Chin. Ann. Math. Ser. B
46(6), 2025, 807–824
DOI: 10.1007/s11401-025-0062-8

Chinese Annals of Mathematics, Series B

Siegel Disks Whose Boundaries are Jordan Curves with Positive Area*

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Abstract In this paper, the authors construct a univalent function having a relatively compact Siegel disk whose boundary is a Jordan curve of positive area. The construction is based on a general scheme in which Chéritat added Runge's theorem, to construct a relatively compact Siegel disk and Osgood's method for constructing a Jordan curve of positive area.

Keywords Univalent functions, Siegel disks, Runge's theorem, A Jordan curve of positive area
2020 MR Subject Classification 37F50

1 Introduction

A Siegel disk is the biggest simply connected rotation domain of a holomorphic function. In general, the boundary of a simply connected domain may have very complicated structure. However, how complicated may the structure of a Siegel disk boundary be? For Siegel disks of rational functions, it is conjectured that the topological structure is simple. See the following well-known open question by Douady and Sullivan (see [5, 11]): "Is a Siegel disk boundary of a rational function always a Jordan curve?" In the past few decades, there have been some breakthroughs on this question (see [3, 6, 12, 14–16]). From the perspective of measure or dimension, there is an open question that people are also concerned about (see [10] for quadratic polynomials): "Can a Siegel disk boundary of a rational function have Hausdorff dimension 2?" In this paper, instead of studying Siegel disks of rational functions directly, we adapt these two questions for a general class of holomorphic functions as follows.

Question Can a simple topological structure and a large Hausdorff dimension (equal to 2) coexist for a relatively compact Siegel disk boundary of a holomorphic function?

Manuscript received June 30, 2025. Revised September 3, 2025.
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*This work was supported by the National Natural Science Foundation of China (Nos. 12301102, 12471084), the Fundamental Research Funds for the Central Universities, Undergraduate General Education Reform Project of Beijing University of Posts and Telecommunications (No. 2025YB63) and the Graduate General Education Reform Project of Beijing University of Posts and Telecommunications (No. 2025YY024).

Our following main result gives this question an affirmative answer.

Main Theorem *There exists a univalent function fixing* 0 *having a relatively compact Siegel disk centered at* 0 *whose boundary is a Jordan curve of positive area.*

Our proof is to adopt a general scheme for constructing a relatively compact Siegel disk. Precisely, in the same spirit as Handel's or Herman's producing a diffeomorphism with a pseudo disk (see [7–8]), Pérez-Marco used tube-log Riemann surfaces to construct examples of injective holomorphic maps having a relatively compact Siegel disk whose boundary is a smooth Jordan curve. Later, Biswas used Pérez-Marco's construction to produce a set of interesting examples (see [1–2]). In [4], Chéritat added Runge's theorem to this construction and got a univalent function having a relatively compact Siegel disk with non-locally connected boundary.[1] In [13], Sun and Qu produced a univalent function having a relatively compact Siegel disk with positive area boundary based on the construction having Runge's theorem by Chéritat. In this paper, together with Osgood's method constructing a Jordan curve of positive area (see [9]), we also use the general scheme having Runge's theorem to prove our main theorem.

2 A General Scheme to Construct Siegel Disks

In this section, a general scheme for constructing univalent holomorphic maps with relatively compact Siegel disks is briefly presented. This content is not new, refer [4] for more details.

Different from Chéritat's convention, we immediately work in the complex plane $\mathbb{C}$. The following notations are needed:

- For any real number θ, define $R_\theta : \mathbb{C} \to \mathbb{C}, z \mapsto \mathrm{e}^{2\pi\mathrm{i}\theta} z$.
- For all $t > 0$, define $\mathbb{D}_t := \{z \in \mathbb{C} : |z| < t\}$, here we call a standard disk, in particular, we also write $\mathbb{D} = \mathbb{D}_1$.
- For all $t > 0$ and $z \in \mathbb{C}$, define

$$\mathbb{B}_t(z) := \{w \in \mathbb{C} : |w - z| < t\}.$$

- For all $0 < r < r'$ and $\theta < \theta'$, we set

$$H_{r,r'} := \{z : r < |z| < r'\},$$
$$l_{\theta,r} := \{\rho \mathrm{e}^{\mathrm{i}\theta} : r < \rho < 1\}$$

and

$$\Omega_{\theta,\theta'}^{r,r'} := \{z = \rho \mathrm{e}^{2\pi\mathrm{i}\alpha} : r < \rho < r', \theta < \alpha < \theta'\},$$

in particular, we write also $\Omega_{\theta,\theta'}^{r,1}$ as $\Omega_{\theta,\theta'}^{r}$ and write also $H_{r,1}$ as H_r.

- We denote by $\mathrm{Dom}(f)$ the domain of a map f.

[1] In [4], Chéritat, based on the new flexibility allowed by Runge's theorem, proposed the following two claims.

Proposal 1 Prove that there exists an injective holomorphic map f defined in a simply connected open subset U of $\mathbb{C}$ containing the origin, fixing 0 and having at 0 a hedgehog of positive Lebesgue measure compactly contained in U.

Proposal 2 Prove that there exists an injective holomorphic map f defined in a doubly connected open subset U of $\mathbb{C}$ and a Jordan curve J with positive Lebesgue measure contained in U that is invariant by f, and carries an invariant line field.

• We denote by $\mathcal{H}$ the set of entire functions satisfying that for all $\beta \in \mathcal{H}$, $\beta(z) = 0 \Leftrightarrow z = 0$, $\beta'(0) > 0$, $\beta'(z) \neq 0$ for all $z \in \mathbb{C}$ and there exists a Jordan domain $\mathcal{D}(\beta)$ $(\subseteq \mathbb{D})$ containing 0 such that $\beta|_{\mathcal{D}(\beta)}$ is a conformal map from $\mathcal{D}(\beta)$ to $\mathbb{D}$. We write $\mathcal{D}_r(\beta) := (\beta|_{\mathcal{D}(\beta)})^{-1}(H_r)$ for all $0 < r < 1$.

Given a sequence $\{q_n\}_{n=1}^{+\infty}$ of positive integers and a sequence $\{\beta_n\}_{n=1}^{+\infty}$ of elements in $\mathcal{H}$, for all $n \geq 1$, we let $\mathcal{R}_n$ be the single-valued analytic branch of $\sqrt[q_1 q_2 \cdots q_n]{\beta_n(z^{q_1 q_2 \cdots q_{n-1}})}$ (set $q_0 = 1$) such that $(\mathcal{R}_n)'(0) > 0$ and G_n be the holomorphic lift of R_{θ_n} with $\theta_n := \frac{1}{q_1 q_2 \cdots q_n}$, defined in a neighborhood of 0, under $\varphi_n := \mathcal{R}_n \circ \mathcal{R}_{n-1} \circ \cdots \circ \mathcal{R}_1$, that is, $\varphi_n \circ G_n = R_{\theta_n} \circ \varphi_n$. In general, the domain of G_n is indefinite, but if q_n is large enough, then $\mathrm{Dom}(G_n)$ contains any given standard disk and $\mathrm{Dom}(G_n)$ is arbitrarily close to the identity map on the given standard disk. Fix $\varepsilon > 0$ and let $f_n := G_n \circ G_{n-1} \circ \cdots \circ G_1$.

Assume 2.1 q_n, $n \geq 1$ are large enough so that $\mathrm{Dom}(G_n) \supseteq \mathbb{D}_{1+\varepsilon}$, $\mathrm{Dom}(f_n) \supseteq \mathbb{D}_{1+\varepsilon}$ and both G_n and f_n are univalent on $\mathbb{D}_{1+\varepsilon}$ for some fixed $\varepsilon > 0$.

The following facts are easily checked:

(a) For all $n \geq 1$, $(f_n)'(0) = \mathrm{e}^{2\pi \mathrm{i} \sum_{j=1}^{n} \theta_j}$.

(b) For all $n \geq 1$, $\mathcal{R}_n \circ R_{\theta_{n-1}} = R_{\theta_{n-1}} \circ \mathcal{R}_n$ and $G_n^{\circ q_n} = G_{n-1}$ with $G_0 = \mathrm{Identity}$.

(c) The map φ_n is semi-conjugate from f_n to $R_{\sum_{j=1}^{n} \theta_n}$ on $\mathbb{D}_{1+\varepsilon}$, that is,

$$\varphi_n \circ f_n(z) = R_{\sum_{j=1}^{n} \theta_n} \circ \varphi_n(z)$$

for all $z \in \mathbb{D}_{1+\varepsilon}$.

It is easy to see that $\varphi_n \in \mathcal{H}$. Then $\mathcal{D}(\varphi_n)$ $(\subseteq \mathbb{D})$ is well-defined.

Assume 2.2 As $n \to +\infty$, $\mathcal{D}(\varphi_n)$ converges to a simply connected region $\mathcal{D} \subseteq \mathbb{D}$ containing 0 in the sense of Carathéodory.

Let φ be the Riemann mapping with $\varphi(0) = 0$ and $\varphi'(0) > 0$ from $\mathcal{D}$ to $\mathbb{D}$. By Assume 2.2 and a theorem of Carathéodory, $\varphi_n|_{\mathcal{D}(\varphi_n)}$ converges locally uniformly to φ on $\mathcal{D}$. Furthermore, the following lemma can be proved.

Lemma 2.1 (see [4]) *As long as $q_n (n \geq 1)$ are large enough, f_n converges locally uniformly on $\mathbb{D}_{1+\varepsilon}$ to f defined on $\mathbb{D}_{1+\varepsilon}$ and satisfies*

- *f is univalent,*
- *there is an irrational number $\theta = \sum_{j=1}^{+\infty} \frac{1}{q_1 q_2 \cdots q_j}$ such that $\varphi \circ f \circ \varphi^{-1}(z) = R_\theta(z)$ on $\mathbb{D}$.*

As seen, $\widetilde{f}$ is exactly what we want with a Siegel disk $\mathcal{D}$ if its boundary is not an analytic curve. Thus to complete the proof, we only need to construct $\{\beta_n\}$ such that Assume 2.2 holds and the boundary of $\mathcal{D}$ is a Jordan curve of positive area. At last, we need the following lemma by Chéritat whose proof is based on Runge's theorem (see [4, Lemma 5]), which will be used to construct β_n.

Lemma 2.2 (see [4]) *Assume $W \subseteq \mathbb{C}$ is a simply connected region containing 0 and ϕ is a conformal bijection from W to $\mathbb{D}$ fixing 0. Then there exists a sequence of holomorphic maps $\psi_n : \mathbb{C} \to \mathbb{C}$ such that*

- $\psi_n(z) = 0 \Leftrightarrow z = 0$,
- ψ_n' *does not vanish,*
- *as n tends to $+\infty$, ψ_n tends uniformly to ϕ on every compact subset of W.*

3 Technique Preparation for the Construction of $\{\beta_n\}$

For all $\delta > 0$, a Jordan curve $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is called δ-analytic if

- there exist $0 = a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$ such that $\operatorname{diam}(\gamma([a_j, a_{j+1}])) < \delta$ for all $j = 1, 2, \cdots, n$,
- for all $1 \le j \le n$, there exists an ε_j-neighborhood $(a_j - \varepsilon, a_j + \varepsilon)$ of a_j such that $\gamma|_{(a_j-\varepsilon_j, a_j+\varepsilon_j)}$ is an analytic arc.

Moreover, we say that a Jordan curve γ is 0-analytic if γ is δ-analytic for any $\delta > 0$. Let H be a 2-connected region in $\mathbb{C}$ bounded by two Jordan curves γ_1 and γ_2. Then there exist $r > 0$ and a conformal map ϕ from H_r to H. We say that ϕ is δ-admissible if there exist $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_{n+1} = \theta_1 + 1$ such that for all $1 \le j \le n$ and all $z, z' \in \Omega^r_{\theta_j, \theta_{j+1}}$, $|\phi(z) - \phi(z')| \le \delta$.

Lemma 3.1 *Let γ be a δ-analytic Jordan curve in $H_{r'}$ such that γ surrounds 0. Then there exist $\beta \in \mathcal{H}$ and $0 < r'' < 1$ such that*

- $\gamma \subseteq \mathcal{D}_{r''}(\beta)$ *and* $\overline{\mathcal{D}_{r''}(\beta)} \subseteq H_{r'}$,
- $(\beta|_{\mathcal{D}_{r''}(\beta)})^{-1}$ *is 14δ-admissible.*

Proof Let D_1 be the component of $\widehat{\mathbb{C}} \setminus \gamma$ containing 0 and D_2 be the component of $\widehat{\mathbb{C}} \setminus \gamma$ containing ∞. Let ϕ_1 be the conformal map from $\mathbb{D}$ to D_1 such that $\phi_1(0) = 0$ and $\phi_1'(0) > 0$. Let ϕ_2 be the conformal map from $\mathbb{D}$ to D_2 such that $\phi_2(0) = \infty$ and $\lim\limits_{z\to 0} z\phi_2(z) > 0$. For all $0 < r < 1$, we denote by D_1^r the component of $\mathbb{C} \setminus \phi_2(\partial\mathbb{D}_r)$ containing 0. Let ϕ_r be the conformal map from $\mathbb{D}$ to D_1^r such that $\phi_r(0) = 0$ and $(\phi_r)'(0) > 0$. Since D_1^r converges to D_1 in the sense of Carathéodory, by a theorem of Carathéodory we have that ϕ_r^{-1} converges uniformly to ϕ_1^{-1} on every compact subset of D_1 as $r \to 1^-$. For all $z \in \mathbb{D}$, we choose $\varepsilon > 0$ sufficiently small so that $\overline{\mathbb{B}_\varepsilon(\phi_1(z))} \subseteq D_1$. Then $\phi_r(w) = \frac{1}{2\pi \mathrm{i}} \int_{|\xi - \phi_r(z)| = \varepsilon} \frac{\xi(\phi_r^{-1})'(\xi)}{\phi_r^{-1}(\xi) - w} \mathrm{d}z$ converges uniformly to $\phi_1(w) = \frac{1}{2\pi \mathrm{i}} \int_{|\xi - \phi_1(z)| = \varepsilon} \frac{\xi(\phi_1^{-1})'(\xi)}{\phi_1^{-1}(\xi) - w} \mathrm{d}z$ on some neighborhood of z as $r \to 1^-$. Thus ϕ_r converges uniformly to ϕ_1 on every compact subset of $\mathbb{D}$ as $r \to 1^-$.

We denote also by ϕ_1 the extension of ϕ_1 to $\overline{\mathbb{D}}$. In this case, $\phi_1(\partial\mathbb{D}) = \gamma([0, 1])$. Since γ is a δ-analytic Jordan curve, we have that there exist $0 = a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$ such that

- $\operatorname{diam}(\{\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}) < \delta$ for all $j = 1, 2, \cdots, n$,
- for all $1 \le j \le n$, there exists an ε_j-neighborhood $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ of a_j such that $\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})$ is an analytic arc in this neighborhood (thanks to the reflection principle).

We denote also by ϕ_2 the extension of ϕ_2 to $\overline{\mathbb{D}}$. Let $b_1 > b_2 > \cdots > b_n > b_{n+1} = b_1 - 1$

correspond to $0 = a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$, that is $\phi_2(\mathrm{e}^{2\pi \mathrm{i} b_j}) = \phi_1(\mathrm{e}^{2\pi \mathrm{i} a_j})$ for $j = 1, 2, \cdots, n+1$. Then

- $\mathrm{diam}(\{\phi_2(\mathrm{e}^{2\pi \mathrm{i}\theta}) : b_j \geq \theta \geq b_{j+1}\}) < \delta$ for all $j = 1, 2, \cdots, n$,
- for all $1 \leq j \leq n$, there exists an ε'_j-neighborhood $(b_j - \varepsilon'_j, b_j + \varepsilon'_j)$ of b_j such that $\phi_2(\mathrm{e}^{2\pi \mathrm{i}\theta})$ is an analytic arc in this neighborhood (thanks to the reflection principle).

We denote also by ϕ_r the extension of ϕ_r to $\overline{\mathbb{D}}$. We set

$$\widehat{\mathbb{D}} = \mathbb{D} \cup \Big\{ \mathrm{e}^{2\pi \mathrm{i}\theta} : \theta \in \bigcup_{j=1}^{n} (a_j - \varepsilon_j, a_j + \varepsilon_j) \Big\}.$$

We claim that ϕ_r converges uniformly to ϕ_1 on every compact subset of $\widehat{\mathbb{D}}$ as $r \to 1^-$. Indeed, for all $j \in \{1, 2, \cdots, n\}$ and $\theta \in (a_j - \varepsilon_j, a_j + \varepsilon_j)$, since $\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})$ is an analytic arc on $(a_j - \varepsilon_j, a_j + \varepsilon_j)$, there exists a ball $\mathbb{B}_\varepsilon(\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})))$ $(0 < \varepsilon \ll 1)$ such that ϕ_2 can be homeomorphically extended to $\overline{\mathbb{D} \cup \mathbb{B}_\varepsilon(\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})))}$ and univalent on $\mathbb{D} \cup \mathbb{B}_\varepsilon(\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})))$, written as $\widetilde{\phi}_2$. It follows that $\widetilde{\phi}_2$ is univalent on $\mathbb{D}_r \cup \mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})))$ for all $0 < r \leq 1$. Then $\phi_r^{-1} \circ \widetilde{\phi}_2$ is univalent on $\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \overline{\mathbb{D}_r}$ and continuous on

$$(\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \overline{\mathbb{D}_r}) \cup (\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \cap \partial\mathbb{D}_r) = \mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \mathbb{D}_r.$$

Again, observe that $\phi_r^{-1} \circ \widetilde{\phi}_2$ maps $\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \overline{\mathbb{D}_r}$ into $\mathbb{D}$ and maps $\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})))$ $\cap\, \partial\mathbb{D}_r$ into $\partial\mathbb{D}$. Thus the reflection principle gives that $\phi_r^{-1} \circ \widetilde{\phi}_2$ can be univalently extended to

$$E_{r,2} := (\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \mathbb{D}_r) \cup \iota(\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \mathbb{D}_r),$$

written as $\phi_{r,2}$, where $\iota(z) = \frac{1}{\bar{z}}$. The composition $\phi_{r,2} \circ \widetilde{\phi}_2^{-1}$ is univalent on $\widetilde{\phi}_2(E_{r,2})$ and coincides with $\phi_r^{-1}(z)$ on $\widetilde{\phi}_2(\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \mathbb{D}_r)$ $(\subseteq \overline{D_1^r})$. Since $\phi_{r,2}$ maps $\iota(\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta})))$ $\setminus \overline{\mathbb{D}_r})$ into $\mathbb{C} \setminus \overline{\mathbb{D}}$, we have that $\phi_{r,2} \circ \widetilde{\phi}_2^{-1}$ maps $\widetilde{\phi}_2 \circ \iota(\mathbb{B}_\varepsilon(r\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi \mathrm{i}\theta}))) \setminus \overline{\mathbb{D}_r})$ into $\mathbb{C} \setminus \overline{\mathbb{D}}$. Thus

$$\Phi_r(z) = \begin{cases} \phi_{r,2} \circ \widetilde{\phi}_2^{-1}, & z \in \widetilde{\phi}_2(E_{r,2}), \\ \phi_r^{-1}(z), & z \in D_1^r \end{cases}$$

is univalent on $D_1^r \cup \widetilde{\phi}_2(E_{r,2})$. It follows from $\varepsilon \ll 1$ that for all r sufficiently close 1, there exists a fixed neighborhood V of 0 contained in D_1^r and not intersecting $\widetilde{\phi}_2(E_{r,2})$, and hence

$$\phi_{r,2} \circ \widetilde{\phi}_2^{-1}(\widetilde{\phi}_2(E_{r,2})) \cap \Phi_r(V) = \Phi_r(\widetilde{\phi}_2(E_{r,2})) \cap \Phi_r(V) = \emptyset.$$

Montel's theorem gives that $\phi_{r,2} \circ \widetilde{\phi}_2^{-1}$ (r sufficiently close to 1) is a normal family on $\widetilde{\phi}_2(E_{r,2})$. Again, since $\phi_{r,2} \circ \widetilde{\phi}_2^{-1}$ converges locally uniformly to ϕ_1^{-1} on $\widetilde{\phi}_2(\mathbb{B}_\varepsilon(\phi_2^{-1}(\phi_1(\mathrm{e}^{2\pi i\theta}))) \setminus \overline{\mathbb{D}})$ $(\subseteq D_1)$ as $r \to 1^-$, we have that $\phi_{r,2} \circ \widetilde{\phi}_2^{-1}$ converges locally uniformly to $\phi_{1,2} \circ \widetilde{\phi}_2^{-1}$ on $\widetilde{\phi}_2(E_{1,2})$ as $r \to 1^-$. Similar to the proof that ϕ_r converges locally uniformly to ϕ_1, we have that $(\phi_{r,2} \circ \widetilde{\phi}_2^{-1})^{-1}$ converges locally uniformly to $(\phi_{1,2} \circ \widetilde{\phi}_2^{-1})^{-1}$ on $\phi_{1,2}(E_{1,2})$. Since $\mathrm{e}^{2\pi \mathrm{i}\theta} \in \phi_{1,2}(E_{1,2})$, we have that $(\phi_{r,2} \circ \widetilde{\phi}_2^{-1})^{-1}$ converges uniformly to $(\phi_{1,2} \circ \widetilde{\phi}_2^{-1})^{-1}$ on some neighborhood U of $\mathrm{e}^{2\pi \mathrm{i}\theta}$ as $r \to 1^-$. It follows that $\phi_r|_{U \cap \overline{\mathbb{D}}} = (\phi_{r,2} \circ \widetilde{\phi}_2^{-1})^{-1}|_{U \cap \overline{\mathbb{D}}}$ converges uniformly to $\phi_1|_{U \cap \overline{\mathbb{D}}} = (\phi_{1,2} \circ \widetilde{\phi}_2^{-1})^{-1}|_{U \cap \overline{\mathbb{D}}}$ on $U \cap \overline{\mathbb{D}}$ as $r \to 1^-$. Again, since ϕ_r converges locally uniformly to ϕ_1 on $\mathbb{D}$ as $r \to 1^-$. Thus ϕ_r converges locally uniformly to ϕ_1 on $\widehat{\mathbb{D}}$ as $r \to 1^-$.

We choose $0 < r_0 < 1$ sufficiently close to 1 so that $\phi_1(\partial\mathbb{D}_{r_0}) \subseteq H_{r'}$ and for all $0 \le \theta < 1$ and $r_0 \le r \le 1$,

$$|\phi_1(r\mathrm{e}^{2\pi\mathrm{i}\theta}) - \phi_1(\mathrm{e}^{2\pi\mathrm{i}\theta})| < \delta. \tag{3.1}$$

For all $1 \le j \le n$, there exists a unique θ_j with $|\theta_j - b_j| < 1$ such that $\phi_r(\mathrm{e}^{2\pi\mathrm{i}a_j}) = \phi_2(r\mathrm{e}^{2\pi\mathrm{i}\theta_j})$. By the above claim, $\phi_r(\mathrm{e}^{2\pi\mathrm{i}a_j}) = \phi_2(r\mathrm{e}^{2\pi\mathrm{i}\theta_j})$ converges to $\phi_1(\mathrm{e}^{2\pi\mathrm{i}a_j}) = \phi_2(\mathrm{e}^{2\pi\mathrm{i}b_j})$ as $r \to 1^-$ and hence $\theta_j \to b_j$ as $r \to 1^-$.

We choose r sufficiently close to 1 so that

(1) $\phi_r(\partial\mathbb{D}) = \phi_2(\partial\mathbb{D}_r) \subseteq H_{r'}$,

(2) for all $0 \le \theta < 1$, $|\phi_2(r\mathrm{e}^{2\pi\mathrm{i}\theta}) - \phi_2(\mathrm{e}^{2\pi\mathrm{i}\theta})| < \delta$,

(3) for all $0 \le \theta < 1$, $|\phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta}) - \phi_1(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})| < \delta$ (thanks to that $\phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})$ converges uniformly to $\phi_1(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})$ for $\theta \in [0,1)$ as $r \to 1^-$),

(4) $\phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})$ is contained in the intersection of $H_{r'}$ and the bounded component of $\mathbb{C} \setminus \gamma$ (thanks to that $\phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})$ converges uniformly to $\phi_1(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})$ for $\theta \in [0,1)$ as $r \to 1^-$),

(5) for all $\theta \in \bigcup_{j=1}^{n} \left[a_j - \frac{\varepsilon_j}{2}, a_j + \frac{\varepsilon_j}{2}\right]$ and $r_0 \le s \le 1$, $|\phi_r(s\mathrm{e}^{2\pi\mathrm{i}\theta}) - \phi_r(\mathrm{e}^{2\pi\mathrm{i}\theta})| < \delta$ (thanks to the above claim and (3.1)),

(6) for all $1 \le j \le n$, $|\theta_j - b_j| < c := \min\{|b_1 - b_2|, \cdots, |b_j - b_{j+1}|, \cdots, |b_n - b_{n+1}|\}$ (thanks to that $\theta_j \to b_j$ as $r \to 1^-$).

Since γ is contained in the bounded component of $\mathbb{C} \setminus \phi_r(\partial\mathbb{D})$ $(= \mathbb{C} \setminus \phi_2(\partial\mathbb{D}_r))$ and at the same time by (4) γ is contained in the unbounded component of $\mathbb{C} \setminus \phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta})$, we have $\gamma \subseteq \phi_r(H_{r_0})$. (1) and (4) give $\phi_r(\partial\mathbb{D}) \cup \phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta}) \subseteq H_{r'}$ and together with $0 \notin \phi_r(H_{r_0})$, we have $\overline{\phi_r(H_{r_0})} \subseteq H_{r'}$. Next, we prove that $\phi_r|_{H_{r_0}}$ is 14δ-admissible. Indeed, for all $j \in \{1, 2, \cdots, n\}$, $\phi_r(\Omega^{r_0}_{a_j,a_{j+1}})$ is bounded by $\{\phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}$, $\phi_r(l_{2\pi a_j, r_0})$, $\phi_r(l_{2\pi a_{j+1}, r_0})$ and $\{\phi_r(\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}$. By (3), (3.1) and $\mathrm{diam}(\{\phi_1(\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}) < \delta$ $(j = 1, 2, \cdots, n)$, we have that for all $j = 1, 2, \cdots, n$,

$$\begin{aligned}\mathrm{diam}(\{\phi_r(r_0\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}) &\le \mathrm{diam}(\{\phi_1(r_0\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}) + 2\delta \\ &\le \mathrm{diam}(\{\phi_1(\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}) + 4\delta \\ &< 5\delta.\end{aligned}$$

It follows from (5) that $\mathrm{diam}(\phi_r(l_{2\pi a_j, r_0})) < 2\delta$ and $\mathrm{diam}(\phi_r(l_{2\pi a_{j+1}, r_0})) < 2\delta$. By (2), (6) and

$$\mathrm{diam}(\{\phi_2(\mathrm{e}^{2\pi\mathrm{i}\theta}) : b_j \ge \theta \ge b_{j+1}\}) < \delta$$

$(j = 1, 2, \cdots, n)$, we have

$$\begin{aligned}\mathrm{diam}(\{\phi_r(\mathrm{e}^{2\pi\mathrm{i}\theta}) : a_j \le \theta \le a_{j+1}\}) &\le \mathrm{diam}(\{\phi_2(r\mathrm{e}^{2\pi\mathrm{i}\theta}) : b_{j+1} - c \le \theta \le b_j + c\}) \\ &\le \mathrm{diam}(\{\phi_2(\mathrm{e}^{2\pi\mathrm{i}\theta}) : b_{j+1} - c \le \theta \le b_j + c\}) + 2\delta \\ &< 5\delta.\end{aligned}$$

Thus $\mathrm{diam}(\phi_r(\Omega^{r_0}_{a_j,a_{j+1}})) < 14\delta$.

By Lemma 2.2 there exists a sequence of holomorphic maps $\psi_n : \mathbb{C} \to \mathbb{C}$ such that

- $\psi_n(z) = 0 \Leftrightarrow z = 0$,
- ψ_n' does not vanish,
- as n tends to $+\infty$, ψ_n tends uniformly to ϕ_r^{-1} on every compact subset of $\phi_r(\mathbb{D})$ $(= D_1^r)$.

We choose r_1 with $r_0<r_1<1$ sufficiently close to 1 so that $\gamma \subseteq \phi_r(H_{r_0,r_1})$, $\overline{\phi_r(H_{r_0,r_1})} \subseteq H_{r'}$ and $\phi_r|_{H_{r_0,r_1}}(r_1 z) : H_{\frac{r_0}{r_1}} \to \phi_r(H_{r_0,r_1})$ is 14δ-admissible. Since ψ_n converges uniformly to ϕ_r^{-1} on every compact subset of $\phi_r(\mathbb{D})$ as $n \to \infty$, we have that ψ_n converges uniformly to ϕ_r^{-1} on $\phi_r(\mathbb{D}_{\frac{1+r_1}{2}})$ as $n \to \infty$ and for sufficiently large n, $\psi_n|_{\phi_r(\mathbb{D}_{\frac{1+r_1}{2}})}$ is univalent. For all $z \in \mathbb{D}_{\frac{1+r_1}{2}}$, we choose $\varepsilon > 0$ sufficiently small so that $\overline{\mathbb{B}_\varepsilon(\phi_r(z))} \subseteq \phi_r(\mathbb{D}_{\frac{1+r_1}{2}})$. Then $\psi_n^{-1}(w) = \frac{1}{2\pi \mathrm{i}} \int_{|\xi - \phi_r(z)|=\varepsilon} \frac{\xi(\psi_n)'(\xi)}{\psi_n(\xi)-w} \mathrm{d}z$ converges uniformly to $\phi_r(w) = \frac{1}{2\pi \mathrm{i}} \int_{|\xi - \phi_r(z)|=\varepsilon} \frac{\xi(\phi_r^{-1})'(\xi)}{\phi_r^{-1}(\xi)-w} \mathrm{d}z$ on some neighborhood of z as $r \to 1^-$. Thus $\psi_n^{-1}|_{\mathbb{D}_{\frac{1+r_1}{2}}}$ converges locally uniformly to $\phi_r|_{\mathbb{D}_{\frac{1+r_1}{2}}}$ on $\mathbb{D}_{\frac{1+r_1}{2}}$. Thus $\psi_n^{-1}|_{H_{r_0,r_1}}(r_1 z) : H_{\frac{r_0}{r_1}} \to \psi_n^{-1}(H_{r_0,r_1})$ converges uniformly to $\phi_r|_{H_{r_0,r_1}}(r_1 z)$ on $H_{\frac{r_0}{r_1}}$. Thus we can take n sufficiently large so that

$$\psi_n^{-1}|_{H_{r_0,r_1}}(r_1 H_{\frac{r_0}{r_1}}) = \psi_n^{-1}(H_{r_0,r_1}) \supseteq \gamma, \quad \overline{\psi_n^{-1}|_{H_{r_0,r_1}}(r_1 H_{\frac{r_0}{r_1}})} = \overline{\psi_n^{-1}(H_{r_0,r_1})} \subseteq H_{r'}$$

and $\psi_n^{-1}|_{H_{r_0,r_1}}(r_1 z)$ is 14δ-admissible. At last, we take $\beta(z) = \frac{\psi_n(z)}{r_1}$ and $r'' = \frac{r_0}{r_1} < 1$. This completes the proof.

For all $0 < r < 1$, we let γ be a δ-analytic Jordan curve in H_r such that γ surrounds 0. Then by Lemma 3.1 there exist $\beta \in \mathcal{H}$ and $0 < r' < 1$ such that

- $\gamma \subseteq \mathcal{D}_{r'}(\beta)$ and $\overline{\mathcal{D}_{r'}(\beta)} \subseteq H_r$,
- $(\beta|_{\mathcal{D}_{r'}(\beta)})^{-1}$ is 14δ-admissible.

For all positive integer q, we denote by $\mathcal{R}$ the single-value analytic branch of $\sqrt[q]{\beta(z^q)}$ such that $(\mathcal{R})'(0) > 0$ and denote by $\widetilde{\gamma}$ the preimage of γ under z^q. Then

- $\widetilde{\gamma}$ is a $\frac{\delta}{qr}$-analytic Jordan curve in $H_{\sqrt[q]{r}}$ and surrounds 0,
- $\widetilde{\gamma} \subseteq \mathcal{D}_{\sqrt[q]{r'}}(\mathcal{R})$ and $\overline{\mathcal{D}_{\sqrt[q]{r'}}(\mathcal{R})} \subseteq H_{\sqrt[q]{r}}$,
- $\widetilde{\gamma} = \mathrm{e}^{\frac{1}{q}2\pi \mathrm{i}}\widetilde{\gamma}$,
- $(\mathcal{R}|_{\mathcal{D}_{\sqrt[q]{r'}}(\mathcal{R})})^{-1}$ is $\frac{14\delta}{qr}$-admissible.

Thus we have the following corollary.

Corollary 3.1 *Let δ, r be two positive real numbers with $0 < r < 1$, q be a positive integer and $b = |b|\mathrm{e}^{2\pi \mathrm{i}\theta} \in H_r$. If γ is a 0-analytic Jordan curve in H_r such that γ surrounds 0, $\gamma \cap l_{(\frac{k}{q}+\theta)2\pi,r} = \{\mathrm{e}^{\frac{k}{q}2\pi \mathrm{i}} b\}$, $0 \le k \le q-1$ and $\mathrm{e}^{\frac{1}{q}2\pi \mathrm{i}}\gamma = \gamma$, then there exist $\beta \in \mathcal{H}$ and $0 < r' < 1$ such that*

- *$\gamma \subseteq \mathcal{D}_{r'}(\mathcal{R})$ and $\overline{\mathcal{D}_{r'}(\mathcal{R})} \subseteq H_r$,*
- *the inverse $(\mathcal{R}|_{\mathcal{D}_{r'}(\mathcal{R})})^{-1}$ is δ-admissible,*

where $\mathcal{R}$ is the single-value analytic branch of $\sqrt[q]{\beta(z^q)}$ such that $(\mathcal{R})'(0) > 0$.

Proof Let γ_1 be the subarc of γ connecting b and $b\mathrm{e}^{\frac{1}{q}2\pi \mathrm{i}}$ contained in $\Omega^r_{\theta,\theta+\frac{1}{q}}$. Then γ_1^q is a Jordan curve in H_{r^q} surrounding 0 and passing through b^q. Since γ is 0-analytic, we have that

γ_1 is 0-analytic and hence γ_1^q is 0-analytic. Applying Lemma 3.1 and $\sqrt[q]{z}$ to γ_1^q, we can obtain the conclusion.

By applying Osgood's method for constructing a Jordan curve of positive area [9], the following lemma is easy to be proved.

Lemma 3.2 *Let r be a positive real numbers with $r < 1$, q be a positive integer and $b = |b|\mathrm{e}^{2\pi\mathrm{i}\theta} \in H_r$. Then for all $\varepsilon > 0$, there exists a 0-analytic Jordan curve γ in H_r such that γ surrounds 0, $\gamma \cap l_{(\frac{k}{q}+\theta)2\pi,r} = \{\mathrm{e}^{\frac{k}{q}2\pi\mathrm{i}}b\}$, $0 \le k \le q-1$, $\mathrm{e}^{\frac{1}{q}2\pi\mathrm{i}}\gamma = \gamma$ and $\mathrm{dens}_{H_r}(H_r \setminus \gamma) < \varepsilon$.*

Proof We choose $\varepsilon_1 > 0$ sufficiently small so that

$$\mathrm{area}(\Omega^{r+\varepsilon_1,1-\varepsilon_1}_{\theta+\varepsilon_1,\theta+\frac{1}{q}-\varepsilon_1}) > \left(1-\frac{\varepsilon}{2}\right)\cdot \mathrm{area}(\Omega^{r,1}_{\theta,\theta+\frac{1}{q}}).$$

By Osgood's method (see [9]), one can construct a simple curve $\gamma_1^{(2)}$ connecting $(1-\varepsilon_1)\mathrm{e}^{2\pi\mathrm{i}(\theta+\varepsilon_1)}$ and $(r+\varepsilon_1)\mathrm{e}^{2\pi\mathrm{i}(\theta+\frac{1}{q}-\varepsilon_1)}$ such that

- except for the two endpoints, $\gamma_1^{(2)}$ is contained in $\Omega^{r+\varepsilon_1,1-\varepsilon_1}_{\theta+\varepsilon_1,\theta+\frac{1}{q}-\varepsilon_1}$,
- $\gamma_1^{(2)}$ is 0-analytic,
- $\mathrm{area}(\gamma_1^{(2)}) > \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\cdot \mathrm{area}(\Omega^{r+\varepsilon_1,1-\varepsilon_1}_{\theta+\varepsilon_1,\theta+\frac{1}{q}-\varepsilon_1})$.

Let $\gamma_1^{(1)}$ be a simple analytic arc connecting b and $(1-\varepsilon_1)\mathrm{e}^{2\pi\mathrm{i}(\theta+\varepsilon_1)}$ such that except for the two endpoints, $\gamma_1^{(1)}$ is contained in $\Omega^{r,1}_{\theta,\theta+\varepsilon_1}$. Let $\gamma_1^{(3)}$ be a simple analytic arc connecting $(r+\varepsilon_1)\mathrm{e}^{2\pi\mathrm{i}(\theta+\frac{1}{q}-\varepsilon_1)}$ and $b\mathrm{e}^{2\pi\mathrm{i}\frac{1}{q}}$ such that except for the two endpoints, $\gamma_1^{(3)}$ is contained in $\Omega^{r,1}_{\theta+\frac{1}{q}-\varepsilon_1,\theta+\frac{1}{q}}$. We denote by γ_1 the simple arc obtained by connecting $\gamma_1^{(1)}$, $\gamma_1^{(2)}$ and $\gamma_1^{(3)}$ in sequence. Then γ_1 connects b and $b\mathrm{e}^{2\pi\mathrm{i}\frac{1}{q}}$; except for the two endpoints, γ_1 is contained in $\Omega^{r,1}_{\theta,\theta+\frac{1}{q}}$; γ_1 is 0-analytic; $\mathrm{area}(\gamma_1) > (1-\varepsilon)\cdot\mathrm{area}(\Omega^{r,1}_{\theta,\theta+\frac{1}{q}})$. We set $\gamma_k = \mathrm{e}^{2\pi\mathrm{i}\frac{k-1}{q}}\gamma_1$ for all $1 \le k \le q$. Then γ_k connects $b\mathrm{e}^{2\pi\mathrm{i}\frac{k-1}{q}}$ and $b\mathrm{e}^{2\pi\mathrm{i}\frac{k}{q}}$; except for the two endpoints, γ_k is contained in $\Omega^{r,1}_{\theta+\frac{k-1}{q},\theta+\frac{k}{q}}$; γ_k is 0-analytic; $\mathrm{area}(\gamma_k) > (1-\varepsilon)\cdot\mathrm{area}(\Omega^{r,1}_{\theta+\frac{k-1}{q},\theta+\frac{k}{q}})$. We denote by γ the Jordan curve obtained by connecting $\gamma_1, \gamma_2, \cdots, \gamma_q$ end-to-end in sequence. Then γ is exactly what we want.

4 Completing the Proof of Main Theorem

4.1 Construct $\{\beta_n\}$ step by step

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $0 < \varepsilon_n < 1$ and $\sum\limits_{n=1}^{\infty}\varepsilon_n < 1$. We fix a $0 < r_1 < 1$. It follows from Lemma 3.2 that there exists a Jordan curve γ_1 in H_{r_1} such that γ_1 surrounds 0, $\gamma_1 \cap l_{0,r_1} = \{\frac{1+r_1}{2}\}$ and $\mathrm{dens}_{H_{r_1}}(H_{r_1}\setminus\gamma_1) < \varepsilon_1$. We write $a_0 := \frac{1+r_1}{2}$.

By Lemma 3.1 there exist $\beta_1 \in \mathcal{H}$ and $0 < r_2 < 1$ such that $\gamma_1 \subseteq \mathcal{D}_{r_2}(\mathcal{R}_1)$ and $\overline{\mathcal{D}_{r_2}(\mathcal{R}_1)} \subseteq H_{r_1}$. Then

$$\mathrm{dens}_{H_{r_1}}(H_{r_1}\setminus\mathcal{D}_{r_2}(\mathcal{R}_1)) = \mathrm{dens}_{H_{r_1}}(H_{r_1}\setminus\mathcal{D}_{r_2}(\beta_1)) < \mathrm{dens}_{H_{r_1}}(H_{r_1}\setminus\gamma_1) < \varepsilon_1.$$

Observe $\mathcal{R}_1(\mathcal{D}_{r_2}(\mathcal{R}_1)) = H_{r_2}$ and $a_0 \in \gamma_1 \subseteq \mathcal{D}_{r_2}(\mathcal{R}_1)$. We write

$$a_{\frac{k}{q_1}} := ((\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1} \circ R_{\frac{k}{q_1}} \circ \mathcal{R}_1)(a_0)$$

for all $0 \leq k \leq q_1 - 1$. For all $0 \leq k \leq q_1 - 1$, $a_{\frac{k}{q_1}} \in \mathcal{D}_{r_2}(\mathcal{R}_1) \subseteq \mathbb{D}$ and

$$f_1(a_{\frac{k}{q_1}}) = G_1(a_{\frac{k}{q_1}}) = ((\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1} \circ R_{\frac{1}{q_1}} \circ \mathcal{R}_1) \circ ((\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1} \circ R_{\frac{k}{q_1}} \circ \mathcal{R}_1)(a_0) = a_{\frac{k+1}{q_1}},$$

where $a_1 = a_0$. Since $(\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}$ has a homeomorphism extension from $\overline{\mathbb{D}}$ to $\overline{\mathcal{D}(\mathcal{R}_1)}$, there exists a $\delta_2 > 0$ such that for all $z_1, z_2 \in H_{r_2}$ with $|z_1 - z_2| < \delta_2$,

$$|(\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(z_1) - (\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(z_2)| < \frac{1}{2^2}. \tag{4.1}$$

We take $0 < \delta_2 \ll \frac{1}{q_1}$. By Lemma 3.2 there exists a 0-analytic Jordan curve γ_2 in H_{r_2} such that

- γ_2 surrounds 0,
- $\gamma_2 \cap l_{\frac{k}{q_1}2\pi+\arg(\mathcal{R}_1(a_0)),r_2} = \{\mathrm{e}^{\frac{k}{q_1}2\pi\mathrm{i}}\mathcal{R}_1(a_0)\}$, $0 \leq k \leq q_1 - 1$,
- $\mathrm{e}^{\frac{1}{q_1}2\pi\mathrm{i}}\gamma_2 = \gamma_2$,
- $\mathrm{dens}_{H_{r_2}}(H_{r_2} \setminus \gamma_2)$ is sufficiently small so that

$$\mathrm{dens}_{H_{r_1}}(H_{r_1} \setminus (\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(\gamma_2)) < \varepsilon_1 + \varepsilon_2.$$

By Corollary 3.1 there exist $\beta_2 \in \mathcal{H}$ and $0 < r_3 < 1$ such that

- $\gamma_2 \subseteq \mathcal{D}_{r_3}(\mathcal{R}_2)$ and $\overline{\mathcal{D}_{r_3}(\mathcal{R}_2)} \subseteq H_{r_2}$,
- $(\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}$ is $\frac{\delta_2}{3}$-admissible.

Since $(\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}$ is $\frac{\delta_2}{3}$-admissible and (4.1), we have that for all $z \in (\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(\mathcal{D}_{r_3}(\mathcal{R}_2))$, both two boundary components of $(\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(\mathcal{D}_{r_3}(\mathcal{R}_2))$ intersect $\mathbb{B}_{\frac{1}{2^2}}(z)$.

Since $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{H}$, we have $\varphi_2 = \mathcal{R}_2 \circ \mathcal{R}_1 \in \mathcal{H}$. Since

$$\mathcal{R}_1(a_0) \in \gamma_2 \subseteq \mathcal{D}_{r_3}(\mathcal{R}_2),$$

we have $\varphi_2(a_0) \in H_{r_3}$. We write

$$a_{\frac{k}{q_1 q_2}} := (\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1}(\mathrm{e}^{\frac{k}{q_1 q_2}2\pi\mathrm{i}}\varphi_2(a_0))$$

for all $0 \leq k \leq q_1 q_2 - 1$. Since $\mathcal{R}_2(\mathrm{e}^{\frac{1}{q_1}2\pi\mathrm{i}}z) = \mathrm{e}^{\frac{1}{q_1}2\pi\mathrm{i}}\mathcal{R}_2(z)$, we have that $0 \leq k \leq q_1 - 1$,

$$((\mathcal{R}_2 \circ \mathcal{R}_1)|_{\mathcal{D}(\mathcal{R}_2\circ\mathcal{R}_1)})^{-1}(\mathrm{e}^{\frac{kq_2}{q_1 q_2}2\pi\mathrm{i}}\mathcal{R}_2 \circ \mathcal{R}_1(a_0)) = (\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_2\circ\mathcal{R}_1)})^{-1}(\mathrm{e}^{\frac{k}{q_1}2\pi\mathrm{i}}\mathcal{R}_1(a_0)).$$

So the two definitions coincide. Since $(\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}$ is $\frac{\delta_2}{3}$-admissible, we can take q_2 large enough so that for all $\theta \in \mathbb{R}$ and all $z, z' \in \Omega^{r_3,1}_{\theta,\theta+\frac{1}{q_1 q_2}}$,

$$|(\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}(z) - (\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}(z')| \leq \frac{2\delta_2}{3}.$$

Since $\varphi_2(a_0) \in H_{r_3}$, we have that for all $z \in H_{r_3}$, there exist θ and k such that $z, R_{\frac{k}{q_1 q_2}} \circ \varphi_2(a_0) \in \Omega^{r_3,1}_{\theta,\theta+\frac{1}{q_1 q_2}}$. Thus

$$|(\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}(z) - (\mathcal{R}_2|_{\mathcal{D}_{r_3}(\mathcal{R}_2)})^{-1}(R_{\frac{k}{q_1 q_2}} \circ \varphi_2(a_0))| \leq \frac{2\delta_2}{3}.$$

Again, since for all $z_1, z_2 \in H_{r_2}$ with $|z_1 - z_2| < \delta_2$, $|(\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(z_1) - (\mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_1)})^{-1}(z_2)| < \frac{1}{2^2}$ holds, we have

$$|(\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1}(z) - (\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1}(R_{\frac{k}{q_1q_2}} \circ \varphi_2(a_0))| < \frac{1}{2^2},$$

that is,

$$|(\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1}(z) - a_{\frac{k}{q_1q_2}}| < \frac{1}{2^2}.$$

For all $0 \le k \le q_1q_2 - 1$, $a_{\frac{k}{q_1q_2}} \in \mathcal{D}_{r_3}(\varphi_2) \subseteq \mathbb{D}$ and by $G_j^{\circ q_j} = G_{j-1}$ $(j \ge 1)$,

$$\begin{aligned} f_2(a_{\frac{k}{q_1q_2}}) &= G_2 \circ G_1(a_{\frac{k}{q_1q_2}}) \\ &= G_2^{\circ(1+q_2)}(a_{\frac{k}{q_1q_2}}) \\ &= (\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1} \circ R_{\frac{1}{q_1q_2}+\frac{1}{q_1}} \circ \varphi_2)(a_{\frac{k}{q_1q_2}}) \\ &\in \{a_{\frac{k}{q_1q_2}} : k \in \{0,1,2,\cdots,q_1q_2-1\}\}. \end{aligned}$$

Since $((\mathcal{R}_2 \circ \mathcal{R}_1)|_{\mathcal{D}(\mathcal{R}_2\circ\mathcal{R}_1)})^{-1}$ has a homeomorphism extension from $\overline{\mathbb{D}}$ to $\overline{\mathcal{D}(\mathcal{R}_2 \circ \mathcal{R}_1)}$, there exists a $\delta_3 > 0$ such that for all $z_1, z_2 \in H_{r_3}$ with $|z_1 - z_2| < \delta_3$,

$$|((\mathcal{R}_2 \circ \mathcal{R}_1)|_{\mathcal{D}(\mathcal{R}_2\circ\mathcal{R}_1)})^{-1}(z_1) - ((\mathcal{R}_2 \circ \mathcal{R}_1)|_{\mathcal{D}(\mathcal{R}_2\circ\mathcal{R}_1)})^{-1}(z_2)| < \frac{1}{2^3}. \tag{4.2}$$

We take $0 < \delta_3 \ll \frac{1}{q_1q_2}$. It follows from Lemma 3.2 that there exists a 0-analytic Jordan curve γ_3 in H_{r_3} such that

- γ_3 surrounds 0,
- $\gamma_3 \cap l_{\frac{k}{q_1q_2}2\pi+\arg(\mathcal{R}_2\circ\mathcal{R}_1(a_0)),r_3} = \{e^{\frac{k}{q_1q_2}2\pi i}\mathcal{R}_2 \circ \mathcal{R}_1(a_0)\}$, $0 \le k \le q_1q_2 - 1$,
- $e^{\frac{1}{q_1q_2}2\pi i}\gamma_3 = \gamma_3$,
- $\mathrm{dens}_{H_{r_3}}(H_{r_3} \setminus \gamma_3)$ is sufficiently small so that

$$\mathrm{dens}_{H_{r_1}}(H_{r_1} \setminus (\mathcal{R}_2 \circ \mathcal{R}_1|_{\mathcal{D}(\mathcal{R}_2\circ\mathcal{R}_1)})^{-1}(\gamma_3)) < \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

By Corollary 3.1 there exist $\beta_3 \in \mathcal{H}$ and $0 < r_4 < 1$ such that

- $\gamma_3 \subseteq \mathcal{D}_{r_4}(\mathcal{R}_3)$ and $\overline{\mathcal{D}_{r_4}(\mathcal{R}_3)} \subseteq H_{r_3}$,
- $(\mathcal{R}_3|_{\mathcal{D}_{r_4}(\mathcal{R}_3)})^{-1}$ is $\frac{\delta_3}{3}$-admissible.

Since $(\mathcal{R}_3|_{\mathcal{D}_{r_4}(\mathcal{R}_3)})^{-1}$ is $\frac{\delta_3}{3}$-admissible and (4.2), we have that for all $z \in (\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1}(\mathcal{D}_{r_4}(\mathcal{R}_3))$, both two boundary components of $(\varphi_2|_{\mathcal{D}(\varphi_2)})^{-1}(\mathcal{D}_{r_4}(\mathcal{R}_3))$ intersects $\mathbb{B}_{\frac{1}{2^3}}(z)$. Since $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathcal{H}$, we have $\varphi_3 = \mathcal{R}_3 \circ \mathcal{R}_2 \circ \mathcal{R}_1 \in \mathcal{H}$. Since

$$\varphi_2(a_0) \in \gamma_3 \subseteq \mathcal{D}_{r_4}(\mathcal{R}_3),$$

we have $\varphi_3(a_0) \in H_{r_4}$.

In general, $\varphi_n = \mathcal{R}_n \circ \cdots \circ \mathcal{R}_2 \circ \mathcal{R}_1 \in \mathcal{H}$ and $\varphi_n(a_0) \in H_{r_{n+1}}$. We write

$$a_{\frac{k}{q_1q_2\cdots q_n}} := (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1} \circ R_{\frac{k}{q_1q_2\cdots q_n}} \circ \varphi_n(a_0)$$

for all $0\le k\le q_1q_2\cdots q_n-1$. The definition coincides with the above ones. Since $(\mathcal{R}_n|_{\mathcal{D}_{r_{n+1}}(\mathcal{R}_n)})^{-1}$ is $\frac{\delta_n}{3}$-admissible, we can take q_n large enough so that for all $\theta\in\mathbb{R}$ and all $z,z'\in\Omega^{r_{n+1},1}_{\theta,\theta+\frac{1}{q_1q_2\cdots q_n}}$,

$$|(\mathcal{R}_n|_{\mathcal{D}_{r_{n+1}}(\mathcal{R}_n)})^{-1}(z)-(\mathcal{R}_n|_{\mathcal{D}_{r_{n+1}}(\mathcal{R}_n)})^{-1}(z')|\le\frac{2\delta_n}{3}. \tag{4.3}$$

Since $\varphi_n(a_0)\in H_{r_{n+1}}$, we have that for all $z\in H_{r_{n+1}}$, there exist θ and k such that $z,R_{\frac{k}{q_1q_2\cdots q_n}}\circ\varphi_n(a_0)\in\Omega^{r_{n+1},1}_{\theta,\theta+\frac{1}{q_1q_2\cdots q_n}}$. Thus

$$|(\mathcal{R}_n|_{\mathcal{D}_{r_{n+1}}(\mathcal{R}_n)})^{-1}(z)-(\mathcal{R}_n|_{\mathcal{D}_{r_{n+1}}(\mathcal{R}_n)})^{-1}(R_{\frac{k}{q_1q_2\cdots q_n}}\circ\varphi_n(a_0))|\le\frac{2\delta_n}{3}.$$

Again, since for all $z_1,z_2\in H_{r_n}$ with $|z_1-z_2|<\delta_n$,

$$|(\varphi_{n-1}|_{\mathcal{D}(\varphi_{n-1})})^{-1}(z_1)-(\varphi_{n-1}|_{\mathcal{D}(\varphi_{n-1})})^{-1}(z_2)|<\frac{1}{2^n}$$

holds, we have

$$|(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(z)-(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(R_{\frac{k}{q_1q_2\cdots q_n}}\circ\varphi_n(a_0))|<\frac{1}{2^n},$$

that is,

$$|(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(z)-a_{\frac{k}{q_1q_2\cdots q_n}}|<\frac{1}{2^n}. \tag{4.4}$$

For all $0\le k\le q_1q_2\cdots q_n-1$, $a_{\frac{k}{q_1q_2\cdots q_n}}\in\mathcal{D}_{r_{n+1}}(\varphi_n)\subseteq\mathbb{D}$ and by $G_j^{\circ q_j}=G_{j-1}$ $(j\ge1)$,

$$\begin{aligned}f_n(a_{\frac{k}{q_1q_2\cdots q_n}})&=G_n\circ\cdots\circ G_2\circ G_1(a_{\frac{k}{q_1q_2\cdots q_n}})\\&=G_n^{\circ(1+q_n+q_nq_{n-1}+\cdots+q_nq_{n-1}\cdots q_2)}(a_{\frac{k}{q_1q_2\cdots q_n}})\\&=((\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}\circ R_{\frac{1}{q_1q_2\cdots q_n}+\cdots+\frac{1}{q_1}}\circ\varphi_n)(a_{\frac{k}{q_1q_2\cdots q_n}})\\&\in\{a_{\frac{k}{q_1q_2\cdots q_n}}:k\in\{0,1,2,\cdots,q_1q_2\cdots q_n-1\}\},\end{aligned}$$

that is,

$$f_n(a_{\frac{k}{q_1q_2\cdots q_n}})\in\{a_{\frac{k}{q_1q_2\cdots q_n}}:k\in\{0,1,2,\cdots,q_1q_2\cdots q_n-1\}\}. \tag{4.5}$$

Since $(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}$ has a homeomorphism extension from $\overline{\mathbb{D}}$ to $\overline{\mathcal{D}(\varphi_n)}$, there exists a $\delta_{n+1}>0$ such that for all $z_1,z_2\in H_{r_{n+1}}$ with $|z_1-z_2|<\delta_{n+1}$,

$$|(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(z_1)-(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(z_2)|<\frac{1}{2^{n+1}}. \tag{4.6}$$

We take $0<\delta_{n+1}\ll\frac{1}{q_1q_2\cdots q_n}$. It follows from Lemma 3.2 that there exists a 0-analytic Jordan curve γ_{n+1} in $H_{r_{n+1}}$ such that

- γ_{n+1} surrounds 0,
- $\gamma_{n+1}\cap l_{\frac{k}{q_1q_2\cdots q_n}2\pi+\arg(\varphi_n(a_0)),r_{n+1}}=\{e^{\frac{k}{q_1q_2\cdots q_n}2\pi i}\varphi_n(a_0)\}$, $0\le k\le q_1q_2\cdots q_n-1$,
- $e^{\frac{1}{q_1q_2\cdots q_n}2\pi i}\gamma_{n+1}=\gamma_{n+1}$,

- $\mathrm{dens}_{H_{r_{n+1}}}(H_{r_{n+1}} \setminus \gamma_{n+1})$ is sufficiently small so that

$$\mathrm{dens}_{H_{r_1}}(H_{r_1} \setminus (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\gamma_{n+1})) < \sum_{j=1}^{n+1} \varepsilon_j. \tag{4.7}$$

By Corollary 3.1 there exist $\beta_{n+1} \in \mathcal{H}$ and $0 < r_{n+2} < 1$ such that

- $\gamma_{n+1} \subseteq \mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})$ and $\overline{\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})} \subseteq H_{r_{n+1}}$,
- $(\mathcal{R}_{n+1}|_{\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})})^{-1}$ is $\frac{\delta_{n+1}}{3}$-admissible.

Since $(\mathcal{R}_{n+1}|_{\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})})^{-1}$ is $\frac{\delta_{n+1}}{3}$-admissible and (4.6), we have that for all

$$z \in (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})),$$

both two boundary components of $(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$ intersects $\mathbb{B}_{\frac{1}{2^{n+1}}}(z)$. Since $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_{n+1} \in \mathcal{H}$, we have $\varphi_{n+1} = \mathcal{R}_{n+1} \circ \cdots \circ \mathcal{R}_2 \circ \mathcal{R}_1 \in \mathcal{H}$. Since

$$\varphi_n(a_0) \in \gamma_{n+1} \subseteq \mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}),$$

we have $\varphi_{n+1}(a_0) \in H_{r_{n+2}}$.

4.2 A good nest

We set

$$\mathcal{I} := \left\{ \frac{k}{q_1 q_2 \cdots q_n} : n \geq 1, k \in \mathbb{Z}, 0 \leq k \leq q_1 q_2 \cdots q_n \right\}$$

and

$$\mathcal{A} := \{a_x : x \in \mathcal{I}\}.$$

For all $k \in \mathbb{Z}$, we set

$$\Omega^k_{q_1 q_2 \cdots q_n} := \left\{ z : r_{n+1} < |z| < 1, \frac{k}{q_1 q_2 \cdots q_n} 2\pi \leq \arg\left(\frac{z}{\varphi_n(a_0)}\right) \leq \frac{k+1}{q_1 q_2 \cdots q_n} 2\pi \right\}$$

and

$$U^k_{q_1 q_2 \cdots q_n} := (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\Omega^k_{q_1 q_2 \cdots q_n}).$$

It is easy to see that $\Omega^{k+q_1 q_2 \cdots q_n}_{q_1 q_2 \cdots q_n} = \Omega^k_{q_1 q_2 \cdots q_n}$ and $U^{k+q_1 q_2 \cdots q_n}_{q_1 q_2 \cdots q_n} = U^k_{q_1 q_2 \cdots q_n}$ for all $k \in \mathbb{Z}$. Furthermore, we have the following properties:

(a) $a_{\frac{k}{q_1 q_2 \cdots q_n}} \in U^k_{q_1 q_2 \cdots q_n}$ for all $0 \leq k \leq q_1 q_2 \cdots q_n$;

(b) for all $s = kq_{n+1} + t$ with $0 \leq t \leq q_{n+1} - 1$,

$$U^s_{q_1 q_2 \cdots q_{n+1}} \subseteq U^{k-1}_{q_1 q_2 \cdots q_n} \cup U^k_{q_1 q_2 \cdots q_n} \cup U^{k+1}_{q_1 q_2 \cdots q_n};$$

in particular,

$$U^{kq_{n+1}+q_{n+1}-1}_{q_1 q_2 \cdots q_{n+1}} \subseteq U^k_{q_1 q_2 \cdots q_n} \cup U^{k+1}_{q_1 q_2 \cdots q_n}$$

and

$$U^{kq_{n+1}}_{q_1 q_2 \cdots q_{n+1}} \subseteq U^{k-1}_{q_1 q_2 \cdots q_n} \cup U^k_{q_1 q_2 \cdots q_n};$$

(c) for all $x \in \mathcal{I}$ with $\left|x - \frac{k}{q_1 q_2 \cdots q_n}\right| < \frac{1}{q_1 q_2 \cdots q_n}$, we have

$$a_x \in \bigcup_{s=k-2}^{k+1} U^s_{q_1 q_2 \cdots q_n};$$

(d) $\operatorname{diam}(U^k_{q_1 q_2 \cdots q_n}) \le \frac{1}{2^n}$.

Proof (a) Since $\varphi_n(a_0) \in H_{r_{n+1}}$, we have $R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0) \in \Omega^k_{q_1 q_2 \cdots q_n}$ and hence

$$a_{\frac{k}{q_1 q_2 \cdots q_n}} = \varphi_n|^{-1}_{\mathcal{D}(\varphi_n)} \circ R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0) \in \varphi_n|^{-1}_{\mathcal{D}(\varphi_n)}(\Omega^k_{q_1 q_2 \cdots q_n}) = U^k_{q_1 q_2 \cdots q_n}.$$

(b) Let γ^k_{n+1} be the subarc of γ_{n+1} in $\Omega^k_{q_1 q_2 \cdots q_n}$ connecting $R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0)$ and $R_{\frac{k+1}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0)$. Then $(\gamma^k_{n+1})^{q_1 q_2 \cdots q_n}$ is a Jordan curve in $H_{r^{q_1 q_2 \cdots q_n}_{n+1}}$ surrounding 0 and passing through

$$(R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0))^{q_1 q_2 \cdots q_n} = (R_{\frac{k+1}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0))^{q_1 q_2 \cdots q_n}.$$

Since $\mathcal{R}_{n+1}|_{\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})}$ is a conformal map from $\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})$ to $H_{r_{n+2}}$, we have that β_{n+1} is a conformal map from $(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))^{q_1 q_2 \cdots q_n} = \mathcal{D}_{r^{q_1 q_2 \cdots q_n}_{n+2}}(\beta_{n+1})$ to $H_{r^{q_1 q_2 \cdots q_n}_{n+2}}$. Since $\gamma^k_{n+1} \subseteq \gamma_{n+1} \subseteq \mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}) \subseteq \mathcal{D}(\mathcal{R}_{n+1})$, we have $(\gamma^k_{n+1})^{q_1 q_2 \cdots q_n} \subseteq (\mathcal{D}(\mathcal{R}_{n+1}))^{q_1 q_2 \cdots q_n}$. It follows that $\beta_{n+1}((\gamma^k_{n+1})^{q_1 q_2 \cdots q_n})$ is a Jordan curve in $H_{r^{q_1 q_2 \cdots q_n}_{n+2}}$ surrounding 0 and passing through

$$\beta_{n+1}((R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0))^{q_1 q_2 \cdots q_n}) = \beta_{n+1}((R_{\frac{k+1}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0))^{q_1 q_2 \cdots q_n}).$$

Thus $\mathcal{R}_{n+1}(\gamma^k_{n+1}) = \sqrt[q_1 q_2 \cdots q_n]{\beta_{n+1}((\gamma^k_{n+1})^{q_1 q_2 \cdots q_n})}$ is an arc in $H_{r_{n+2}}$ connecting $R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_{n+1}(a_0)$ and $R_{\frac{k+1}{q_1 q_2 \cdots q_n}} \circ \varphi_{n+1}(a_0)$, and at the same time homotopic to $R_\theta \circ \varphi_{n+1}(a_0)$, $\frac{k}{q_1 q_2 \cdots q_n} \le \theta \le \frac{k+1}{q_1 q_2 \cdots q_n}$ in $H_{r_{n+2}}$. This implies that $\mathcal{R}_{n+1}(\gamma^k_{n+1})$ intersects each $\Omega^s_{q_1 q_2 \cdots q_{n+1}}$, $kq_{n+1} \le s \le kq_{n+1} + q_{n+1} - 1$. Again, since $\gamma^k_{n+1} \subseteq \Omega^k_{q_1 q_2 \cdots q_n}$, we have that each $(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\Omega^s_{q_1 q_2 \cdots q_{n+1}})$, $kq_{n+1} \le s \le kq_{n+1} + q_{n+1} - 1$ intersects $\Omega^k_{q_1 q_2 \cdots q_n}$. It follows from (4.3) and $0 < \delta_{n+1} \ll \frac{1}{q_1 q_2 \cdots q_n}$ that

$$(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\Omega^s_{q_1 q_2 \cdots q_{n+1}}) \subseteq \Omega^{k-1}_{q_1 q_2 \cdots q_n} \cup \Omega^k_{q_1 q_2 \cdots q_n} \cup \Omega^{k+1}_{q_1 q_2 \cdots q_n}$$

for all $kq_{n+1} \le s \le kq_{n+1} + q_{n+1} - 1$. Applying $(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}$ to the above formula, we obtain that

$$U^s_{q_1 q_2 \cdots q_{n+1}} \subseteq U^{k-1}_{q_1 q_2 \cdots q_n} \cup U^k_{q_1 q_2 \cdots q_n} \cup U^{k+1}_{q_1 q_2 \cdots q_n}$$

for all $kq_{n+1} \le s \le kq_{n+1} + q_{n+1} - 1$.

Since $R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_{n+1}(a_0) \in \Omega^{kq_{n+1}}_{q_1 q_2 \cdots q_{n+1}}$ and $R_{\frac{k+1}{q_1 q_2 \cdots q_n}} \circ \varphi_{n+1}(a_0) \in \Omega^{kq_{n+1}+q_{n+1}-1}_{q_1 q_2 \cdots q_{n+1}}$, we have that

$$R_{\frac{k}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0) \in (\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\Omega^{kq_{n+1}}_{q_1 q_2 \cdots q_{n+1}})$$

and

$$R_{\frac{k+1}{q_1 q_2 \cdots q_n}} \circ \varphi_n(a_0) \in (\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\Omega^{kq_{n+1}+q_{n+1}-1}_{q_1 q_2 \cdots q_{n+1}}).$$

Similarly, by (4.3) and $0 < \delta_{n+1} \ll \frac{1}{q_1 q_2 \cdots q_n}$, we have

$$(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\Omega^{kq_{n+1}}_{q_1 q_2 \cdots q_{n+1}}) \subseteq \Omega^{k-1}_{q_1 q_2 \cdots q_n} \cup \Omega^k_{q_1 q_2 \cdots q_n}$$

and

$$(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\Omega^{kq_{n+1}+q_{n+1}-1}_{q_1q_2\cdots q_{n+1}}) \subseteq \Omega^{k}_{q_1q_2\cdots q_n} \cup \Omega^{k+1}_{q_1q_2\cdots q_n}.$$

Applying $(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}$ to the above formula, we obtain that

$$U^{kq_{n+1}}_{q_1q_2\cdots q_{n+1}} \subseteq U^{k-1}_{q_1q_2\cdots q_n} \cup U^{k}_{q_1q_2\cdots q_n}$$

and

$$U^{kq_{n+1}+q_{n+1}-1}_{q_1q_2\cdots q_{n+1}} \subseteq U^{k}_{q_1q_2\cdots q_n} \cup U^{k+1}_{q_1q_2\cdots q_n}.$$

(c) Since $x \in \mathcal{I}$ and $|x - \frac{k}{q_1q_2\cdots q_n}| < \frac{1}{q_1q_2\cdots q_n}$, we have

$$x = \frac{s}{q_1q_2\cdots q_n\cdots q_{n+m}}$$

with

$$(k-1)q_{n+1}\cdots q_{n+m}+1 \le s \le (k+1)q_{n+1}\cdots q_{n+m}-1$$

for some positive integer m and s. Then (a)–(b) give

$$\begin{aligned}
a_x &\in \bigcup_{s=(k-1)q_{n+1}\cdots q_{n+m}+1}^{(k+1)q_{n+1}\cdots q_{n+m}-1} U^{s}_{q_1\cdots q_n\cdots q_{n+m}}\\
&\subseteq \bigcup_{s=(k-1)q_{n+1}\cdots q_{n+m-1}-1}^{(k+1)q_{n+1}\cdots q_{n+m-1}} U^{s}_{q_1\cdots q_n\cdots q_{n+m-1}}\\
&\subseteq \bigcup_{s=(k-1)q_{n+1}\cdots q_{n+m-2}-1}^{(k+1)q_{n+1}\cdots q_{n+m-2}} U^{s}_{q_1\cdots q_n\cdots q_{n+m-2}}\\
&\cdots\cdots\\
&\subseteq \bigcup_{s=(k-1)q_{n+1}-1}^{(k+1)q_{n+1}} U^{s}_{q_1\cdots q_nq_{n+1}}\\
&\subseteq \bigcup_{s=k-2}^{k+1} U^{s}_{q_1\cdots q_n}.
\end{aligned}$$

(d) Since (4.3) holds and for all $z_1, z_2 \in H_{r_n}$ with

$$|z_1 - z_2| < \delta_n,$$

$$|(\varphi_{n-1}|_{\mathcal{D}(\varphi_{n-1})})^{-1}(z_1) - (\varphi_{n-1}|_{\mathcal{D}(\varphi_{n-1})})^{-1}(z_2)| < \frac{1}{2^n}$$

holds, we have that for all $z_1, z_2 \in \Omega^{k}_{q_1q_2\cdots q_n}$,

$$|(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(z_1) - (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(z_2)| < \frac{1}{2^n}.$$

Thus (d) holds.

4.3 $\partial\mathcal{D}$ has positive area

Since $\overline{\mathcal{D}_{r_{n+3}}(\mathcal{R}_{n+2})} \subseteq H_{r_{n+2}}$,

$$(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\overline{\mathcal{D}_{r_{n+3}}(\mathcal{R}_{n+2})}) \subseteq (\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(H_{r_{n+2}}),$$

that is,

$$(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\overline{\mathcal{D}_{r_{n+3}}(\mathcal{R}_{n+2})}) \subseteq \mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}).$$

Together with $\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}) \subseteq \mathbb{D}$, it follows that

$$\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}((\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\overline{\mathcal{D}_{r_{n+3}}(\mathcal{R}_{n+2})})) \subseteq \varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})),$$

that is,

$$\overline{\varphi_{n+1}|_{\mathcal{D}(\varphi_{n+1})}^{-1}(\mathcal{D}_{r_{n+3}}(\mathcal{R}_{n+2}))} \subseteq \varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})).$$

Thus $\bigcap_{n=1}^{\infty} \varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$ is a nonempty compact set. Since $\overline{\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})} \subseteq H_{r_{n+1}}$, we have that $\overline{\mathcal{D}(\mathcal{R}_{n+1})} \subseteq \mathbb{D}$ and $(\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\mathbb{D}_{r_{n+2}}) \supseteq \overline{\mathbb{D}_{r_{n+1}}}$. Thus

$$\mathcal{D}(\varphi_n) = (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathbb{D}) \supseteq (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathcal{D}(\mathcal{R}_{n+1})}) = (\varphi_{n+1}|_{\mathcal{D}(\varphi_{n+1})})^{-1}(\overline{\mathbb{D}}) = \overline{\mathcal{D}(\varphi_{n+1})}$$

and

$$(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}}) \subseteq (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}((\mathcal{R}_{n+1}|_{\mathcal{D}(\mathcal{R}_{n+1})})^{-1}(\mathbb{D}_{r_{n+2}})) = (\varphi_{n+1}|_{\mathcal{D}(\varphi_{n+1})})^{-1}(\mathbb{D}_{r_{n+2}}).$$

Again, observe that

$$\begin{aligned}\mathcal{D}(\varphi_n) &= (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}}) \cup (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(H_{r_{n+1}}) \\ &= (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}}) \cup (\varphi_{n-1}|_{\mathcal{D}(\varphi_{n-1})})^{-1}(\mathcal{D}_{r_{n+1}}(\mathcal{R}_n)).\end{aligned}$$

Thus

$$\bigcap_{n=1}^{\infty} \mathcal{D}(\varphi_n) = \Big(\bigcup_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}})\Big) \cup \Big(\bigcap_{n=1}^{\infty} \varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))\Big)$$

and

$$\Big(\bigcup_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}})\Big) \cap \Big(\bigcap_{n=1}^{\infty} \varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))\Big) = \emptyset.$$

Since $\overline{\mathcal{D}(\varphi_{n+1})} \subseteq \mathcal{D}(\varphi_n)$ for all $n \geq 1$, we have

$$\bigcap_{n=1}^{\infty} \mathcal{D}(\varphi_n) \subseteq \bigcap_{n=2}^{\infty} \overline{\mathcal{D}(\varphi_n)} \subseteq \bigcap_{n=1}^{\infty} \mathcal{D}(\varphi_n)$$

and hence $\bigcap_{n=1}^{\infty} \mathcal{D}(\varphi_n) = \bigcap_{n=2}^{\infty} \overline{\mathcal{D}(\varphi_n)}$ is a closed set. Since

$$(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}}) \subseteq (\varphi_{n+1}|_{\mathcal{D}(\varphi_{n+1})})^{-1}(\mathbb{D}_{r_{n+2}}),$$

we have

$$\bigcup_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}}) \subseteq \bigcup_{n=1}^{\infty} (\varphi_{n+1}|_{\mathcal{D}(\varphi_{n+1})})^{-1}(\mathbb{D}_{r_{n+2}}) \subseteq \bigcup_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathbb{D}_{r_{n+1}})$$

and hence

$$\bigcup_{n=1}^{\infty}(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}}) = \bigcup_{n=1}^{\infty}(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathbb{D}_{r_{n+1}})$$

is a connected open set. Since for all $z \in \varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$, both two boundary components of $\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$ intersects $\mathbb{B}_{\frac{1}{2^{n+1}}}(z)$, we have that for all

$$z \in \bigcap_{n=1}^{\infty}\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$$

and all $n \geq 1$,

$$\mathbb{B}_{\frac{1}{2^{n+1}}}(z) \nsubseteq \Big(\bigcap_{n=1}^{\infty}\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))\Big).$$

Thus $\bigcap\limits_{n=1}^{\infty}\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$ does not contain any interior points. Then we have that

$$\mathcal{D} = \bigcup_{n=1}^{\infty}(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\overline{\mathbb{D}_{r_{n+1}}})$$

and

$$\partial\mathcal{D} = \bigcap_{n=1}^{\infty}\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})).$$

Then it follows from $\gamma_{n+1} \subseteq \mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})$ and (4.7) that

$$\begin{aligned}\operatorname{area}(\partial\mathcal{D}) &= \lim_{n\to\infty}\operatorname{area}(\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})))\\ &\geq \lim_{n\to\infty}\operatorname{area}(H_{r_1})\cdot\operatorname{dens}_{H_{r_1}}\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))\\ &\geq \limsup_{n\to\infty}\ \operatorname{area}(H_{r_1})\cdot\operatorname{dens}_{H_{r_1}}\varphi_n|_{\mathcal{D}(\varphi_n)}^{-1}(\gamma_{n+1})\\ &\geq \operatorname{area}(H_{r_1})\cdot\Big(1-\sum_{n=1}^{\infty}\varepsilon_n\Big).\\ &> 0.\end{aligned}$$

4.4 $\partial\mathcal{D}$ is a Jordan curve

Observe that for all $0 \leq k \leq q_1q_2\cdots q_n - 1$,

$$\mathrm{e}^{\frac{k}{q_1q_2\cdots q_n}2\pi\mathrm{i}}\varphi_n(a_0) \in \gamma_{n+1} \subseteq \mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}).$$

Then for all $0 \leq k \leq q_1q_2\cdots q_n - 1$,

$$a_{\frac{k}{q_1q_2\cdots q_n}} = (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathrm{e}^{\frac{k}{q_1q_2\cdots q_n}2\pi\mathrm{i}}\varphi_n(a_0)) \in (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$$

and hence for all $m \geq n$,

$$a_{\frac{k}{q_1q_2\cdots q_n}} = a_{\frac{kq_{n+1}\cdots q_m}{q_1q_2\cdots q_n\cdots q_m}} \in (\varphi_m|_{\mathcal{D}(\varphi_m)})^{-1}(\mathcal{D}_{r_{m+2}}(\mathcal{R}_{m+1})).$$

Again, since $\{(\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))\}_{n=1}^{\infty}$ is decreasing, we have that

$$\mathcal{A} \subseteq \bigcap_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})).$$

By (4.4) we have that $\mathcal{A}$ is dense in $\bigcap_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1}))$. Thus

$$\overline{\mathcal{A}} = \bigcap_{n=1}^{\infty} (\varphi_n|_{\mathcal{D}(\varphi_n)})^{-1}(\mathcal{D}_{r_{n+2}}(\mathcal{R}_{n+1})) = \partial\mathcal{D}.$$

We define a map

$$\tau : \mathcal{I} \to \mathbb{C}, \ x \mapsto a_x.$$

For all $x_1, x_2 \in \mathcal{I}$ with $|x_1 - x_2| < \frac{1}{q_1 q_2 \cdots q_n}$, there exists a positive integer k with $0 \le k \le q_1 q_2 \cdots q_n$ such that $\left|x_1 - \frac{k}{q_1 q_2 \cdots q_n}\right| < \frac{1}{q_1 q_2 \cdots q_n}$ and $\left|x_2 - \frac{k}{q_1 q_2 \cdots q_n}\right| < \frac{1}{q_1 q_2 \cdots q_n}$. By (c) we have $a_{x_1}, a_{x_2} \in \bigcup_{s=k-2}^{k+1} U^s_{q_1 q_2 \cdots q_n}$. By (d) we have $|a_{x_1} - a_{x_2}| < \frac{4}{2^n}$. Thus τ is uniformly continuous and hence τ can be continuously extended to $[0,1]$, still denoted by τ. Evidently, $\tau([0,1]) = \overline{\mathcal{A}}$.

At last, we only need to prove that τ is a Jordan curve. Indeed, for any $x \neq y \in [0,1)$, we choose a sufficiently large n such that $x = \lim_{j\to\infty} \frac{k_j}{q_1 q_2 \cdots q_j}$ and $y = \lim_{j\to\infty} \frac{k'_j}{q_1 q_2 \cdots q_j}$, where

$$|k_n - k'_n|_{\mathbb{Z}/(q_1 q_2 \cdots q_n)} > 8,$$

and for all $j \ge n$,

$$\left|\frac{k_j}{q_1 q_2 \cdots q_j} - x\right|_{\mathbb{R}/\mathbb{Z}} = \min_{0 \le s \le q_1 q_2 \cdots q_j - 1} \left\{\left|\frac{s}{q_1 q_2 \cdots q_j} - x\right|_{\mathbb{R}/\mathbb{Z}}\right\} \tag{4.8}$$

and

$$\left|\frac{k'_j}{q_1 q_2 \cdots q_j} - y\right|_{\mathbb{R}/\mathbb{Z}} = \min_{0 \le s \le q_1 q_2 \cdots q_j - 1} \left\{\left|\frac{s}{q_1 q_2 \cdots q_j} - y\right|_{\mathbb{R}/\mathbb{Z}}\right\}. \tag{4.9}$$

We write $x_j := \frac{k_j}{q_1 q_2 \cdots q_j}$ and $y_j := \frac{k'_j}{q_1 q_2 \cdots q_j}$ for all $j \ge n$. Since $|k_n - k'_n|_{\mathbb{Z}/(q_1 q_2 \cdots q_n)} > 8$, we have that $a_{x_n} \in U^{k_n}_{q_1 q_2 \cdots q_n}$ and $a_{y_n} \in U^{k'_n}_{q_1 q_2 \cdots q_n}$ with

$$\overline{\Big(\bigcup_{j=k_n-3}^{k_n+3} U^j_{q_1 q_2 \cdots q_n}\Big)} \cap \overline{\Big(\bigcup_{j=k'_n-3}^{k'_n+3} U^j_{q_1 q_2 \cdots q_n}\Big)} = \emptyset. \tag{4.10}$$

By (4.8)–(4.9) we have that for all $j \ge n$, $|x_j - \frac{k_n}{q_1 q_2 \cdots q_n}| < \frac{1}{q_1 q_2 \cdots q_n}$ and $|y_j - \frac{k'_n}{q_1 q_2 \cdots q_n}| < \frac{1}{q_1 q_2 \cdots q_n}$. Then by (c) we have that for all $j \ge n$,

$$a_{x_j} \in \bigcup_{s=k_n-2}^{k_n+1} U^s_{q_1 q_2 \cdots q_n} \quad \text{and} \quad a_{y_j} \in \bigcup_{s=k'_n-2}^{k'_n+1} U^s_{q_1 q_2 \cdots q_n}.$$

Thus

$$\tau(x) = \lim_{j\to\infty} a_{x_j} \in \overline{\bigcup_{s=k_n-2}^{k_n+1} U^s_{q_1 q_2 \cdots q_n}} \subseteq \overline{\bigcup_{j=k_n-3}^{k_n+3} U^j_{q_1 q_2 \cdots q_n}}$$

and

$$\tau(y)=\lim_{j\to\infty}a_{y_j}\in\overline{\bigcup_{s=k'_n-2}^{k'_n+1}U^s_{q_1q_2\cdots q_n}}\subseteq\overline{\bigcup_{j=k'_n-3}^{k'_n+3}U^j_{q_1q_2\cdots q_n}}.$$

Together with (4.10), we have $\tau(x)\neq\tau(y)$.

Declarations

Conflicts of interest Jianyong QIAO is an editorial board member for Chinese Annals of Mathematics Series B and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no conflicts of interest.

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