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On Meromorphic Solutions of Non-linear Differential Equations and Their Applications*

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Abstract In this paper, the authors consider meromorphic solutions of nonhomogeneous differential equation

$$f^n(f'+af)+P_d(z,f)=u(z)\mathrm{e}^{v(z)},$$

where n is a positive integer, a is a nonzero constant, $P_d(z,f)$ is a differential polynomial in $f(z)$ of degree d with rational functions as its coefficients and $d\le n-1$, $u(z)$ is a nonzero rational function, $v(z)$ is a nonconstant polynomial with $v'(z)\not\equiv(n+1)a$, $v'(z)\not\equiv -na$ and $v'(z)\not\equiv -\frac{(n+1)^2}{n}a$. They prove that if it admits a meromorphic solution $f(z)$ with finitely many poles, then

$$f(z)=s(z)\mathrm{e}^{\frac{v(z)}{n+1}}\quad\text{and}\quad P_d(z,f)\equiv 0,$$

where $s(z)$ is a rational function and $s^n[(n+1)s'+sv']+(n+1)as^{n+1}=(n+1)u$. Using this result, they also prove that if $f(z)$ is a transcendental entire function, then $f^n(f'+af)+q_m(f)$ assumes every complex number α infinitely many times, except for a possible value $q_m(0)$, where n, m are positive integers with $n\ge m+1$ and $q_m(f)$ is a polynomial in $f(z)$ with degree m.

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1 Introduction

In the following, let $\mathbb{C}$ denote the complex plane and $f(z)$ be a meromorphic function on $\mathbb{C}$. Throughout this paper, we assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory (see [3–4, 17–18]), such as $T(r,f(z))$, $m(r,f(z))$, $N(r,f(z))$, $\cdots$. The term $S(r,f(z))$ always has the property that $S(r,f(z))=o\{T(r,f(z))\}$ as $r\to\infty$, possibly outside an exceptional set of finite linear measure. Let $f(z)$ and $a(z)$ be meromorphic functions, $a(z)$ is said to be a small function of $f(z)$ if and only if $T(r,a(z))=S(r,f(z))$. We use $S(f)$ to denote the family of all meromorphic functions $a(z)$ satisfying $T(r,a(z))=S(r,f(z))$. For a meromorphic function $f(z)$, we define its order in terms of

$$\rho(f(z))=\overline{\lim_{r\to\infty}}\frac{\log T(r,f(z))}{\log r}.$$

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If $\rho(f(z)) < \infty$, then we say that $f(z)$ is a meromorphic function of finite order. If $\rho(f(z)) = \infty$, then we say that $f(z)$ is a meromorphic function of infinite order.

Let I be a finite set of multi-indices $\lambda = (i_0, \cdots, i_n)$ and let a differential polynomial

$$P_d(z, f) = \sum_{\lambda \in I} a_\lambda(z) f^{i_0} (f')^{i_1} \cdots (f^{(n)})^{i_n} \tag{1.1}$$

be a polynomial of $f(z)$ and its derivatives with degree d and meromorphic functions $a_\lambda(z)$ as its coefficients. Standard notations to be used below are as follows. The degree $|\lambda|$ of a single term in (1.1) will be defined by $|\lambda| = i_0 + i_1 + \cdots + i_n$. In a natural way, the degree of $P_d(z, f)$ will be defined by $d = \max_{\lambda \in I} |\lambda|$. Following the above, $P_d(z, f)$ is said to be a differential polynomial in $f(z)$ of degree d.

In 1964, a generalization of the theorem of Tumura-Clunie [1, 12] given by Hayman [3] states the following theorem.

Theorem 1.1 *Let $n \geq 2$ be an integer and $P(z, f)$ be a differential polynomial in $f(z)$ of degree $\leq n - 1$. If a nonconstant meromorphic function $f(z)$ satisfies*

$$f^n(z) + P(z, f) = g(z)$$

and $N(r, f) + N\big(r, \frac{1}{g}\big) = S(r, f)$, then there is a small function $a(z)$ of $f(z)$ such that $(f(z) - a(z))^n = g(z)$.

It has always been an interesting and quite difficult problem to prove the existence of the entire or meromorphic solution of a given non-linear differential equation (see [6–10, 13–15, 19, 21]) in the past few decades.

In 2013, Zhang and Liao [22] proved the following result.

Theorem 1.2 *If the algebraic differential equation $P(z, f) = 0$, where $P(z, f)$ is a differential polynomial in $f(z)$ with polynomial coefficients, has only one dominant term, then it has no admissible transcendental meromorphic solutions satisfying $N(r, f) = S(r, f)$.*

In 2014, Liao and Ye [11] considered the meromorphic solutions of the algebraic differential equation $f^n f' + Q_d(z, f) = u(z)e^{v(z)}$ and obtained the following result.

Theorem 1.3 *Let $Q_d(z, f)$ be a differential polynomial in $f(z)$ of degree d with rational function coefficients. Suppose that $u(z)$ is a nonzero rational function and $v(z)$ is a nonconstant polynomial. If $n \geq d + 1$ and the differential equation*

$$f^n f' + Q_d(z, f) = u(z)e^{v(z)}$$

admits a meromorphic solution $f(z)$ with finitely many poles, then $f(z)$ has the following form:

$$f(z) = s(z)e^{\frac{v(z)}{n+1}} \quad \text{and} \quad Q_d(z, f) \equiv 0,$$

where $s(z)$ is a rational function with $s^n((n+1)s' + v's) = (n+1)u$. In particular, if $u(z)$ is a polynomial, then $s(z)$ is a polynomial, too.

At the same time, Liao and Ye [11] obtained the following theorem.

Theorem 1.4 *Let $f(z)$ be a transcendental entire function, $q_m(f) = b_m f^m + \cdots + b_1 f + b_0$ be a polynomial with degree m and n be a positive integer with $n \geq m+1$. Then $f'f^n + q_m(f)$ assumes every complex number α infinitely many times, except for a possible value $b_0 = q_m(0)$. On the other hand, if $f'f^n + q_m(f)$ assumes $b_0 = q_m(0)$ finitely many times, then $q_m(f) \equiv b_0$, f and f' have only finitely many zeros.*

In this paper, we change $f^n f'$ in Theorem 1.3 to $f^n(f' + af)$ and prove the following result.

Theorem 1.5 *Let n be a positive integer, a be a nonzero constant, and $P_d(z, f)$ be a differential polynomial in $f(z)$ of degree d with rational functions as its coefficients. Suppose that $u(z)$ is a nonzero rational function, $v(z)$ is a nonconstant polynomial with $v'(z) \neq (n+1)a$, $v'(z) \neq -na$ and $v'(z) \neq -\frac{(n+1)^2}{n}a$. If $n \geq d+1$ and the differential equation*

$$f^n(f' + af) + P_d(z, f) = u(z)e^{v(z)}$$

admits a meromorphic solution $f(z)$ with finitely many poles, then $f(z)$ has the following form:

$$f(z) = s(z)e^{\frac{v(z)}{n+1}} \quad \text{and} \quad P_d(z, f) \equiv 0,$$

where $s(z)$ is a rational function and

$$s^n(z)[(n+1)s'(z) + s(z)v'(z)] + (n+1)as^{n+1}(z) = (n+1)u(z).$$

In particular, if $u(z)$ is a polynomial, then $s(z)$ is a polynomial, too.

Remark 1.1 In Theorem 1.5, the conditions $v'(z) \neq -na$ and $v'(z) \neq -\frac{(n+1)^2}{n}a$ are necessary, we have the following three examples to show this. But we are not sure whether the condition $v'(z) \neq (n+1)a$ is necessary or not.

Example 1.1 It is easy to check that $f(z) = e^{-2z+1} + 1$ satisfies the differential equation

$$f^3(z)(f'(z) + 2f(z)) + 3f(z)f'(z) - 2 = 2e^{-6z+3}.$$

In this differential equation, $n = 3$, $a = 2$ and $P_d(z, f) = 3f(z)f'(z) - 2$, $d = 2 \leq n-1$, but $v'(z) = -na = -6$ and the solution $f(z)$ is not the form in Theorem 1.5.

Example 1.2 It is easy to check that $f(z) = e^{-z} + z^2 + 1$ satisfies the differential equation

$$f^2(z)(f'(z) + f(z)) - 2(z^2+1)(z+1)^2 f(z) + (z^2+1)^2(z+1)^2 = (z+1)^2 e^{-2z}.$$

In this differential equation, $n = 2$, $a = 1$ and $P_d(z, f) = -2(z^2+1)(z+1)^2 f(z) + (z^2+1)^2(z+1)^2$, $d = 1 \leq n-1$, but $v'(z) = -na = -2$ and the solution $f(z)$ is not the form in Theorem 1.5.

Example 1.3 It is easy to check that $f(z) = 2e^{-5z+\frac{1}{5}} - 1$ satisfies the differential equation

$$f^4(z)(f'(z) + 4f(z)) + 2f'(z)f^2(z) - 5f(z) - 1 = -32e^{-25z+1}.$$

In this differential equation, $n = 4$, $a = 4$ and $P_d(z, f) = 2f'(z)f^2(z) - 5f(z) - 1$, $d = 3 \leq n-1$, but $v'(z) = -\frac{(n+1)^2}{n}a = -25$ and the solution $f(z)$ is not the form in Theorem 1.5.

With Theorem 1.5 in hand, we get the following result similar to Theorem 1.4.

Theorem 1.6 *Let $f(z)$ be a transcendental entire function, $q_m(f) = b_m f^m + \cdots + b_1 f + b_0$ be a polynomial with degree m, n be a positive integer with $n \geq m+1$ and a be a nonzero constant. Then $f^n(f' + af) + q_m(f)$ assumes every complex number α infinitely many times, except for a possible value $b_0 = q_m(0)$. Furthermore, if $f^n(f'+af)+q_m(f)$ assumes $b_0 = q_m(0)$ finitely many times, then $q_m(f) \equiv b_0$, f and $f' + af$ have only finitely many zeros.*

2 Some Lemmas

To prove the theorems, we need the following lemmas.

Lemma 2.1 (see [4]) *Let $k \geq 1$ be an integer and $f(z)$ be a transcendental meromorphic function, then*

$$m\Big(r, \frac{f^{(k)}(z)}{f(z)}\Big) = S(r, f(z)).$$

If $f(z)$ is of finite order of growth, then

$$m\Big(r, \frac{f^{(k)}(z)}{f(z)}\Big) = O(\log r),$$

and if $f(z)$ is of infinite order of growth, then

$$m\Big(r, \frac{f^{(k)}(z)}{f(z)}\Big) = O(\log rT(r, f)),$$

outside of a possible exceptional set E of finite linear measure.

Lemma 2.2 (see [5]) *Let $P_d(z, f)$ be a differential polynomial in $f(z)$ of degree d with small functions of $f(z)$ as its coefficients. Then we have*

$$m(r, P_d(z, f)) \leq dm(r, f) + S(r, f).$$

Lemma 2.3 (see [2, 16]) *Let $f(z)$ be a transcendental meromorphic function in the complex plane and satisfy*

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in $f(z)$ and its derivatives with meromorphic coefficients, say $\{a_\lambda \mid \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is at most n, then

$$m(r, P(z, f)) = O(\log r),$$

if $f(z)$ is of finite order, and

$$m(r, P(z, f)) = O(\log rT(r, f)),$$

outside of a possible exceptional set E of finite linear measure, if $f(z)$ is of infinite order.

Lemma 2.4 (see [17]) *Let $a_j(z)$ $(j = 1, 2, \cdots, n)$ be entire functions of finite order $\leq \rho$. Let $g_j(z)$ be entire and $g_k(z) - g_j(z)$ $(j \neq k)$ be a transcendental entire function or polynomial of degree greater than ρ. Then*

$$\sum_{j=1}^{n} a_j(z)\mathrm{e}^{g_j(z)} = a_0(z)$$

holds only when

$$a_0(z) = a_1(z) = \cdots = a_n(z) \equiv 0.$$

Lemma 2.5 *Let n be a positive integer, a be a nonzero constant and $P_d(z, f)$ be an algebraic differential polynomial in $f(z)$ of degree $d \leq n-1$ with small functions of $f(z)$ as its coefficients. If $p(z)$ is a small function of $f(z)$, $\alpha(z)$ is a nonconstant polynomial and $f(z)$ is a meromorphic solution of the equation*

$$f^n(f' + af) + P_d(z, f) = p(z)\mathrm{e}^{\alpha(z)}$$

and $N(r, f) = S(r, f)$, then $f(z)$ is a transcendental meromorphic function of finite order.

Proof Because $f(z)$ is a meromorphic solution of the equation

$$f^n(f' + af) + P_d(z, f) = p(z)\mathrm{e}^{\alpha(z)},$$

$f(z)$ must be transcendental.

Denote $\deg(\alpha(z)) = m$. By the first fundamental theorem, Lemmas 2.1–2.2 and $N(r, f) = S(r, f)$, we have

$$\begin{aligned}
(n+1)T(r, f) &= T(r, f^{n+1}) = T\Big(r, \frac{1}{f^{n+1}}\Big) + S(r, f) \\
&= m\Big(r, \frac{1}{f^n(f' + af)} \cdot \frac{f' + af}{f}\Big) + N\Big(r, \frac{1}{f^n(f' + af)} \cdot \frac{f' + af}{f}\Big) + S(r, f) \\
&\leq m\Big(r, \frac{1}{f^n(f' + af)}\Big) + m\Big(r, \frac{f'}{f}\Big) + N\Big(r, \frac{1}{f^n(f' + af)}\Big) + N\Big(r, \frac{f'}{f}\Big) + S(r, f) \\
&= T\Big(r, \frac{1}{f^n(f' + af)}\Big) + \overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}(r, f) + S(r, f) \\
&\leq T(r, f^n(f' + af)) + N\Big(r, \frac{1}{f}\Big) + S(r, f) \\
&= m(r, f^n(f' + af)) + N\Big(r, \frac{1}{f}\Big) + S(r, f) \\
&= m(r, p(z)\mathrm{e}^{\alpha(z)} - P_d(z, f)) + N\Big(r, \frac{1}{f}\Big) + S(r, f) \\
&\leq m(r, p(z)\mathrm{e}^{\alpha(z)}) + m(r, P_d(z, f)) + N\Big(r, \frac{1}{f}\Big) + S(r, f) \\
&\leq T(r, p(z)\mathrm{e}^{\alpha(z)}) + dT(r, f) + T(r, f) + S(r, f) \\
&\leq Ar^m + (d+1)T(r, f) + S(r, f),
\end{aligned}$$

where A is a positive constant. Thus

$$(n - d)T(r, f) \leq Ar^m + S(r, f)$$

and $f(z)$ is of finite order.

Lemma 2.6 (see [11]) *Let $f(z)$ be a meromorphic function. If $f(z)$ is of infinite order, then there exists a sequence $\{z_k\}$ with $\lim_{k\to\infty} z_k = \infty$ such that $\{f(z_k+z)\}_{k=1}^{\infty}$ is not normal at $z=0$.*

Lemma 2.7 (see [20]) *Let $\mathscr{F}$ be a family of meromorphic functions on $D=\{|z|<1\}$ and α be a real number satisfying $-1<\alpha<1$. Then $\mathscr{F}$ is not normal in D if and only if there exist*:

(1) *A number r, $0<r<1$,*

(2) *a sequence of points z_k, $|z_k|<r$,*

(3) *a positive sequence ρ_k, $\lim_{k\to\infty}\rho_k=0$,*

(4) *a sequence $\{f\}_{n\in N}\subset\mathscr{F}$ such that $\rho_k^{\alpha} f_k(z_k+\rho_k x)\to g(x)$ spherically uniformly on compact subsets of $\mathbb{C}$, where $g(x)$ is a nonconstant meromorphic function of order $\rho(g(x))\le 2$.*

3 Proof of Theorem 1.5

Let $f(z)$ be a meromorphic solution of

$$f^n(f'+af)+P_d(z,f)=u(z)\mathrm{e}^{v(z)} \tag{3.1}$$

with finitely many poles. It follows from Lemma 2.5 that $f(z)$ is a transcendental meromorphic function of finite order. Denote $F(z)=f^n(z)(f'(z)+af(z))$ and $G(z)=P_d(z,f)$, then (3.1) can be simplified to

$$F(z)+G(z)=u(z)\mathrm{e}^{v(z)}. \tag{3.2}$$

By differentiating (3.2), we have

$$F'(z)+G'(z)=(u'(z)+u(z)v'(z))\mathrm{e}^{v(z)}. \tag{3.3}$$

It follows from (3.2)–(3.3) that

$$u(z)F'(z)-(u'(z)+u(z)v'(z))F(z)+u(z)G'(z)-(u'(z)+u(z)v'(z))G(z)=0. \tag{3.4}$$

Next, we discuss the following two cases.

Case 1 If $u(z)F'(z)-(u'(z)+u(z)v'(z))F(z)\equiv 0$, then

$$\frac{F'(z)}{F(z)}=\frac{u'(z)}{u(z)}+v'(z).$$

By integrating the above equation, we derive

$$F(z)=cu(z)\mathrm{e}^{v(z)}, \tag{3.5}$$

where c is a nonzero complex number. Substituting $F(z)=f^n(z)(f'(z)+af(z))$ into (3.5), we obtain

$$f^n(z)(f'(z)+af(z))=cu(z)\mathrm{e}^{v(z)}.$$

From this, $f(z)$ has only finitely many zeros. Hence,

$$f(z) = s(z)\mathrm{e}^{t(z)}, \tag{3.6}$$

where $s(z)$ is a rational function and $t(z)$ is a nonconstant polynomial. Substituting (3.6) into (3.1) and by Lemma 2.4, we get

$$P_d(z, f) \equiv 0$$

and

$$s^n(z)\mathrm{e}^{nt(z)}(s'(z) + s(z)t'(z) + as(z))\mathrm{e}^{t(z)} = u(z)\mathrm{e}^{v(z)}.$$

It follows from above that

$$(n + 1)t(z) = v(z), \quad s^n(z)(s'(z) + s(z)t'(z) + as(z)) = u(z),$$

that is,

$$s^n(z)[(n + 1)s'(z) + s(z)v'(z)] + (n + 1)as^{n+1}(z) = (n + 1)u(z).$$

It also implies that $s(z)$ is a polynomial if $u(z)$ is a polynomial.

Case 2 $u(z)F'(z) - (u'(z) + u(z)v'(z))F(z) \not\equiv 0$. If $f(z)$ has only finitely many zeros, then by the similar argument in Case 1, we have

$$f(z) = s(z)\mathrm{e}^{\frac{v(z)}{n+1}} \quad \text{and} \quad P_d(z, f) \equiv 0,$$

where $s(z)$ is a rational function and

$$s^n(z)[(n + 1)s'(z) + s(z)v'(z)] + (n + 1)as^{n+1}(z) = (n + 1)u(z).$$

Now we assume that $f(z)$ has infinitely many zeros. By substituting $F(z) = f^n(z)(f'(z)+af(z))$ into (3.4), we obtain

$$\begin{aligned} G^*(z) = {} & [a(u'(z) + u(z)v'(z))f^2(z) + (u'(z) + u(z)v'(z) - (n + 1)au(z))f(z)f'(z) \\ & - nu(z)(f'(z))^2 - u(z)f(z)f''(z)]f^{n-1}(z), \end{aligned} \tag{3.7}$$

where $G^*(z) = u(z)G'(z) - (u'(z) + u(z)v'(z))G(z)$ is a differential polynomial in $f(z)$ of degree d with rational functions as its coefficients. Denote

$$\begin{aligned} \psi(z) = {} & a(u'(z) + u(z)v'(z))f^2 + (u'(z) + u(z)v'(z) \\ & - (n + 1)au(z))ff' - nu(z)(f')^2 - u(z)ff'', \end{aligned} \tag{3.8}$$

then (3.7) can be simplified to

$$f^{n-1}(z)\psi(z) = G^*(z).$$

It follows from Lemma 2.3 and $d \le n - 1$ that

$$m(r, \psi(z)) = S(r, f).$$

Since $f(z)$ has finitely many poles, $\psi(z)$ is a rational function. First, we prove $\psi(z) \not\equiv 0$. If $\psi(z) \equiv 0$, then (3.8) can be simplified to

$$\begin{aligned} &a(u'(z)+u(z)v'(z))f^2 \\ &= [(n+1)au(z)-(u'(z)+u(z)v'(z))]ff' + nu(z)(f')^2 + u(z)ff''. \end{aligned} \tag{3.9}$$

Let z_0 be a zero of $f(z)$ with multiplicity m which is not a zero or pole of $u(z)$, $v(z)$ and $(n+1)au(z)-(u'(z)+u(z)v'(z))$, then it follows from (3.9) that $f'(z_0)=0$. Thus, z_0 is a zero of $f(z)$ with multiplicity $m \geq 2$. By comparing the multiplicities at z_0 of both sides of (3.9), we know that z_0 is a zero of the left side of (3.9) with multiplicity $2m$, and a zero of the right side of (3.9) with multiplicity $2m-1$, which is a contradiction. Hence, $\psi(z) \not\equiv 0$. By differentiating (3.8), we derive

$$\begin{aligned} \psi'(z) = {} & a(u'(z)+u(z)v'(z))'f^2 + [2a(u'(z)+u(z)v'(z)) + (u'(z)+u(z)v'(z))' \\ & - (n+1)au'(z)]ff' + [(u'(z)+u(z)v'(z)) - (n+1)au(z) - nu'(z)](f')^2 \\ & + [(u'(z)+u(z)v'(z)) - (n+1)au(z) - u'(z)]ff'' \\ & - (2n+1)u(z)f'f'' - u(z)ff'''. \end{aligned} \tag{3.10}$$

Multiplying (3.8) by $\psi'(z)$ and (3.10) by $\psi(z)$, and then subtracting the resulting equation

$$(B_1(z)f + B_2(z)f' + B_3(z)f'' + B_4(z)f''')f = (A_1(z)f' + A_2(z)f'')f', \tag{3.11}$$

where

$$\begin{aligned} B_1(z) &= a(u'(z)+u(z)v'(z))'\psi(z) - a(u'(z)+u(z)v'(z))\psi'(z), \\ B_2(z) &= [2a(u'(z)+u(z)v'(z)) + (u'(z)+u(z)v'(z))' - (n+1)au'(z)]\psi(z) \\ &\quad - [(u'(z)+u(z)v'(z)) - (n+1)au(z)]\psi'(z), \\ B_3(z) &= [(u'(z)+u(z)v'(z)) - (n+1)au(z) - u'(z)]\psi(z) + u(z)\psi'(z), \\ B_4(z) &= -u(z)\psi(z), \\ A_1(z) &= [(n+1)au(z) + (n-1)u'(z) - u(z)v'(z)]\psi(z) - nu(z)\psi'(z), \\ A_2(z) &= (2n+1)u(z)\psi(z). \end{aligned}$$

Let z_1 be a zero of $f(z)$ which is not a zero or pole of $u(z)$, $v(z)$ and $\psi(z)$, then by (3.8), we have $(f'(z_1))^2 = -\frac{\psi(z_1)}{nu(z_1)} \neq 0$. Thus, z_1 is a simple zero of $f(z)$. Meanwhile, z_1 is a zero of $A_1(z)f'(z) + A_2(z)f''(z)$ from (3.11), that is

$$A_1(z_1)f'(z_1) + A_2(z_1)f''(z_1) = 0.$$

Because $f(z)$ has finitely many poles,

$$\alpha(z) = \frac{A_1(z)f'(z) + A_2(z)f''(z)}{f(z)} \tag{3.12}$$

has only finitely many poles. By Lemma 2.1, we obtain

$$m(r, \alpha(z)) = m\Big(r, \frac{A_1(z)f'(z) + A_2(z)f''(z)}{f(z)}\Big) = O(\log r).$$

Therefore, $\alpha(z)$ is a rational function. (3.12) can be changed to

$$A_1(z)f'(z) + A_2(z)f''(z) - \alpha(z)f(z) = 0. \tag{3.13}$$

Next, we discuss two cases.

Case 2.1 If $\alpha(z) \equiv 0$, then it follows from (3.13) and $v'(z) \neq (n+1)a$ that

$$\frac{f''(z)}{f'(z)} = -\frac{A_1(z)}{A_2(z)} = \frac{n}{2n+1}\frac{\psi'(z)}{\psi(z)} - \frac{n-1}{2n+1}\frac{u'(z)}{u(z)} + \frac{1}{2n+1}v'(z) - \frac{n+1}{2n+1}a. \tag{3.14}$$

By integrating (3.14), we get

$$f'(z) = \beta_1(z)\mathrm{e}^{\frac{v(z)-(n+1)az}{2n+1}}, \tag{3.15}$$

where $\beta_1(z) = c_1\big(\frac{\psi^n(z)}{u^{n-1}(z)}\big)^{\frac{1}{2n+1}}$ (c_1 is a nonzero complex number) is a rational function. Differentiating (3.15), we derive

$$f''(z) = \Big(\beta_1'(z) + \beta_1(z)\cdot\frac{v'(z)-(n+1)a}{2n+1}\Big)\mathrm{e}^{\frac{v(z)-(n+1)az}{2n+1}}. \tag{3.16}$$

Substituting (3.15)–(3.16) into (3.8), we have

$$\begin{aligned} &a(u'(z)+u(z)v'(z))f^2 + \Big[(u'(z)+u(z)v'(z)-(n+1)au(z))\beta_1(z) - u(z)\Big(\beta_1'(z)+\beta_1(z) \\ &\cdot\frac{v'(z)-(n+1)a}{2n+1}\Big)\Big]\mathrm{e}^{\frac{v(z)-(n+1)az}{2n+1}}\cdot f - (nu(z)\beta_1^2(z)\mathrm{e}^{\frac{2(v(z)-(n+1)az)}{2n+1}} + \psi(z)) = 0. \end{aligned} \tag{3.17}$$

Denote $H(z) = \frac{v(z)-(n+1)az}{2n+1}$, then $v'(z)-(n+1)a = (2n+1)H'(z)$. Therefore, (3.17) can be simplified to

$$aa_2(z)f^2(z) + a_1(z)\mathrm{e}^{H(z)}\cdot f(z) - (nu(z)\beta_1^2(z)\mathrm{e}^{2H(z)} + \psi(z)) = 0,$$

where

$$\begin{aligned} a_1(z) &= (u'(z)+(2n+1)u(z)H'(z))\beta_1(z) - u(z)(\beta_1'(z)+\beta_1(z)H'(z)), \\ a_2(z) &= u'(z)+u(z)[(n+1)a+(2n+1)H'(z)]. \end{aligned}$$

Solving the above equation, we obtain

$$f(z) = \frac{-a_1(z)\mathrm{e}^{H(z)} + \sqrt{a_1^2(z)\mathrm{e}^{2H(z)} + 4aa_2(z)(nu(z)\beta_1^2(z)\mathrm{e}^{2H(z)} + \psi(z))}}{2aa_2(z)}. \tag{3.18}$$

We denote

$$\begin{aligned} &a_1^2(z)\mathrm{e}^{2H(z)} + 4aa_2(z)(nu(z)\beta_1^2(z)\mathrm{e}^{2H(z)} + \psi(z)) \\ &= (a_1^2(z)+4nau(z)a_2(z)\beta_1^2(z))\mathrm{e}^{2H(z)} + 4aa_2(z)\psi(z) \\ &= A(z)\mathrm{e}^{2H(z)} + B(z), \end{aligned} \tag{3.19}$$

where $A(z)$ and $B(z)$ are rational functions. If $A(z) \equiv 0$, then

$$a_1^2(z) + 4nau(z)a_2(z)\beta_1^2(z) \equiv 0.$$

Substituting the expressions for $a_1(z)$ and $a_2(z)$ into the above equation, we get

$$\begin{aligned}&[(u'(z)+(2n+1)u(z)H'(z))^2+4nau(z)(u'(z)+u(z)((n+1)a\\&+(2n+1)H'(z)))]\beta_1^2(z)+u^2(z)(\beta_1'(z)+\beta_1(z)H'(z))^2\\&-2u(z)\beta_1(z)(u'(z)+(2n+1)u(z)H'(z))(\beta_1'(z)+\beta_1(z)H'(z))=0.\end{aligned}$$

Since $\beta_1(z)\neq 0$ and $u(z)\neq 0$, the above equation can be simplified to

$$\begin{aligned}&\Big(\frac{u'(z)}{u(z)}\Big)^2+4n\frac{u'(z)}{u(z)}H'(z)+4n^2(H'(z))^2+4na((n+1)a+(2n+1)H'(z))+\Big(\frac{\beta_1'(z)}{\beta_1(z)}\Big)^2\\&-4n\frac{\beta_1'(z)}{\beta_1(z)}H'(z)-2\frac{u'(z)}{u(z)}\frac{\beta_1'(z)}{\beta_1(z)}+4na\frac{u'(z)}{u(z)}=0,\end{aligned}$$

that is,

$$\begin{aligned}&\Big(\frac{u'(z)}{u(z)}-\frac{\beta_1'(z)}{\beta_1(z)}\Big)^2+4na\frac{u'(z)}{u(z)}+4n(n+1)a^2\\&+4n\Big(\frac{u'(z)}{u(z)}-\frac{\beta_1'(z)}{\beta_1(z)}+nH'(z)+(2n+1)a\Big)H'(z)=0.\end{aligned}\tag{3.20}$$

If $H(z)$ is a polynomial with degree $\geqslant 2$, then $H'(z)$ is a nonconstant polynomial. This is impossible, it follows from (3.20) that the left-hand side tends to infinity as z tends to infinity, but the right-hand side is zero. Hence, $H(z)$ is a linear polynomial and $H'(z)$ is a nonzero constant. It follows from (3.20) that the left-hand side tends to $4n(n+1)a^2+4n^2(H'(z))^2+4n(2n+1)aH'(z)$ and the right-hand side is zero as z tends to infinity. If $4n(n+1)a^2+4n^2(H'(z))^2+4n(2n+1)aH'(z)=0$, then substituting $H'(z)=\frac{v'(z)-(n+1)a}{2n+1}$ into this equation, we have

$$n(v'(z))^2+(2n^2+2n+1)av'(z)+n(n+1)^2a^2=0.\tag{3.21}$$

It follows from (3.21) that $v'(z)=-na$ or $v'(z)=-\frac{(n+1)^2}{n}a$, a contradiction. Therefore,

$$4n(n+1)a^2+4n^2(H'(z))^2+4n(2n+1)aH'(z)\neq 0,$$

which is a contradiction. Thus, $A(z)\not\equiv 0$. If $B(z)\equiv 0$, then

$$4aa_2(z)\psi(z)\equiv 0.$$

Substituting the expression for $a_2(z)$ and $H'(z)=\frac{v'(z)-(n+1)a}{2n+1}$ into the above equation, we obtain

$$4a[u'(z)+u(z)((n+1)a+(2n+1)H'(z))]\psi(z)=4a(u'(z)+u(z)v'(z))\psi(z)\equiv 0.$$

Because $a\neq 0$ and $\psi(z)\not\equiv 0$, $u'(z)+u(z)v'(z)\equiv 0$. By integrating this equation, we have $u(z)=c_2\mathrm{e}^{v(z)}$ (c_2 is a nonzero complex number), which is a contradiction. Hence, $B(z)\not\equiv 0$. We can get that $A(z)\mathrm{e}^{2H(z)}+B(z)$ has infinitely many zeros. Now we assume that $A(z)\mathrm{e}^{2H(z)}+B(z)$

has no simple zeros. Let z_2 be a zero of $A(z)e^{2H(z)}+B(z)$, then $e^{2H(z_2)}=-\frac{B(z_2)}{A(z_2)}$. By differentiating $A(z)e^{2H(z)}+B(z)$, we derive $(A'(z)+2A(z)H'(z))e^{2H(z)}+B'(z)$. Meanwhile,

$$(A'(z_2)+2A(z_2)H'(z_2))e^{2H(z_2)}+B'(z_2)=(A'(z_2)+2A(z_2)H'(z_2))\Big(-\frac{B(z_2)}{A(z_2)}\Big)+B'(z_2)=0.$$

The above equation can be changed to

$$-\frac{A'(z_2)}{A(z_2)}B(z_2)-2B(z_2)H'(z_2)+B'(z_2)=0.$$

$-\frac{A'(z)}{A(z)}B(z)-2B(z)H'(z)+B'(z)$ is a rational function and not equal to zero. If $-\frac{A'(z)}{A(z)}B(z)-2B(z)H'(z)+B'(z)\equiv 0$, then $B(z)=c_3A(z)e^{2H(z)}$ (c_3 is a nonzero complex number), which is a contradiction. Thus, $-\frac{A'(z)}{A(z)}B(z)-2B(z)H'(z)+B'(z)\not\equiv 0$ and it has only finitely many zeros. This contradicts the fact that $A(z)e^{2H(z)}+B(z)$ has infinitely many zeros. Hence, $A(z)e^{2H(z)}+B(z)$ has a simple zero z_3 at least, and it is an algebraic branching point of $\sqrt{A(z)e^{2H(z)}+B(z)}$. Furthermore, it follows from (3.18) that z_3 is also an algebraic branching point of $f(z)$. It contradicts the fact that $f(z)$ is a meromorphic function.

Case 2.2 If $\alpha(z)\not\equiv 0$, then we also have

$$\alpha(z)=\frac{B_1(z)f(z)+B_2(z)f'(z)+B_3(z)f''(z)+B_4(z)f'''(z)}{f'(z)} \tag{3.22}$$

from (3.11)–(3.12). (3.22) can be changed to

$$B_4(z)f'''(z)+B_3(z)f''(z)+(B_2(z)-\alpha(z))f'(z)+B_1(z)f(z)=0. \tag{3.23}$$

By differentiating (3.13), we derive

$$A_2(z)f'''(z)+(A_2'(z)+A_1(z))f''(z)+(A_1'(z)-\alpha(z))f'(z)-\alpha'(z)f(z)=0. \tag{3.24}$$

It follows from (3.23)–(3.24) and the expressions for $A_2(z)$, $B_4(z)$ that

$$\begin{aligned}&[(2n+1)B_3(z)+A_2'(z)+A_1(z)]f''(z)+[(2n+1)(B_2(z)-\alpha(z))+A_1'(z)-\alpha(z)]f'(z)\\&+[(2n+1)B_1(z)-\alpha'(z)]f(z)=0.\end{aligned} \tag{3.25}$$

By eliminating $f''(z)$ from (3.13) and (3.25), we obtain

$$Q_1(z)f'(z)+Q_2(z)f(z)=0, \tag{3.26}$$

where

$$\begin{aligned}Q_1(z)=&[(2n+1)(B_2(z)-\alpha(z))+A_1'(z)-\alpha(z)]A_2(z)\\&-[(2n+1)B_3(z)+A_2'(z)+A_1(z)]A_1(z),\end{aligned} \tag{3.27}$$

$$Q_2(z)=[(2n+1)B_3(z)+A_2'(z)+A_1(z)]\alpha(z)+[(2n+1)B_1(z)-\alpha'(z)]A_2(z). \tag{3.28}$$

We assume that $Q_1(z)\equiv 0$, then $Q_2(z)\equiv 0$. By (3.27)–(3.28), we have

$$\frac{(2n+1)(B_2(z)-\alpha(z))+A_1'(z)-\alpha(z)}{A_1(z)}=\frac{(2n+1)B_3(z)+A_2'(z)+A_1(z)}{A_2(z)} \tag{3.29}$$

and

$$\frac{(2n+1)B_3(z)+A_2'(z)+A_1(z)}{A_2(z)}=\frac{\alpha'(z)-(2n+1)B_1(z)}{\alpha(z)}. \tag{3.30}$$

Combining the expressions for $A_1(z)$, $A_2(z)$ and $B_3(z)$, (3.29)–(3.30) can be changed to

$$(2n+2)\frac{\alpha(z)}{A_1(z)}=(2n+1)\frac{B_2(z)}{A_1(z)}+\frac{A_1'(z)}{A_1(z)}-\frac{A_2'(z)}{A_2(z)}-\frac{n-1}{2n+1}\frac{u'(z)}{u(z)}-\frac{n+1}{2n+1}\frac{\psi'(z)}{\psi(z)}$$
$$+\frac{2n}{2n+1}((n+1)a-v'(z))$$

and

$$(2n+1)\frac{B_1(z)}{\alpha(z)}=\frac{\alpha'(z)}{\alpha(z)}-\frac{A_2'(z)}{A_2(z)}-\frac{n-1}{2n+1}\frac{u'(z)}{u(z)}-\frac{n+1}{2n+1}\frac{\psi'(z)}{\psi(z)}+\frac{2n}{2n+1}((n+1)a-v'(z)).$$

Multiplying the above two equations, we get

$$\begin{aligned}&(2n+2)(2n+1)\frac{B_1(z)}{A_1(z)}\\&=\Big[(2n+1)\frac{B_2(z)}{A_1(z)}+\frac{A_1'(z)}{A_1(z)}-\frac{A_2'(z)}{A_2(z)}-\frac{n-1}{2n+1}\frac{u'(z)}{u(z)}-\frac{n+1}{2n+1}\frac{\psi'(z)}{\psi(z)}\\&\quad+\frac{2n}{2n+1}((n+1)a-v'(z))\Big]\cdot\Big[\frac{\alpha'(z)}{\alpha(z)}-\frac{A_2'(z)}{A_2(z)}-\frac{n-1}{2n+1}\frac{u'(z)}{u(z)}\\&\quad-\frac{n+1}{2n+1}\frac{\psi'(z)}{\psi(z)}+\frac{2n}{2n+1}((n+1)a-v'(z))\Big].\end{aligned} \tag{3.31}$$

It follows from the expressions for $A_1(z)$, $B_1(z)$ and $B_2(z)$ that

$$\begin{aligned}\frac{B_1(z)}{A_1(z)}&=\frac{a(u'+uv')'\psi-a(u'+uv')\psi'}{[(n+1)au+nu'-(u'+uv')]\psi-nu\psi'}\\&=\frac{a\frac{(u'+uv')'}{u'+uv'}-a\frac{\psi'}{\psi}}{(n+1)a\frac{u}{u'+uv'}+n\frac{u'}{u'+uv'}-1-n\frac{u}{u'+uv'}\frac{\psi'}{\psi}}\\&=\frac{a\frac{(u'+uv')'}{u'+uv'}-a\frac{\psi'}{\psi}}{(n+1)a\frac{1}{\frac{u'}{u}+v'}+n\frac{\frac{u'}{u}}{\frac{u'}{u}+v'}-1-n\frac{1}{\frac{u'}{u}+v'}\frac{\psi'}{\psi}}\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\frac{B_2(z)}{A_1(z)}&=\frac{[2a(u'+uv')+(u'+uv')'-(n+1)au']\psi-[(u'+uv')-(n+1)au]\psi'}{[(n+1)au+nu'-(u'+uv')]\psi-nu\psi'}\\&=\frac{2a+\frac{(u'+uv')'}{u'+uv'}-(n+1)a\frac{u'}{u'+uv'}-\frac{\psi'}{\psi}+(n+1)a\frac{u}{u'+uv'}\frac{\psi'}{\psi}}{(n+1)a\frac{u}{u'+uv'}+n\frac{u'}{u'+uv'}-1-n\frac{u}{u'+uv'}\frac{\psi'}{\psi}}\\&=\frac{2a+\frac{(u'+uv')'}{u'+uv'}-(n+1)a\frac{\frac{u'}{u}}{\frac{u'}{u}+v'}-\frac{\psi'}{\psi}+(n+1)a\frac{1}{\frac{u'}{u}+v'}\frac{\psi'}{\psi}}{(n+1)a\frac{1}{\frac{u'}{u}+v'}+n\frac{\frac{u'}{u}}{\frac{u'}{u}+v'}-1-n\frac{1}{\frac{u'}{u}+v'}\frac{\psi'}{\psi}}.\end{aligned} \tag{3.33}$$

If $v(z)$ is a linear polynomial, then $v'(z)$ is a nonzero constant. Because $v'(z)\neq(n+1)a$, by (3.32)–(3.33), we have $\frac{B_1(z)}{A_1(z)}\to 0$ and $\frac{B_2(z)}{A_1(z)}\to\frac{2a}{\frac{(n+1)a}{v'}-1}$ as z tends to infinity. Meanwhile, it follows from (3.31) that the left-hand side tends to zero and the right-hand side tends to

$$\frac{2(2n+1)a}{\frac{(n+1)a}{v'}-1}\frac{2n}{2n+1}((n+1)a-v')+\Big[\frac{2n}{2n+1}((n+1)a-v')\Big]^2.$$

Now we assume that

$$\frac{2(2n+1)a}{\frac{(n+1)a}{v'}-1}\frac{2n}{2n+1}((n+1)a-v')+\Big[\frac{2n}{2n+1}((n+1)a-v')\Big]^2=0, \tag{3.34}$$

then

$$(v'(z)-(n+1)a)^2=-\frac{(2n+1)^2}{n}av'(z). \tag{3.35}$$

It follows from (3.35) that

$$n(v'(z))^2+(2n^2+2n+1)av'(z)+n(n+1)^2a^2=0. \tag{3.36}$$

By (3.36), we get $v'(z)=-na$ or $v'(z)=-\frac{(n+1)^2}{n}a$. This contradicts the condition of the theorem. Hence, $\frac{2(2n+1)a}{\frac{(n+1)a}{v'}-1}\frac{2n}{2n+1}((n+1)a-v')+[\frac{2n}{2n+1}((n+1)a-v')]^2\neq 0$, which is a contradiction. If $v(z)$ is a polynomial with degree ≥ 2, then $v'(z)$ is a nonconstant polynomial. By (3.32) and (3.33), we have $\frac{B_1(z)}{A_1(z)}\to 0$ and $\frac{B_2(z)}{A_1(z)}\to -2a$ as z tends to infinity. Meanwhile, it follows from (3.31) that the left-hand side tends to zero and the right-hand side tends to infinity, this is impossible.

This shows that $Q_1(z)\not\equiv 0$. By (3.26), we have

$$\frac{f'(z)}{f(z)}=-\frac{Q_2(z)}{Q_1(z)}.$$

Since $\frac{f'(z)}{f(z)}$ has only simple poles, $\frac{Q_2(z)}{Q_1(z)}$ also has only simple poles. Thus, $f(z)=s(z)\mathrm{e}^{t(z)}$, where $s(z)$ is a rational function and $t(z)$ is a polynomial. Substituting the expression for $f(z)$ into (3.1), we obtain

$$s^n(z)\mathrm{e}^{nt(z)}[(s'(z)+s(z)t'(z))\mathrm{e}^{t(z)}+as(z)\mathrm{e}^{t(z)}]+P_d(z,f)=u(z)\mathrm{e}^{v(z)}, \tag{3.37}$$

where $P_d(z,f)=\sum\limits_{i=0}^{d}q_i(z)\mathrm{e}^{it(z)}$, $q_i(z)$ $(i=0,1,\cdots,d)$ are rational functions. By (3.37) and Lemma 2.4, we get $P_d(z,f)\equiv 0$, $(n+1)t(z)=v(z)$ and $s^n(z)[(n+1)s'(z)+s(z)v'(z)]+(n+1)as^{n+1}(z)=(n+1)u(z)$. In particular, $s(z)$ is a polynomial if $u(z)$ is a polynomial.

This completes the proof of Theorem 1.5.

4 Proof of Theorem 1.6

We prove Theorem 1.6 by contradiction.

Suppose that $f^n(f'+af)+q_m(f)$ assumes a complex number $\alpha_1\neq b_0=q_m(0)$ finitely many times. We discuss the following two cases.

Case 1 If $f(z)$ is of finite order, then $f(z)$ is an entire function solution of the following differential equation:

$$f^n(z)(f'(z)+af(z))+q_m(f(z))-\alpha_1=\alpha(z)\mathrm{e}^{\beta(z)},$$

where $\alpha(z)$ and $\beta(z)$ are polynomials. Hence, it follows from Theorem 1.5 that

$$f(z) = \gamma(z)\mathrm{e}^{\frac{\beta(z)}{n+1}} \quad \textit{and} \quad q_m(f(z)) - \alpha_1 \equiv 0,$$

where $\gamma(z)$ is a polynomial, and from which Lemma 2.4 implies $q_m(0) - \alpha_1 = b_0 - \alpha_1 = 0$. This is a contradiction.

Case 2 If $f(z)$ is of infinite order, then it follows from Lemma 2.6 that there exists a sequence $\{z_k\}$ with $\lim\limits_{k\to\infty} z_k = \infty$ such that $\{g_k(z) = f(z_k + z)\}_{k=1}^{\infty}$ is not normal at $z = 0$. Thus, by Lemma 2.7, there exists a sequence of complex points $\{w_k\}_{k=1}^{\infty}$ with $|w_k| < 1$ and a positive sequence of ρ_k with $\lim\limits_{k\to\infty} \rho_k = 0$ such that

$$h_k(z) = \rho_k^{-\frac{1}{n+1}} g_k(w_k + \rho_k z) = \rho_k^{-\frac{1}{n+1}} f(w_k + z_k + \rho_k z) \to g(z)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(z)$ is a nonconstant entire function of order $\rho(g(z)) \le 2$. Hence,

$$\begin{aligned} F_k(z) &= f^n(w_k + z_k + \rho_k z)(f'(w_k + z_k + \rho_k z) + af(w_k + z_k + \rho_k z)) + q_m(f(w_k + z_k + \rho_k z)) \\ &= h_k^n(z)(h_k'(z) + a\rho_k h(z)) + b_m \rho_k^{\frac{m}{n+1}} h_k^m(z) + \cdots + b_1 \rho_k^{\frac{1}{n+1}} h_k(z) + b_0 \end{aligned}$$

converges to $g^n(z)g'(z) + b_0$ spherically uniformly on compact subsets of $\mathbb{C}$. Next, we consider two cases.

Case 2.1 We assume y_0 is a zero of $g^n(z)g'(z) + b_0 - \alpha_1$. By Hurwitz's theorem, there are a sequence of complex numbers $\{y_k\}$ and a sufficiently large integer N such that

$$F_k(y_k) - \alpha_1 = 0, \quad k \ge N \quad \text{and} \quad \lim_{k\to\infty} y_k = y_0.$$

Therefore

$$f^n(x_k)(f'(x_k) + af(x_k)) + q_m(f(x_k)) - \alpha_1 = 0,$$

where $x_k = w_k + z_k + \rho_k y_k$. Because $|w_k| < 1$, $\lim\limits_{k\to\infty} z_k = \infty$, $\lim\limits_{k\to\infty} \rho_k = 0$ and $\lim\limits_{k\to\infty} y_k = y_0$, we can choose subsequences $\{z_{k_i}\}$, $\{w_{k_i}\}$ and $\{\rho_{k_i}\}$ such that

$$|z_{k_i} - z_{k_j}| > 3 \quad (k_i \ne k_j), \quad |w_{k_i}| < 1 \quad \text{and} \quad |\rho_{k_i} y_{k_i}| < \frac{1}{2}.$$

Hence, $x_{k_i} = w_{k_i} + z_{k_i} + \rho_{k_i} y_{k_i}$ $(i = 1, 2, \cdots)$ are distinct zeros of $f^n(z)(f'(z) + af(z)) + q_m(f(z)) - \alpha_1$. This is a contradiction.

Case 2.2 If $g^n(z)g'(z) + b_0 - \alpha_1$ does not have a zero, then $g(z)$ is a transcendental entire function. Since $\rho(g(z)) \le 2$, $g(z)$ is a solution of the differential equation

$$g^n(z)g'(z) + b_0 - \alpha_1 = \mathrm{e}^{\mu(z)},$$

where $\mu(z)$ is a polynomial. From Theorem 1.3 (see [11]), we have $b_0 - \alpha_1 = 0$, which is a contradiction. Thus, $f^n(z)(f'(z) + af(z)) + q_m(f(z))$ takes every complex number α infinitely many times, except for a possible value $b_0 = q_m(0)$.

Furthermore, if $f^n(z)(f'(z)+af(z))+q_m(f(z))$ assumes $b_0=q_m(0)$ finitely many times, then

$$\begin{aligned} & f^n(z)(f'(z)+af(z))+q_m(f(z))-q_m(0) \\ = & f(z)[f^{n-1}(z)(f'(z)+af(z))+b_m f^{m-1}(z)+\cdots+b_1] \end{aligned}$$

has only finitely many zeros. Hence, $f(z)$ and $f^{n-1}(z)(f'(z)+af(z))+b_m f^{m-1}(z)+\cdots+b_1$ have only finitely many zeros. That is, $f^{n-1}(z)(f'(z)+af(z))+b_m f^{m-1}(z)+\cdots+b_1$ takes the value 0 finitely many times. By similar arguments as we have done with $f^n(z)(f'(z)+af(z))+q_m(f(z))$ assumes $\alpha=0$ finitely many times above and we got $b_0=\alpha$. Thus, from $f^{n-1}(z)(f'(z)+af(z))+b_m f^{m-1}(z)+\cdots+b_1$ takes the value 0 finitely many times that $b_1=0$. Next, by $f^2(z)(f^{n-2}(z)(f'(z)+af(z))+b_m f^{m-2}+\cdots+b_2)$ has only finitely many zeros, we have $b_2=0$. Continuing the arguments, we can get $b_1=b_2=\cdots=b_m=0$. Hence, $q_m(f(z))\equiv b_0$ and $f^n(z)(f'(z)+af(z))$ has only finitely many zeros, so $f(z)$ and $f'(z)+af(z)$ also have only finitely many zeros.

This completes the proof of Theorem 1.6.

Finally, it follows from the proof of Theorem 1.6 that the following corollary holds.

Corollary 4.1 *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, $q_m(f)=b_m f^m+\cdots+b_1 f+b_0$ be a polynomial with degree m, n be positive integer with $n\geq m+1$ and a be a nonzero constant. If $f^n(f'+af)+q_m(f)$ dose not assume a complex number $\alpha\neq q_m(0)$ for every function $f\in\mathscr{F}$, then $\mathscr{F}$ is a normal family in the domain D.*

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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