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Some Soliton Structures on the Cotangent Bundle with Respect to the Modified Riemannian Extension

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Abstract Let (M, J, g) be an anti-Kähler manifold of dimension $n = 2k$ with an almost complex structure J and a pseudo-Riemannian metric g and let T^*M be its cotangent bundle with modified Riemannian extension metric $\widetilde{g}_{\nabla,G}$. The modified Riemannian extension metric $\widetilde{g}_{\nabla,G}$ is obtained by deformation in the horizontal part of the Riemannian extension known in the literature by means of the twin Norden metric G. The paper aims first to examine the curvature properties of the cotangent bundle T^*M with modified Riemannian extension metric $\widetilde{g}_{\nabla,G}$ and second to study some geometric solitons on the cotangent bundle T^*M according to the modified Riemannian extension metric $\widetilde{g}_{\nabla,G}$.

Keywords Cotangent bundle, Modified Riemannian extension, Soliton structure
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1 Introduction

Let M be an n-dimensional Riemannian manifold and let T^*M be its cotangent bundle with a torsion-free affine connection ∇. Patterson and Walker [16] introduced the notion of Riemannian extensions and they showed how to construct a pseudo-Riemannian metric $\widetilde{g}_\nabla$ on the $2n$-dimensional cotangent bundle of any n-dimensional differentiable manifold with a torsion-free connection ∇. Afifi [1] studied the local properties of the Riemannian extension of connected affine spaces. Riemannian extensions were also studied by Garcia-Rio et al. [8]. They established the following very nice characterization: $(T^*M, \widetilde{g}_\nabla)$ satisfies the Osserman condition if and only if (M, ∇) is an affine Osserman space. Since Riemannian extensions provide a link between affine and pseudo-Riemannian geometries, some properties of the affine connection ∇ can be investigated by means of corresponding properties of the Riemannian extension $\widetilde{g}_\nabla$. The different properties of the Riemannian extension $\widetilde{g}_\nabla$ were studied by some authors (see [2–3, 7, 14–15, 19–22]). Later, in [4–5], the authors considered the deformations of the Riemannian extension $\widetilde{g}_\nabla$ on the cotangent bundle T^*M; the first one is given by $\widetilde{g}_{\nabla,c} = \widetilde{g}_\nabla + \pi^*c$ for a symmetric $(0, 2)$-tensor field c; the second one is defined by $\widetilde{g}_{\nabla,c,T,S} = \widetilde{g}_\nabla + \pi^*c + \imath T \circ \imath S$ for a symmetric $(0, 2)$ -tensor field c and $(1, 1)$-tensor fields T, S. The authors referred to the deformation metrics as modified Riemannian extensions. In [9], the authors studied some

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properties of the modified Riemannian extension $\widetilde{g}_{\nabla,c} = \widetilde{g}_{\nabla} + \pi^* c$.

Let J be an almost complex structure on a smooth manifold, i.e., $J^2 = -\mathrm{id}$. Then the pair (M, J) is called a $2n$-dimensional almost complex manifold. If the pseudo-Riemannian metric g of neutral signature (n, n) satisfies

$$g(JX, JY) = g(X, Y)$$

or equivalent to this equation

$$g(JX, Y) = g(X, JY) \quad \text{(purity condition)}$$

for any vector fields X, Y on M, then this metric is referred to as a Norden (also known as anti-Hermitian) metric (see [3]). The $2n$-dimensional manifold M equipped with an almost complex structure and an anti-Hermitian metric g is referred to as an almost anti-Hermitian manifold or an almost Norden manifold. It is also defined as an anti-Kähler or a Kähler-Norden if the almost complex structure J is parallel with regard to the Levi-Civita connection ∇ ($\nabla J = 0$). It is well known that the condition $\nabla J = 0$ is equivalent to the holomorphicity of the Norden metric g, that is, $\Phi_J g = 0$, where $(\Phi_J g)(X, Y, Z) = (L_{JX} g - L_X G)(Y, Z)$. $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$ is the twin Norden metric (see [17]). It is a remarkable fact that (M, J, g) is Kähler-Norden if and only if (M, J, G) is Kähler-Norden. This is of special significance for Kähler-Norden metrics since in such case g and G share the same Levi-Civita connection ($\nabla g = \nabla G = 0$) (see [18]).

Salimov and Cakan considered, on the cotangent bundle T^*M, a special type of the modified Riemannian extension given in [18]. They constructed this metric on the cotangent bundle over a Kähler-Norden manifold (M, J, g) using the twin Norden metric G and studied some properties. This metric is of the form: $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + \pi^* G$, where ∇ is the Levi-Civita connection of g. In this paper, we shall consider the special type of the modified Riemannian extension: $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + \pi^* G$. In Section 1, we introduce essential notions, definitions, and preliminary results concerning the deformed Riemannian extension, which serve as a foundation for the subsequent analysis. Section 2 is devoted to the study of various soliton structures, including Ricci solitons, generalized Ricci-Yamabe solitons, generalized gradient Ricci-Yamabe solitons, as well as Riemannian and gradient Riemannian solitons.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im^p_q(M)$ the set of all tensor fields of type (p, q) on M, and by $\Im^p_q(T^*M)$ the corresponding set on the cotangent bundle T^*M. The Einstein summation convention is used, the range of the indices i, j, s being always $\{1, 2, \cdots, n\}$.

1.1 The cotangent bundle

Let M be an n-dimensional differentiable manifold with a torsion-free affine connection ∇, T^*M be its cotangent bundle and π be the natural projection $T^*M \to M$. A system of local coordinates (U, x^i), $i = 1, \cdots, n$ in M induces on T^*M a system of local coordinates

$(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} = n+i = n+1, \cdots, 2n$, where $x^{\bar{i}} = p_i$ are components of covectors p in each cotangent space $T^*_x M$, $x \in U$ with respect to the natural coframe $\{\mathrm{d}x^i\}$.

Let $X = X^i \partial_i$ and $\omega = \omega_i \mathrm{d}x^i$ be the local expressions in U of a vector field X and covector (1-form) field ω on M, respectively. Then, the vertical lift ${}^V\omega$ of ω, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of X are given, with respect to the induced coordinates, by

$$ {}^V\omega = \omega_i \partial_{\bar{i}}, \tag{1.1} $$

$$ {}^HX = X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_{\bar{i}} \tag{1.2} $$

and

$$ {}^CX = X^i \partial_i - p_h \partial_i X^h \partial_{\bar{i}}, $$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$ and Γ^h_{ij} are the coefficients of a symmetric (torsion-free) affine connection ∇ in M.

The Lie bracket operation of vertical and horizontal vector fields on T^*M are given by the formulas

$$ \begin{aligned} [{}^HX, {}^HY] &= {}^H[X,Y] + {}^V(p \circ R(X,Y)), \\ [{}^HX, {}^V\omega] &= {}^V(\nabla_X \omega), \\ [{}^V\theta, {}^V\omega] &= 0 \end{aligned} \tag{1.3} $$

for any $X, Y \in \Im^1_0(M)$ and $\theta, \omega \in \Im^0_1(M)$, where R is the Riemannian curvature tensor of the symmetric affine connection ∇ defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (for details, see [23]).

1.2 Expressions in the adapted frame

We will use the adapted frame that simplifies our tensor calculations in the cotangent bundle T^*M. With the symmetric affine connection ∇ in M, we can introduce the adapted frame on each induced coordinate neighbourhood $\pi^{-1}(U)$ of T^*M. In each local chart $U \subset M$, if we write $X_{(j)} = \frac{\partial}{\partial x^j}$, $\theta^{(j)} = \mathrm{d}x^j$, $j = 1, \cdots, n$, then from (1.1)–(1.2), we can see that these vector fields have, respectively, the local expressions

$$ \begin{aligned} {}^HX_{(j)} &= \partial_j + p_a \Gamma^a_{hj} \partial_{\bar{h}}, \\ {}^V\theta^{(j)} &= \partial_{\bar{j}} \end{aligned} $$

with respect to the natural frame $\{\partial_j, \partial_{\bar{j}}\}$. These $2n$-vector fields are linearly independent and they generate the horizontal distribution of ∇ and the vertical distribution of T^*M, respectively. The set $\{{}^HX_{(j)}, {}^V\theta^{(j)}\}$ is called the frame adapted to ∇ in $\pi^{-1}(U) \subset T^*M$. By putting

$$ \begin{aligned} E_j &= {}^HX_{(j)}, \\ E_{\bar{j}} &= {}^V\theta^{(j)}, \end{aligned} \tag{1.4} $$

we can write the adapted frame as $\{E_\alpha\} = \{E_j, E_{\bar{j}}\}$. Using (1.1)–(1.2) and (1.4), we have

$$ {}^V\omega = \omega_j E_{\bar{j}} \tag{1.5} $$

and

$$^{H}X = X^{j} E_{j} \tag{1.6}$$

with respect to the adapted frame $\{E_\alpha\}$ (for details, see [23]). By straightforward calculations, we have the lemma below.

Lemma 1.1 *The Lie brackets of the adapted frame of T^*M satisfy the following identities*

$$\begin{aligned}
&[E_i, E_j] = p_s R^s_{ijl} E_{\bar{l}},\\
&[E_i, E_{\bar{j}}] = -\Gamma^j_{il} E_{\bar{l}},\\
&[E_{\bar{i}}, E_{\bar{j}}] = 0,
\end{aligned}$$

where R^s_{ijl} denotes the components of the curvature tensor of the symmetric affine connection ∇ in M (see [23]).

1.3 The modified Riemannian extension on the cotangent bundle

Let (M, J, g) be an almost anti-Hermitian manifold of dimension $n = 2k$. $G(X, Y) = g(JX, Y)$ is the twin Norden metric, which is locally expressed as $G_{ij} = g_{ih} J^h_j$. The modified Riemannian extension, denoted by $\widetilde{g}_{\nabla,G}$, is a pseudo-Riemannian metric of neutral signature (n, n) on the cotangent bundle, given by $\widetilde{g}_{\nabla,G} = \widetilde{g}_\nabla + {}^V G^*$. The pseudo-Riemannian metric $\widetilde{g}_{\nabla,G}$ has the components

$$((\widetilde{g}_{\nabla,G})_{IJ}) = \begin{pmatrix} G_{ij} - 2p_s\Gamma^s_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}$$

with respect to the induced coordinates (x^i, p_i), where Γ^s_{ij} are the coefficients of the Levi-Civita connection ∇ (see [5]). Here, ${}^V G^*$ has the form

$$({}^V G^*) = \begin{pmatrix} G_{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

The modified Riemannian extension and its inverse have the following components with respect to the adapted frame $\{E_\alpha\}$, respectively,

$$((\widetilde{g}_{\nabla,G})_{IJ}) = \begin{pmatrix} G_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}$$

and

$$((\widetilde{g}_{\nabla,G})^{IJ}) = \begin{pmatrix} 0 & \delta^j_i \\ \delta^i_j & -G_{ij} \end{pmatrix}. \tag{1.7}$$

We will now compute the Levi-Civita connection $\widetilde{\widetilde{\nabla}}$ of the modified Riemannian extension $\widetilde{g}_{\nabla,G}$. The coefficients of the Levi-Civita connection $\widetilde{\widetilde{\nabla}}$ can be determined by

$$\widetilde{\Gamma}^\alpha_{\gamma\beta} = \frac{1}{2}\widetilde{g}^{\alpha\varepsilon}(\eta_\gamma \widetilde{g}_{\varepsilon\beta} + \eta_\beta \widetilde{g}_{\gamma\varepsilon} - \eta_\varepsilon \widetilde{g}_{\gamma\beta}) + \frac{1}{2}(\Omega^\alpha_{\gamma\beta} + \Omega^\alpha_{\gamma\beta} + \Omega^\alpha_{\beta\gamma}),$$

where

$$\Omega^{\alpha}_{\gamma\beta} = \widetilde{g}^{\alpha\varepsilon}\widetilde{g}_{\delta\beta}\Omega^{\delta}_{\varepsilon\gamma},$$
$$\Omega^{\overline{h}}_{ji} = p_s R^s_{jih},$$
$$\Omega^{\overline{h}}_{j\overline{i}} = -\Omega^{\overline{h}}_{\overline{i}j} = -\Gamma^i_{jh}$$

and it will be used as $\gamma = j, \overline{j}$, $\beta = i, \overline{i}$, $\alpha = h, \overline{h}$, $\varepsilon = k, \overline{k}$ and $\delta = m, \overline{m}$.

For the Levi-Civita connection $\widetilde{\nabla}$ of the modified Riemannian extension $\widetilde{g}_{\nabla,G}$, we give the following proposition.

Proposition 1.1 (see [18]) *Let (M, J, g) be an anti-Kähler manifold and $(T^*M, \widetilde{g}_{\nabla,G})$ be its cotangent bundle with the modified Riemannian extension. The Levi-Civita connection $\widetilde{\nabla}$ of the modified Riemannian extension $\widetilde{g}_{\nabla,G}$ on T^*M is locally given by*

$$\widetilde{\nabla}_{E_i}E_j = \Gamma^k_{ij}E_k + p_s R^s_{kji}E_{\overline{k}},$$
$$\widetilde{\nabla}_{E_i}E_{\overline{j}} = -\Gamma^j_{ik}E_{\overline{k}},$$
$$\widetilde{\nabla}_{E_{\overline{i}}}E_j = 0,$$
$$\widetilde{\nabla}_{E_{\overline{i}}}E_{\overline{j}} = 0,$$

where R is the Riemannian curvature tensor of g.

The Riemannian curvature tensor is obtained by

$$\widetilde{R}^{\alpha}_{\delta\gamma\beta} = E_\delta\widetilde{\Gamma}^{\alpha}_{\gamma\beta} - E_\gamma\widetilde{\Gamma}^{\alpha}_{\delta\beta} + \widetilde{\Gamma}^{\alpha}_{\delta\varepsilon}\widetilde{\Gamma}^{\varepsilon}_{\gamma\beta} - \widetilde{\Gamma}^{\alpha}_{\gamma\varepsilon}\widetilde{\Gamma}^{\varepsilon}_{\delta\beta} - \Omega^{\varepsilon}_{\delta\gamma}\widetilde{\Gamma}^{\alpha}_{\varepsilon\beta}.$$

Here, we use the following notation: $\gamma = j, \overline{j}$, $\beta = i, \overline{i}$, $\alpha = h, \overline{h}$, $\varepsilon = k, \overline{k}$ and $\delta = m, \overline{m}$. In the subsequent proposition, we present the components of the Riemannian curvature tensor associated with the modified Riemannian extension $\widetilde{g}_{\nabla,G}$.

Proposition 1.2 *Let (M, J, g) be an anti-Kähler manifold and $(T^*M, \widetilde{g}_{\nabla,G})$ be its cotangent bundle with the modified Riemannian extension. Then, the corresponding Riemannian curvature tensor $\widetilde{R}$ is locally given by*

$$\widetilde{R}^h_{ijk} = R^h_{ijk},$$
$$\widetilde{R}^{\overline{h}}_{ijk} = p_s(\nabla_i R^s_{hkj} - \nabla_j R^s_{hki}) - \frac{1}{2}(R^m_{ijk}G_{mh} + R^m_{ijh}G_{km}),$$
$$\widetilde{R}^{\overline{h}}_{ij\overline{k}} = R^k_{jih},$$
$$\widetilde{R}^{\overline{h}}_{i\overline{j}k} = R^j_{hki},$$
$$\widetilde{R}^{\overline{h}}_{\overline{i}jk} = R^i_{hkj}$$

and other components are zero (*see also* [9]).

We now examine the Ricci and scalar curvature tensors. Utilizing Proposition 1.2 and performing the standard calculations, we obtain the following result.

Proposition 1.3 *Let (M, J, g) be an anti-Kähler manifold and $(T^*M, \widetilde{g}_{\nabla,G})$ be its cotangent bundle with the deformed Riemannian extension. Then, the corresponding Ricci curvature tensor is given by*

$$\begin{aligned} \widetilde{R}_{jk} &= R_{jk} + R_{kj} = 2R_{jk}, \\ \widetilde{R}_{\overline{j}k} &= \widetilde{R}_{j\overline{k}} = \widetilde{R}_{\overline{j}\overline{k}} = 0. \end{aligned} \tag{1.8}$$

Using (1.7)–(1.8), we calculate

$$\begin{aligned} \widetilde{r} &= (\widetilde{g}_{\nabla,G})^{\alpha\beta}\widetilde{R}_{\alpha\beta} = (\widetilde{g}_{\nabla,G})^{jk}\widetilde{R}_{jk} + (\widetilde{g}_{\nabla,G})^{\overline{j}k}\widetilde{R}_{\overline{j}k} \\ &\quad + (\widetilde{g}_{\nabla,G})^{j\overline{k}}\widetilde{R}_{j\overline{k}} + (\widetilde{g}_{\nabla,G})^{\overline{j}\overline{k}}\widetilde{R}_{\overline{j}\overline{k}} \\ &= 0. \end{aligned}$$

Hence, we have the following result.

Proposition 1.4 *Let (M, J, g) be an anti-Kähler manifold and $(T^*M, \widetilde{g}_{\nabla,G})$ be its cotangent bundle with the modified Riemannian extension. Then, the corresponding scalar curvature $\widetilde{r}$ is zero.*

2 Soliton Structures on the Cotangent Bundle with the Modified Riemannian Extension

Before examining the soliton structures, we first present the Lie derivative of the modified Riemannian extension $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^V G^*$, as well as the Hessian operator of the smooth function ${}^V f$ on T^*M with respect to $\widetilde{g}_{\nabla,G}$. These concepts will be utilized in the subsequent study of soliton structures. Let $L_{\widetilde{X}}$ denote the Lie derivative with respect to a vector field $\widetilde{X}$. A vector field $\widetilde{X}$, expressed in components as $(v^h, v^{\overline{h}})$, is characterized as fibre-preserving if and only if the functions v^h depend exclusively on the base coordinates (x^h).

Proposition 2.1 *Let $\widetilde{X}$ be a fibre-preserving vector field of T^*M with components $(v^h, v^{\overline{h}})$. Then, the Lie derivatives of the adapted frame and dual basis are given as follows*

$$\begin{aligned} &\text{(i)} \quad L_{\widetilde{X}} E_i = -(E_i v^k)E_k - (v^a p_s R^s_{iak} + E_i v^{\overline{k}} - v^{\overline{a}}\Gamma^a_{ik})E_{\overline{k}}, \\ &\text{(ii)} \quad L_{\widetilde{X}} E_{\overline{i}} = -(v^a \Gamma^i_{ak} + E_{\overline{i}} v^{\overline{k}})E_{\overline{k}}, \\ &\text{(iii)} \quad L_{\widetilde{X}} dx^h = (E_m v^h)\mathrm{d}x^m, \\ &\text{(iv)} \quad L_{\widetilde{X}} \delta p_h = [v^a p_s R^s_{mah} - v^{\overline{a}}\Gamma^a_{mh} + (E_m v^{\overline{k}})\delta^k_h]\mathrm{d}x^m \\ &\qquad\qquad + [v^a \Gamma^m_{ah} + (E_{\overline{m}} v^{\overline{k}})\delta^k_h]\delta p_m, \end{aligned}$$

where $\delta p_h = \mathrm{d}p_h - p_s \Gamma^s_{ih}\mathrm{d}x^i$.

Now, let us give the following lemma which are needed later on.

Proposition 2.2 *The Lie derivative of $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^V G^*$ with respect to the fibre-preserving vector field $\widetilde{X}$ is given as follows*

$$L_{\widetilde{X}}(\widetilde{g}_{\nabla,G}) = [L_X G_{ij} + 2v^a p_s R^s_{iaj} - 2v^{\overline{a}}\Gamma^a_{ij} + 2(E_i v^{\overline{j}})]\mathrm{d}x^i \mathrm{d}x^j$$

$$+\,2[\nabla_i v^j + E_{\bar{j}} v^{\bar{i}}]\mathrm{d}x^i \delta p_j,$$

where $L_{\widetilde{X}}(\widetilde{g}_{\nabla,G})$ *and* $L_X G$ *denote the components of the Lie derivative* $\widetilde{g}_{\nabla,G} = \widetilde{g}_\nabla + {}^V G^*$ *and the twin Norden metric* G*, respectively.*

For any smooth function f on M, the vertical lift ${}^V f$ of f to T^*M is defined by ${}^V f = f \circ \pi$. To introduce our main notion, we denote as usually the Hessian operator of the vertical lift ${}^V f$ of any smooth function f on M with respect to the modified Riemannian extension $\widetilde{g}_{\nabla,G}$, by

$$(\mathrm{Hess}_{\widetilde{g}_{\nabla,G}}\ {}^V f)(X,Y) = XY\,{}^V f - (\widetilde{\nabla}_X Y)\,{}^V f, \quad \forall X, Y \in \Im^1_0(T^*M)$$

or in local coordinates the above expression can be expressed as

$$(\widetilde{\nabla}^2\ {}^V f)_{\beta\gamma} = \partial_\beta \partial_\gamma\ {}^V f - \widetilde{\Gamma}^{\varepsilon}_{\beta\gamma} \partial_\varepsilon\ {}^V f$$

for $\gamma = j, \bar{j}$ and $\beta = i, \bar{i}$.

Proposition 2.3 *Let* f *be a smooth function on an anti-Kähler manifold* (M, J, g)*. Then, the Hessian (with respect to the modified Riemannian extension* $\widetilde{g}_{\nabla,G}$*) of its vertical lift is expressed by*

$$\begin{aligned}
&\text{(i)} && (\widetilde{\nabla}^2 f)_{ij} = \partial_i \partial_j f - \widetilde{\Gamma}^k_{ij} \partial_k f = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f = (\nabla^2 f)_{ij},\\
&\text{(ii)} && (\widetilde{\nabla}^2 f)_{i\bar{j}} = \partial_i \partial_{\bar{j}} f - \widetilde{\Gamma}^k_{i\bar{j}} \partial_k f = 0,\\
&\text{(iii)} && (\widetilde{\nabla}^2 f)_{\bar{i}j} = \partial_{\bar{i}} \partial_j f - \widetilde{\Gamma}^k_{\bar{i}j} \partial_k f = 0,\\
&\text{(iv)} && (\widetilde{\nabla}^2 f)_{\bar{i}\bar{j}} = \partial_{\bar{i}} \partial_{\bar{j}} f - \widetilde{\Gamma}^k_{\bar{i}\bar{j}} \partial_k f = 0.
\end{aligned}$$

2.1 Ricci solitons with respect to the modified Riemannian extension

The notion of Ricci flow was introduced by Hamilton [11] and is given by the evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(t),$$

where Ric denotes the Ricci tensor associated with the metric $g(t)$. A Ricci soliton is defined as a self-similar solution to this flow. Ricci solitons play a fundamental role in geometric analysis, particularly due to their crucial contribution to the resolution of the Poincaré conjecture. Let M be a smooth manifold of dimension $n \geq 2$. In the sense of Hamilton's definition, a Ricci soliton on M is a triple (g, X, λ), where g is a pseudo-Riemannian metric on M, Ric is the corresponding Ricci tensor, X is a vector field on M, and $\lambda \in \mathbb{R}$ is a constant, such that the following equation is satisfied

$$\mathrm{Ric} + \frac{1}{2} L_X g = \lambda g, \tag{2.1}$$

where L_X is the Lie derivative along X. The Ricci soliton is said to be either shrinking, steady, or expanding, according as λ is negative, zero, or positive, respectively.

Theorem 2.1 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$ and $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^{V}G^{*}$ be the modified Riemannian extension metric on the cotangent bundle $T^{*}M$. The quadruple $(T^{*}M, \widetilde{g}_{\nabla,G}, \widetilde{X}, \lambda)$ is a Ricci soliton if and only if the following conditions are satisfied*

$$\begin{aligned}
&\text{(i)} \quad \widetilde{X} = (v^{a}, v^{\overline{a}}) = (v^{a}, p_{s}A_{i}^{s} + B_{i}),\\
&\text{(ii)} \quad \lambda = \frac{1}{n}(E_{i}v^{i} + v^{a}\Gamma_{ai}^{i} + A_{i}^{i}),\\
&\text{(iii)} \quad \lambda G_{ij} = 2R_{ij} + \frac{1}{2}L_{V}G_{ij} + \nabla_{i}B_{j},\\
&\text{(iv)} \quad v^{a}R_{iaj}^{s} + \nabla_{i}A_{j}^{s} = 0,
\end{aligned}$$

where $\widetilde{X} = v^{a}E_{a} + v^{\overline{a}}E_{\overline{a}}$ is a fibre-preserving vector field on $T^{*}M$, $\lambda \in \mathbb{R}$, $B = (B_{i})$ and $A = (A_{s}^{h})$ are $(0,1)$ and $(1,1)$ tensor fields on M, respectively.

Proof We will show that the existence of the scalar λ. If the expression of $L_{\widetilde{X}}(\widetilde{g}_{\nabla,G})$ in Proposition 2.2 is used in (2.1), we have

$$2R_{ij} + \frac{1}{2}[L_{V}G_{ij} + 2v^{a}p_{s}R_{iaj}^{s} - 2v^{\overline{a}}\Gamma_{ij}^{a} + 2(E_{i}v^{\overline{j}})] = \lambda G_{ij} \tag{2.2}$$

and

$$E_{i}v^{j} + v^{a}\Gamma_{ai}^{j} + E_{\overline{j}}v^{\overline{i}} = \lambda\delta_{i}^{j}. \tag{2.3}$$

Applying $E_{\overline{k}}$ to both sides of (2.3), we obtain

$$E_{\overline{k}}E_{\overline{j}}v^{\overline{i}} = 0, \quad v^{\overline{i}} = p_{s}A_{i}^{s} + B_{i}. \tag{2.4}$$

Substituting (2.4) into (2.3), we get

$$E_{i}v^{j} + v^{a}\Gamma_{ai}^{j} + A_{i}^{i} = \lambda\delta_{i}^{j}$$

and contracting with the last equation by δ_{j}^{i}, we have

$$\lambda = \frac{1}{n}(E_{i}v^{i} + v^{a}\Gamma_{ai}^{i} + A_{i}^{i}).$$

Substituting (2.4) into (2.2), we find

$$\lambda G_{ij} = 2R_{ij} + \frac{1}{2}L_{V}G_{ij} + \nabla_{i}B^{j} \tag{2.5}$$

and

$$v^{a}R_{iaj}^{s} + \nabla_{i}A_{j}^{s} = 0.$$

Conversely, by a routine calculation, the sufficiency of the theorem can be easily checked under conditions (i)–(iv).

As concerning Theorem 2.1, we have the following conclusion.

Corollary 2.1 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$, $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^{V}G^{*}$ be the modified Riemannian extension metric on the cotangent bundle $T^{*}M$ and the quadruple $(T^{*}M, \widetilde{g}_{\nabla,G}, \widetilde{X}, \lambda)$ be a Ricci soliton. Then, the quadruple (M, G, V, λ) is a Ricci soliton if and only if*

$$R_{ij} + \nabla_i B_j = 0,$$

where $B = B_j \in \Im_1^0(M)$.

Proof Let the triplets $(\widetilde{g}_{\nabla,G}, \widetilde{X}, \lambda)$ and (G, V, λ) be Ricci soliton structures. From (2.5), we get

$$R_{ij} + \nabla_i B_j = 0.$$

Conversely, if $R_{ij} + \nabla_i B_j = 0$, from (2.5) we have

$$R_{ij} + \frac{1}{2} L_V G_{ij} = \lambda G_{ij},$$

which means that (M, G, V, λ) is a Ricci soliton. So the proof is completed.

Let $\widetilde{X}$ be a vector field on $T^{*}M$ with components $(v^a, v^{\overline{a}})$ with respect to the adapted frame. Then $\widetilde{X}$ is a vertical vector field on $T^{*}M$ if and only if $v^a = 0$. In the case, the vector field $\widetilde{X}$ in Theorem 2.1 reduces $\widetilde{X} = (v^a, v^{\overline{a}}) = (0, p_s A_i^s + B_i)$. Hence, we obtain the following conclusion.

Corollary 2.2 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$ and $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^{V}G^{*}$ be the modified Riemannian extension metric on the cotangent bundle $T^{*}M$. The quadruple $(T^{*}M, \widetilde{g}_{\nabla,G}, \widetilde{X}, \lambda)$ is a Ricci soliton if and only if the following conditions are satisfied*

$$\begin{aligned}
&\text{(i)} && \widetilde{X} = (v^a, v^{\overline{a}}) = (0, p_s A_i^s + B_i),\\
&\text{(ii)} && \lambda = \frac{1}{n} A_i^i,\\
&\text{(iii)} && \lambda G_{ij} = 2R_{ij} + \frac{1}{2} L_V G_{ij} + \nabla_i B_j,\\
&\text{(iv)} && \nabla_i A_j^s = 0,
\end{aligned}$$

where $\widetilde{X} = v^a E_a + v^{\overline{a}} E_{\overline{a}}$ is a vertical vector field on $T^{*}M$, $\lambda \in \mathbb{R}$, $B = (B_i)$ and $A = (A_s^h)$ are $(0, 1)$ and $(1, 1)$ tensor fields on M, respectively.

2.2 Generalized Ricci-Yamabe and generalized gradient Ricci-Yamabe solitons with respect to the modified Riemannian extension

A Yamabe soliton constitutes a special solution to the Yamabe flow, a geometric evolution equation introduced by Hamilton [11]. The Yamabe flow is described by the following equation

$$\frac{\partial}{\partial t} g(t) = -S(t) g(t),$$

where $g(t)$ is a one-parameter family of metrics on a (pseudo-)Riemannian manifold and $S(t)$ its scalar curvature with respect to $g(t)$.

Let (M, g) be a complete (pseudo-)Riemannian manifold. The (pseudo-)Riemannian metric g is said to admit a Yamabe soliton structure if it satisfies the equation

$$\frac{1}{2}L_V g = (r - \lambda)g,$$

where $\lambda \in \mathbb{R}$, r denotes the scalar curvature of M, and V is a smooth vector field on M, referred to as the soliton vector field. When the soliton vector field V is the gradient of a smooth function $f\colon M \to \mathbb{R}$, the soliton is called a gradient Yamabe soliton, in which case the defining equation reduces to

$$\nabla^2 f = (r - \lambda)g,$$

where $\nabla^2 f$ denotes the Hessian of f. In 2018, Chen and Deshmukh [6] introduced the concept of a quasi-Yamabe soliton, which extends the classical Yamabe soliton framework and is defined on a (pseudo-)Riemannian manifold as follow

$$(L_V g)(X, Y) = 2(\lambda - r)g(X, Y) + 2\gamma V^{\#}(X)V^{\#}(Y), \tag{2.6}$$

where $V^{\#}$ is the dual 1-form of V, λ is a constant and γ is a smooth function. If V is the gradient of a smooth function f, then the soliton is referred to as a quasi-Yamabe gradient soliton, and (2.6) simplifies to

$$\nabla^2 f = (\lambda - r)g + \gamma \mathrm{d}f \otimes \mathrm{d}f, \tag{2.7}$$

where $\nabla^2 f$ is the Hessian of a smooth function f.

In 2019, Güler and Crasmareanu [10] introduced the notion of Ricci-Yamabe flow on a (pseudo-)Riemannian manifold (M, g) by considering a scalar combination of the Ricci flow and Yamabe flow as

$$\frac{\partial g}{\partial t}(t) + 2\alpha \mathrm{Ric}(t) + \beta r(t)g(t) = 0, \tag{2.8}$$

where g, Ric, r are the (pseudo-)Riemannian metric, Ricci tensor and scalar curvature, respectively. Also α, β are two constants. The sign of α and β can be chosen arbitrarily. This freedom of choice is very useful in differential geometry and theory of relativity.

We now intend to extend these notions to a more generalized version as follows.

Definition 2.1 *A (pseudo-)Riemannian manifold (M, g) of dimension $n > 2$ is said to admit a generalized Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta, \gamma)$ if*

$$L_V g + 2\alpha \mathrm{Ric} = (2\lambda - \beta r)g + 2\gamma V^{\#} \otimes V^{\#}, \tag{2.9}$$

where $\lambda, \alpha, \beta, \gamma \in \mathbb{R}$, and $V^{\#}$ denotes the 1-form dual to the vector field V. If the vector field V is the gradient of a smooth function f on M, then the structure is referred to as a generalized gradient Ricci-Yamabe soliton, and (2.9) accordingly takes the form

$$\nabla^2 f + \alpha \mathrm{Ric} = \left(\lambda - \frac{1}{2}\beta r\right)g + \gamma \mathrm{d}f \otimes \mathrm{d}f. \tag{2.10}$$

The generalized (gradient) Ricci-Yamabe soliton is said to be expanding, steady or shrinking according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively. The above concept encompasses a broad class of soliton-type equations. In fact, a generalized Ricci-Yamabe soliton is said to be a

* proper Ricci-Yamabe soliton if $\gamma = 0$ and $\alpha \neq 0, 1$;
* Ricci soliton if $\alpha = 1,\ \beta = \gamma = 0$;
* Yamabe soliton if $\alpha = \gamma = 0,\ \beta = 2$;
* Quasi-Yamabe soliton if $\alpha = 0$ and $\beta = 2$;
* Einstein soliton if $\alpha = 1,\ \beta = -1$ and $\gamma = 0$;
* ρ-Einstein soliton if $\alpha = 1,\ \beta = -2\rho$ and $\gamma = 0$.

Theorem 2.2 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$ and $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^{V}G^{*}$ be the modified Riemannian extension metric on the cotangent bundle $T^{*}M$. The quadruple $(T^{*}M, \widetilde{g}, {}^{C}V, \lambda)$ is a generalized Ricci-Yamabe soliton if and only if the following conditions are satisfied*

$$\begin{aligned}
&\text{(i)} && {}^{C}V = (v^{a}, v^{\overline{a}}) = (v^{a}, -p_{s}\nabla_{a}v^{s}),\\
&\text{(ii)} && \lambda = \frac{\gamma}{n}v^{i}(p_{s}\nabla_{i}v^{s} - G_{ai}v^{a}),\\
&\text{(iii)} && L_{V}G_{ij} = 2\lambda G_{ij} + \frac{1}{2}\partial_{t}G_{ij} + 2\gamma G_{ai}G_{bj}v^{a}v^{b},\\
&\text{(iv)} && \nabla_{i}\nabla_{j}v^{s} = v^{a}R_{iaj}^{s} + \gamma(G_{ai}v^{a}\nabla_{j}v^{s} + G_{bj}v^{b}\nabla_{i}v^{s}),\\
&\text{(v)} && \nabla_{i}v^{s}\nabla_{j}v^{t} = 0,
\end{aligned}$$

where the potential vector field is the complete lift ${}^{C}V$ of a vector field V on M to the cotangent bundle $T^{*}M$.

Proof We will show the existence of the scalar λ. Because of scalar curvature of the modified Riemannian extension is zero, from (2.8) we get

$$4\alpha R_{ij} = -\partial_{t}G_{ij}. \tag{2.11}$$

If the expression of $L_{V}(\widetilde{g}_{\nabla,G})$ in Proposition 2.2 and (2.11) are used in (2.9), we have

$$L_{V}G_{ij} + 2v^{a}p_{s}R_{iaj}{}^{s} + 2v^{\overline{a}}\Gamma_{ij}^{a} + 2(E_{i}v^{\overline{j}}) - \partial_{t}G_{ij} = 2\lambda G_{ij} + 2\gamma V^{\#}\otimes V^{\#} \tag{2.12}$$

and

$$E_{i}v^{j} + v^{a}\Gamma_{ai}^{j} + (E_{\overline{j}}v^{\overline{i}}) = 2\lambda\delta_{i}^{j} + 2\gamma V^{\#}\otimes V^{\#}, \tag{2.13}$$

where the potential vector field ${}^{C}V$ and its dual 1-form are expressed as follows

$${}^{C}V = \begin{pmatrix} v^{m} \\ v^{\overline{m}} \end{pmatrix} = \begin{pmatrix} v^{m} \\ -p_{s}\nabla_{m}v^{s} \end{pmatrix} \tag{2.14}$$

and

$$V_{j}^{\#} = V^{I}\widetilde{g}_{Ij} = v^{i}\widetilde{g}_{ij} + v^{\overline{i}}\widetilde{g}_{\overline{i}j} = G_{ij}v^{i} - p_{s}\nabla_{i}v^{s}\delta_{i}^{j} = G_{ij}v^{i} - p_{s}\nabla_{j}v^{s},$$

$$V^{\#}_{\bar{j}} = V^I \widetilde{g}_{I\bar{j}} = v^i \widetilde{g}_{i\bar{j}} + v^{\bar{i}} \widetilde{g}_{\bar{i}\bar{j}} = v^i \delta^j_i = v^j,$$

from which

$$(V^{\#}) = \begin{pmatrix} G_{ij} v^i - p_s \nabla_j v^s \\ v^j \end{pmatrix}. \tag{2.15}$$

If (2.14)–(2.15) are substituted into (2.13), we get

$$E_i v^j + v^a \Gamma^j_{ai} + E_{\bar{j}}(-p_s \nabla_i v^s) = 2\lambda \delta^j_i + 2\gamma V^{\#}_i \otimes V^{\#}_{\bar{j}},$$

from which

$$0 = 2\lambda \delta^j_i + 2\gamma (G_{ai} v^a - p_s \nabla_i v^s) v^j.$$

Contracting with δ^i_j both sides of the last equation, we have

$$\lambda = \frac{\gamma}{n}(p_s \nabla_i v^s - G_{ai} v^a).$$

If (2.14)–(2.15) are placed in (2.12), we find

$$\begin{aligned} &L_V G_{ij} + 2v^a p_s R^s_{iaj} + 2p_s \nabla_a v^s \Gamma^a_{ij} - 2E_i(p_s \nabla_j v^s) - \frac{1}{2}\partial_t G_{ij} \\ &= 2\lambda G_{ij} + 2\gamma V^{\#}_i \otimes V^{\#}_j, \end{aligned}$$

from which

$$\begin{aligned} L_V G_{ij} = &-2p_s[v^a R^s_{iaj} + \nabla_a v^s \Gamma^a_{ij} - E_i(\nabla_j v^s)] \\ &+ \frac{1}{2}\partial_t G_{ij} + 2\lambda G_{ij} + 2\gamma G_{ai} G_{bj} v^a v^b \\ &- 2\gamma p_s (G_{ai} v^a \nabla_j v^s + G_{bj} v^b \nabla_i v^s) \\ &+ 2\gamma p_s p_t \nabla_i v^s \nabla_j v^t. \end{aligned}$$

From the above equation, we have

$$\begin{aligned} \lambda &= \frac{\gamma}{n}(p_s \nabla_i v^s - G_{ai} v^a), \\ L_V G_{ij} - 2\lambda G_{ij} - \frac{1}{2}\partial_t G_{ij} - 2\gamma G_{ai} G_{bj} v^a v^b &= 0, \\ v^a R^s_{iaj} - \nabla_i \nabla_j v^s + \gamma (G_{ai} v^a \nabla_j v^s + G_{bj} v^b \nabla_i v^s) &= 0 \end{aligned}$$

and

$$\nabla_i v^s \nabla_j v^t = 0.$$

If the above calculations are followed in reverse, the sufficiency of the theorem can be easily proved.

Theorem 2.3 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$ and $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^{V}G^*$ be the modified Riemannian extension metric on the cotangent bundle T^*M. The quadruple $(T^*M, \widetilde{g}_{\nabla,G}, \lambda)$ is a generalized gradient Ricci-Yamabe soliton if and only if the following condition are satisfied*

$$\lambda = \frac{1}{n}[G^{ij}(\nabla^2 f)_{ij} + 2G^{ij}R_{ij} - \gamma G^{ij}\partial_i f\,\partial_j f].$$

Proof If the expression of $\widetilde{\nabla}^2 f$ in Proposition 2.3 is used in (2.10), we have

$$(\nabla^2 f)_{ij} + 2R_{ij} = \lambda G_{ij} + \gamma\partial_i f\,\partial_j f.$$

Contracting with G^{ij} both sides of the last equation, we have

$$\lambda = \frac{1}{n}[G^{ij}(\nabla^2 f)_{ij} + 2G^{ij}R_{ij} - \gamma G^{ij}\partial_i f\,\partial_j f],$$

which completes the proof.

2.3 Riemannian and gradient Riemannian solitons with respect to the modified Riemannian extension

As a natural generalization of Ricci flow, the concept of Riemannian flow is defined by the equation $\frac{\partial}{\partial t}G(t) = -2Rg(t)$, where R is the Riemannian curvature tensor and $\wedge$ denotes the Kulkarni-Nomizu product. For $C, D \in \Im_2^0(M)$, the Kulkarni-Nomizu product is given by

$$\begin{aligned}(C \wedge D)(W, X, Y, Z) = {} & C(W, Z)D(X, Y) + C(X, Y)D(W, Z) \\ & - C(W, Y)D(X, Z) - C(X, Z)D(W, Y),\end{aligned}$$

or locally

$$C \wedge D = C_{il}D_{jk} + C_{jk}D_{il} - C_{ik}D_{jl} - C_{jl}D_{ik}.$$

In a manner similar to Ricci solitons, the concept of a Riemannian soliton was introduced by Hirica and Udriste [12]. A Riemannian metric g on a smooth manifold M is called a Riemannian soliton if there exists a differentiable vector field X and a real constant λ such that

$$\mathrm{Ric} + \lambda K + \frac{1}{2}g \wedge L_X g = 0,$$

where L_X denotes the Lie derivative along X, λ is a constant, and Ric is the Ricci tensor of g. The vector field X is known as the potential of the soliton. A Riemannian soliton is classified as shrinking when $\lambda < 0$, steady when $\lambda = 0$, and expanding when $\lambda > 0$. If X is the gradient of a smooth function f on M, the concept described above is referred to as a gradient Riemannian soliton. In this case, the equation can be rewritten as

$$R + \lambda K + g \wedge \nabla^2 f = 0,$$

where f is a smooth potential function on M, and $\nabla^2 f$ represents the Hessian of f. Additionally, $R = R_{ijkl}$ denotes the Riemannian curvature tensor, and $K = K_{ijkl} = g \wedge g = g_{ik}g_{jl} - g_{il}g_{jk}$.

Theorem 2.4 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$ and $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^{V}G^{*}$ be the modified Riemannian extension metric on the cotangent bundle $T^{*}M$. The quadruple $(T^{*}M, \widetilde{g}, \widetilde{X}, \lambda)$ is a Riemannian soliton if and only if the following conditions are satisfied*

$$\begin{aligned}
&\text{(i)} \quad \widetilde{X} = (v^{l}, v^{\overline{l}}) = (v^{l}, p_{s}A_{l}^{s} + B_{l}),\\
&\text{(ii)} \quad \lambda = \frac{1}{n}G^{jl}\Big(L_{X}G_{jl} + \frac{2}{n-1}R_{jl} + 2\nabla_{j}B_{l}\Big),\\
&\text{(iii)} \quad v^{a}R_{j\alpha l}^{s} + \nabla_{j}A_{l}^{s} \ = 0,
\end{aligned}$$

where $\widetilde{X} = v^{a}E_{a} + v^{\overline{a}}E_{\overline{a}}$ is a fibre-preserving vector field on $T^{*}M$, $\lambda \in \mathbb{R}$, $B = (B_{i})$ and $A = (A_{s}^{h})$ are the $(0,1)$ and $(1,1)$ tensor fields on M, respectively.

Proof We begin by stating that only the following equations will be explicitly computed

$$\begin{aligned}
&\text{(i)} \quad \widetilde{R}_{ijkl} + \lambda\widetilde{K}_{ijkl} + \frac{1}{2}(\widetilde{g}_{\nabla,G})_{ij} \wedge (L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{kl} = 0,\\
&\text{(ii)} \quad \widetilde{R}_{\overline{i}jkl} + \lambda\widetilde{K}_{\overline{i}jkl} + \frac{1}{2}(\widetilde{g}_{\nabla,G})_{\overline{i}j} \wedge (L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{kl} = 0,\\
&\text{(iii)} \quad \widetilde{R}_{\overline{i}jk\overline{l}} + \lambda\widetilde{K}_{\overline{i}jk\overline{l}} + \frac{1}{2}(\widetilde{g}_{\nabla,G})_{\overline{i}j} \wedge (L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{k\overline{l}} = 0,\\
&\text{(iv)} \quad \widetilde{R}_{\overline{i}\overline{j}kl} + \lambda\widetilde{K}_{\overline{i}\overline{j}kl} + \frac{1}{2}(\widetilde{g}_{\nabla,G})_{\overline{i}\overline{j}} \wedge (L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{kl} = 0.
\end{aligned}$$

As all other components lead to analogous results, it suffices to restrict our attention to these four equations.

Since $\widetilde{K} = \widetilde{K}_{IJKL} = \widetilde{g}_{\nabla,G} \wedge \widetilde{g}_{\nabla,G} = (\widetilde{g}_{\nabla,G})_{IK}(\widetilde{g}_{\nabla,G})_{JL} - (\widetilde{g}_{\nabla,G})_{IL}(\widetilde{g}_{\nabla,G})_{JK}$, the Kulkarni-Nomizu products that we will use in our proof are as follows

$$\begin{aligned}
&\widetilde{K}_{ijkl} = G_{ik}G_{jl} - G_{il}G_{jk}, && \widetilde{K}_{ij\overline{k}l} = \delta_{i}^{k}G_{jl} - \delta_{j}^{k}G_{il}, && \widetilde{K}_{\overline{i}jk\overline{l}} = \delta_{k}^{i}\delta_{j}^{l},\\
&\widetilde{K}_{\overline{i}jkl} = \delta_{k}^{i}G_{jl} - \delta_{l}^{i}G_{jk}, && \widetilde{K}_{ijk\overline{l}} = \delta_{j}^{l}G_{ik} - \delta_{i}^{l}G_{jk}, && \widetilde{K}_{i\overline{j}\overline{k}l} = \delta_{i}^{k}\delta_{l}^{j},\\
&\widetilde{K}_{i\overline{j}kl} = \delta_{l}^{j}G_{ik} - \delta_{k}^{j}G_{il}, && \widetilde{K}_{\overline{i}j\overline{k}l} = -\delta_{l}^{i}\delta_{j}^{k}, && \widetilde{K}_{i\overline{j}k\overline{l}} = -\delta_{k}^{j}\delta_{i}^{l},
\end{aligned}$$

and other components are zero. From

$$\widetilde{R}_{ijkl} + \lambda\widetilde{K}_{ijkl} + \frac{1}{2}(\widetilde{g}_{\nabla,G})_{ij} \wedge (L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{kl} = 0,$$

we have

$$\begin{aligned}
0 = {}& p_{s}(\nabla_{i}R_{lkj}^{s} - \nabla_{j}R_{lki}^{s}) + \frac{1}{2}(R_{ijk}^{m}G_{ml} - R_{ijl}^{m}G_{km}) + \lambda(G_{ik}G_{jl} - G_{il}G_{lk})\\
&+ \frac{1}{2}[G_{il}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{jk} + G_{jk}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{il} - G_{ik}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{jl} - G_{jl}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{ik}].
\end{aligned}$$

Contracting with G^{ij} both sides of the above equation, we get

$$p_{s}G^{ij}(\nabla_{i}R_{lkj}^{s} - \nabla_{j}R_{lki}^{s}) = 0.$$

From

$$\widetilde{R}_{\bar{i}jkl} + \lambda \widetilde{K}_{\bar{i}jkl} + \frac{1}{2}(\widetilde{g}_{\nabla,G})_{\bar{i}j} \wedge (L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{kl} = 0,$$

we have

$$\begin{aligned} 0 = \frac{1}{2}[&(\widetilde{g}_{\nabla,G})_{\bar{i}l}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{jk} + (\widetilde{g}_{\nabla,G})_{jk}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{\bar{i}l} \\ &- (\widetilde{g}_{\nabla,G})_{\bar{i}k}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{jl} - (\widetilde{g}_{\nabla,G})_{jl}(L_{\widetilde{X}}\widetilde{g}_{\nabla,G})_{\bar{i}k}] \\ &+ R^i_{lkj} + \lambda(\delta^i_k G_{jl} - \delta^i_l G_{jk}). \end{aligned}$$

Contracting with δ^k_i in the last equation and after then again contracting with $\frac{1}{n(n-1)}G^{jl}$ in the resulting equation, we get

$$\lambda = \frac{1}{n(n-1)}G^{jl}R_{jl} + \frac{1}{2n}G^{jl}(L_{\widetilde{X}}\widetilde{g})_{jl} + \frac{1}{2n}(\nabla_i v^i + E_{\bar{i}}v^{\bar{i}}). \tag{2.16}$$

From

$$\widetilde{R}_{\bar{i}jk\bar{l}} + \lambda \widetilde{G}_{\bar{i}jk\bar{l}} + \frac{1}{2}\widetilde{g}_{\bar{i}j} \wedge (L_{\widetilde{X}}\widetilde{g})_{k\bar{l}} = 0,$$

we have

$$\lambda \delta^i_k \delta^l_j + \frac{1}{2}[-\widetilde{g}_{\bar{i}k}(L_{\widetilde{X}}\widetilde{g})_{j\bar{l}} - \widetilde{g}_{j\bar{l}}(L_{\widetilde{X}}\widetilde{g})_{\bar{i}k}] = 0$$

and contracting with $\delta^k_i \delta^j_l$ in the above equation, we have

$$\lambda = \frac{1}{n}(\nabla_l v^l + E_{\bar{l}}v^{\bar{l}}). \tag{2.17}$$

If $E_{\overline{h}}$ is applied to both sides in (2.17), we obtain

$$v^{\bar{l}} = p_s A^s_l + B_l. \tag{2.18}$$

If (2.18) is substituted into (2.16)–(2.17) and also the equality of (2.16)–(2.17) is used, the following equation is obtained

$$\frac{1}{n-1}G^{jl}R_{jl} + \frac{1}{2}G^{jl}(L_{\widetilde{X}}\widetilde{g})_{jl} = \frac{1}{2}(\nabla_l v^l + A^l_l). \tag{2.19}$$

From Proposition 2.2, we get

$$\begin{aligned} \frac{1}{2}(\nabla_l v^l + A^l_l) = \frac{1}{n-1}&G^{jl}R_{jl} + \frac{1}{2}G^{jl}L_X G_{jl} + G^{jl}v^a p_s R^s_{jal} \\ &- G^{jl}v^{\bar{a}}\Gamma^a_{jl} + G^{jl}(E_j v^{\bar{l}}). \end{aligned} \tag{2.20}$$

Substituting (2.18) into (2.20), we find

$$\begin{aligned} \frac{1}{2}(\nabla_l v^l + A^l_l) = G^{jl}&\Big(\frac{1}{2}L_X G_{jl} + \frac{1}{n-1}R_{jl} + \nabla_j B_l\Big) \\ &+ p_s G^{jl}(v^a R^s_{jal} + \nabla_j A^s_l), \end{aligned}$$

from which we can write

$$v^a R^s_{jal} + \nabla_j A^s_l = 0 \tag{2.21}$$

and

$$G^{jl}\Big(L_X G_{jl} + \frac{2}{n-1} R_{jl} + 2\nabla_j B_l\Big) = \nabla_l v^l + A^l_l. \tag{2.22}$$

When (2.17) and (2.22) are evaluated together, we can write the following equation

$$\lambda = \frac{1}{n} G^{jl}\Big(L_X G_{jl} + \frac{2}{n-1} R_{jl} + 2\nabla_j B_l\Big).$$

It is seen that the last equation

$$\widetilde{R}_{\overline{i}\overline{j}kl} + \lambda \widetilde{G}_{\overline{i}\overline{j}kl} + \frac{1}{2}\widetilde{g}_{\overline{i}\overline{j}} \wedge (L_X \widetilde{g})_{kl} = 0$$

is easily provided with the obtained findings which completes the proof of the necessity of the theorem.

The sufficiency of the theorem can be easily proved by following the above calculations in reverse.

Theorem 2.5 *Let (M, J, g) be an anti-Kähler manifold of dimension $n > 2$ and $\widetilde{g}_{\nabla,G} = \widetilde{g}_{\nabla} + {}^V G^*$ be the modified Riemannian extension metric on the cotangent bundle T^*M. The quadruple $(T^*M, \widetilde{g}_{\nabla,G}, \lambda)$ is a gradient Riemannian soliton structure if and only if the following conditions are satisfied*

$$\text{(i)} \quad \lambda = \frac{1}{n(n-1)} G^{jk} R_{jk},$$

$$\text{(ii)} \quad (\nabla^2 f)_{jk} = 0,$$

where $\nabla^2 f$ is the Hessian operator.

Proof Let us define $\widetilde{g}_{IJ} \wedge (\widetilde{\nabla}^2 f)_{KL}$ as the Kulkarni-Nomizu product that will be used in our proof, which is given as follows

$$\text{(i)} \quad \widetilde{g}_{ij} \wedge (\widetilde{\nabla}^2 f)_{kl} = G_{il}(\nabla^2 f)_{jk} + G_{jk}(\nabla^2 f)_{il} - G_{ik}(\nabla^2 f)_{jl} - G_{jl}(\nabla^2 f)_{ik},$$

$$\text{(ii)} \quad \widetilde{g}_{\overline{i}j} \wedge (\widetilde{\nabla}^2 f)_{kl} = \delta^i_l(\nabla^2 f)_{jk} - \delta^i_k(\nabla^2 f)_{jl},$$

$$\text{(iii)} \quad \widetilde{g}_{i\overline{j}} \wedge (\widetilde{\nabla}^2 f)_{kl} = \delta^j_k(\nabla^2 f)_{il} - \delta^j_l(\nabla^2 f)_{ik},$$

$$\text{(iv)} \quad \widetilde{g}_{ij} \wedge (\widetilde{\nabla}^2 f)_{\overline{k}l} = \delta^k_j(\nabla^2 f)_{il} - \delta^k_i(\nabla^2 f)_{jl},$$

$$\text{(v)} \quad \widetilde{g}_{ij} \wedge (\widetilde{\nabla}^2 f)_{k\overline{l}} = \delta^l_i(\nabla^2 f)_{jk} - \delta^l_j(\nabla^2 f)_{ik}.$$

All other components of these Kulkarni-Nomizu products are zero. From

$$\widetilde{R}_{ijkl} + \lambda \widetilde{G}_{ijkl} + \widetilde{g}_{ij} \wedge (\widetilde{\nabla}^2 f)_{kl} = 0,$$

we have

$$0 = p_s(\nabla_i R^s_{lkj} - \nabla_j R^s_{lki})$$

$$+\frac{1}{2}(R^m_{ijk}G_{ml} - R^m_{ijl}G_{km}) + G_{il}(\nabla^2 f)_{jk}$$
$$+ G_{jk}(\nabla^2 f)_{il} - G_{ik}(\nabla^2 f)_{jl} - G_{jl}(\nabla^2 f)_{ik}$$
$$+ \lambda(G_{ik}G_{jl} - G_{il}G_{jk}).$$

If the last equation is contracted by G^{jk} and after then the obtaining expression is again contracted by G^{jl}, the following result is obtained

$$\begin{aligned}0 = {} & p_s G^{ik}G^{jl}(\nabla_i R^s_{lkj} - \nabla_j R^s_{lki}) \\ & - G^{jl}R_{jl} + \lambda n(n-1) \\ & - 2(n-1)G^{jl}(\nabla^2 f)_{jl},\end{aligned}$$

from which we get

$$\lambda = \frac{1}{n(n-1)}G^{jl}R_{jl} + \frac{2}{n}G^{jl}(\nabla^2 f)_{jl}. \tag{2.23}$$

From

$$\widetilde{R}_{ijk\bar{l}} + \lambda\widetilde{G}_{ijk\bar{l}} + \widetilde{g}_{ij} \wedge (\widetilde{\nabla}^2 f)_{k\bar{l}} = 0,$$

we have

$$R^l_{ijk} + \lambda(\delta^l_j G_{ik} - \delta^l_i G_{jk}) + \delta^l_i(\nabla^2 f)_{jk} - \delta^l_j(\nabla^2 f)_{ik} = 0$$

and multiplying the last equation by δ^i_l, we get

$$(\nabla^2 f)_{jk} = \lambda G_{jl} - \frac{1}{n-1}R_{jk}. \tag{2.24}$$

If (2.23) is placed in (2.24), the following equation is

$$(\nabla^2 f)_{jk} = \Big[\frac{1}{n(n-1)}G^{il}R_{il} + \frac{2}{n}G^{il}(\nabla^2 f)_{il}\Big]G_{jk} - \frac{1}{n-1}R_{jk}.$$

When it is multiplied both sides of the above equation by G^{jk}, we get

$$(\nabla^2 f)_{jk} = 0. \tag{2.25}$$

Substituting (2.25) into (2.23), we find

$$\lambda = \frac{1}{n(n-1)}G^{jl}R_{jl}.$$

Since the same results are obtained from other components, it is not included in the proof of the theorem.

Conversely, by a routine calculation, we can obtain the sufficient condition of the theorem. So the proof is completed.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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