilde{U} \cong \widetilde{U} \times \wedge^*_C\big(\frac{n}{2}\big)$. Let $\{f_1, \cdots, f_n\}$ be the orthonormal basis of $\wedge^*_C\big(\frac{n}{2}\big)$. Take a spin frame field $\sigma : \widetilde{U} \to \mathrm{Spin}(M)$ such that $\pi\sigma = \{\widetilde{e_1}, \cdots, \widetilde{e_n}\}$ where $\pi : \mathrm{Spin}(M) \to O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \le i \le 4\}$ is an orthonormal frame of $\wedge^*(T^*M)|_{\widetilde{U}}$. In the following, since the global form Ψ is independent of the choice of the local frame, we can compute $\mathrm{trace}_{\wedge^*(T^*M)}$ in the frame $\{[\sigma, f_i], 1 \le i \le 4\}$. Let $\{E_1, \cdots, E_n\}$ be the canonical basis of R^n and $c(E_i) \in \mathrm{cl}_C(n) \cong \mathrm{Hom}\big(\wedge^*_C\big(\frac{n}{2}\big), \wedge^*_C\big(\frac{n}{2}\big)\big)$ be the Clifford action. By [12], then

$$c(\widetilde{e_i}) = [(\sigma, c(E_i))], \quad c(\widetilde{e_i})[(\sigma, f_i)] = [\sigma, (c(E_i))f_i], \quad \frac{\partial}{\partial x_i} = \Big[\Big(\sigma, \frac{\partial}{\partial x_i}\Big)\Big], \tag{4.21}$$

we have $\frac{\partial}{\partial x_i}c(\widetilde{e_i}) = 0$ in the above frame. By [12, Lemma 2.2], we have the following lemma.

Lemma 4.3

$$\partial_{x_j}(|\xi|^2_{g^M})(x_0) = \begin{cases} 0, & \text{if } j < n; \\ h'(0)|\xi'|^2_{g^{\partial M}}, & \text{if } j = n, \end{cases} \tag{4.22}$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n; \\ \partial_{x_n}(c(\xi'))(x_0), & \text{if } j = n, \end{cases} \tag{4.23}$$

where $\xi = \xi' + \xi_n \mathrm{d}x_n$.

Then an application of [12, Lemma 2.3] shows the following lemma.

Lemma 4.4 *The symbol of the twisted signature operators* $\widehat{D}_F^*, \widehat{D}_F$ *are as follows*:

$$\sigma_0(\widehat{D}_F^*) = \theta + \vartheta^*, \tag{4.24}$$

$$\sigma_0(\widehat{D}_F) = \theta + \vartheta, \tag{4.25}$$

where

$$\begin{aligned}\theta &= -\frac{5}{4}h'(0)c(\mathrm{d}x_n) + \frac{1}{4}h'(0)\sum_{i=1}^{n-1} c(\widetilde{e_i})\widehat{c}(\widetilde{e_n})\widehat{c}(\widetilde{e_i})(x_0)\otimes \mathrm{id}_F,\\ \vartheta^* &= \sum_{i=1}^{n} c(\widetilde{e_i})\sigma_i^{F,e} - \frac{1}{2}\sum_{i=1}^{n}\widehat{c}(e_i)\omega^*(F,g^F)(e_i),\\ \vartheta &= \sum_{i=1}^{n} c(\widetilde{e_i})\sigma_i^{F,e} - \frac{1}{2}\sum_{i=1}^{n}\widehat{c}(e_i)\omega(F,g^F)(e_i).\end{aligned} \tag{4.26}$$

In order to get the symbol of operators $\widehat{D}_F^* f\cdot\widehat{D}_F f^{-1}\cdot\widehat{D}_F^* f$, similar to (3.19)–(3.23), we give the specification of $\widehat{D}_F^* f\cdot\widehat{D}_F f^{-1}\cdot\widehat{D}_F^* f$.

Combining (4.11) and (4.12), we have

$$\begin{aligned}&\widehat{D}_F^* f\cdot\widehat{D}_F f^{-1}\cdot\widehat{D}_F^* f\\ &= f\cdot\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^* + c(\mathrm{d}f)\widehat{D}_F\widehat{D}_F^* - \widehat{D}_F^*\widehat{D}_F f\cdot c(\mathrm{d}f^{-1})\cdot f + \widehat{D}_F^*\cdot c(\mathrm{d}f)c(\mathrm{d}f^{-1})f\\ &= f\cdot\Big\{\sum_{i,j,l=1}^{n}\sum_{r=1}^{n} c(e_r)\langle e_r,\mathrm{d}x_l\rangle(-g^{ij}\partial_l\partial_i\partial_j) + \sum_{r,l=1}^{n} c(e_r)\langle e_r,\mathrm{d}x_l\rangle\Big\{-\sum_{i,j=1}^{n}(\partial_l g^{ij})\partial_i\partial_j\\ &\quad - \sum_{i,k,j=1}^{n} g^{ij}(4\sigma_i^{\wedge^*(T^*M)\otimes F}\partial_j - 2\Gamma_{ij}^k\times\partial_k)\partial_l\Big\} + \sigma_0(\widehat{D}_F^*)\Big(-\sum_{i,j=1}^{n} g^{ij}\partial_i\partial_j\Big) - \sum_{r,l=1}^{n} c(e_r)\\ &\quad\times\langle e_r,\mathrm{d}x_l\rangle\sum_{j,k=1}^{n}[\widehat{c}(w)c(e_j) + c(e_j)\widehat{c}(w^*)]\langle e_j,\mathrm{d}x^k\rangle\times\partial_l\partial_k + \sum_{r,l=1}^{n} c(e_r)\langle e_r,\mathrm{d}x_l\rangle\partial_l\\ &\quad\times\Big\{-\sum_{i,j,k=1}^{n} g^{ij}[(\partial_i\sigma_{\wedge^*(T^*M)\otimes F}^{j,e}) + \sigma_{\wedge^*(T^*M)\otimes F}^{i}\sigma_{\wedge^*(T^*M)\otimes F,e}^{j,e} - \Gamma_{ij}^k\sigma_{\wedge^*(T^*M)\otimes F}^{k}] + \frac{1}{4}s\\ &\quad - \frac{1}{2}\sum_{j=1}^{n}\widehat{c}(\omega)c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e} - \frac{1}{2}\sum_{j=1}^{n} c(e_j)e_j(\widehat{c}(\omega^*)) + \frac{1}{2}\sum_{i\neq j} R^{F,e}(e_i,e_j)c(e_i)c(e_j)\\ &\quad + \frac{1}{4}\widehat{c}(\omega)\widehat{c}(\omega^*) - \frac{1}{2}\sum_{j=1}^{n} c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e}\widehat{c}(\omega^*)\Big\} + \sigma_0(\widehat{D}_F^*)\Big\{-2\sigma_{\wedge^*(T^*M)\otimes F}^{j}\partial_j + \Gamma^k\partial_k\\ &\quad - \frac{1}{2}\sum_{j=1}^{n}[\widehat{c}(\omega)c(e_j) + c(e_j)\widehat{c}(\omega^*)]e_j - g^{ij}[(\partial_i\sigma_{\wedge^*(T^*M)\otimes F}^{j,e}) + \sigma_{\wedge^*(T^*M)\otimes F}^{i}\sigma_{\wedge^*(T^*M)\otimes F,e}^{j,e}\\ &\quad - \Gamma_{ij}^k\sigma_{\wedge^*(T^*M)\otimes F}^{k}] + \frac{1}{4}\widehat{c}(\omega)\widehat{c}(\omega^*) - \frac{1}{2}\sum_{j=1}^{n}\widehat{c}(\omega)c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e} - \frac{1}{2}\sum_{j=1}^{n} c(e_j)e_j(\widehat{c}(\omega^*))\\ &\quad - \frac{1}{2}\sum_{j=1}^{n} c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e}\widehat{c}(\omega^*) + \frac{1}{4}s + \frac{1}{2}\sum_{i\neq j} R^{F,e}(e_i,e_j)c(e_i)c(e_j)\Big\} + \sum_{r,l=1}^{n} c(e_r)\langle e_r,\mathrm{d}x_l\rangle\\ &\quad\times\Big\{\sum_{i,j,k=1}^{n} g^{ij}(\partial_l\Gamma_{ij}^k)\partial_k - 2\sum_{i,j=1}^{n} g^{ij}(\partial_l\sigma_i^{\wedge^*(T^*M)\otimes F})\partial_j - 2\sum_{i,j=1}^{n}(\partial_l g^{ij})\sigma_i^{\wedge^*(T^*M)\otimes F}\partial_j\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{j,k=1}^{n}[\partial_l(\widehat{c}(w)c(e_j)+c(e_j)\widehat{c}(w^*))]\langle e_j,\mathrm{d}x^k\rangle\partial_k+\sum_{i,j,k=1}^{n}(\partial_l g^{ij})\Gamma_{ij}^k\partial_k-\frac{1}{2}\sum_{j,k=1}^{n}(\widehat{c}(w)c(e_j)\\
&+c(e_j)\widehat{c}(w^*))[\partial_l\langle e_j,\mathrm{d}x^k\rangle]\partial_k\Big\}\Big\}+c(\mathrm{d}f)\Big\{-g^{ij}\partial_i\partial_j-2\sigma^j_{\wedge^*(T^*M)\otimes F}\partial_j+\Gamma^k\partial_k\\
&-\frac{1}{2}\sum_j[\widehat{c}(\omega)\times c(e_j)+c(e_j)\widehat{c}(\omega^*)]e_j-g^{ij}[(\partial_i\sigma^{j,e}_{\wedge^*(T^*M)\otimes F})+\sigma^i_{\wedge^*(T^*M)\otimes F}\sigma^{j,e}_{\wedge^*(T^*M)\otimes F,e}\\
&-\Gamma_{ij}^k\sigma^k_{\wedge^*(T^*M)\otimes F}]-\frac{1}{2}\sum_j\widehat{c}(\omega)c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e}-\frac{1}{2}\sum_j c(e_j)e_j(\widehat{c}(\omega^*))\\
&+\frac{1}{4}s+\frac{1}{4}\widehat{c}(\omega)\widehat{c}(\omega^*)-\frac{1}{2}\sum_j c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e}\widehat{c}(\omega^*)+\frac{1}{2}\sum_{i\neq j}R^{F,e}(e_i,e_j)c(e_i)c(e_j)\Big\}\\
&-\Big\{-g^{ij}\partial_i\partial_j-2\sigma^j_{\wedge^*(T^*M)\otimes F}\partial_j+\Gamma^k\partial_k-\frac{1}{2}\sum_j[\widehat{c}(\omega)c(e_j)+c(e_j)\widehat{c}(\omega^*)]e_j\\
&-g^{ij}[(\partial_i\sigma^{j,e}_{\wedge^*(T^*M)\otimes F})+\sigma^i_{\wedge^*(T^*M)\otimes F}\sigma^{j,e}_{\wedge^*(T^*M)\otimes F,e}-\Gamma_{ij}^k\times\sigma^k_{\wedge^*(T^*M)\otimes F}]\\
&-\frac{1}{2}\sum_j\widehat{c}(\omega)c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e}-\frac{1}{2}\sum_j c(e_j)e_j(\widehat{c}(\omega^*))+\frac{1}{4}s+\frac{1}{4}\widehat{c}(\omega)\widehat{c}(\omega^*)\\
&-\frac{1}{2}\sum_j c(e_j)\sigma_j^{\wedge^*(T^*M)\otimes F,e}\widehat{c}(\omega^*)+\frac{1}{2}\sum_{i\neq j}R^{F,e}(e_i,e_j)c(e_i)c(e_j)\Big\}f\cdot c(\mathrm{d}f^{-1})\cdot f\\
&+\Big\{\sum_{i,j=1}^{n}g^{ij}\times c(\partial_i)\Big[\partial_j+\Big(\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\widehat{c}(\widetilde{e_s})\widehat{c}(\widetilde{e_t})-c(\widetilde{e_s})c(\widetilde{e_t})]\otimes\mathrm{id}_F\\
&+\mathrm{id}_{\wedge^*(T^*M)}\otimes\sigma_i^{F,e}\Big)\Big]-\frac{\widehat{c}(\omega^*)}{2}\Big\}\times c(\mathrm{d}f)c(\mathrm{d}f^{-1})f.
\end{aligned}
\tag{4.27}
$$

By the above composition formulas, we obtain the following lemma.

Lemma 4.5 *Let $\widehat{D}_F^*,\widehat{D}_F$ be the twisted signature operators on $\Gamma(\wedge^*(T^*M)\otimes F)$, then*

$$\sigma_3(\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1}\cdot\widehat{D}_F^*f)=f\sigma_3(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)=f\sqrt{-1}c(\xi)|\xi|^2,\tag{4.28}$$

$$\sigma_2(\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1}\cdot\widehat{D}_F^*f)=f\sigma_2(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)+2c(\mathrm{d}f)|\xi|^2,\tag{4.29}$$

where $\sigma_2(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)=c(\xi)(4\sigma^k-2\Gamma^k)\xi_k-\frac{1}{4}|\xi|^2h'(0)c(\mathrm{d}x_n)+|\xi|^2\big(\frac{1}{4}h'(0)\sum_{i=1}^{5}c(\widetilde{e_i})\widehat{c}(\widetilde{e_n})\widehat{c}(\widetilde{e_i})(x_0)+\vartheta^*-\widehat{c}(w^*)\big)+c(\xi)\widehat{c}(w)c(\xi)$.

For convenience, we write that $\sigma_2(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)=G+|\xi|^2(p+\vartheta^*-\widehat{c}(w^*))+c(\xi)\widehat{c}(w)c(\xi)$. By (4.28)–(4.29), [12, Lemma 2.1] and the composition formula of pseudodifferential operators, similar to (3.26)–(3.28), we obtain the following lemma.

Lemma 4.6 *Let $\widehat{D}_F^*,\widehat{D}_F$ be the generalized twisted signature operators on $\Gamma(\wedge^*(T^*M)\otimes F)$, then*

$$\sigma_{-3}(\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1}\cdot\widehat{D}_F^*f)^{-1}=\frac{\sqrt{-1}c(\xi)}{f|\xi|^4},\tag{4.30}$$

$$
\begin{aligned}
\sigma_{-4}(\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1}\cdot\widehat{D}_F^*f)^{-1}&=f^{-1}\sigma_{-4}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})+\frac{2c(\xi)c(\mathrm{d}f)c(\xi)}{f^2|\xi|^6}\\
&\quad+\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8},
\end{aligned}
\tag{4.31}
$$

where

$$\begin{aligned}
&\sigma_{-4}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})\\
&=\frac{c(\xi)\sigma_2(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)c(\xi)}{|\xi|^8}+\frac{c(\xi)}{|\xi|^{10}}\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)][\partial_{x_j}[c(\xi)]|\xi|^2-2c(\xi)\partial_{x_j}(|\xi|^2)]\\
&=\frac{c(\xi)Gc(\xi)}{|\xi|^8}+\frac{c(\xi)(p+\vartheta^*-\widehat{c}(w^*))c(\xi)}{|\xi|^6}+\frac{\widehat{c}(w)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^{10}}\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]\\
&\quad\times[\partial_{x_j}[c(\xi)]|\xi|^2-2c(\xi)\partial_{x_j}(|\xi|^2)]. \qquad (4.32)
\end{aligned}$$

Hence we cite the following theorem.

Theorem 4.1 (see [19]) *For even n-dimensional oriented compact Riemannian manifolds without boundary, the following equality holds:*

$$\begin{aligned}
&\mathrm{Wres}(\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1})^{\left(\frac{-n+2}{2}\right)}\\
&=\frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2}-2)!}\int_M\Big\{\mathrm{trace}\Big[-\frac{s}{12}+\frac{n}{16}[\widehat{c}(\omega^*)-\widehat{c}(\omega)]^2-\frac14\widehat{c}(\omega^*)\widehat{c}(\omega)-\frac14\sum_j\nabla_{e_j}^F(\widehat{c}(\omega^*))c(e_j)\\
&\quad+\frac14\sum_jc(e_j)\nabla_{e_j}^F(\widehat{c}(\omega))\Big]+4f^{-1}\Delta(f)+8\langle\mathrm{grad}_Mf,\mathrm{grad}_M(f^{-1})\rangle-5f^{-2}[|\mathrm{grad}_Mf|^2\\
&\quad+2\Delta f]\Big\}\mathrm{d\,vol}_M. \qquad (4.33)
\end{aligned}$$

5 Conformal Perturbations of Twisted Signature Operators and Noncommutative Residue

In the following, we will compute the more general case $\widetilde{\mathrm{Wres}}[\pi^+(f\widehat{D}_F^{-1})\circ\pi^+(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})]$ for nonzero smooth functions $f,\ f^{-1}$. An application of [14, (2.1.4)] shows that

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+(f\widehat{D}_F^{-1})\circ\pi^+(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})]\\
&=\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*(T^*M)\otimes F}((\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1})^{-2})\sigma(\xi)\mathrm{d}x+\int_{\partial M}\Psi, \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
\Psi&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-\mathrm{i})^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!}\mathrm{trace}_{\wedge^*(T^*M)\otimes F}[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+(f\,\widehat{D}_F^{-1})(x',0,\xi',\xi_n)\\
&\quad\times\partial_{x_n}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})(x',0,\xi',\xi_n)]\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x', \qquad (5.2)
\end{aligned}$$

and the sum is taken over $r-k+|\alpha|+\ell-j-1=-n, r\le-1,\ell\le-1$.

Locally we can use Theorem 4.7 to compute the interior term of (5.1), then

$$\begin{aligned}
&\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*(T^*M)\otimes F}[\sigma_{-4}((\widehat{D}_F^*f\cdot\widehat{D}_Ff^{-1})^{-2})]\sigma(\xi)\mathrm{d}x\\
&=8\pi^3\int_M\Big\{\mathrm{trace}\Big[-\frac{s}{12}+\frac38[\widehat{c}(\omega^*)-\widehat{c}(\omega)]^2-\frac14\widehat{c}(\omega^*)\widehat{c}(\omega)-\frac14\sum_j\nabla_{e_j}^F(\widehat{c}(\omega^*))c(e_j)
\end{aligned}$$

$$+\frac{1}{4}\sum_j c(e_j)\nabla^F_{e_j}(\widehat{c}(\omega))\Big]+4f^{-1}\Delta(f)+8\langle \mathrm{grad}_M(f),\mathrm{grad}_M(f^{-1})\rangle-5f^{-2}[|\mathrm{grad}_M(f)|^2$$
$$+2\Delta(f)]\Big\}\mathrm{d\,vol}_M. \tag{5.3}$$

So we only need to compute $\int_{\partial M}\Psi$. From the remark above, now we can compute Ψ (see formula (5.2) for the definition of Ψ). Since the sum is taken over $r+\ell-k-j-|\alpha|-1=-6,\ r\leq -1,\ \ell\leq -3$, we have $\int_{\partial_M}\Psi$ is the sum of the following five cases:

Case (a) (I) $r=-1,l=-3,j=k=0,|\alpha|=1$.

By (5.2), we get

$$\begin{aligned}&\text{case (a) (I)}\\&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(f\widehat{D}_F^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-3}(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\\&\quad\times f^{-1}(\widehat{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&\quad-f\sum_{j<n}\partial_j(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\\&\quad\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.\end{aligned}\tag{5.4}$$

By (3.24) and (4.29), we have $\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})=\sigma_{-3}((\widetilde{D}_F^*\widetilde{D}_F\widetilde{D}_F^*)^{-1})$.

By (3.34) and [12, Lemma 2.2], for $i<n$, we have

$$\partial_{x_i}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})(x_0)=0.\tag{5.5}$$

Thus we have

$$\begin{aligned}&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\\&\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'=0.\end{aligned}\tag{5.6}$$

By (3.12) and (4.16), we have $\sigma_{-1}(\widehat{D}_F)^{-1}=\sigma_{-1}(\widetilde{D}_F)^{-1}$. Similar to (3.36)–(3.38), for $i<n$, we have

$$\begin{aligned}&\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\\&=-\xi_i\mathrm{trace}\Big[\frac{c(\mathrm{d}x_n)^2}{2(\xi_n-\sqrt{-1})^2}\Big]-4\sqrt{-1}\xi_n\xi_i\mathrm{trace}\Big[\frac{c(\mathrm{d}x_i)^2}{2(\xi_n-\sqrt{-1})|\xi|^6}\Big]+4\sqrt{-1}\xi_n\xi_i(\xi_n\\&\quad-2\sqrt{-1})\mathrm{trace}\Big[\frac{c(\xi')^2}{2(\xi_n-\sqrt{-1})^2|\xi|^6}\Big]+4\sqrt{-1}\xi_n^2\xi_i\mathrm{trace}\Big[\frac{c(\mathrm{d}x_n)^2}{2(\xi_n-\sqrt{-1})^2|\xi|^6}\Big].\end{aligned}\tag{5.7}$$

We note that $i<n$, $\int_{|\xi'|=1}\xi_i\sigma(\xi')=0$, so

$$\begin{aligned}&-f\sum_{j<n}\partial_j(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\\&\times\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'=0.\end{aligned}\tag{5.8}$$

Then we have case (a) (I) $= 0$.

Case (a) (II) $r=-1$, $l=-3$, $|\alpha|=k=0$, $j=1$.

By (5.2), we have

$$\begin{aligned}
&\text{case (a) (II)}\\
&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_{-1}(f\widehat{D}_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(f^{-1}(\widehat{D}_F^{*})^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^{*})^{-1})](x_0)\\
&\quad\times \mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(\widehat{D}_F^{*}\widehat{D}_F\widehat{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&\quad-\frac{1}{2}f^{-1}\partial_{x_n}(f)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^{+}_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(\widehat{D}_F^{*}\widehat{D}_F\widehat{D}_F^{*})^{-1}](x_0)\\
&\quad\times \mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned}\tag{5.9}$$

Since $n=6$, $\mathrm{trace}_{\wedge^*(T^*M)}[-\mathrm{id}]=-64\dim F$. By the relation of the Clifford action and $\mathrm{trace}\,PQ=\mathrm{trace}\,QP$, we have

$$\begin{aligned}
&\mathrm{trace}[c(\xi')c(\mathrm{d}x_n)]=0,\quad \mathrm{trace}[c(\mathrm{d}x_n)^2]=-64\dim F,\quad \mathrm{trace}\,[c(\xi')^2](x_0)|_{|\xi'|=1}=-64\dim F,\\
&\mathrm{trace}[\partial_{x_n}[c(\xi')]c(\mathrm{d}x_n)]=0,\quad \mathrm{trace}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1}=-32h'(0)\dim F.
\end{aligned}\tag{5.10}$$

Similar to (3.41)–(3.45), we obtain

$$\begin{aligned}
&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(\widehat{D}_F^{*}\widehat{D}_F\widehat{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}8h'(0)\dim F\frac{-8-24\xi_n\mathrm{i}+40\xi_n^2+24\mathrm{i}\xi_n^3}{(\xi_n-\mathrm{i})^6(\xi_n+\mathrm{i})^4}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=8h'(0)\dim F\,\Omega_4\frac{\pi\mathrm{i}}{5!}\Big[\frac{8+24\xi_n\mathrm{i}-40\xi_n^2-24\mathrm{i}\xi_n^3}{(\xi_n+\mathrm{i})^4}\Big]^{(5)}\Big|_{\xi_n=\mathrm{i}}\mathrm{d}x'\\
&=-\frac{15}{2}\pi h'(0)\Omega_4\dim F\mathrm{d}x'.
\end{aligned}\tag{5.11}$$

Similar to (3.47)–(3.48), we obtain

$$\begin{aligned}
&-\frac{1}{2}f^{-1}\partial_{x_n}(f)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^{+}_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(\widehat{D}_F^{*}\widehat{D}_F\widehat{D}_F^{*})^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&=(10\pi\mathrm{i}+88\pi)\Omega_4\dim F\cdot f^{-1}\partial_{x_n}(f)\mathrm{d}x',
\end{aligned}\tag{5.12}$$

where Ω_4 is the canonical volume of S_4.

Then

$$\text{case (a) (II)}=-\frac{15}{2}\pi h'(0)\Omega_4\dim F\mathrm{d}x'+(10\pi\mathrm{i}+88\pi)\Omega_4\dim F\cdot f^{-1}\partial_{x_n}(f)\mathrm{d}x',\tag{5.13}$$

where Ω_4 is the canonical volume of S_4.

Case (a) (III) $r=-1$, $l=-3$, $|\alpha|=j=0$, $k=1$.

By (5.2) and an integration by parts, we have

case (a) (III)

$$= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(f\widehat{D}_F^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})](x_0)$$
$$\times\,\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}(\sigma_{-1}(\widehat{D}_F^{-1}))\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$-\frac{1}{2}f\partial_{x_n}(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\mathrm{d}\xi_n$$
$$\times\,\sigma(\xi')\mathrm{d}x'. \tag{5.14}$$

Similar to (3.52)–(3.53) and combining (5.10), we have

$$\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})](x_0)|_{|\xi'|=1}$$
$$= 8h'(0)\dim F\frac{8\mathrm{i}-32\xi_n-8\mathrm{i}\xi_n^2}{(\xi_n-\mathrm{i})^5(\xi+\mathrm{i})^4}. \tag{5.15}$$

Then

$$-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}(\sigma_{-1}(\widehat{D}_F^{-1}))\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}8h'(0)\dim F\frac{8\mathrm{i}-32\xi_n-8\mathrm{i}\xi_n^2}{(\xi_n-\mathrm{i})^5(\xi+\mathrm{i})^4}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -8h'(0)\dim F\Omega_4\frac{\pi\mathrm{i}}{4!}\Big[\frac{8\mathrm{i}-32\xi_n-8\mathrm{i}\xi_n^2}{(\xi+\mathrm{i})^4}\Big]^{(4)}\Big|_{\xi_n=\mathrm{i}}\mathrm{d}x'$$
$$= \frac{25}{2}\pi h'(0)\Omega_4\dim F\mathrm{d}x' \tag{5.16}$$

and

$$-\frac{1}{2}f\partial_{x_n}(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}](x_0)$$
$$\times\,\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= \frac{\pi\mathrm{i}}{2}\cdot f\cdot\partial_{x_n}(f^{-1})\Omega_4\dim F\mathrm{d}x', \tag{5.17}$$

where Ω_4 is the canonical volume of S_4. Then

$$\text{case (a) (III)} = \frac{25}{2}\pi h'(0)\Omega_4\dim F\mathrm{d}x' + \frac{\pi\mathrm{i}}{2}\cdot f\cdot\partial_{x_n}(f^{-1})\Omega_4\dim F\mathrm{d}x'. \tag{5.18}$$

Case (b) $r=-2$, $l=-3$, $|\alpha|=j=k=0$.
By (5.2) and an integration by parts, we have

$$\text{case (b)}$$
$$= -\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-2}(f\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})](x_0)$$
$$\times\,\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$= -\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-2}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'. \tag{5.19}$$

Then an application of Lemma 4.3 shows

$$\begin{aligned}
&\sigma_{-2}(\widehat{D}_F^{-1})(x_0)\\
&= \frac{c(\xi)\sigma_0(\widehat{D}_F)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)](x_0)\\
&= \frac{c(\xi)\sigma_0(\widehat{D}_F)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}c(\mathrm{d}x_n)[\partial_{x_n}(c(\xi'))(x_0) - c(\xi)h'(0)|\xi'|^2_{g^{\partial M}}]. \qquad (5.20)
\end{aligned}$$

Hence,

$$\pi^+_{\xi_n}\sigma_{-2}((\widehat{D}_F)^{-1})(x_0) := B_1 + B_2 + B_3 + B_4, \tag{5.21}$$

where

$$\begin{aligned}
B_1 &= \frac{-1}{4(\xi_n - \mathrm{i})^2}\Big[(2+\mathrm{i}\xi_n)c(\xi')\Big(-\frac{5}{4}h'(0)c(\mathrm{d}x_n)\Big)c(\xi') + \mathrm{i}\xi_n c(\mathrm{d}x_n)\Big(-\frac{5}{4}h'(0)c(\mathrm{d}x_n)\Big)c(\mathrm{d}x_n)\\
&\quad + (2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi') + \mathrm{i}c(\mathrm{d}x_n)\Big(-\frac{5}{4}h'(0)c(\mathrm{d}x_n)\Big)c(\xi') + \mathrm{i}c(\xi')\Big(-\frac{5}{4}h'(0)\\
&\quad \times c(\mathrm{d}x_n)\Big)c(\mathrm{d}x_n) - \mathrm{i}\partial_{x_n}c(\xi')\Big]\\
&= \frac{1}{4(\xi_n - \mathrm{i})^2}\Big[\frac{5}{2}h'(0)c(\mathrm{d}x_n) - \frac{5\mathrm{i}}{2}h'(0)c(\xi') - (2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{\xi_n}c(\xi')\\
&\quad + \mathrm{i}\partial_{\xi_n}c(\xi')\Big], \qquad (5.22)\\
B_2 &= -\frac{h'(0)}{2}\Big[\frac{c(\mathrm{d}x_n)}{4\mathrm{i}(\xi_n - i)} + \frac{c(\mathrm{d}x_n) - \mathrm{i}c(\xi')}{8(\xi_n - \mathrm{i})^2} + \frac{3\xi_n - 7\mathrm{i}}{8(\xi_n - \mathrm{i})^3}[\mathrm{i}c(\xi') - c(\mathrm{d}x_n)]\Big], \qquad (5.23)\\
B_3 &= \frac{-1}{4(\xi_n - \mathrm{i})^2}[(2+\mathrm{i}\xi_n)c(\xi')pc(\xi') + \mathrm{i}\xi_n c(\mathrm{d}x_n)pc(\mathrm{d}x_n) + (2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi')\\
&\quad + \mathrm{i}c(\mathrm{d}x_n)pc(\xi') + \mathrm{i}c(\xi')pc(\mathrm{d}x_n) - \mathrm{i}\partial_{x_n}c(\xi')], \qquad (5.24)\\
B_4 &= \frac{-1}{4(\xi_n - \mathrm{i})^2}[(2+\mathrm{i}\xi_n)c(\xi')\vartheta c(\xi') + \mathrm{i}\xi_n c(\mathrm{d}x_n)\vartheta c(\mathrm{d}x_n) + \mathrm{i}c(\mathrm{d}x_n)\vartheta c(\xi')\\
&\quad + \mathrm{i}c(\xi')\vartheta c(\mathrm{d}x_n)]. \qquad (5.25)
\end{aligned}$$

On the other hand,

$$\partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}) = \frac{-4\mathrm{i}\xi_n c(\xi')}{(1+\xi_n^2)^3} + \frac{\mathrm{i}(1-3\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}. \tag{5.26}$$

From (5.22) and (5.26), we have

$$\begin{aligned}
&\mathrm{trace}[B_1 \times \partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})(x_0)]|_{|\xi'|=1}\\
&= \mathrm{tr}\Big\{\frac{1}{4(\xi_n - \mathrm{i})^2}\Big[\frac{5}{2}h'(0)c(\mathrm{d}x_n) - \frac{5\mathrm{i}}{2}h'(0)c(\xi') - (2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{\xi_n}c(\xi') + \mathrm{i}\partial_{\xi_n}c(\xi')\Big]\\
&\quad \times \frac{-4\mathrm{i}\xi_n c(\xi') + (\mathrm{i} - 3\mathrm{i}\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\Big\}\\
&= 8h'(0)\frac{3 + 12\mathrm{i}\xi_n + 3\xi_n^2}{(\xi_n - \mathrm{i})^4(\xi_n + \mathrm{i})^3}. \qquad (5.27)
\end{aligned}$$

Similarly, we obtain

$$\mathrm{trace}[B_2 \times \partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})(x_0)]|_{|\xi'|=1}$$

$$
\begin{aligned}
&= \mathrm{tr}\Big\{-\frac{h'(0)}{2}\Big[\frac{c(\mathrm{d}x_n)}{4\mathrm{i}(\xi_n-\mathrm{i})}+\frac{c(\mathrm{d}x_n)-\mathrm{i}c(\xi')}{8(\xi_n-\mathrm{i})^2}+\frac{3\xi_n-7\mathrm{i}}{8(\xi_n-\mathrm{i})^3}[\mathrm{i}c(\xi')-c(\mathrm{d}x_n)]\Big]\\
&\quad\times\frac{-4\mathrm{i}\xi_n c(\xi')+(\mathrm{i}-3\mathrm{i}\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\Big\}\\
&= -8h'(0)\frac{4\mathrm{i}-11\xi_n-6\mathrm{i}\xi_n^2+3\xi_n^3}{(\xi_n-\mathrm{i})^5(\xi_n+\mathrm{i})^3}.
\end{aligned}
\tag{5.28}
$$

For the signature operator case,

$$
\mathrm{trace}[c(\xi')pc(\xi')c(\mathrm{d}x_n)](x_0)=\mathrm{trace}[pc(\xi')c(\mathrm{d}x_n)c(\xi')](x_0)=|\xi'|^2\mathrm{trace}[p(x_0)c(\mathrm{d}x_n)] \tag{5.29}
$$

and

$$
\begin{aligned}
&c(\mathrm{d}x_n)p(x_0)\\
&= -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(\widetilde{e_i})\widehat{c}(\widetilde{e_i})c(\widetilde{e_n})\widehat{c}(\widetilde{e_n})\\
&= -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}[\varepsilon(\widetilde{e_i*})\iota(\widetilde{e_i*})-\iota(\widetilde{e_i*})\varepsilon(\widetilde{e_i*})][\varepsilon(\widetilde{e_n*})\iota(\widetilde{e_n*})-\iota(\widetilde{e_n*})\varepsilon(\widetilde{e_n*})].
\end{aligned}
\tag{5.30}
$$

By [12, Section 3], we have

$$
\begin{aligned}
&\mathrm{trace}_{\wedge^m(T^*M)}\{[\varepsilon(e_i*)\iota(e_i*)-\iota(e_i*)\varepsilon(e_i*)][\varepsilon(e_n*)\iota(e_n*)-\iota(e_n*)\varepsilon(e_n*)]\}\\
&= a_{n,m}\langle e_i*,e_n*\rangle^2+b_{n,m}|e_i*|^2|e_n*|^2=b_{n,m},
\end{aligned}
\tag{5.31}
$$

where $b_{6,m}=\binom{4}{m-2}+\binom{4}{m}-2\binom{4}{m-1}$.

Then

$$
\mathrm{tr}_{\wedge^*(T^*M)}\{[\varepsilon(\widetilde{e_i*})\iota(\widetilde{e_i*})-\iota(\widetilde{e_i*})\varepsilon(\widetilde{e_i*})][\varepsilon(\widetilde{e_n*})\iota(\widetilde{e_n*})-\iota(\widetilde{e_n*})\varepsilon(\widetilde{e_n*})]\}=\sum_{m=0}^{6}b_{6,m}=0. \tag{5.32}
$$

Hence in this case,

$$
\mathrm{trace}_{\wedge^*(T^*M)}[c(\mathrm{d}x_n)p(x_0)]=0. \tag{5.33}
$$

We note that $\int_{|\xi'|=1}\xi_1\cdots\xi_{2q+1}\sigma(\xi')=0$, then $\mathrm{trace}_{\wedge^*(T^*M)}[c(\xi')p(x_0)]$ has no contribution for computing case (b).

So, we obtain

$$
\begin{aligned}
&\mathrm{trace}[B_3\times\partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})(x_0)]|_{|\xi'|=1}\\
&= \mathrm{trace}\Big\{\frac{-1}{4(\xi_n-\mathrm{i})^2}[(2+\mathrm{i}\xi_n)c(\xi')pc(\xi')+\mathrm{i}\xi_n c(\mathrm{d}x_n)pc(\mathrm{d}x_n)+(2+\mathrm{i}\xi_n)c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi')\\
&\quad+\mathrm{i}c(\mathrm{d}x_n)pc(\xi')+\mathrm{i}c(\xi')pc(\mathrm{d}x_n)-\mathrm{i}\partial_{x_n}c(\xi')]\times\frac{-4\mathrm{i}\xi_n c(\xi')+(\mathrm{i}-3\mathrm{i}\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}\Big\}\\
&= 8h'(0)\dim F\frac{3\xi_n^2-3\mathrm{i}\xi_n-2}{(\xi_n-\mathrm{i})^4(\xi_n+\mathrm{i})^3}.
\end{aligned}
\tag{5.34}
$$

Then, we have

$$
\mathrm{trace}[(B_1+B_2+B_3)\times\partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'
$$

$$= 8h'(0)\dim F\frac{3\xi_n^3 + 9\mathrm{i}\xi_n^2 + 21\xi_n - 5\mathrm{i}}{(\xi_n - \mathrm{i})^5(\xi_n + \mathrm{i})^3}. \tag{5.35}$$

By the relation of the Clifford action and $\mathrm{trace}\,PQ = \mathrm{trace}\,QP$, we have the equalities

$$\mathrm{trace}[c(\widetilde{e}_i)c(\mathrm{d}x_n)] = 0, \quad i < n; \quad \mathrm{trace}[c(\widetilde{e}_i)c(\mathrm{d}x_n)] = -64\dim F, \quad i = n; \tag{5.36}$$

$$\mathrm{trace}[\widehat{c}(\widetilde{e}_i)c(\xi')] = \mathrm{trace}[\widehat{c}(\widetilde{e}_i)c(\mathrm{d}x_n)] = 0. \tag{5.37}$$

Then $\mathrm{trace}[\vartheta c(\xi')]$ has no contribution for computing case (b).

Then, we have

$$\begin{aligned}
&\mathrm{trace}[B_4 \times \partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})]|_{|\xi'|=1} \\
&= \mathrm{trace}\Big\{\frac{-1}{4(\xi_n - \mathrm{i})^2}[(2 + \mathrm{i}\xi_n)c(\xi')\vartheta c(\xi') + \mathrm{i}\xi_n c(\mathrm{d}x_n)\vartheta c(\mathrm{d}x_n) + \mathrm{i}c(\mathrm{d}x_n)\vartheta c(\xi') \\
&\quad + \mathrm{i}c(\xi')\vartheta c(\mathrm{d}x_n)] \times \frac{-4\mathrm{i}\xi_n c(\xi') + (\mathrm{i} - 3\mathrm{i}\xi_n^2)c(\mathrm{d}x_n)}{(1 + \xi_n^2)^3}\Big\} \\
&= \frac{\mathrm{i}(3\xi_n - \mathrm{i})}{2(\xi_n - \mathrm{i})^4(\xi_n + \mathrm{i})^3}\mathrm{trace}[c(\mathrm{d}x_n)\vartheta] \\
&= -32\dim F\frac{1 + 3\xi_n\mathrm{i}}{(\xi_n - \mathrm{i})^4(\xi_n + \mathrm{i})^3}\mathrm{trace}[\sigma_n^{F,e}].
\end{aligned} \tag{5.38}$$

From (5.35), we obtain

$$\begin{aligned}
&-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[(B_1 + B_2 + B_3) \times \partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&= -8\mathrm{i}\dim F h'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3\xi_n^3 + 9\xi_n^2\mathrm{i} + 21\xi_n - 5\mathrm{i}}{(\xi_n - \mathrm{i})^5(\xi_n + \mathrm{i})^3}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&= -8\mathrm{i}\dim F h'(0)\frac{2\pi\mathrm{i}}{4!}\Big[\frac{3\xi_n^3 + 9\xi_n^2\mathrm{i} + 21\xi_n - 5\mathrm{i}}{(\xi_n + \mathrm{i})^3}\Big]^{(4)}\Big|_{\xi_n=\mathrm{i}}\Omega_4\mathrm{d}x' \\
&= \frac{45}{2}\dim F\pi h'(0)\Omega_4\mathrm{d}x'.
\end{aligned} \tag{5.39}$$

From (5.38), we obtain

$$\begin{aligned}
&-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[B_4 \times \partial_{\xi_n}\sigma_{-3}((\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&= 32\mathrm{i}\dim F\mathrm{trace}[\sigma_n^{F,e}]\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{1 + 3\xi_n}{(\xi_n - \mathrm{i})^4(\xi_n + \mathrm{i})^3}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&= 32\mathrm{i}\dim F\mathrm{trace}[\sigma_n^{F,e}]\frac{2\pi\mathrm{i}}{3!}\Big[\frac{1 + 3\xi_n}{(\xi_n + \mathrm{i})^3}\Big]^{(4)}\Big|_{\xi_n=\mathrm{i}}\Omega_4\mathrm{d}x' \\
&= -16\dim F\mathrm{trace}[\sigma_n^{F,e}]\Omega_4\mathrm{d}x'.
\end{aligned} \tag{5.40}$$

Combining (5.19) and (5.39)–(5.40), we have

$$\text{case (b)} = \Big[\frac{45}{2}h'(0) - 16\mathrm{trace}(\sigma_n^{F,e})\Big]\pi\dim F\Omega_4\mathrm{d}x'. \tag{5.41}$$

Case (c) $r = -1$, $l = -4$, $|\alpha| = j = k = 0$.

By (5.2) and an integration by parts, we have

case (c)

$$
\begin{aligned}
&= -\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}[\pi^+_{\xi_n}\sigma_{-1}(f\widehat{D}_F^{-1})\times\partial_{\xi_n}\sigma_{-4}(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})](x_0)\\
&\quad\times \mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&= -\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\Big(\sigma_{-4}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}\Big)\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&\quad+2\mathrm{i}f^{-1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big)\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'. \qquad (5.42)
\end{aligned}
$$

By direct calculations, we have

$$
\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1}) = -\frac{c(\xi')+\mathrm{i}c(\mathrm{d}x_n)}{2(\xi_n-\mathrm{i})}. \qquad (5.43)
$$

In the normal coordinate, $g^{ij}(x_0)=\delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<n$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta^\alpha_\beta$, if $j=n$. So by [12, Lemma A.2], we have $\Gamma^n(x_0)=\frac{5}{2}h'(0)$ and $\Gamma^k(x_0)=0$ for $k<n$. By the definition of δ^k and [12, Lemma 2.3], we have $\delta^n(x_0)=0$ and $\delta^k=\frac{1}{4}h'(0)c(\widetilde{e_k})c(\widetilde{e_n})$ for $k<n$. By [19, (3.15)], we obtain

$$
\begin{aligned}
&\sigma_{-4}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1}(x_0)\\
&= \frac{-17-9\xi_n^2}{4(1+\xi_n^2)^4}h'(0)c(\xi')c(\mathrm{d}x_n)c(\xi')+\frac{33\xi_n+17\xi_n^3}{2(1+\xi_n^2)^4}h'(0)c(\xi')+\frac{49\xi_n^2+25\xi_n^4}{2(1+\xi_n^2)^4}h'(0)c(\mathrm{d}x_n)\\
&\quad+\frac{1}{(1+\xi_n^2)^3}c(\xi')c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')](x_0)-\frac{3\xi_n}{(1+\xi_n^2)^3}\partial_{x_n}[c(\xi')](x_0)-\frac{2\xi_n}{(1+\xi_n^2)^3}h'(0)\xi_nc(\xi')\\
&\quad+\frac{1-\xi_n^2}{(1+\xi_n^2)^3}h'(0)c(\mathrm{d}x_n)+\frac{c(\xi)(p+\vartheta^*-\widehat{c}(w^*))c(\xi)}{|\xi|^6}+\frac{\widehat{c}(w)}{|\xi|^4}. \qquad (5.44)
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}(\sigma_{-4}(\widehat{D}_F^*\widehat{D}_F\widehat{D}_F^*)^{-1})](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\
&= -\frac{129}{2}\pi h'(0)\dim F\Omega_4\mathrm{d}x'+12\pi\operatorname{trace}[\sigma_n^{F,e}]\dim F\Omega_4\mathrm{d}x'+4\pi\operatorname{trace}[w(F,g^F)(e_n)]\dim F\Omega_4\mathrm{d}x'\\
&\quad-12\pi\operatorname{trace}[w^*(F,g^F)(e_n)]\dim F\Omega_4\mathrm{d}x'. \qquad (5.45)
\end{aligned}
$$

By $\sigma_{-1}(\widehat{D}_F^{-1})=\sigma_{-1}(\widetilde{D}_F^{-1})$, similar to case (b) in Section 3, and we get

$$
\begin{aligned}
&\operatorname{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big)\Big](x_0)\\
&= \frac{(4\xi_n\mathrm{i}+2)\mathrm{i}}{2(\xi_n+\mathrm{i})(1+\xi_n^2)^3}\operatorname{trace}[c(\xi')c(\mathrm{d}f)]+\frac{4\xi_n\mathrm{i}+2}{2(\xi_n+\mathrm{i})(1+\xi_n^2)^3}\operatorname{trace}[c(\mathrm{d}x_n)c(\mathrm{d}f)]
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\Big](x_0)\\
&= \frac{(3\xi_n-\mathrm{i})\mathrm{i}}{(\xi_n+\mathrm{i})(1+\xi_n^2)^4}\operatorname{trace}\Big[c(\xi')\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]D_{x_j}(f^{-1})\Big]\\
&\quad+\frac{3\xi_n-\mathrm{i}}{(\xi_n+\mathrm{i})(1+\xi_n^2)^4}\operatorname{trace}\Big[c(\mathrm{d}x_n)\sum_j[c(\mathrm{d}x_j)|\xi|^2+2\xi_jc(\xi)]D_{x_j}(f^{-1})\Big]. \qquad (5.46)
\end{aligned}
$$

By the relation of the Clifford action and $\mathrm{trace}\,QP = \mathrm{trace}\,PQ$, we have the following equalities

$$\mathrm{trace}[c(\mathrm{d}x_n)c(\mathrm{d}f)] = -g(\mathrm{d}x_n, \mathrm{d}f)$$

and

$$\begin{aligned}&\mathrm{trace}\Big[c(\mathrm{d}x_n)\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})\Big]\\ &= \mathrm{trace}(-\mathrm{id})|\xi|^2(-\mathrm{i}\partial_{x_n}(f)f^{-1}) + 2\sum_j \xi_j\xi_n \mathrm{trace}(-\mathrm{id})(-\mathrm{i}\partial_{x_j}(f)f^{-1})\\ &= -64\dim F|\xi|^2(-\mathrm{i}\partial_{x_n}(f)f^{-1}) + 2\sum_j \xi_j\xi_n \mathrm{trace}(-\mathrm{id})(-\mathrm{i}\partial_{x_j}(f)f^{-1}).\end{aligned}$$

We note that $i < n$, $\int_{|\xi'|=1}\xi_i\sigma(\xi') = 0$, so $\mathrm{trace}[c(\xi')c(\mathrm{d}f)]$, $\mathrm{trace}\big[c(\xi')\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})\big]$ and $2\mathrm{i}\sum_j \xi_j\xi_n\partial_{x_j}(f)f^{-1}\mathrm{trace}[-\mathrm{id}]$ have no contribution for computing case (b). Then we obtain

$$\begin{aligned}&-2\mathrm{i}f^{-1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{c(\xi)c(\mathrm{d}f)c(\xi)}{|\xi|^6}\Big)\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\ &= \frac{3}{8f}\pi g[\mathrm{d}x_n, \mathrm{d}f]\Omega_4\mathrm{d}x' \end{aligned}\tag{5.47}$$

and

$$\begin{aligned}&-f\mathrm{i}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widehat{D}_F^{-1})\times\partial_{\xi_n}\Big(\frac{\mathrm{i}c(\xi)\sum_j[c(\mathrm{d}x_j)|\xi|^2 + 2\xi_j c(\xi)]D_{x_j}(f^{-1})c(\xi)}{|\xi|^8}\Big)\Big]\\ &\times(x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\ &= -60\mathrm{i}\partial_{x_n}(f)\pi\dim F\Omega_4\mathrm{d}x'.\end{aligned}\tag{5.48}$$

Then we have

$$\begin{aligned}\text{case (c)} = &\Big\{12\mathrm{trace}[\sigma_n^{F,e}] - \frac{129}{2}h'(0) + 4\mathrm{trace}[w(F,g^F)(e_n)] - 12\mathrm{trace}[w^*(F,g^F)(e_n)]\\ &- 60\mathrm{i}\partial_{x_n}(f)\Big\}\pi\dim F\Omega_4\mathrm{d}x' + \frac{3}{8f}g(\mathrm{d}x_n, \mathrm{d}f)\pi\Omega_4\mathrm{d}x'.\end{aligned}\tag{5.49}$$

Now Ψ is the sum of the case (a), case (b) and case (c), then

$$\begin{aligned}\Psi = &\Big\{4\mathrm{trace}[w(F,g^F)(e_n)] - 37h'(0) - 4\mathrm{trace}[\sigma_n^{F,e}] - 12\mathrm{trace}[w^*(F,g^F)(e_n)]\\ &+\Big(\frac{19\mathrm{i}}{22f} + \frac{88}{f} - 60\mathrm{i}\Big)\partial_{x_n}(f)\Big\}\pi\Omega_4\dim F\mathrm{d}x' + \frac{3}{8f}g(\mathrm{d}x_n, \mathrm{d}f)\pi\Omega_4\mathrm{d}x'.\end{aligned}\tag{5.50}$$

By [12, (4.2)], we have

$$K = \sum_{1\le i,j\le n-1} K_{i,j}g^{i,j}_{\partial M}, \quad K_{i,j} = -\Gamma^n_{i,j},$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. For $n = 6$, we have

$$K(x_0) = \sum_{1\le i,j\le n-1} K_{i,j}(x_0)g^{i,j}_{\partial M}(x_0) = \sum_{i=1}^{5} K_{i,i}(x_0) = -\frac{5}{2}h'(0). \tag{5.51}$$

Hence we conclude the following theorem.

Theorem 5.1 *Let M be a six-dimensional compact manifolds with the boundary ∂M. Then*

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+(f\widehat{D}_F^{-1})\circ\pi^+(f^{-1}(\widehat{D}_F^*)^{-1}\cdot f\widehat{D}_F^{-1}\cdot f^{-1}(\widehat{D}_F^*)^{-1})]\\
&=8\pi^3\int_M\Big\{\mathrm{trace}\Big[-\frac{s}{12}+\frac{3}{8}[\widehat{c}(\omega^*)-\widehat{c}(\omega)]^2-\frac{1}{4}\widehat{c}(\omega^*)\widehat{c}(\omega)-\frac{1}{4}\sum_j\nabla^F_{e_j}(\widehat{c}(\omega^*))c(e_j)\\
&\quad+\frac{1}{4}\sum_j c(e_j)\nabla^F_{e_j}(\widehat{c}(\omega))\Big]+4f^{-1}\Delta(f)+8\langle\mathrm{grad}_M(f),\mathrm{grad}_M(f^{-1})\rangle-5f^{-2}[|\mathrm{grad}_M(f)|^2\\
&\quad+2\Delta(f)]\Big\}\mathrm{d\,vol}_M+\int_{\partial M}\Big\{\Big\{4\mathrm{trace}[w(F,g^F)(e_n)]-4\mathrm{trace}[\sigma_n^{F,e}]-12\mathrm{trace}[w^*(F,g^F)(e_n)]\\
&\quad-37h'(0)+\Big(\frac{19\mathrm{i}}{22f}+\frac{88}{f}-60\mathrm{i}\Big)\partial_{x_n}(f)\Big\}\dim F+\frac{3}{8f}g(\mathrm{d}x_n,\mathrm{d}f)\Big\}\pi\Omega_4\mathrm{d\,vol}_M,
\end{aligned}\tag{5.52}$$

where s is the scalar curvature.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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