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On the $N = 1$ Bondi-Metzner-Sachs Lie Conformal Superalgebra*

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Abstract This paper constructs a finite Lie conformal superalgebra $\mathfrak{R}$ associated to the $N = 1$ Bondi-Metzner-Sachs (BMS for short) superalgebra. The authors completely determine conformal derivations, the automorphism group, and the second cohomology with coefficients in trivial module. They also classify free conformal modules of rank $(1 + 1)$ and finite irreducible conformal modules over $\mathfrak{R}$.

Keywords Conformal derivation, Automorphism, Second cohomology, Conformal module
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1 Introduction

The notion of Lie conformal superalgebras was introduced by Kac as a formal language describing the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory. It is closely connected to the Lie superalgebra spanned by the coefficients of a family of mutually local formal distributions. In fact, a Lie conformal superalgebra canonically associates a maximal formal distribution Lie superalgebra, which establishes an equivalence between the category of Lie conformal superalgebras and the category of equivalence classes of formal distribution Lie superalgebras (see [15]).

The classification of finite simple Lie conformal superalgebras was completed in [13]. The list consists of current Lie conformal superalgebras $\mathrm{Cur}\,\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra, four series of "Virasoro like" Lie conformal superalgebras W_n $(n \geq 0)$, $S_{n,b}$ and $\widetilde{S}_n$ $(n \geq 2, b \in \mathbb{C})$, K_n $(n \geq 0, n \neq 4)$, and the exceptional Lie conformal superalgebras CK_6 and K_4'. The structure theory and finite irreducible representations of simple Lie conformal superalgebras were developed in a series of papers (see [1, 3–6, 8, 13, 18, 20]).

It is interesting to study non-simple or even non-semisimple Lie conformal (super)algebras since the conformal analogues of the Levi theorem fail, although there exists conformal version

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of Lie's Theorem (see [14, 17]). What we mainly consider in this paper is a non-semisimple finite Lie conformal superalgebra

$$\mathfrak{R} = \mathfrak{R}_{\bar{0}} \oplus \mathfrak{R}_{\bar{1}}, \quad \mathfrak{R}_{\bar{0}} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]M, \quad \mathfrak{R}_{\bar{1}} = \mathbb{C}[\partial]Q$$

satisfying λ-brackets

$$[L\,_\lambda L] = (\partial + 2\lambda)L, \qquad [L\,_\lambda M] = (\partial + 2\lambda)M,$$
$$[L\,_\lambda Q] = \Big(\partial + \frac{3}{2}\lambda\Big)Q, \quad [Q_\lambda Q] = 2M, \quad [Q\,_\lambda M] = [M\,_\lambda M] = 0.$$

Clearly, $\mathfrak{R}$ contains a subalgebra Vir $= \mathbb{C}[\partial]L$ which is isomorphic to the Virasoro Lie conformal algebra. The even part $\mathfrak{R}_{\bar{0}}$ is exactly the $W(2,2)$ Lie conformal algebra (see [23, 25]), and the subalgebra generated by M and Q is a solvable ideal (and thus $\mathfrak{R}$ is non-semisimple).

We shall see in Section 2 that the annihilation superalgebra of $\mathfrak{R}$ is a maximal subalgebra of the $N = 1$ Bondi-Metzner-Sachs (BMS for short) superalgebra

$$\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}},$$

where

$$\mathcal{G}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{L_n, M_n \mid n \in \mathbb{Z}\}, \quad \mathcal{G}_{\bar{1}} = \mathrm{span}_{\mathbb{C}}\Big\{Q_r \mid r \in \frac{1}{2} + \mathbb{Z}\Big\}$$

with the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n}, \qquad [L_m, M_n] = (m - n)M_{m+n},$$
$$[L_m, Q_r] = \Big(\frac{m}{2} - r\Big)Q_{m+r}, \quad [Q_r, Q_s] = 2M_{r+s}, \quad [Q_r, M_n] = [M_m, M_n] = 0$$

for any $m, n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$. Hence, we refer to this Lie conformal superalgebra $\mathfrak{R}$ as the $N = 1$ BMS Lie conformal superalgebra. The $N = 1$ BMS superalgebra has a close relation with the $N = 1$ Neveu-Schwarz algebra and plays a key role in describing asymptotic supergravity in three-dimensional flat spacetime (see [2, 10–12, 19]). Note that the even part $\mathcal{G}_{\bar{0}}$ corresponds to the centerless W-algebra $W(2,2)$ introduced in [26]. Thus, it is interesting to study the structure and representation theory of the Lie conformal superalgebra $\mathfrak{R}$ associated to the $N = 1$ BMS superalgebra. This is also our motivation to present this paper.

The paper is organized as follows. In Section 2, we introduce some basic definitions, and then construct Lie conformal superalgebra $\mathfrak{R}$ by generating relations. In Section 3, we determine conformal derivations, the automorphism group and the second cohomology group of $\mathfrak{R}$ with coefficients in trivial module. In Sections 4–5, we classify free conformal modules of rank $(1+1)$ over $\mathfrak{R}$ and obtain that all finite irreducible conformal modules are simply irreducible ones over the Virasoro Lie conformal algebra, where some results in [22, 24] will be used.

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$ and $\mathbb{N}$ the sets of complex numbers, non-zero complex numbers, integers and non-negative integers, respectively. Let $\mathbb{C}[\partial]$ be the ring of polynomials in the indeterminate ∂.

2 Preliminaries

In this section, we recall some definitions related to a Lie conformal superalgebra (see [6, 16]) and construct Lie conformal superalgebra $\mathfrak{R}$ by some generating relations.

2.1 Basic definitions

Let $\mathfrak{g}$ be a Lie superalgebra. A $\mathfrak{g}$-valued formal distribution in one indeterminate z is a formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathfrak{g}.$$

Two $\mathfrak{g}$-valued formal distributions $a(z)$ and $b(z)$ are called mutually local if

$$(z-w)^N [a(z), b(w)] = 0$$

for some $N \in \mathbb{N}$.

Define $\mathrm{Res}_z a(z) = a_{(0)}$ and

$$\delta(z-w) = z^{-1} \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n.$$

The bracket of two local formal distributions is given by the formula

$$[a(z), b(w)] = \sum_{n\in\mathbb{N}} (a_{(n)}b)(w) \frac{\partial_w^n \delta(z-w)}{n!},$$

where $(a_{(n)}b)(w) = \mathrm{Res}_z (z-w)^n [a(z), b(w)]$. This is called the operator product expansion.

Definition 2.1 *The Lie superalgebra $\mathfrak{g}$ is called a Lie superalgebra of formal distributions if there exists a family of pairwise local formal distributions whose coefficients span $\mathfrak{g}$.*

Example 2.1 (see [6]) The Virasoro algebra has a basis L_n $(n \in \mathbb{Z})$ and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}.$$

It is spanned by the local formal distribution $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, since one has

$$[L(z), L(w)] = \partial_w L(w)\delta(z-w) + 2L(w)\partial_w \delta(z-w).$$

Example 2.2 (see [6]) The $N = 1$ Neveu-Schwarz algebra, apart from even basis Virasoro elements L_n, has odd basis elements $G_r, r \in \frac{1}{2} + \mathbb{Z}$ with commutation relations:

$$[L_n, G_r] = \left(\frac{n}{2} - r\right) G_{n+r}, \quad [G_r, G_s] = 2L_{r+s}.$$

It is spanned by the following family of pairwise local formal distributions

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad G(z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}.$$

Note that one has

$$L_{(0)}G = \partial G, \quad G_{(0)}L = \frac{1}{2}\partial G, \quad L_{(1)}G = G_{(1)}L = \frac{3}{2}G, \quad G_{(0)}G = 2L,$$

where other products are zero.

For a Lie superalgebra of formal distributions, we define the following λ-bracket (see [9, 15]):

$$[a(w)_\lambda b(w)] = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}\mathrm{Res}_z(z-w)^n[a(z), b(w)].$$

The properties of this λ-bracket lead to the following definition.

Definition 2.2 *A Lie conformal superalgebra $R = R_{\bar{0}} \oplus R_{\bar{1}}$ is a $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module (satisfying $[R_{\alpha\,\lambda} R_\beta] \subseteq R_{\alpha+\beta}$ for any $\alpha, \beta \in \{\bar{0}, \bar{1}\}$) endowed with a λ-bracket, that is a $\mathbb{C}$-linear map $R\otimes R \to \mathbb{C}[\lambda]\otimes R$, denoted by $a\otimes b \mapsto [a_\lambda b]$, satisfying the following properties $(a, b, c \in R)$:*

$$\begin{aligned}
&[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial+\lambda)[a_\lambda b] \quad \text{(conformal sesquilinearity)},\\
&[a_\lambda b] = -(-1)^{|a||b|}[b_{-\lambda-\partial} a] \quad \text{(skew-commutativity)},\\
&[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{|a||b|}[b_\mu[a_\lambda c]] \quad \text{(Jacobi identity)}.
\end{aligned}$$

Here and further, we use the notation $|a| \in \mathbb{Z}_2$ to denote the parity of a, and we always assume that a is homogeneous if $|a|$ appears in an expression. If there exists a finite generating subset $S \subset R$ such that S generates R as a $\mathbb{C}[\partial]$-module, then we call that R is a finite rank Lie conformal superalgebra. Otherwise, it is called infinite.

Let R be a Lie conformal superalgebra. We may associate to R a Lie superalgebra of formal distributions (see [6]). This leads to the following definition.

Definition 2.3 *The annihilation superalgebra $\mathcal{A}(R)$ of a Lie conformal superalgebra R is a Lie superalgebra with $\mathbb{C}$-basis $\{a_{(n)} \mid a \in R, n \in \mathbb{N}\}$ and relations*

$$[a_{(m)}, b_{(n)}] = \sum_{k\in\mathbb{N}} \binom{m}{k} (a_{(k)}b)_{(m+n-k)}, \quad (\partial a)_{(n)} = -na_{(n-1)}, \tag{2.1}$$

where $a_{(k)}b$ is given by

$$[a_\lambda b] = \sum_{k\in\mathbb{N}} \lambda^{(k)} a_{(k)} b, \quad \lambda^{(k)} = \frac{\lambda^k}{k!}.$$

Here, the reason why $\mathcal{A}(R)$ is a Lie superalgebra can be found in the book by Kac [15, p. 41–42]. The parity $|a_{(n)}|$ of $a_{(n)} \in \mathcal{A}(R)$ is the same as $|a|$ for any $a \in R$ and $n \in \mathbb{N}$. Note that $\mathcal{A}(R)$ admits a derivation T given by $T(a_{(n)}) = -na_{(n-1)}$ for $a_{(n)} \in \mathcal{A}(R)$. Denote by $\mathcal{A}(R)^e$ the semidirect sum of $\mathbb{C}T$ and $\mathcal{A}(R)$ with the commutation relations given by (2.1) and

$$[T, a_{(n)}] = -na_{(n-1)}, \quad a_{(n)} \in \mathcal{A}(R),\ n \in \mathbb{N}. \tag{2.2}$$

Then $\mathcal{A}(R)^e = \mathbb{C}T \ltimes \mathcal{A}(R)$ forms a Lie superalgebra called the extended annihilation superalgebra.

2.2 $N = 1$ BMS Lie conformal superalgebra

Recall that the commutation relations over the $N = 1$ BMS superalgebra are defined by

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [L_m, M_n] = (m-n)M_{m+n}, \tag{2.3}$$

$$[L_m, Q_r] = \Big(\frac{m}{2} - r\Big) Q_{m+r}, \quad [Q_r, Q_s] = 2M_{r+s}, \quad [Q_r, M_n] = [M_m, M_n] = 0, \tag{2.4}$$

where $m, n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z}$. Introducing the following local formal distributions

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad M(z) = \sum_{n\in\mathbb{Z}} M_n z^{-n-2}, \quad Q(z) = \sum_{n\in\mathbb{Z}} Q_{\frac{1}{2}+n} z^{-n-2},$$

one can check that

$$\begin{aligned}
&[L(z), L(w)] = \partial_w L(w)\delta(z-w) + 2L(w)\partial_w \delta(z-w),\\
&[L(z), M(w)] = \partial_w M(w)\delta(z-w) + 2M(w)\partial_w \delta(z-w),\\
&[L(z), Q(w)] = \partial_w Q(w)\delta(z-w) + \frac{3}{2} Q(w)\partial_w \delta(z-w),\\
&[Q(z), Q(w)] = 2M(w)\delta(z-w),\\
&[Q(z), M(w)] = [M(z), M(w)] = 0.
\end{aligned}$$

From these generating relations above, we have the following theorem.

Theorem 2.1 *Let* $\mathfrak{R} = \mathfrak{R}_{\bar{0}} \oplus \mathfrak{R}_{\bar{1}}$ *with*

$$\mathfrak{R}_{\bar{0}} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]M, \quad \mathfrak{R}_{\bar{1}} = \mathbb{C}[\partial]Q.$$

Define the λ*-brackets over* $\mathfrak{R}$ *by*

$$\begin{aligned}
&[L\,{}_\lambda L] = (\partial + 2\lambda)L, \qquad [L\,{}_\lambda M] = (\partial + 2\lambda)M,\\
&[L\,{}_\lambda Q] = \Big(\partial + \frac{3}{2}\lambda\Big)Q, \quad [Q_\lambda Q] = 2M, \quad [Q\,{}_\lambda M] = [M\,{}_\lambda M] = 0.
\end{aligned}$$

Then $\mathfrak{R}$ *becomes a Lie conformal superalgebra.*

Next we determine the generators and relations of the (extended) annihilation superalgebra of $\mathfrak{R}$ (see Proposition 2.1). First, it follows from Definition 2.3 and the λ-brackets over $\mathfrak{R}$ that $\mathcal{A}(\mathfrak{R})$ has a $\mathbb{C}$-basis $\{L_{(n)}, M_{(n)}, Q_{(n)} \mid n \in \mathbb{N}\}$ with nonvanishing relations

$$[L_{(m)}, L_{(n)}] = (m-n)L_{(m+n-1)}, \qquad [L_{(m)}, M_{(n)}] = (m-n)L_{(m+n-1)}, \tag{2.5}$$

$$[L_{(m)}, Q_{(n)}] = \Big(\frac{m}{2} - n\Big)Q_{(m+n-1)}, \quad [Q_{(m)}, Q_{(n)}] = 2M_{(m+n)}, \tag{2.6}$$

where $m, n \in \mathbb{N}$. In addition, by (2.2), we see that $\mathcal{A}(\mathfrak{R})$ admits a derivation T given by

$$[T, X_{(n)}] = -nX_{(n-1)} \quad \text{for } X \in \{L, M, Q\}. \tag{2.7}$$

Now define $L_n = L_{(n+1)}$, $M_n = M_{(n+1)}$ and $Q_r = Q_{(r+\frac{1}{2})}$, where $n \in \mathbb{Z}_{\geq -1}$ and $r \in \frac{1}{2} + \mathbb{Z}_{\geq -1}$. Then by (2.5)–(2.7), we obtain the following proposition.

Proposition 2.1 *Let* $\mathcal{A}(\mathfrak{R})$ *be the annihilation superalgebra of* $\mathfrak{R}$ *and* T *be a derivation of* $\mathfrak{R}$ *given by* (2.7)*. Then* $\mathcal{A}(\mathfrak{R})$ *admits a* $\mathbb{C}$*-basis* $\big\{L_n, M_n, Q_r \mid n \in \mathbb{Z}_{\geq -1}, r \in \frac{1}{2} + \mathbb{Z}_{\geq -1}\big\}$ *with super-commutation relations given by* (2.3)–(2.4)*. The extended annihilation superalgebra* $\mathcal{A}(\mathfrak{R})^e = \mathbb{C}T \ltimes \mathcal{A}(\mathfrak{R})$ *is generated by* T *and* $\mathcal{A}(\mathfrak{R})$ *with the additional relations*

$$[T, L_n] = -(n+1)L_{n-1}, \quad [T, M_n] = -(n+1)M_{n-1}, \quad [T, Q_r] = -\Big(r + \frac{1}{2}\Big)Q_{r-1}. \tag{2.8}$$

One sees that the annihilation superalgebra $\mathcal{A}(\mathfrak{R})$ of $\mathfrak{R}$ is a maximal subalgebra of the $N = 1$ BMS Lie superalgebra. Hence, we refer to this Lie conformal superalgebra $\mathfrak{R}$ as the $N = 1$ BMS Lie conformal superalgebra.

3 Derivations and Automorphisms and Central Extensions of $\mathfrak{R}$

In this section, we shall study the structure theory of Lie conformal superalgebra $\mathfrak{R}$. Concretely, we investigate conformal derivations, automorphism group and the second cohomology group with coefficients in trivial module of $\mathfrak{R}$.

3.1 Conformal derivations

Let V and W be $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-modules. A conformal linear map between V and W is a $\mathbb{C}$-linear map $\phi_\lambda : V \to W[\lambda]$ such that

$$\phi_\lambda(\partial v) = (\partial+\lambda)(\phi_\lambda v), \quad \forall v \in V.$$

Let $\mathrm{Chom}(V, W)$ be the $\mathbb{C}$-vector space of all conformal linear maps from V to W. It has a $\mathbb{C}[\partial]$-module structure by

$$(\partial\phi)_\lambda v = -\lambda\phi_\lambda v.$$

Denote $\mathrm{Chom}(V, V)$ by $\mathrm{Cend}\, V$. When V is finite over $\mathbb{C}[\partial]$, $\mathrm{Cend}\, V$ becomes an associative conformal superalgebra with the λ-product $(a_\lambda b)_\mu v = a_\lambda(b_{\mu-\lambda} v)$ for any $a, b \in \mathrm{Cend}\, V$. Furthermore, an element $\phi_\lambda \in \mathrm{Cend}\, V$ is called a conformal linear map of degree α if it satisfies $\phi_\lambda(V_\beta) \subseteq V_{\alpha+\beta}[\lambda]$ for any $\alpha, \beta \in \mathbb{Z}_2$. Denote by $(\mathrm{Cend}\, V)_\alpha$ the space of all conformal linear maps of degree α. Then $\mathrm{Cend}\, V = (\mathrm{Cend}\, V)_{\bar{0}} \oplus (\mathrm{Cend}\, V)_{\bar{1}}$. Similarly, we use the notation $|\phi_\lambda| \in \mathbb{Z}_2$ to denote the parity of ϕ_λ.

Definition 3.1 *A conformal derivation of a Lie conformal superalgebra R is a conformal endomorphism $D_\lambda \in \mathrm{Cend}\, R$ such that for any homogeneous $a, b \in R$,*

$$D_\lambda[a_\mu b] = [(D_\lambda a)_{\lambda+\mu} b] + (-1)^{|D||a|}[a_\mu(D_\lambda b)].$$

Denote by $\mathrm{CDer}\, R$ the space of all conformal derivations of R. Then it is obvious that $\mathrm{CDer}\, R = (\mathrm{CDer}\, R)_{\bar{0}} \oplus (\mathrm{CDer}\, R)_{\bar{1}}$ is a subalgebra of $\mathrm{Cend}\, R$. Note that, for any $a \in R$, one can define a conformal derivation $\mathrm{ad}\, a$ of R by $(\mathrm{ad}\, a)_\lambda b = [a_\lambda b]$ for $b \in R$. All conformal derivations of this kind are called inner. We denote by $\mathrm{CInder}\, R$ the space of all inner conformal derivations.

Lemma 3.1 *Let $D_\lambda \in (\mathrm{CDer}\, \mathfrak{R})_{\bar{0}}$. Then there exist $a_1(\lambda), b_1(\lambda) \in \mathbb{C}[\lambda]$ such that*

$$D_\lambda L = a_1(\lambda)(\partial+2\lambda)L + b_1(\lambda)(\partial+2\lambda)M,$$
$$D_\lambda M = a_1(\lambda)(\partial+2\lambda)M, \quad D_\lambda Q = a_1(\lambda)\Big(\partial+\frac{3}{2}\lambda\Big)Q.$$

Proof Since D_λ is an even conformal derivation, we can set

$$D_\lambda L = a_1(\partial,\lambda)L + b_1(\partial,\lambda)M, \quad D_\lambda M = a_2(\partial,\lambda)L + b_2(\partial,\lambda)M, \quad D_\lambda Q = a_3(\partial,\lambda)Q,$$

where $x(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$ for $x \in \{a_1, b_1, a_2, b_2, a_3\}$. By using $D_\lambda[L_\mu L] = [(D_\lambda L)_{\lambda+\mu} L] + [L_\mu(D_\lambda L)]$, we have

$$(\partial+2\mu)x(\partial+\mu,\lambda) + (\partial+2\lambda+2\mu)x(-\lambda-\mu,\lambda) = (\partial+\lambda+2\mu)x(\partial,\lambda)$$

for $x \in \{a_1, b_1\}$. Thus by [9, Lemma 6.1], we can set $x(\partial,\lambda) = x(\lambda)(\partial+2\lambda)$ for $x(\lambda) \in \mathbb{C}[\lambda]$ and $x \in \{a_1, b_1\}$. Furthermore, applying D_λ to $[L_\mu M] = (\partial+2\mu)M$, we get

$$\begin{aligned}
&(\partial+2\mu)a_2(\partial+\mu,\lambda) = (\partial+\lambda+2\mu)a_2(\partial,\lambda),\\
&(\partial+2\mu)b_2(\partial+\mu,\lambda) + (\partial+2\lambda+2\mu)a_1(\lambda)(\lambda-\mu) = (\partial+\lambda+2\mu)b_2(\partial,\lambda),
\end{aligned}$$

which give $a_2(\partial,\lambda) = 0$ and $b_2(\partial,\lambda) = a_1(\lambda)(\partial+2\lambda)$, respectively. Thus we have $D_\lambda M = a_1(\lambda)(\partial+2\lambda)M$. Now, by $[(D_\lambda Q)_{\lambda+\mu}Q] + [Q_\mu(D_\lambda Q)] = 2D_\lambda M$, we have

$$a_3(-\lambda-\mu,\lambda) + a_3(\partial+\mu,\lambda) = a_1(\lambda)(\partial+2\lambda). \tag{3.1}$$

Then we can assume that $a_3(\partial,\lambda) = a_3(\lambda) + a_1(\lambda)\partial$ for some $a_3(\lambda) \in \mathbb{C}[\lambda]$. This together with (3.1) gives $a_3(\partial,\lambda) = a_1(\lambda)\big(\partial + \frac{3}{2}\lambda\big)$. Hence, the lemma follows.

Lemma 3.2 *Let $D_\lambda \in (\mathrm{CDer}\,\mathfrak{R})_{\bar{1}}$. Then there exists $c_1(\lambda) \in \mathbb{C}[\lambda]$ such that*

$$D_\lambda L = c_1(\lambda)(\partial+3\lambda)Q, \quad D_\lambda M = 0, \quad D_\lambda Q = 4c_1(\lambda)M.$$

Proof Assume that

$$D_\lambda L = c_1(\partial,\lambda)Q, \quad D_\lambda M = c_2(\partial,\lambda)Q, \quad D_\lambda Q = c_3(\partial,\lambda)L + d_3(\partial,\lambda)M.$$

Then, by direct computation, we can get the lemma.

Theorem 3.1 *For the $N=1$ BMS Lie conformal superalgebra $\mathfrak{R}$, we have*

$$\mathrm{CDer}\,\mathfrak{R} = \mathrm{CInder}\,\mathfrak{R}.$$

Proof Let $D_\lambda = D_0 + D_1 \in \mathrm{CDer}\,\mathfrak{R}$ with $D_0 \in (\mathrm{CDer}\,\mathfrak{R})_{\bar{0}}$ and $D_1 \in (\mathrm{CDer}\,\mathfrak{R})_{\bar{1}}$. By Lemmas 3.1–3.2, we get $D_0 = \mathrm{ad}\,(a_1(-\partial)L + b_1(-\partial)M)$ and $D_1 = \mathrm{ad}\,(2c_1(-\partial)Q)$. Hence, D_λ is an inner conformal derivation induced by the element $a_1(-\partial)L + b_1(-\partial)M + 2c_1(-\partial)Q$. This completes the proof.

3.2 Automorphism group

Denote by $\mathrm{Aut}(\mathfrak{R})$ the automorphism group of $\mathfrak{R}$. For any $c \in \mathbb{C}$ and $b \in \mathbb{C}^*$, define the $\mathbb{C}[\partial]$-linear maps $\sigma_c, \tau_b : \mathfrak{R} \to \mathfrak{R}$ by

$$\sigma_c(L) = L + c\partial M, \quad \sigma_c(M) = M, \quad \sigma_c(Q) = Q, \tag{3.2}$$

$$\tau_b(L) = L, \quad \tau_b(M) = b^2 M, \quad \tau_b(Q) = bQ. \tag{3.3}$$

One can verify that $\sigma_c, \tau_b \in \mathrm{Aut}(\mathfrak{R})$ and

$$\tau_b\sigma_c = \sigma_{b^2c}\tau_b, \quad \sigma_{c_1}\sigma_{c_2} = \sigma_{c_1+c_2}, \quad \tau_{b_1}\tau_{b_2} = \tau_{b_1b_2}, \tag{3.4}$$

where $c_1, c_2 \in \mathbb{C}$, $b_1, b_2 \in \mathbb{C}^*$. Denote by

$$H = \{\sigma_c \mid c \in \mathbb{C}\}, \quad K = \{\tau_b \mid b \in \mathbb{C}^*\}. \tag{3.5}$$

Clearly, H, K and HK are subgroups of $\mathrm{Aut}(\mathfrak{R})$. Next, we shall describe the structure of $\mathrm{Aut}(\mathfrak{R})$.

Take $\sigma \in \mathrm{Aut}(\mathfrak{R})$. Since $\mathfrak{R} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]M \oplus \mathbb{C}[\partial]Q$, we may assume that

$$\sigma(L) = f_L(\partial)L + g_L(\partial)M + h_L(\partial)Q,$$

where $f_L(\partial), g_L(\partial), h_L(\partial) \in \mathbb{C}[\partial]$. Note that $\mathbb{C}[\partial]M \oplus \mathbb{C}[\partial]Q$ is a maximal ideal of $\mathfrak{R}$, and any maximal ideal of $\mathfrak{R}$ does not contain an element of the form $k(\partial)L$ with $k(\partial) \in \mathbb{C}[\partial]$. Thus we can safely set

$$\sigma(X) = a_X(\partial)M + b_X(\partial)Q, \quad a_X(\partial),\ b_X(\partial) \in \mathbb{C}[\partial],\ X \in \{M, Q\}.$$

Lemma 3.3 *Using notations as above, we have*

$$f_L(\partial) = 1, \quad g_L(\partial) = c\partial, \quad h_L(\partial) = 0,$$

where c is a complex number.

Proof By applying σ to $[L_\lambda L] = (\partial + 2\lambda)L$, we get

$$f_L(\partial) = f_L(-\lambda)f_L(\partial + \lambda), \tag{3.6}$$

$$2(\partial + 2\lambda)h_L(\partial) = (2\partial + 3\lambda)f_L(-\lambda)h_L(\lambda + \partial) + (\partial + 3\lambda)h_L(-\lambda)f_L(\lambda + \partial), \tag{3.7}$$

$$2h_L(-\lambda)h_L(\partial + \lambda) = (\partial + 2\lambda)(g_L(\partial) - f_L(-\lambda)g_L(\partial + \lambda) - g_L(-\lambda)f_L(\partial + \lambda)). \tag{3.8}$$

From (3.6), we get $f_L(\partial) = 1$. Using this in (3.7), we have

$$2(\partial + 2\lambda)h_L(\partial) = (2\partial + 3\lambda)h_L(\lambda + \partial) + (\partial + 3\lambda)h_L(-\lambda). \tag{3.9}$$

Assume that $h_L(\partial) = \sum_{i=0}^{n} c_i\partial^i \in \mathbb{C}[\partial]$ with $c_n \neq 0$ and $n > 1$. Comparing the coefficients of λ^{n+1}, $\lambda^n\partial$ and λ^n, respectively, we have

$$c_n(1 + (-1)^n) = c_n(2 + 3n + (-1)^n) = c_{n-1}(1 + (-1)^{n-1}) = 0. \tag{3.10}$$

Since $n > 1$, we get $c_n = 0$, a contradiction. Thus, it follows from (3.10) that $h_L(\partial) = c_1\partial$. Using this and $f_L(\partial) = 1$ in (3.8), we have

$$2c_1^2\lambda(\partial + \lambda) = (\partial + 2\lambda)(g_L(\partial + \lambda) + g_L(-\lambda) - g_L(\partial)). \tag{3.11}$$

If $c_1 = 0$, we have $g_L(\partial + \lambda) + g_L(-\lambda) - g_L(\partial) = 0$, which gives $g_L(\partial) = c\partial$ for some $c \in \mathbb{C}$. Assume that $c_1 \neq 0$ and set $g_L(\partial) = \sum_{i=0}^{n} e_i\partial^i \in \mathbb{C}[\partial]$ with $e_n \neq 0$ and $n > 1$. Comparing the coefficients of $\partial\lambda^n$, we get $e_n(1 + (-1)^n + 2n) = 0$. Since $n > 1$, we get $e_n = 0$, a contradiction. Thus, we can set $g_L(\partial) = e_0 + e_1\partial$. This together with (3.11) gives $c_1 = 0$. Thus the case $c_1 \neq 0$ does not occur. This completes the proof.

Lemma 3.4 *Using notations as above, we have*

$$a_Q(\partial) = b_M(\partial) = 0, \quad a_M(\partial) = b^2, \quad b_Q(\partial) = b,$$

where $b \in \mathbb{C}^*$.

Proof Recall that $h_L(\partial)=0$ by Lemma 3.3. By applying σ to $[L_\lambda M]=(\partial+2\lambda)M$, we get

$$2(\partial+2\lambda)b_M(\partial)=(2\partial+3\lambda)b_M(\partial+\lambda), \tag{3.12}$$
$$(\partial+2\lambda)a_M(\partial)=(\partial+2\lambda)a_M(\partial+\lambda). \tag{3.13}$$

Considering the highest degree of λ in (3.12), we deduce $b_M(\partial)\in\mathbb{C}$. Using this in (3.12), we have $b_M(\partial)=0$. It is obvious that $a_M(\partial)\in\mathbb{C}$ by (3.13). Furthermore, by applying σ to $[L_\lambda Q]=\big(\partial+\frac{3}{2}\lambda\big)M$, we get

$$b_Q(\partial+\lambda)=b_Q(\partial), \tag{3.14}$$
$$(2\partial+3\lambda)a_Q(\partial)=2(\partial+2\lambda)a_Q(\partial+\lambda). \tag{3.15}$$

From (3.14), we get $b_Q(\partial)\in\mathbb{C}$. By (3.15), we can deduce that $a_Q(\partial)=0$. Besides, by applying σ to $[Q_\lambda Q]=2M$, we get $b_M(\partial)=0$ and $b_Q(-\lambda)b_Q(\partial+\lambda)=a_M(\partial)$. Since σ is an automorphism, we get $b_Q(\partial)\neq 0$. Thus the lemma follows by setting $b_Q(\partial)=b\in\mathbb{C}^*$.

The following theorem is our main result of this subsection.

Theorem 3.2 *For the $N=1$ BMS Lie conformal superalgebra $\mathfrak{R}$, we have*

$$\mathrm{Aut}(\mathfrak{R})\cong\mathbb{C}\rtimes\mathbb{C}^*,\quad (c_1,b_1)(c_2,b_2)=(c_1+b_1^2c_2,b_1b_2),$$

where $c_1,c_2\in\mathbb{C},b_1,b_2\in\mathbb{C}^*$.

Proof Let $\sigma\in\mathrm{Aut}(\mathfrak{R})$. It follows from Lemmas 3.3–3.4 that there exists some $c\in\mathbb{C}$ and $b\in\mathbb{C}^*$ such that $\sigma(L)=L+c\partial M$, $\sigma(M)=b^2M$ and $\sigma(Q)=bQ$. Thus $\sigma=\sigma_c\tau_b$, where σ_c and τ_b are defined in (3.2) and (3.3), respectively. Let H and K be as those defined in (3.5). Then by the multiplicative relations given in (3.4), we get

$$\mathrm{Aut}(\mathfrak{R})=H\rtimes K,\quad (\sigma_{c_1}\tau_{b_1})(\sigma_{c_2}\tau_{b_2})=\sigma_{c_1+b_1^2c_2}\tau_{b_1b_2},$$

where $\sigma_{c_1},\sigma_{c_2}\in H$, $\tau_{b_1},\tau_{b_2}\in K$. Thus the theorem follows.

3.3 Second cohomology group

In this subsection, we discuss central extensions of $\mathfrak{R}$. Note that an equivalence class in the second cohomology group defines a central extension, and vice versa. Thus it is sufficient to determine second cohomology group of $\mathfrak{R}$.

Let R be a Lie conformal superalgebra. It is obvious that one-dimensional vector space $\mathbb{C}$ can be regarded as a trivial R-module with the action of ∂ and R being zero.

Definition 3.2 *A 2-cocycle of R is a $\mathbb{C}$-linear map $\phi_\lambda:R^{\otimes 2}\to\mathbb{C}[\lambda]$, denoted by $a\otimes b\mapsto\phi_\lambda(a,b)$, satisfying the following conditions $(a,b,c\in R)$:*

$$\begin{aligned}&\phi_\lambda(\partial a,b)=-\lambda\phi_\lambda(a,b),\quad \phi_\lambda(a,\partial b)=\lambda\phi_\lambda(a,b),\\ &\phi_\lambda(b,a)=-(-1)^{|a||b|}\phi_{-\lambda}(a,b),\\ &\phi_{\lambda+\mu}([a_\lambda b],c)=\phi_\lambda(a,[b_\mu c])-(-1)^{|a||b|}\phi_\mu(b,[a_\lambda c]).\end{aligned} \tag{3.16}$$

For a 2-cocycle ϕ_λ, if there exists a $\mathbb{C}[\partial]$-linear map $f : R \to \mathbb{C}$ such that the condition $\phi_\lambda(a,b) = -f([a_\lambda b])$ holds, then the 2-cocycle ϕ_λ is called a 2-coboundary or a trivial 2-cocycle. Let $C^2(R,\mathbb{C})$ and $B^2(R,\mathbb{C})$ denote the spaces of 2-cocycles and 2-coboundaries, respectively. Then the second cohomology group of R with trivial coefficients $\mathbb{C}$ is defined by

$$H^2(R,\mathbb{C}) = C^2(R,\mathbb{C})/B^2(R,\mathbb{C}).$$

In the following part, we shall compute the group $H^2(\mathfrak{R},\mathbb{C})$.

Lemma 3.5 *Let $\psi_\lambda \in C^2(\mathfrak{R},\mathbb{C})$. There exist some $\overline{a}, a, \overline{b}, b \in \mathbb{C}$ such that*

$$\psi_\lambda(L,L) = \overline{a}\lambda + a\lambda^3, \quad \psi_\lambda(L,M) = \overline{b}\lambda + b\lambda^3, \quad \psi_\lambda(Q,Q) = \overline{b} + 4b\lambda^2,$$

where all other terms are vanishing.

Proof Since L generates the Virasoro conformal algebra, there exist some $\overline{a}, a \in \mathbb{C}$ such that $\psi_\lambda(L,L) = \overline{a}\lambda + a\lambda^3$ (see [21, Section 4]). Since $[L_\lambda M] = (\partial + 2\lambda)M$, one can safely set $\psi_\lambda(L,M) = \overline{b}\lambda + b\lambda^3$ for some $\overline{b}, b \in \mathbb{C}$. Now, by (3.16) for triple (L,L,Q), we get

$$2(\lambda-\mu)\psi_{\lambda+\mu}(L,Q) = (2\lambda+3\mu)\psi_\lambda(L,Q) - (2\mu+3\lambda)\psi_\mu(L,Q).$$

Set $\psi_\lambda(L,Q) = \sum\limits_{i=0}^{n} c_i\lambda^i \in \mathbb{C}[\lambda]$. Then by comparing the coefficients of λ^n, one has $(2n-3)c_n\mu = 0$, which gives $c_n = 0$ for any $n \geq 0$. Thus, $\psi_\lambda(L,Q) = 0$. Furthermore, by (3.16) for triple (L,M,M), we get $(\mu-\lambda)\psi_{\lambda+\mu}(M,M) = (\mu+2\lambda)\psi_\mu(M,M)$, which forces $\psi_\mu(M,M) = 0$ by comparing the coefficients of λ. Now, by (3.16) for triple (L,M,Q), we get $2(\mu-\lambda)\psi_{\lambda+\mu}(M,Q) = (2\mu+3\lambda)\psi_\mu(M,Q)$. Thus, $\psi_\mu(M,Q) = 0$.

Similarly, by (3.16) for triple (L,Q,Q), and using $\psi_\lambda(L,M) = \overline{b}\lambda + b\lambda^3$, we get

$$(\lambda-2\mu)\psi_{\lambda+\mu}(Q,Q) = 4(\overline{b}\lambda + b\lambda^3) - (2\mu+3\lambda)\psi_\mu(Q,Q). \tag{3.17}$$

Considering the degree of λ, one can set $\psi_\lambda(Q,Q) = e_0 + e_1\lambda + e_2\lambda^2$ for some $e_0, e_1, e_2 \in \mathbb{C}$. Using this in (3.17) and comparing the coefficients of $\lambda^3, \lambda^2, \lambda$, respectively, we obtain $e_0 = \overline{b}$, $e_1 = 0$ and $e_2 = 4b$. Hence, we get $\psi_\lambda(Q,Q) = \overline{b} + 4b\lambda^2$. This completes the proof.

By direct computation, we have the following non-trivial 2-cocycles $\overline{\psi}$ and $\widehat{\psi}$ defined by

$$\overline{\psi}_\lambda(L,L) = \lambda^3, \quad \widehat{\psi}_\lambda(L,M) = \lambda^3, \quad \widehat{\psi}_\lambda(Q,Q) = 4\lambda^2, \tag{3.18}$$

where all other terms are vanishing.

Theorem 3.3 *Let $[\overline{\psi}_\lambda]$ and $[\widehat{\psi}_\lambda]$ denote the equivalence classes of $\overline{\psi}_\lambda$ and $\widehat{\psi}_\lambda$ in $H^2(\mathfrak{R},\mathbb{C})$, respectively. We have*

$$H^2(\mathfrak{R},\mathbb{C}) = \mathbb{C}[\overline{\psi}_\lambda] \oplus \mathbb{C}[\widehat{\psi}_\lambda].$$

Proof Assume that ψ_λ is a 2-cocycle on $\mathfrak{R}$. It follows from Lemma 3.5 that there exist some $\overline{a}, a, \overline{b}, b \in \mathbb{C}$ such that (any other terms are vanishing)

$$\psi_\lambda(L,L) = \overline{a}\lambda + a\lambda^3, \quad \psi_\lambda(L,M) = \overline{b}\lambda + b\lambda^3, \quad \psi_\lambda(Q,Q) = \overline{b} + 4b\lambda^2.$$

Thus, $\psi_\lambda = a\overline{\psi}_\lambda + b\widehat{\psi}_\lambda + \overline{a}\phi_\lambda^1 + \overline{b}\phi_\lambda^2$, where $\overline{\psi}_\lambda$ and $\widehat{\psi}_\lambda$ are defined in (3.18). The two trivial cocycles ϕ_λ^1 and ϕ_λ^2 are defined by $\phi_\lambda^i(X,Y) = -f_i([X,Y])$ with $i \in \{1,2\}$ and $X, Y \in \{L, M, Q\}$, where

$$f_1(L) = f_2(M) = -\frac{1}{2}, \quad f_1(M) = f_1(Q) = f_2(L) = f_2(Q) = 0.$$

This completes the proof.

4 Free Conformal Modules of Rank $(1+1)$

In this section, we shall classify free conformal modules of rank $(1+1)$ over $\mathfrak{R}$.

Definition 4.1 *A conformal module $V = V_{\overline{0}} \oplus V_{\overline{1}}$ over a Lie conformal superalgebra R is a $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes V \to \mathbb{C}[\lambda] \otimes V$, $a \times v \mapsto a_\lambda v$ called λ-action, such that* $(a, b \in R, v \in V)$

$$(\partial a)_\lambda v = -\lambda a_\lambda v, \quad a_\lambda(\partial v) = (\partial + \lambda) a_\lambda v, \quad [a_\lambda b]_{\lambda+\mu} v = a_\lambda(b_\mu v) - (-1)^{|a||b|} b_\mu(a_\lambda v).$$

Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$. We call V finite if it is finitely generated over $\mathbb{C}[\partial]$. As $\mathbb{C}[\partial]$-modules, if $V_{\overline{0}}$ has rank m and $V_{\overline{1}}$ has rank n, we say that V has rank $(m+n)$, denoted by $\mathrm{rank}(V) = m + n$.

Recall that $\mathfrak{R}$ contains a Virasoro conformal subalgebra Vir. It is known that all free non-trivial Vir-modules of rank 1 over $\mathbb{C}[\partial]$ are the following ones (Δ, $\alpha \in \mathbb{C}$):

$$V_{\Delta,\alpha} = \mathbb{C}[\partial]v, \quad L_\lambda v = (\partial + \alpha + \Delta\lambda)v.$$

The modules $V_{\Delta,\alpha}$ with $\Delta \neq 0$ exhaust all finite irreducible non-trivial Vir-modules (see [6]).

It is clear that for any $c_0, c_1, d_0, d_1 \in \mathbb{C}$, one sees that the following λ-actions define a rank $(1+1)$ free conformal $\mathfrak{R}$-module:

$$\begin{aligned} &L_\lambda v_{\overline{0}} = (\partial + c_0 + d_0\lambda)v_{\overline{0}}, \quad L_\lambda v_{\overline{1}} = (\partial + c_1 + d_1\lambda)v_{\overline{1}}, \\ &X_\lambda v_{\overline{0}} = X_\lambda v_{\overline{1}} = 0, \quad X \in \{Q, M\}. \end{aligned} \tag{4.1}$$

Furthermore, for complex numbers Δ, α, $\beta \neq 0$, up to parity change, we construct four classes of rank $(1+1)$ conformal $\mathfrak{R}$-modules as follows:

(1) The $\mathfrak{R}$-module $V^{(1)}_{\Delta,\alpha,\beta}$ defined by

$$L_\lambda v_{\overline{0}} = (\overline{\alpha} + \Delta\lambda)v_{\overline{0}}, \quad L_\lambda v_{\overline{1}} = \Big(\overline{\alpha} + \Big(\Delta + \frac{1}{2}\Big)\lambda\Big)v_{\overline{1}}, \quad Q_\lambda v_{\overline{0}} = \beta v_{\overline{1}}; \tag{4.2}$$

(2) the $\mathfrak{R}$-module $V^{(2)}_{\Delta,\alpha,\beta}$ defined by

$$L_\lambda v_{\overline{0}} = \Big(\overline{\alpha} + \Big(\Delta + \frac{1}{2}\Big)\lambda\Big)v_{\overline{0}}, \quad L_\lambda v_{\overline{1}} = (\overline{\alpha} + \Delta\lambda)v_{\overline{1}}, \quad Q_\lambda v_{\overline{0}} = \beta(\overline{\alpha} + 2\Delta\lambda)v_{\overline{1}}; \tag{4.3}$$

(3) the $\mathfrak{R}$-module $V^{(3)}_{\alpha,\beta}$ defined by

$$L_\lambda v_{\overline{0}} = (\overline{\alpha} + \lambda)v_{\overline{0}}, \quad L_\lambda v_{\overline{1}} = \Big(\overline{\alpha} - \frac{1}{2}\lambda\Big)v_{\overline{1}}, \quad Q_\lambda v_{\overline{0}} = \beta(\overline{\alpha} + \lambda)(\overline{\alpha} - \lambda)v_{\overline{1}}; \tag{4.4}$$

(4) the $\mathfrak{R}$-module $V^{(4)}_{\alpha,\beta}$ defined by

$$L_\lambda v_{\overline{0}} = \Big(\overline{\alpha} + \frac{3}{2}\lambda\Big) v_{\overline{0}}, \quad L_\lambda v_{\overline{1}} = \overline{\alpha} v_{\overline{1}}, \quad Q_\lambda v_{\overline{0}} = \beta\overline{\alpha}(\overline{\alpha} + 2\lambda) v_{\overline{1}}, \tag{4.5}$$

and in all cases $M_\lambda v_{\overline{0}} = M_\lambda v_{\overline{1}} = Q_\lambda v_{\overline{1}} = 0$ and $\overline{\alpha} = \partial + \alpha$. It is obvious that these $\mathfrak{R}$-modules above are all reducible.

Now, assume that $V = \mathbb{C}[\partial]v_{\overline{0}} \oplus \mathbb{C}[\partial]v_{\overline{1}}$ is a free conformal module of rank $(1+1)$ over $\mathfrak{R}$. Then, there exist some $\alpha_0, \alpha_1, \Delta_0, \Delta_1 \in \mathbb{C}$ such that

$$\begin{aligned} L_\lambda v_{\overline{0}} &= (\partial + \alpha_0 + \Delta_0\lambda) v_{\overline{0}}, \quad M_\lambda v_{\overline{0}} = f_M(\partial,\lambda) v_{\overline{0}}, \quad Q_\lambda v_{\overline{0}} = f_Q(\partial,\lambda) v_{\overline{1}}, \\ L_\lambda v_{\overline{1}} &= (\partial + \alpha_1 + \Delta_1\lambda) v_{\overline{1}}, \quad M_\lambda v_{\overline{1}} = H_M(\partial,\lambda) v_{\overline{1}}, \quad Q_\lambda v_{\overline{1}} = H_Q(\partial,\lambda) v_{\overline{0}}, \end{aligned}$$

where $f_M(\partial,\lambda), f_Q(\partial,\lambda), H_M(\partial,\lambda), H_Q(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$.

In the following part, we shall determine the polynomials $f_X(\partial,\lambda), H_X(\partial,\lambda)$ with $X \in \{M, Q\}$, and thus obtain the main result of this subsection (see Theorem 4.1).

Lemma 4.1 *Using notations as above, we have* $f_M(\partial,\mu) = H_M(\partial,\mu) = 0$.

Proof By $[M_\lambda M]_{\lambda+\mu} v_{\overline{0}} = 0$, we get $f_M(\partial+\mu,\lambda) f_M(\partial,\mu) = f_M(\partial+\lambda,\mu) f_M(\partial,\lambda)$. This gives $\deg_\partial f_M(\partial,\lambda) = 0$, where $\deg_\partial f_M(\lambda,\partial)$ stands for the degree of ∂ in $f_M(\partial,\lambda)$. Furthermore, by applying $[L_\lambda M] = (\partial + 2\lambda)M$ to $v_{\overline{0}}$, we obtain $\mu f_M(\partial,\mu) = (\mu - \lambda) f_M(\partial, \lambda+\mu)$. Thus $f_M(\partial,\mu) = 0$. Similarly, one gets $H_M(\partial,\mu) = 0$.

Remark 4.1 By Lemma 4.1, we see that the problem reduces to the same problem for the Lie conformal superalgebra $S\big(\frac{3}{2}, 0\big)$. The results for general $S(a,b)$ have been given in [22]. Here, for the convenience of the reader, we retain the remaining proof details (see Lemma 4.2).

Lemma 4.2 *Using notations as above, we have* (*up to scalars*)

(1) *if* $\alpha_0 \neq \alpha_1$, *then* $f_Q(\partial,\lambda) = 0$;

(2) *if* $\alpha_0 = \alpha_1$, *then*

$$f_Q(\partial,\lambda) = \begin{cases} 1, & \text{if } \Delta_0 - \Delta_1 = -\dfrac{1}{2}, \\ \partial + \alpha_1 + 2\Delta_1\lambda, & \text{if } \Delta_0 - \Delta_1 = \dfrac{1}{2}, \\ (\partial + \alpha_1 + \lambda)(\partial + \alpha_1 - \lambda), & \text{if } (\Delta_0, \Delta_1) = \Big(1, -\dfrac{1}{2}\Big), \\ (\partial + \alpha_1)(\partial + \alpha_1 + 2\lambda), & \text{if } (\Delta_0, \Delta_1) = \Big(\dfrac{3}{2}, 0\Big), \\ 0, & \text{otherwise}. \end{cases} \tag{4.6}$$

Proof By applying $[L_\lambda Q] = \big(\partial + \frac{3}{2}\lambda\big)Q$ to $v_{\overline{0}}$, we get

$$(\partial + \alpha_1 + \Delta_1\lambda) f_Q(\partial+\lambda,\mu) - (\partial + \mu + \alpha_0 + \Delta_0\lambda) f_Q(\partial,\mu) = \Big(\frac{1}{2}\lambda - \mu\Big) f_Q(\partial, \lambda+\mu). \tag{4.7}$$

Taking $\lambda = 0$ in (4.7), we have $(\alpha_1 - \alpha_0) f_Q(\partial,\mu) = 0$. Thus $f_Q(\partial,\mu) = 0$ for $\alpha_0 \neq \alpha_1$.

Assume that $\alpha_0 = \alpha_1$. By letting $\mu = 0$ in (4.7), we have

$$f_Q(\partial,\lambda) = \frac{2}{\lambda}((\partial + \alpha_1 + \Delta_1\lambda)(f_Q(\partial+\lambda,0) - f_Q(\partial,0)) + (\Delta_1 - \Delta_0)\lambda f_Q(\partial,0)). \tag{4.8}$$

Taking $\lambda \to 0$, we get

$$(1 + 2(\Delta_0 - \Delta_1))f_Q(\partial, 0) = 2(\partial + \alpha_1)\frac{\mathrm{d}}{\mathrm{d}\partial}f_Q(\partial, 0). \tag{4.9}$$

Case 1 $\Delta_0 - \Delta_1 = -\frac{1}{2}$. In this case, (4.9) forces $f_Q(\partial, 0) \in \mathbb{C}$. Thus, $f_Q(\partial, \lambda) = 1$ (up to scalars) by (4.8).

Case 2 $\Delta_0 - \Delta_1 \neq -\frac{1}{2}$. In this case, we can check that $f_Q(\partial, 0) = k\sqrt{(\partial + \alpha_1)^{1+2(\Delta_0 - \Delta_1)}}$ with $k \neq 0$. Since $f_Q(\partial, 0) \in \mathbb{C}[\partial]$, it is necessary that $\Delta_0 - \Delta_1 \in \frac{1}{2} + \mathbb{N}$. Thus we can set $f_Q(\partial, 0) = k(\partial + \alpha_1)^n$ with $n = \frac{1}{2} + (\Delta_0 - \Delta_1)$. Using this in (4.8), together with (4.7), we get the lemma (see also [24, Lemma 4.1]).

Note that by applying $[L_\lambda Q] = \left(\partial + \frac{3}{2}\lambda\right)Q$ to $v_{\bar{1}}$, we get

$$(\partial + \alpha_0 + \Delta_0\lambda)H_Q(\partial + \lambda, \mu) - (\partial + \mu + \alpha_1 + \Delta_1\lambda)H_Q(\partial, \mu) = \left(\frac{1}{2}\lambda - \mu\right)H_Q(\partial, \lambda + \mu).$$

Thus, by the same argument as that of Lemma 4.2, we obtain the expression of $H_Q(\partial, \lambda)$ by replacing (Δ_0, Δ_1) by (Δ_1, Δ_0) in Lemma 4.2. Furthermore, by applying $[Q_\lambda Q] = 2M$ to $v_{\bar{0}}$, we get

$$f_Q(\partial + \lambda, \mu)H_Q(\partial, \lambda) + f_Q(\partial + \mu, \lambda)H_Q(\partial, \mu) = 0.$$

This gives $X_Q(\partial, \lambda) = 0$ if $Y_Q(\partial, \lambda) \in \mathbb{C}^*$, where $(X, Y) \in \{(f, H), (H, f)\}$. Combining these observations and Lemma 4.2, we see that if $f_Q(\partial, \lambda)$ has nontrivial forms of (4.6), then $H_Q(\partial, \lambda) = 0$, and vice versa. Namely, we have proved the following result.

Theorem 4.1 *Let V be a free rank $(1+1)$ conformal $\mathfrak{R}$-module. Then, up to parity change, V is one of the modules defined by* (4.1)–(4.5).

5 Finite Irreducible Conformal Modules

First, we recall the following proposition (see [6, Proposition 2.1]).

Proposition 5.1 *A conformal module V over a Lie conformal superalgebra R is precisely a module over the Lie superalgebra $\mathcal{A}(R)^e$ satisfying $a_n v = 0$ for $a \in R$, $v \in V$, $n \gg 0$, where $\mathcal{A}(R)^e$ is the extended annihilation superalgebra of R.*

5.1 Irreducible representations of the quotient algebra $\mathfrak{g}_K$

Consider a subalgebra $\mathcal{A}(\mathfrak{R})_+$ of $\mathcal{A}(\mathfrak{R})$:

$$\mathcal{A}(\mathfrak{R})_+ = \mathrm{span}_{\mathbb{C}}\left\{L_m, M_m, Q_r \mid m \in \mathbb{N}, r \in \frac{1}{2} + \mathbb{N}\right\}.$$

For any fixed $K \in \mathbb{N}$, the set defined by

$$\mathcal{I}_K = \mathrm{span}_{\mathbb{C}}\left\{L_m, M_m, Q_r \mid m > K, r > \frac{1}{2} + K\right\}$$

is an ideal of $\mathcal{A}(\mathfrak{R})_+$. Thus we obtain a subquotient algebra of $\mathcal{A}(\mathfrak{R})_+$, denoted by

$$\mathfrak{g}_K = \mathcal{A}(\mathfrak{R})_+/\mathcal{I}_K \quad \text{and} \quad \overline{x} = x + \mathcal{I}_K, \quad \forall x \in \mathcal{A}(\mathfrak{R})_+. \tag{5.1}$$

The following key lemma is due to Cheng and Kac [6–7].

Lemma 5.1 *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and $\mathfrak{n}$ be a solvable ideal of $\mathfrak{g}$. Let $\mathfrak{a}$ be an even subalgebra of $\mathfrak{g}$ such that $\mathfrak{n}$ is a completely reducible* $\mathrm{ad}\mathfrak{a}$*-module with no trivial summand. Then $\mathfrak{n}$ acts trivially on any irreducible finite-dimensional $\mathfrak{g}$-module V.*

Now we shall describe finite-dimensional irreducible modules over $\mathfrak{g}_K$ defined in (5.1).

Theorem 5.1 *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a nontrivial finite-dimensional irreducible module over $\mathfrak{g}_K$. We have $\dim V_s \le 1$ for any $s \in \mathbb{Z}_2$.*

Proof From (5.1), we see that $\mathfrak{g}_K = \mathfrak{a} + \mathfrak{n}$ with $K \in \mathbb{N}$, where

$$\begin{aligned} \mathfrak{a} &= \mathrm{span}_{\mathbb{C}}\{\overline{L}_0, \overline{M}_0\}, \\ \mathfrak{n} &= \mathrm{span}_{\mathbb{C}}\Big\{\overline{L}'_n, \overline{M}_n, \overline{Q}_r \mid 1 \le n \le K,\ \frac{1}{2} \le r \le \frac{1}{2} + K\Big\}, \\ \overline{L}'_n &= \overline{L}_n + \overline{M}_n. \end{aligned}$$

Note that

$$\begin{aligned} &[\overline{L}_0, \overline{L}'_n] = (-n)\overline{L}'_n, \quad [\overline{L}_0, \overline{M}_n] = (-n)\overline{M}_n, \quad [\overline{L}_0, \overline{Q}_r] = (-r)\overline{Q}_r, \\ &[\overline{M}_0, \overline{L}'_n] = n\overline{M}_n, \qquad [\overline{M}_0, \overline{M}_n] = [\overline{M}_0, \overline{Q}_r] = 0, \end{aligned}$$

where $1 \le n \le K, \frac{1}{2} \le r \le \frac{1}{2} + K$. One checks that $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{g}_K$. Then $\mathfrak{n}$ is a completely reducible $\mathrm{ad}\mathfrak{a}$-module with no trivial summand. Thus $\mathfrak{n}$ acts trivially on V by Lemma 5.1. It follows that V can be viewed as a nontrivial finite-dimensional irreducible module over $\mathfrak{a} = \mathrm{span}_{\mathbb{C}}\{\overline{L}_0, \overline{M}_0\}$. Since $\mathfrak{a}$ is an Abelian Lie algebra, we have $\dim V_s \le 1$ for any $s \in \mathbb{Z}_2$.

5.2 Finite irreducible conformal modules over $\mathfrak{R}$

In this subsection, we shall classify all finite irreducible conformal modules over $\mathfrak{R}$ based on the results in Section 4.

Lemma 5.2 *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a nontrivial finite irreducible conformal module over $\mathfrak{R}$. We have $\dim V_s \le 1$ for any $s \in \mathbb{Z}_2$.*

Proof Proposition 5.1 shows that V can be viewed as a module over the extended annihilation algebra $\mathcal{A}(\mathfrak{R})^e$ satisfying

$$L_i v = M_i v = Q_{i+\frac{1}{2}} v = 0 \tag{5.2}$$

for any $v \in V$, $0 \ll i \in \mathbb{Z}_{\ge -1}$. Note that $\mathcal{A}(\mathfrak{R})^e = \mathcal{A}(\mathfrak{R}) \oplus \mathbb{C}T$, where

$$\mathcal{A}(\mathfrak{R}) = \mathrm{span}_{\mathbb{C}}\Big\{L_n, M_n, Q_r \mid n \in \mathbb{Z}_{\ge -1}, r \in \frac{1}{2} + \mathbb{Z}_{\ge -1}\Big\}.$$

For a fixed $\alpha \in \mathbb{Z}_+$, let $\mathcal{A}(\mathfrak{R})_\alpha = \mathrm{span}_{\mathbb{C}}\{L_{n-1}, M_{n-1}, Q_{n-\frac{1}{2}} \mid n \ge \alpha\}$. Then $\mathcal{A}(\mathfrak{R})_\alpha$ is a subalgebra of $\mathcal{A}(\mathfrak{R})$ and

$$\mathcal{A}(\mathfrak{R})^e \supset \mathcal{A}(\mathfrak{R})_0 \supset \cdots \supset \mathcal{A}(\mathfrak{R})_n \supset \cdots.$$

Furthermore, we see that $[T, \mathcal{A}(\mathfrak{R})_\alpha] = \mathcal{A}(\mathfrak{R})_{\alpha-1}$ for $\alpha \geq 1$. Now, let $V_i = \{v \in V \mid \mathcal{A}(\mathfrak{R})_i v = 0\}$ for $i \in \mathbb{Z}_+$. Then by (5.2), $V_i \neq 0$ for $i \gg 0$. Assume that β is the smallest integer satisfying $V_\beta \neq 0$.

Case 1 Suppose that $\beta = 0$ and let $0 \neq v \in M_0$. Then $U(\mathcal{A}(\mathfrak{R})^e)v = \mathbb{C}[T]U(\mathcal{A}(\mathfrak{R})_0)v = \mathbb{C}[T]v$. So, $V = \mathbb{C}[T]v$ by the irreducibility of M. Note that $\mathcal{A}(\mathfrak{R})_0$ is an ideal of $\mathcal{A}(\mathfrak{R})^e$. Then $\mathcal{A}(\mathfrak{R})_0$ acts trivially on V. Thus V is an irreducible $\mathbb{C}[T]$-module. It follows that V is one-dimensional since T is even. Equivalently, V is a one-dimensional trivial conformal $\mathfrak{R}$-module, a contradiction.

Case 2 Assume that $\beta \geq 1$. Then by the equalities (2.8), we have that $T - L_{-1}$ is an even central element, so $T - L_{-1}$ acts on V as a scalar and $\mathcal{A}(\mathfrak{R})_0$ acts irreducibly on V. Note that

$$L_{-1} = [L_0, L_{-1}], \quad M_{-1} = [M_0, L_{-1}], \quad Q_{-\frac{1}{2}} = [Q_{\frac{1}{2}}, L_{-1}].$$

Then the action of $\mathcal{A}(\mathfrak{R})_0$ is determined by L_{-1} and $\mathcal{A}(\mathfrak{R})_1$, that is, is determined by T and $\mathcal{A}(\mathfrak{R})_1$. Note that V_β is an $\mathcal{A}(\mathfrak{R})_1$-module. Thus by the irreducibility of V and [6, Lemma 3.1], we have $V = \mathbb{C}[T]V_\beta = \mathbb{C}[T] \otimes_{\mathbb{C}} V_\beta$ and V_β is a nontrivial irreducible finite-dimensional $\mathcal{A}(\mathfrak{R})_1$-module. If $\beta = 1$, then by the definition of V_1, we have $\mathcal{A}(\mathfrak{R})_1 v = 0$ for any $v \in V_1$. Thus V_1 is a trivial $\mathcal{A}(\mathfrak{R})_1$-module, a contradiction. Now, suppose that $\beta > 1$. Note that $\mathcal{A}(\mathfrak{R})_\beta$ is an ideal of $\mathcal{A}(\mathfrak{R})_1$. Thus V_β is an $\mathcal{A}(\mathfrak{R})_1/\mathcal{A}(\mathfrak{R})_\beta$-module. Since $\mathcal{A}(\mathfrak{R})_1/\mathcal{A}(\mathfrak{R})_\beta \cong \mathfrak{g}_{\beta-1}$ defined as in (5.1), we have $\dim V_s \leq 1$ for any $s \in \mathbb{Z}_2$ by Theorem 5.1.

The main theorem of this subsection is as follows.

Theorem 5.2 *Let V be a finite irreducible conformal module over $\mathfrak{R}$. Then V is simply a finite irreducible conformal module over* Vir.

Proof Assume that $V = V_{\bar{0}} \oplus V_{\bar{1}}$. It follows from Lemma 5.2 that $\dim V_s \leq 1$ for any $s \in \mathbb{Z}_2$. If $\dim V_{\bar{0}} = \dim V_{\bar{1}} = 1$, then from Theorem 4.1, V is isomorphic to one of the $\mathfrak{R}$-modules defined in (4.1)–(4.5), which are all reducible. Thus $\mathrm{rank}(V) = (1 + 0)$ or $\mathrm{rank}(V) = (0 + 1)$. These imply our results.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Bagnoli, L. and Caselli, F., Classification of finite irreducible conformal modules for K'_4, *J. Math. Phys.*, **63**, 2022, 091701, 41 pp.

[2] Barnich, G., Donnay, L., Matulich, J. and Troncoso, R., Asymptotic symmetries and dynamics of three-dimensional flat supergravity, *J. High Energy Phys.*, **2014**, 2014, Paper No. 71, 18 pp.

[3] Boyallian, C., Kac, V. G. and Liberati, J. I., Irreducible modules over finite simple Lie conformal superalgebras of type K, *J. Math. Phys.*, **51**, 2010, 063507, 37 pp.

[4] Boyallian, C., Kac, V. G. and Liberati, J. I., Classification of finite irreducible modules over the Lie conformal superalgebra CK_6, *Comm. Math. Phys.*, **317**, 2013, 503–546.

[5] Boyallian, C., Kac, V. G., Liberati, J. I. and Rudakov, A., Representations of simple finite Lie conformal superalgebras of type W and S, *J. Math. Phys.*, **47**, 2006, 043513, 25 pp.

[6] Cheng, S. J. and Kac, V. G., Conformal modules, *Asian J. Math.*, **1**, 1997, 181–193.

[7] Cheng, S. J. and Kac, V. G., Erratum: Conformal modules, *Asian J. Math.*, **2**, 1998, 153–156.

[8] Cheng, S. J. and Lam, N., Finite conformal modules over $N = 2, 3, 4$ superconformal algebras, *J. Math. Phys.*, **42**, 2001, 906–933.

[9] D'Andrea, A. and Kac, V. G., Structure theory of finite conformal algebras, *Selecta Math. New Ser.*, **4**, 1998, 377–418.

[10] Dilxat, M., Chen, L. Y. and Liu, D., Classification of simple Harish-Chandra modules over the Ovsienko-Roger superalgebra, *Proc. Roy. Soc. Edinburgh A*, **154**(2), 2024, 483–493.

[11] Dilxat, M., Gao, S. L. and Liu, D., Super-biderivations and post-Lie superalgebras on some Lie superalgebras, *Acta Math. Sinica, English Series*, **39**(9), 2023, 1736–1754.

[12] Dilxat, M., Gao, S. L. and Liu, D., Whittaker modules for the $N = 1$ super-BMS$_3$ algebra, *J. Alg. Appl.*, **23**(5), 2024, 2450088, 16 pp.

[13] Fattori, D., Kac, V. G. and Retakh, A., Structure theory of finite Lie conformal superalgebras, Lie Theory and Its Applications in Physics V, World Sci. Publ., River Edge, NJ, 2004, 27–63.

[14] Han, X., Wang, D. Y. and Xia, C. G., Representations of Lie conformal algebras related to Galilean conformal algebras, *Comm. Algebra*, **50**, 2022, 2427–2438.

[15] Kac, V. G., Vertex Algebras for Beginners, 2nd ed., American Mathematical Society, Providence, RI, 1998.

[16] Kac, V. G., Formal distribution algebras and conformal algebras, XIIth International Congress of Mathematical Physics (ICMP 97), Int. Press, Cambridge, MA, 1999, 80–97.

[17] Kolesnikov, P., The Ado theorem for finite Lie conformal algebras with Levi decomposition, *J. Algebra Appl.*, **15**(7), 2016, 1650130, 13 pp.

[18] Lan, C., Liu, D. and Wu, Q. Y., Local and 2-Local Derivations on the $N = 1$ BMS Superalgebra, *Bull. Iranian Math. Soc.*, **51**, 2025, Paper No. 62, 20 pp.

[19] Liu, D., Pei, Y. F. and Zhao, K. M., Smooth modules over the $N = 1$ Bondi-Metzner-Sachs superalgebra, *Commun. Contemp. Math.*, **27**(4), 2025, 2450021, 30 pp.

[20] Martínez, C. and Zelmanov, E., Irreducible representations of the exceptional Cheng-Kac superalgebra, *Trans. Am. Math. Soc.*, **366**, 2014, 5853–5876.

[21] Wang, W., Xu, Y. and Xia, C. G., A class of Schrödinger-Virasoro type Lie conformal algebras, *Int. J. Math.*, **26**, 2015, 1550058, 12 pp.

[22] Wu, Y., Wang, H. D. and Xia, C. G., Representations of super deformation of Heisenberg-Virasoro type Lie conformal algebras, *Comm. Algebra*, **51**, 2023, 1944–1954.

[23] Wu, H. N. and Yuan, L. M., Classification of finite irreducible conformal modules over some Lie conformal algebras related to the Virasoro conformal algebra, *J. Math. Phys.*, **58**, 2017, 041701, 10 pp.

[24] Xia, C. G., Representations of twisted infinite Lie conformal superalgebras, *J. Algebra*, **596**, 2022, 155–176.

[25] Xu, Y. and Yue, X. Q., $W(a, b)$ Lie conformal algebra and its conformal module of rank one, *Algebra Colloq.*, **22**(3), 2015, 405–412.

[26] Zhang, W. and Dong, C. Y., W-algebra $W(2, 2)$ and the vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$, *Comm. Math. Phys.*, **285**, 2009, 991–1004.