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# Entire Solutions of Certain Types of Delay Differential Equations*

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**Abstract** In this paper, the authors investigate a delay differential equation of the form

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=\frac{P(z,w)}{Q(z,w)},$$

where $a(z)$ is a nonzero rational function, $P(z,w)$ and $Q(z,w)$ are prime polynomials in $w$ with rational coefficients. They remove the restriction that the order of meromorphic solutions of the above difference equation is $\sigma_2(w)<1$, and obtain the growth of transcendental meromorphic solutions. The exact forms of all transcendental entire solutions are obtained when $\deg_w P=\deg_w Q=0$, or $\deg_w P=1$ and $\deg_w Q=0$, respectively. If $\deg_w P\geq 2$ and $\deg_w Q=0$, or $\deg_w Q\geq 1$ and $Q(z,0)\not\equiv 0$, they prove that the above equation has no transcendental entire solution. They show that the existence of transcendental entire solutions of the above equation depends on the degrees of $P(z,w)$ and $Q(z,w)$.

**Keywords** Delay differential equation, Entire solution, Growth, Existence
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## 1 Introduction

In this paper, we use the basic notions of Nevanlinna's theory (see [6, 11]). In addition, we use $\sigma(w)$ (resp. $\sigma_2(w)$), to denote the order (resp. the hyper order), of meromorphic function $w(z)$; $\lambda(w)$ (resp. $\lambda(\frac{1}{w})$), to denote the exponents of convergence of zeros (resp. poles), of $w(z)$. Let $S(r,w)$ denote any quantity satisfying $S(r,w)=o(T(r,w))$ for all $r$ outside of a set with finite logarithmic measure.

Halburd and Korhonen [5] studied delay differential equations and obtained the following theorem.

**Theorem 1.1** (see [5, Theorem 1.1]) *Let $w(z)$ be a non-rational meromorphic solution of*

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=R(z,w)=\frac{P(z,w)}{Q(z,w)},\tag{1.1}$$

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*where $a(z)$ is rational, $P(z,w)$ is a polynomial in $w(z)$ having rational coefficients in $z$, and $Q(z,w)$ is a polynomial in $w$ with roots that are non-zero rational functions of $z$ and not roots of $P(z,w)$. If the hyper-order of $w(z)$ is less than one, then*

$$\deg_w P = \deg_w Q + 1 \leq 3 \quad or \quad \deg_w R = \max\{\deg_w P, \deg_w Q\} \leq 1.$$

Zhang and Huang [12, Theorem 2.1] proved if (1.1) admits a transcendental entire solution with $\sigma_2(w) < 1$, then (1.1) reduces into

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{a_2(z)w(z)^2 + a_1(z)w(z) + a_0(z)}{w(z)}, \tag{1.2}$$

where $a_2(z)(\not\equiv 0), a_1(z)$ and $a_0(z)$ are rational functions. Wang, Long and Wang [9] studied the properties of rational solutions of (1.1) with constant coefficients.

As we all know, the order of meromorphic solutions of difference equations is usually restricted by the condition "$\sigma_2(w) < 1$". Naturally, some interesting problems arise without this restriction.

**Problem 1.1** What is the growth of transcendental meromorphic solutions of (1.1)?

**Problem 1.2** What is the existence of transcendental entire solutions of (1.1); if they exist, what will the entire solutions be presented?

In Section 3, we give answers to Problems 1.1–1.2, and find interesting properties on entire solutions of (1.1) depending on the degree of $P(z,w)$ and $Q(z,w)$.

## 2 Lemmas

Before relating our main results, we prepare some lemmas.

**Lemma 2.1** (see [3, Corollary 2.5]) *Let $f(z)$ be a meromorphic function of finite order $\sigma$ and let $\eta$ be a nonzero complex constant. Then for each $\varepsilon(0 < \varepsilon < 1)$, we have*

$$m\Big(r, \frac{f(z+\eta)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+\eta)}\Big) = O(r^{\sigma-1+\varepsilon}).$$

**Lemma 2.2** (see [3, Theorem 2.1]) *Let $f$ be a meromorphic function with order $\sigma = \sigma(f)$, $\sigma < +\infty$, and let $\eta$ be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

**Lemma 2.3** (see [4, Theorem 3.2, 7, Theorem 2.4]) *Let $w$ be a transcendental meromorphic solution with finite order of difference equation*

$$P(z,w) = 0,$$

*where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a) \not\equiv 0$ for a meromorphic function $a$, where $a$ is a small function with respect to $w$, then*

$$m\Big(r, \frac{1}{w-a}\Big) = S(r,w).$$

**Lemma 2.4** (see [2, Theorem 3]) *Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and satisfy*

$$P_n(z) + \cdots + P_0(z) \not\equiv 0.$$

*Then every finite order transcendental meromorphic solution $f(z) \not\equiv 0$ of difference equation*

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0$$

*satisfies $\sigma(f) \geq 1$, $f(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often, and $\lambda(f-a) = \sigma(f)$.*

**Lemma 2.5** *Let $w(z)$ be a transcendental meromorphic solution of delay differential equation*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = a_0(z), \tag{2.1}$$

*where $a(z)$ and $a_0(z)$ are nonzero rational functions. Then $\sigma(w) \geq 1$.*

**Proof** First, suppose that $w(z)$ has finitely many zeros and poles, then $w(z)$ is of regular order or infinite order. Obviously, $\sigma(w) \geq 1$.

Second, suppose that $w(z)$ has infinitely many zeros. Since $a(z)$ and $a_0(z)$ are rational functions, there exists $R > 0$ such that all zeros and poles of $a(z)$ and $a_0(z)$ lying in the region $D = \{z : |\Re z| < R, |\Im z| < R\}$. The region $\mathbb{C} \setminus D$ can be divided into four regions (see [1]):

$$D_1 = \{z : \Re z \geq R\}, \quad D_2 = \{z : \Re z \leq -R\},$$
$$D_3 = \{z : \Im z \geq R\}, \quad D_4 = \{z : \Im z \leq -R\}.$$

Choosing a zero $z_0$ of $w(z)$ such that $|z_0|$ is large enough and $a(z_0+1) \neq -a(z_0+3)$. Then $z_0 \in \mathbb{C} \setminus D$. Without loss of generality, we assume $z_0 \in D_1$. If $z_0 \in D_j (j = 2, 3, 4)$, the similar results can be obtained. The process will not interrupt, since $a(z)$ and $a_0(z)$ have only finitely many zeros and poles, but $w(z)$ has infinitely many zeros.

For convenience, let $k_i, i \in \mathbb{Z}$ denote the multiplicity of the poles of $w(z)$ at $z_0 + i$. Specially, $k_i = 0$ means $w(z_0 + i) \neq \infty$.

By (2.1), we see $z_0$ is a simple pole of $a(z)\frac{w'(z)}{w(z)}$, then either $z_0 + 1$ or $z_0 - 1$ is a pole of $w(z)$, and $k_1 + k_{-1} > 0$. Again by (2.1), we see if $k_1 \geq 2$, then $k_{-1} \geq 2$. Thus, we can divide the proof into the following three cases: $k_1 = 1,\ k_1 \geq 2,\ k_{-1} = 1$.

**Case 1** $k_1 = 1$.

Iterating (2.1) twice, we obtain

$$w(z+2) = w(z) - a(z+1)\frac{w'(z+1)}{w(z+1)} + a_0(z+1) \tag{2.2}$$

and

$$w(z+3) = w(z+1) - a(z+2)\frac{w'(z+2)}{w(z+2)} + a_0(z+2). \tag{2.3}$$

In (2.2), $w(z_0) = 0$, $z_0 + 1$ is a simple pole of $a(z)\frac{w'(z)}{w(z)}$, $a_0(z_0 + 1) \neq \infty$, then $z_0 + 2$ is a simple pole of $w(z)$. So, $k_2 = 1$. In (2.3), $z_0 + 1, z_0 + 2$ are simple poles of $w(z), a(z)\frac{w'(z)}{w(z)}$ respectively, then $w(z_0 + 3)$ may take three different values as follows.

**Case 1.1** $z_0 + 3$ is a simple pole of $w(z)$.

The next iteration will loop like the first step $k_1 = 1,\ k_2 = 1$.

**Case 1.2** $z_0 + 3$ is a zero of $w(z)$.

We assert $z_0 + 4$ is a pole of $w(z)$. Suppose to the contrary, $w(z_0 + 4) \neq \infty$. Iterating (2.3), we have

$$w(z + 4) = w(z + 2) - a(z + 3)\frac{w'(z + 3)}{w(z + 3)} + a_0(z + 3). \tag{2.4}$$

By $k_2 = 1$ and (2.2), for $z$ near $z_0 + 2$, we have

$$w(z) = \frac{a(z_0 + 1)}{z - (z_0 + 2)} + O(1), \tag{2.5}$$

while by $k_3 = 1, w(z_0 + 4) \neq \infty$ and (2.4), for $z$ near $z_0 + 2$, we have

$$w(z) = \frac{-a(z_0 + 3)}{z - (z_0 + 2)} + O(1).$$

But $a(z_0 + 1) \neq -a(z_0 + 3)$, a contradiction with (2.5). So, $z_0 + 4$ is a pole of $w(z)$. Combining this with $k_1 = k_2 = 1$, we have $k_4 = 1$.

We continue to iterate (2.4), then

$$w(z + 5) = w(z + 3) - a(z + 4)\frac{w'(z + 4)}{w(z + 4)} + a_0(z + 4). \tag{2.6}$$

By $k_3 = 1, k_4 = 1$ and (2.6), we obtain $k_5 = 1$. The next iteration will loop like the first step $k_1 = 1, k_2 = 1$.

**Case 1.3** $z_0 + 3$ is neither a pole nor a zero of $w(z)$.

We also have (2.4) and (2.6). By $k_2 = 1$ and $w(z_0 + 3) \neq 0, \infty$, we obtain $k_4 = 1$ from (2.4). Again by $k_4 = 1$ and $w(z_0 + 3) \neq 0, \infty$, we have $k_5 = 1$. The next iteration will loop like the first one $k_1 = 1, k_2 = 1$.

From above Cases 1.1–1.3, we see $w(z_0 + i)(i \in \mathbb{N})$ are simple poles of $w(z)$ possibly except $z_0 + n_j$, where $n_1 \geq 3,\ n_{j+1} \geq n_j + 3(j \in \mathbb{N})$. Hence, $\sigma(w) \geq \lambda\big(\frac{1}{w}\big) \geq 1$.

**Case 2** $k_1 \geq 2$.

Similar to Case 1, we also have (2.3) and (2.6).

In (2.3), $z_0 + 1$ is a pole of $w(z)$ with multiplicity $k_1 \geq 2$, $z_0 + 2$ is at most a simple pole of $a(z)\frac{w'(z)}{w(z)}$, $a_0(z_0 + 2) \neq \infty$, so $z_0 + 3$ is a pole of $w(z)$ with multiplicity $k_3 = k_1 \geq 2$.

By $k_3 \geq 2$ and (2.6), $z_0 + 4$ is at most a simple pole of $a(z)\frac{w'(z)}{w(z)}$, so $z_0 + 5$ is a pole of $w(z)$ with multiplicity $k_5 = k_3 = k_1 \geq 2$. Continuing this step, we see $z_0 + 2n - 1(n \in \mathbb{N})$ are poles of $w(z)$ with the same multiplicity $k_1 \geq 2$. Thus, $\sigma(w) \geq \lambda\big(\frac{1}{w}\big) \geq 1$.

**Case 3** $k_{-1} = 1$.

The proof is similar to the proof of Case 1, so we omit it.

Third, suppose that $w(z)$ has finitely many zeros. Using the same method similar to Step 3 of Lemma 2.6 below, we also obtain $\sigma(w) \geq 1$.

**Lemma 2.6** *Let $w(z)$ be a transcendental meromorphic solution of equation*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = a_1(z)w(z) + a_0(z), \tag{2.7}$$

*where $a(z),\ a_1(z),\ a_0(z)$ are nonzero rational functions such that $\frac{a(z)}{a_1(z)}$ is not a constant. Then $\sigma(w) \geq 1$.*

**Proof** First, suppose that $w(z)$ has finitely many zeros and poles, then $w(z)$ is of regular order or infinite order. Obviously, $\sigma(w) \geq 1$.

Second, suppose that $w(z)$ has infinitely many zeros. Since $a(z),\ a_1(z)$ and $a_0(z)$ are rational functions, there exists $R > 0$ such that all zeros and poles of $a(z),\ a_1(z)$ and $a_0(z)$, lie in the region $D = \{z : |\Re z| < R,\ |\Im z| < R\}$. The region $\mathbb{C} \setminus D$ can be divided into four regions (see [1]):

$$D_1 = \{z : \Re z \geq R\}, \quad D_2 = \{z : \Re z \leq -R\},$$
$$D_3 = \{z : \Im z \geq R\}, \quad D_4 = \{z : \Im z \leq -R\}.$$

Choosing a zero $z_0$ of $w(z)$, such that $|z_0|$ is large enough and $z_0 \in \mathbb{C} \setminus D$. For convenience, we assume $a(z)$ and $a_1(z)$ are polynomials. Without loss of generality, we assume $z_0 \in D_1$. If $z_0 \in D_j (j = 2, 3, 4)$, the similar results can be obtained. The process will not interrupt, since $a(z),\ a_1(z)$ and $a_0(z)$ have only finitely many zeros and poles, but $w(z)$ has infinitely many zeros.

Substituting $z_0$ into (2.7), $z_0$ is a simple pole of $a(z)\frac{w'(z)}{w(z)}$, then at least one of $z_0 + 1$ and $z_0 - 1$ is a pole of $w(z)$, and $k_1 + k_{-1} \geq 1$. Obviously, there are three cases: $k_1 \geq 2$, $k_1 = 1$ and $k_{-1} = 1$. If $z_0 + n (n \in \mathbb{Z})$ is a pole of $w(z)$ with multiplicity $k_n$, then $w(z)$ can be written as

$$w(z) = \frac{c_{-k_n}}{(z - z_0 - n)^{k_n}} + \frac{c_{-k_n+1}}{(z - z_0 - n)^{k_n - 1}} + \cdots + \frac{c_{-1}}{z - z_0 - n} + \varphi(z),$$

where $c_{-k_n}(\neq 0),\ c_{-k_n+1}, \cdots, c_{-1}$ are constants, $\varphi(z)$ is an analytic function.

We only concern the coefficient $c_{-k_n}$ of the first item in the principal part. In the following proof, it will be seen that $c_{-k_n}$ is the combinations of $a(z_0),\ a_1(z_0)$ and their shifts, which may be regarded as the polynomials $a(z),\ a_1(z)$ and their shifts at the value of $z_0$. That is, $c_{-k_n} = c_{-k_n}(z_0)$, and we discuss $c_{-k_n}(z)$ first.

**Case 1** $k_1 \geq 2$.

Shifting (2.7), we have

$$w(z+2) = w(z) + a_1(z+1)w(z+1) - a(z+1)\frac{w'(z+1)}{w(z+1)} + a_0(z+1). \tag{2.8}$$

In the right side of (2.8), $w(z_0) = 0$, $z_0 + 1$ is a pole of $w(z)$ with multiplicity $k_1 (\geq 2)$, a simple pole of $a(z)\frac{w'(z)}{w(z)}$. So, $z_0 + 2$ is a pole of $w(z)$ with multiplicity $k_1$, and

$$c_{-k_2}(z) = c_{-k_1} a_1(z+1). \tag{2.9}$$

Iterating (2.8), we obtain

$$w(z+3)=w(z+1)+a_1(z+2)w(z+2)-a(z+2)\frac{w'(z+2)}{w(z+2)}+a_0(z+2). \tag{2.10}$$

By (2.10), $k_1=k_2\geq 2$, $z_0+2$ is a simple pole of $a(z)\frac{w'(z)}{w(z)}$, then

$$c_{-k_3}(z)=c_{-k_1}+c_{-k_2}(z)a_1(z+2). \tag{2.11}$$

Shifting (2.7) $n$ ($n\geq 4$) times, we have

$$\begin{aligned}w(z+n)=&\,w(z+n-2)+a_1(z+n-1)w(z+n-1)\\&-a(z+n-1)\frac{w'(z+n-1)}{w(z+n-1)}+a_0(z+n-1).\end{aligned} \tag{2.12}$$

In (2.12), we obtain the recurrence formula

$$c_{-k_n}(z)=c_{-k_{n-2}}(z)+c_{-k_{n-1}}(z)a_1(z+n-1). \tag{2.13}$$

From (2.9), (2.11) and (2.13), we see $c_{-k_n}(z)$ is a polynomial. Assert that

$$\deg c_{-k_n}(z)=(n-1)\deg a_1(z),\quad n\in\mathbb{N}. \tag{2.14}$$

(i) For $n=1$, $c_{-k_1}$ is a constant, so $\deg c_{-k_1}=0=(1-1)\deg a_1(z)$.
(ii) Assume $\deg c_{-k_j}(z)=(j-1)\deg a_1(z)$, $j=2,\cdots,n-1$.
(iii) By (2.13) and $\deg c_{-k_{n-2}}(z)<\deg c_{-k_{n-1}}(z)$, we have

$$\begin{aligned}\deg c_{-k_n}(z)=&\deg c_{-k_{n-1}}(z)+\deg a_1(z+n-1)\\=&(n-2)\deg a_1(z)+\deg a_1(z)=(n-1)\deg a_1(z).\end{aligned}$$

The above (i)–(iii) show (2.14) holds. Obviously, $c_{-k_n}(z_0)\neq 0$ since $z_0$ is large enough. Hence, $z_0+n(n\in\mathbb{N})$ are poles of $w(z)$ with the same multiplicity $k_1$. So, $\sigma(w)\geq\lambda(\frac{1}{w})\geq 1$.

**Case 2** $k_1=1$.

Shifting (2.7), we also have (2.8), (2.10) and (2.12). In the right side of (2.8), $w(z_0)=0$, $z_0+1$ is a simple pole of $w(z)$ and $a(z)\frac{w'(z)}{w(z)}$. So,

$$c_{-2}(z)=c_{-1}a_1(z+1)+a(z+1). \tag{2.15}$$

We have $c_{-2}(z)\not\equiv 0$, by the fact that $\frac{a(z)}{a_1(z)}$ is not a constant. So, $z_0+2$ is a simple pole of $w(z)$. By this and (2.10), we have

$$c_{-3}(z)=c_{-1}+c_{-2}(z)a_1(z+2)+a(z+2). \tag{2.16}$$

By (2.12), we have the recurrence formula

$$c_{-n}(z)=c_{-n+2}(z)+c_{-n+1}(z)a_1(z+n-1)+a(z+n-1),\quad n\geq 3. \tag{2.17}$$

Assert that

$$c_{-n}(z) \not\equiv 0, \quad n \in \mathbb{N}.$$

We deduce from (2.15) that

$$0 \le \deg c_{-2}(z) \le \max\{\deg a_1, \deg a\}.$$

**Case 2.1** $\deg c_{-2}(z) = \max\{\deg a_1, \deg a\}$.

By the assumption and (2.16), we have

$$\deg c_{-3}(z) = \deg(c_{-2}(z)a_1(z+2)) = \deg a_1 + \max\{\deg a_1, \deg a\}.$$

By this and (2.17), for $n \ge 4$, we have

$$\begin{aligned}\deg c_{-n}(z) =& \deg(c_{-n+1}(z)a_1(z+n-1)) = \deg c_{-n+1}(z) + \deg a_1\\ =&(n-2)\deg a_1 + \max\{\deg a_1, \deg a\},\end{aligned}$$

which follows $c_{-n}(z) \not\equiv 0$.

**Case 2.2** $0 < \deg c_{-2}(z) < \max\{\deg a_1, \deg a\}$.

So, $\deg a_1 = \deg a$. By this and (2.16), we have

$$\deg c_{-3}(z) = \deg(c_{-2}(z)a_1(z+2)) = \deg a_1 + \deg c_{-2}.$$

By this and (2.17), for $n \ge 4$, we have

$$\begin{aligned}\deg c_{-n}(z) &= \deg(c_{-n+1}(z)a_1(z+n-1)) = \deg c_{-n+1}(z) + \deg a_1\\ &= (n-2)\deg a_1 + \deg c_{-2},\end{aligned}$$

which follows $c_{-n}(z) \not\equiv 0$.

**Case 2.3** $\deg c_{-2} = 0$.

So, $c_{-2}$ is a nonzero constant. By this and (2.16), we have

$$0 \le \deg c_{-3} \le \max\{\deg a_1, \deg a\}.$$

If $0 < \deg c_{-3} \le \max\{\deg a_1, \deg a\}$, using the same method similar to the above Cases 2.1–2.2, we can obtain $c_{-n}(z) \not\equiv 0$.

We only need to consider $\deg c_{-3} = 0$, which means $c_{-3}$ is a constant. By (2.15)–(2.16) and the fact that $c_{-2}, c_{-3}$ are constants, we have

$$\begin{cases} c_{-2} = c_{-1}a_1(z) + a(z), \\ c_{-3} = c_{-2}a_1(z) + a(z) + c_{-1}. \end{cases}$$

Minus the above equalities, we have

$$c_{-3} - c_{-2} = (c_{-2} - c_{-1})a_1(z) + c_{-1}.$$

Thus, $a_1$ is a constant or $c_{-2} = c_{-1}$. If $a_1$ is a constant, by $\deg a = \deg a_1 = 0$, we see both $a$ and $a_1$ are constants, which contradicts the fact $\frac{a(z)}{a_1(z)}$ is not a constant. So, $c_{-2} = c_{-1}$, which follows $c_{-3} = 2c_{-1}$.

Next we prove $\deg c_{-4} \geq 1$. Assume $c_{-4}$ is a constant, by (2.17), we have

$$c_{-4} = c_{-3}a_1(z) + a(z) + c_{-2}.$$

Together with $c_{-3} = c_{-2}a_1(z) + a(z) + c_{-1}$ and $c_{-3} = 2c_{-1},\ c_{-2} = c_{-1}$, we have

$$c_{-4} - c_{-3} = (c_{-3} - c_{-2})a_1(z) + c_{-2} - c_{-1} = c_{-1}a_1(z),$$

which follows $a_1(z) = \frac{c_{-4}-c_{-3}}{c_{-1}}$ is a constant. By $\deg a = \deg a_1 = 0$, which contradicts the fact $\frac{a(z)}{a_1(z)}$ is not a constant. So $\deg c_{-4} \geq 1$.

By (2.17), we see $c_{-5}(z) = c_{-3} + c_{-4}(z)a_1(z+4) + a(z+4)$, hence,

$$\deg c_{-5} = \deg(c_{-4}(z)a_1(z+4)) = \deg c_{-4} + \deg a_1.$$

Using a same method similar to the above Case 2.2, we have

$$\deg c_{-n} = (n-4)\deg a_1 + \deg c_{-4}, \quad n \geq 4.$$

Combining this with $c_{-1}, c_{-2} = c_{-1}, c_{-3} = 2c_{-1}$, we have $c_n(z) \not\equiv 0,\ n \in \mathbb{N}$.

The above Cases 2.1–2.3 show $c_n(z) \not\equiv 0 (n \in \mathbb{N})$. Since $z_0$ is large enough, $c_{-n}(z_0) \neq 0$, that is, $z_0 + n$ are simple poles of $w(z)$, therefore, $\sigma(w) \geq \lambda\big(\frac{1}{w}\big) \geq 1$.

**Case 3** $k_{-1} = 1$.

Using the same method similar to the above Case 2, we may prove $\sigma(w) \geq \lambda\big(\frac{1}{w}\big) \geq 1$.

Suppose that $w(z)$ has finitely many zeros. Assume $\sigma(w) = \sigma < 1$. By equation (2.7), we have

$$\frac{w(z+1)}{w(z)} - \frac{w(z-1)}{w(z)} + a(z)\frac{w'(z)}{w^2(z)} = a_1(z) + \frac{a_0(z)}{w(z)}.$$

Let $y(z) = \frac{1}{w(z)}$, then $\sigma(y) = \sigma(w) = \sigma < 1$. The last equality shows

$$\frac{y(z)}{y(z+1)} - \frac{y(z)}{y(z-1)} - a(z)y'(z) = a_1(z) + a_0(z)y(z)$$

or

$$a(z)y'(z) + a_0(z)y(z) = c(z), \tag{2.18}$$

where

$$c(z) = \frac{y(z)}{y(z+1)} - \frac{y(z)}{y(z-1)} - a_1(z). \tag{2.19}$$

Applying Lemma 2.1 to (2.19), for any given $\epsilon$ $(0 < \epsilon < 1 - \sigma)$, we have

$$m(r, c(z)) \leq m\Big(r, \frac{y(z)}{y(z+1)}\Big) + m\Big(r, \frac{y(z)}{y(z-1)}\Big) + m(r, a_1(z))$$

$$= O(r^{\sigma-1+\epsilon}) + O(\log r) = O(\log r).$$

Since $w(z)$ has finitely many zeros, $y(z)$ has finitely many poles. By this and (2.18), we have

$$\begin{aligned} N(r, c(z)) &\le N(r, y') + N(r, a(z)) + N(r, a_0(z)) \\ &\le 2N(r, y) + O(\log r) = O(\log r). \end{aligned} \tag{2.20}$$

By (2.20) and $m(r, c) = O(\log r)$, we have

$$T(r, c(z)) = m(r, c(z)) + N(r, c(z)) = O(\log r),$$

that is, $c(z)$ is a rational function, and $c^*(z) = c(z) + a_1(z)$ is a rational function, too. By (2.19), we have

$$\frac{y(z)}{y(z+1)} - \frac{y(z)}{y(z-1)} = c^*(z). \tag{2.21}$$

If $y(z+1) \equiv y(z-1)$, then $y(z+2) \equiv y(z)$, that is, $y(z)$ is a nonconstant periodic function, obviously $\sigma(y) \ge 1$, a contradiction. So, $y(z+1) \not\equiv y(z-1)$. By this and (2.21), we have $c^*(z) \not\equiv 0$.

Set $c^*(z) = \frac{P(z)}{Q(z)}$, where $P(z)$, $Q(z)$ are nonzero polynomials. By this and substituting $y(z) = \frac{1}{w(z)}$ into (2.21), we have

$$\frac{w(z+1)}{w(z)} - \frac{w(z-1)}{w(z)} = \frac{P(z)}{Q(z)}$$

and

$$Q(z)w(z+1) - Q(z)w(z-1) - P(z)w(z) = 0. \tag{2.22}$$

In (2.22), the coefficients satisfy

$$Q(z) + (-Q(z)) + (-P(z)) = -P(z) \not\equiv 0.$$

Applying Lemma 2.4 to (2.22) yields a contradiction.

**Lemma 2.7** (see [10, Lemma 1.9]) *Let $g_j(z)(j = 1, 2, \cdots, n)$ be entire functions and $a_j(z)(j = 0, 1, \cdots, n)$ be meromorphic functions satisfying*

$$T(r, a_j) = o\Big(\sum_{k=1}^{n} T(r, \mathrm{e}^{g_k})\Big), \quad r \to \infty,\ r \notin E,\ j = 0, 1, \cdots, n.$$

*If*

$$\sum_{j=1}^{n} a_j(z)\mathrm{e}^{g_j(z)} \equiv a_0(z),$$

*then there exist constants $c_j, j = 1, 2, \cdots, n$, at least one of them is not zero, such that*

$$\sum_{j=1}^{n} c_j a_j(z)\mathrm{e}^{g_j(z)} \equiv 0.$$

**Lemma 2.8** (see [10, Theorem 1.50]) *Suppose that* $f_1(z), f_2(z), \cdots, f_n(n \geq 2)$ *are meromorphic functions satisfying the following conditions*:

(i) $\sum\limits_{j=1}^{n} C_j f_j(z) \equiv 0$, *where* $C_j (j = 1, 2, \cdots, n)$ *are constants*;

(ii) $f_j(z) \not\equiv 0 (j = 1, 2, \cdots, n)$, *and* $\frac{f_j(z)}{f_k(z)}$ *are not constants for* $1 \leq j < k \leq n$;

(iii) $\sum\limits_{j=1}^{n} \big(N(r, f_j) + N\big(r, \frac{1}{f_j}\big)\big) = o(\tau(r))(r \to \infty, r \notin E)$, *where* $\tau(r) = \min\limits_{1 \leq j < k \leq n} \big\{T\big(r, \frac{f_j}{f_k}\big)\big\}$.

*Then* $C_j = 0 (j = 1, 2, \cdots, n)$.

**Lemma 2.9** (see [8, Lemma 3]) *Suppose that* $h$ *is a nonconstant meromorphic function satisfying*

$$\overline{N}(r, h) + \overline{N}\Big(r, \frac{1}{h}\Big) = S(r, h).$$

*Let* $f = a_p h^p + a_{p-1} h^{p-1} + \cdots + a_0$ *and* $g = b_q h^q + b_{q-1} h^{q-1} + \cdots + b_0$ *be polynomials in* $h$ *with coefficients* $a_0, a_1, \cdots, a_p, b_0, b_1, \cdots, b_q$ *being small functions of* $h$ *and* $a_p b_q a_0 \not\equiv 0$. *If* $q \leq p$, *then* $m\big(r, \frac{g}{f}\big) = S(r, h)$.

**Lemma 2.10** *Let* $w(z) = H(z)\mathrm{e}^{h(z)}$, *where* $H(z)$ *is a nonzero small function of* $\mathrm{e}^{h(z)}$, $P(z, w)$ *and* $Q(z, w)$ *be polynomials in* $w$ *with coefficients being small functions of* $w$ *and* $Q(z, 0) \not\equiv 0$. *If* $p = \deg_w P \leq \deg_w Q = q$, *then*

$$m\Big(r, \frac{P(z, w)}{Q(z, w)}\Big) = S(r, \mathrm{e}^h) = S(r, w).$$

**Proof** Denote

$$\begin{cases} P(z, w) = a_p w^p + a_{p-1} w^{p-1} + \cdots + a_0, \\ Q(z, w) = b_q w^q + b_{q-1} w^{q-1} + \cdots + b_0, \end{cases}$$

where $a_p (\not\equiv 0), \cdots, a_0, b_q (\not\equiv 0),\ b_1, \cdots, b_0$ are small functions of $w$.

Substituting $w(z) = H(z)\mathrm{e}^{h(z)}$ into $P(z, w)$ and $Q(z, w)$ respectively, we obtain

$$\begin{aligned} P(z, w) &= a_p(z) H^p(z) \mathrm{e}^{ph(z)} + \cdots + a_1(z) H(z) \mathrm{e}^{h(z)} + a_0(z) \\ &= a_p^*(z) \mathrm{e}^{ph(z)} + \cdots + a_1^*(z) \mathrm{e}^{h(z)} + a_0^*(z) \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} Q(z, w) &= b_q(z) H^q(z) \mathrm{e}^{qh(z)} + \cdots + b_1(z) H(z) \mathrm{e}^{h(z)} + b_0(z) \\ &= b_q^*(z) \mathrm{e}^{qh(z)} + \cdots + b_1^*(z) \mathrm{e}^{h(z)} + b_0^*(z), \end{aligned} \tag{2.24}$$

where

$$a_j^*(z) = a_j(z) H^j(z), \quad b_k^*(z) = b_k(z) H^k(z), \quad j = 0, \cdots, p,\ k = 0, \cdots, q. \tag{2.25}$$

Obviously, for $j = 0, \cdots, p,\ k = 0, \cdots, q$,

$$T(r, a_j^*) = S(r, w) = S(r, \mathrm{e}^h), \quad T(r, b_k^*) = S(r, w) = S(r, \mathrm{e}^h). \tag{2.26}$$

We note that

$$N(r, \mathrm{e}^h) + N\Big(r, \frac{1}{\mathrm{e}^h}\Big) = 0. \tag{2.27}$$

By (2.23)–(2.27), we see $P(z, w)$ and $Q(z, w)$ can also be regarded as polynomials in $\mathrm{e}^h$ with coefficients $a_0^*, a_1^*, \cdots, a_p^*, b_0^*, b_1^*, \cdots, b_q^*$ being small functions of $\mathrm{e}^h$.

Since $Q(z, 0) \not\equiv 0$, we obtain $b_0(z) \not\equiv 0$. Thus,

$$a_p^*(z)b_q^*(z)b_0^*(z) = a_p(z)b_q(z)b_0(z)H^{p+q}(z) \not\equiv 0.$$

Combining this with $p \le q$ and Lemma 2.9, we have

$$m\Big(r, \frac{P(z, w)}{Q(z, w)}\Big) = S(r, \mathrm{e}^h) = S(r, w).$$

## 3 Main Results

In the following, we give answers to Problems 1.1–1.2, and find interesting properties on entire solutions of (1.1) depending on the degrees of $P(z, w)$ and $Q(z, w)$.

### 3.1 $\deg_w P = \deg_w Q = 0$

(1.1) reduces into

$$w(z + 1) - w(z - 1) + a(z)\frac{w'(z)}{w(z)} = a_0(z),$$

where $a(z)$ and $a_0(z)$ are rational functions.

Halburd and Korhonen [5] pointed out if $a_0(z) = \mathrm{i}k\pi a(z) (k \in \mathbb{N})$, then $w(z) = C\mathrm{e}^{\mathrm{i}k\pi z}$, $C \ne 0$ is a one-parameter family of zero-free entire transcendental finite-order solution of (2.1) for any rational function $a(z)$. The following Theorems 3.1–3.2 deal with Problems 1.1–1.2 for (2.1), respectively.

**Theorem 3.1** *Let $w(z)$ be a transcendental meromorphic solution of delay differential equation (2.1), where $a(z), a_0(z)$ are nonzero rational functions. Then $\sigma(w) \ge 1$.*

**Proof** Theorem 3.1 arrives quickly by Lemma 2.5.

**Theorem 3.2** *All transcendental entire solutions $w(z)$ of (2.1) have the forms*

$$w(z) = C\mathrm{e}^{\mathrm{i}k\pi z}, \quad C \in \mathbb{C} \setminus \{0\},\ k \in \mathbb{Z} \setminus \{0\}.$$

**Proof** Let $w(z)$ be a transcendental entire solution of (2.1). If $w(z)$ has infinitely many zeros, then $a(z)\frac{w'(z)}{w(z)}$ has infinitely many poles, but $w(z+1)-w(z-1)-a_0(z)$ has only finitely many poles, a contradiction. So, $w(z)$ has finitely many zeros. By this, $w(z)$ has the following form

$$w(z) = H(z)\mathrm{e}^{h(z)}, \tag{3.1}$$

where $H(z)$ is a nonzero polynomial, $h(z)$ is a nonconstant entire function.

Substituting (3.1) into (2.1), we have

$$H(z+1)\mathrm{e}^{h(z+1)} - H(z-1)\mathrm{e}^{h(z-1)} = c(z), \tag{3.2}$$

where

$$c(z) = a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big). \tag{3.3}$$

**Case 1** $c(z) \not\equiv 0$.

By (3.2), we have

$$H(z+2)\mathrm{e}^{h(z+2)} + H(z+1)\mathrm{e}^{h(z+1)} - H(z)\mathrm{e}^{h(z)} - H(z-1)\mathrm{e}^{h(z-1)} = c(z+1) + c(z). \tag{3.4}$$

By (3.3), we have

$$T(r, c(z)) = T(r, h'(z)) + O(\log r) = S(r, \mathrm{e}^{h(z)}).$$

So,

$$T(r, c(z+1) + c(z)) = S(r, \mathrm{e}^{h(z+1)}) + S(r, \mathrm{e}^{h(z)}). \tag{3.5}$$

We note that $T(r, H(z+j-1)) = O(\log r),\ j = 0, 1, 2, 3$. We obtain from (3.4)–(3.5) and Lemma 2.7 that there exist constants $c_0, c_1, c_2, c_3$, at least one of them is not zero, such that

$$c_3H(z+2)\mathrm{e}^{h(z+2)} + c_2H(z+1)\mathrm{e}^{h(z+1)} - c_1H(z)\mathrm{e}^{h(z)} - c_0H(z-1)\mathrm{e}^{h(z-1)} = 0. \tag{3.6}$$

Let

$$f_j(z) = H(z+j-1)\mathrm{e}^{h(z+j-1)}, \quad j = 0, 1, 2, 3. \tag{3.7}$$

By (3.6)–(3.7), we have

$$c_3f_3(z) + c_2f_2(z) - c_1f_1(z) - c_0f_0(z) = 0. \tag{3.8}$$

**Case 1.1** Suppose none of $h(z+1)-h(z), h(z+2)-h(z), h(z+2)-h(z-1)$ are constants. Then

$$\frac{f_j(z)}{f_k(z)} = \frac{H(z+j-1)}{H(z+k-1)}\mathrm{e}^{h(z+j)-h(z+k)}, \quad 0 \le j < k \le 3 \text{ are transcendental.}$$

Obviously,

$$\sum_{j=0}^{3}\Big(N(r, f_j) + N\Big(r, \frac{1}{f_j}\Big)\Big) = O(\log r) = o(\tau(r)), \tag{3.9}$$

where $\tau(r) = \min\limits_{0\le j<k\le 3}\big\{T\big(r, \frac{f_j}{f_k}\big)\big\}$. Thus, we obtain from (3.8)–(3.9) and Lemma 2.8 that $c_j = 0,\ j = 0, 1, 2, 3$, a contradiction.

**Case 1.2** Suppose that $h(z+1)-h(z)=d$ is a constant. Then

$$h(z+1)=h(z)+d,\quad h(z-1)=h(z)-d.$$

By (3.2) and $c(z)\not\equiv 0$, we have

$$c(z)=(\mathrm{e}^{d}H(z+1)-\mathrm{e}^{-d}H(z-1))\mathrm{e}^{h(z)}\not\equiv 0. \tag{3.10}$$

Hence $\mathrm{e}^{d}H(z+1)-\mathrm{e}^{-d}H(z-1)\not\equiv 0$.

By (3.3) and (3.10), we obtain

$$T(r,\mathrm{e}^{h(z)})=T(r,c(z))+O(\log r)=T(r,h'(z))+O(\log r)=S(r,\mathrm{e}^{h(z)}),$$

a contradiction.

**Case 1.3** Suppose that $h(z+2)-h(z)=d$ is a constant. Then

$$h(z+2)=h(z)+d,\quad h(z+1)=h(z-1)+d.$$

Substituting the above equalities into (3.4), we have

$$(\mathrm{e}^{d}H(z+2)-H(z))\mathrm{e}^{h(z)}+(\mathrm{e}^{d}H(z+1)-H(z-1))\mathrm{e}^{h(z-1)}=c(z+1)+c(z). \tag{3.11}$$

By (3.2) and $c(z)\not\equiv 0$ , we have

$$c(z)=H(z+1)\mathrm{e}^{h(z+1)}-H(z-1)\mathrm{e}^{h(z-1)}=(\mathrm{e}^{d}H(z+1)-H(z-1))\mathrm{e}^{h(z-1)}\not\equiv 0,$$

and so,

$$\mathrm{e}^{d}H(z+1)-H(z-1)\not\equiv 0,\quad \mathrm{e}^{d}H(z+2)-H(z)\not\equiv 0. \tag{3.12}$$

By $h(z+1)=h(z-1)+d$, we conclude

$$T(r,c(z+1)+c(z))=S(r,\mathrm{e}^{h(z)})+S(r,\mathrm{e}^{h(z+1)})=S(r,\mathrm{e}^{h(z)})+S(r,\mathrm{e}^{h(z-1)}).$$

Applying Lemma 2.7 to (3.11), there exist constants $c_4,c_5$, at least one of them is not zero, such that

$$c_4(\mathrm{e}^{d}H(z+2)-H(z))\mathrm{e}^{h(z)}+c_5(\mathrm{e}^{d}H(z+1)-H(z-1))\mathrm{e}^{h(z-1)}=0. \tag{3.13}$$

By (3.12)–(3.13), $c_4=0$ if and only if $c_5=0$. So $c_4c_5\neq 0$.

If $h(z)-h(z-1)$ is not constant, we have from (3.13) that

$$\mathrm{e}^{h(z)-h(z-1)}=-\frac{c_5}{c_4}\frac{\mathrm{e}^{d}H(z+1)-H(z-1)}{\mathrm{e}^{d}H(z+2)-H(z)},$$

which yields

$$T(r,\mathrm{e}^{h(z)-h(z-1)})=T\Big(r,\frac{\mathrm{e}^{d}H(z+1)-H(z-1)}{\mathrm{e}^{d}H(z+2)-H(z)}\Big)=O(\log r),$$

a contradiction.

Thus, $h(z)-h(z-1)$ is a constant, say $h(z)-h(z-1)=d_1$. Thus, $h(z+1)-h(z)=d_1$. Together with Case 1.2, we again obtain a contradiction.

**Case 1.4** Suppose that $h(z+2)-h(z-1)=d$ is a constant. Then (3.4) can be written as

$$\begin{aligned}&H(z+1)\mathrm{e}^{h(z+1)}-H(z)\mathrm{e}^{h(z)}+(\mathrm{e}^dH(z+2)-H(z-1))\mathrm{e}^{h(z-1)}\\&=c(z+1)+c(z).\end{aligned}\tag{3.14}$$

**Case 1.4.1** $\mathrm{e}^dH(z+2)-H(z-1)\equiv 0$.

(3.14) can be written as

$$H(z+1)\mathrm{e}^{h(z+1)}-H(z)\mathrm{e}^{h(z)}=c(z+1)+c(z).\tag{3.15}$$

By (3.5), applying Lemma 2.7 to (3.15), there exist constants $c_6, c_7$, at least one of them is not zero, such that

$$c_6H(z+1)\mathrm{e}^{h(z+1)}-c_7H(z)\mathrm{e}^{h(z)}=0.\tag{3.16}$$

By (3.16), $c_6=0$ if and only if $c_7=0$. So, $c_6c_7\neq 0$.

If $h(z+1)-h(z)$ is not constant, by (3.16), we have

$$T(r,\mathrm{e}^{h(z+1)-h(z)})=T\Big(r,\frac{H(z)}{H(z+1)}\Big)=O(\log r),$$

a contradiction.

Thus, $h(z+1)-h(z)$ is a constant, say $h(z+1)-h(z)=d_1$. Together with Case 1.2, we also obtain a contradiction.

**Case 1.4.2** $\mathrm{e}^dH(z+2)-H(z-1)\not\equiv 0$.

By (3.5) and applying Lemma 2.7 to (3.14), there exist constants $c_8, c_9, c_{10}$, at least one of them is not zero, such that

$$c_8H(z+1)\mathrm{e}^{h(z+1)}-c_9H(z)\mathrm{e}^{h(z)}+c_{10}(\mathrm{e}^dH(z+2)-H(z-1))\mathrm{e}^{h(z-1)}=0.\tag{3.17}$$

If one of $h(z+1)-h(z), h(z+1)-h(z-1)$ is a constant, by Case 1.2 and Case 1.3, respectively, we obtain a contradiction.

So, neither $h(z+1)-h(z)$ nor $h(z+1)-h(z-1)$ are constants. So is $h(z)-h(z-1)$. Set

$$f_1(z)=H(z+1)\mathrm{e}^{h(z+1)},\quad f_2(z)=H(z)\mathrm{e}^{h(z)},\quad f_3(z)=(\mathrm{e}^dH(z+2)-H(z-1))\mathrm{e}^{h(z-1)}.$$

Thus, (3.17) can be written as

$$c_8f_1(z)-c_9f_2(z)+c_{10}f_3(z)=0.\tag{3.18}$$

Obviously,

$$\frac{f_1(z)}{f_2(z)}=\frac{H(z+1)}{H(z)}\mathrm{e}^{h(z+1)-h(z)},$$

$$\frac{f_1(z)}{f_3(z)}=\frac{H(z+1)}{\mathrm{e}^d H(z+2)-H(z-1)}\mathrm{e}^{h(z+1)-h(z-1)},$$
$$\frac{f_2(z)}{f_3(z)}=\frac{H(z)}{\mathrm{e}^d H(z+2)-H(z-1)}\mathrm{e}^{h(z)-h(z-1)}$$

are transcendental, and so

$$\sum_{j=1}^{3}\Big(N(r,f_j)+N\Big(r,\frac{1}{f_j}\Big)\Big)=O(\log r)=o(\tau(r)),\tag{3.19}$$

where $\tau(r)=\min\limits_{1\le j<k\le 3}\big\{T\big(r,\frac{f_j}{f_k}\big)\big\}$.

Applying Lemma 2.8 to (3.18), we have $c_j=0,\ j=8,9,10$, a contradiction.

**Case 2** $c(z)\equiv 0$.

It follows (3.3) that $h'(z)=\frac{a_0(z)}{a(z)}-\frac{H'(z)}{H(z)}$. Since $h(z)$ is an entire function, $h(z)$ must be a polynomial. By (3.2), we have

$$\mathrm{e}^{h(z+1)-h(z-1)}=\frac{H(z-1)}{H(z+1)}.\tag{3.20}$$

So, $H(z)$ must be a nonzero constant, otherwise, (3.20) shows

$$T(r,\mathrm{e}^{h(z+1)-h(z-1)})=T\Big(r,\frac{H(z-1)}{H(z+1)}\Big)=O(\log r),$$

a contradiction.

Thus, $h(z+1)-h(z-1)$ is also a constant and $\deg h=1$. Set $h(z)=d_1z+d_0$, where $d_1(\neq 0),\ d_0$ are constants. Substituting $h(z)=d_1z+d_0$ into (3.20), we have $\mathrm{e}^{2d_1}=1$ and so $d_1=\mathrm{i}k\pi,\ k\in\mathbb{Z}\setminus\{0\}$. Thus,

$$w(z)=H\mathrm{e}^{h(z)}=H\mathrm{e}^{\mathrm{i}k\pi z+c_4}=C\mathrm{e}^{\mathrm{i}k\pi z},\quad C\in\mathbb{C}\setminus\{0\},\ k\in\mathbb{Z}\setminus\{0\}.$$

### 3.2 $\deg_w P=1,\ \deg_w Q=0$

Halburd and Korhonen [5] obtained the following theorem.

**Theorem 3.3** (see [5, Lemma 3.2]) *Let $w(z)$ be a non-rational meromorphic solution of equation*

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=P(z,w),\tag{3.21}$$

*where $a(z)$ is rational in $z$ and $P(z,w)$ is a polynomial in $w$ and rational in $z$. If the hyper-order of $w$ is less than one, then* $\deg_w P\le 1$.

By Theorem 3.3, (3.21) has the form

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=a_1(z)w(z)+a_0(z),$$

where $a(z),a_1(z),a_0(z)$ are rational functions.

Zhang and Huang [12] investigated the value distribution and representation of entire solution of (2.7) as follows.

**Theorem 3.4** (see [12, Theorem 3.1]) *Let $a(z), a_0(z)$ and $a_1(z)$ be rational functions with $a_1(z) \not\equiv 0$ or $a_0(z) \not\equiv 0$, and let $w(z)$ be a transcendental entire solution of (2.7) with $\sigma_2(w) < 1$.*

(i) *If $a(z) \equiv 0$, then $\sigma(w) \geq 1$;*

(ii) *If $a(z) \not\equiv 0$, then $w(z) = H(z)\mathrm{e}^{dz}$, where $H(z)$ is a polynomial, and $d \neq 0$ is a complex number. Especially, if $a_1(z)$ is a polynomial with $a_1(z) \not\equiv \pm 2\mathrm{i}$, then $w(z) = C\mathrm{e}^{dz}$, where $C \in \mathbb{C} \setminus \{0\}$; if $a_1(z) \equiv \pm 2\mathrm{i}$, then $w(z) = (C_1 z + C_0)\mathrm{e}^{(2k \pm \frac{1}{2})\pi \mathrm{i} z}$, where $k$ is an integer and $C_1, C_0 \in \mathbb{C}$ with $|C_1| + |C_0| \neq 0$.*

Thus, we deal with Problems 1.1–1.2 for (2.7), and obtain the following theorem.

**Theorem 3.5** *Let $w(z)$ be a transcendental meromorphic solution of (2.7), where $a(z)$, $a_1(z)$, $a_0(z)$ are nonzero rational functions such that $\frac{a(z)}{a_1(z)}$ is not a constant. Then $\sigma(w) \geq 1$.*

**Proof** Theorem 3.5 arrives quickly by Lemma 2.6.

**Example 3.1** The function $w(z) = \frac{1}{\mathrm{e}^{\mathrm{i}\pi z} - 1}$ satisfies delay differential equation

$$w(z+1) - w(z-1) + \frac{w'(z)}{w(z)} = -\mathrm{i}\pi z w(z) - \mathrm{i}\pi z,$$

which satisfies conditions and results of Theorem 3.5.

**Theorem 3.6** *All transcendental entire solutions $w(z)$ of (2.7) have the form*

$$w(z) = H(z)\mathrm{e}^{dz}, \quad d \in \mathbb{C} \setminus \{0\}.$$

*Here, $H(z)$ is a nonzero polynomial, and*

$$\frac{a_0(z)}{a(z)} = d + \frac{H'(z)}{H(z)}, \quad a_1(z) = \frac{\mathrm{e}^{d} H(z+1)}{H(z)} - \frac{\mathrm{e}^{-d} H(z-1)}{H(z)}.$$

**Proof** Let $w(z)$ be a transcendental entire solution of (2.7). If $w(z)$ has infinitely many zeros, then $a(z)\frac{w'(z)}{w(z)}$ has infinitely many poles, but $w(z+1) - w(z-1) - a_1(z)w(z) - a_0(z)$ has only finitely many poles, a contradiction. So, $w(z)$ has finitely many zeros and can be written as

$$w(z) = H(z)\mathrm{e}^{h(z)}, \tag{3.22}$$

where $H(z)$ is a nonzero polynomial, $h(z)$ is a transcendental entire function.

Substituting (3.22) into (2.7), we have

$$\begin{aligned} &H(z+1)\mathrm{e}^{h(z+1)} - H(z-1)\mathrm{e}^{h(z-1)} - a_1(z)H(z)\mathrm{e}^{h(z)} \\ &= a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big). \end{aligned} \tag{3.23}$$

We assert that

$$a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big) \equiv 0.$$

Otherwise, if $a_0(z)-a(z)\big(\frac{H'(z)}{H(z)}+h'(z)\big)\not\equiv 0$. We have from (3.23) that

$$\begin{cases} T\Big(r,a_0-a\Big(\dfrac{H'}{H}+h'\Big)\Big)=T(r,h')+O(\log r)=S(r,\mathrm{e}^{h(z)}),\\ T(r,H(z+1))=O(\log r),\quad T(r,H(z-1))=O(\log r),\\ T(r,a_1(z)H(z))=O(\log r). \end{cases} \tag{3.24}$$

Applying Lemma 2.7 to (3.23), there exist constants $c_1,c_2,c_3$, at least one of them is not zero, such that

$$c_1H(z+1)\mathrm{e}^{h(z+1)}-c_2H(z-1)\mathrm{e}^{h(z-1)}-c_3a_1(z)H(z)\mathrm{e}^{h(z)}=0. \tag{3.25}$$

**Case 1** $c_1\neq 0$.

By (3.25), we have

$$H(z+1)\mathrm{e}^{h(z+1)}=\frac{c_2}{c_1}H(z-1)\mathrm{e}^{h(z-1)}+\frac{c_3}{c_1}a_1(z)H(z)\mathrm{e}^{h(z)}. \tag{3.26}$$

Substituting (3.26) into (3.23), we have

$$\begin{aligned}&\Big(\frac{c_2}{c_1}-1\Big)H(z-1)\mathrm{e}^{h(z-1)}+\Big(\frac{c_3}{c_1}-1\Big)a_1(z)H(z)\mathrm{e}^{h(z)}\\ &=a_0(z)-a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big).\end{aligned} \tag{3.27}$$

If $\frac{c_2}{c_1}-1=0$, then (3.27) can be written as

$$\Big(\frac{c_3}{c_1}-1\Big)a_1(z)H(z)\mathrm{e}^{h(z)}=a_0(z)-a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big).$$

By $a_0(z)-a(z)\big(\frac{H'(z)}{H(z)}+h'(z)\big)\not\equiv 0$, we have $\frac{c_3}{c_1}-1\neq 0$. By (3.24) and the last equality, we obtain

$$T(r,\mathrm{e}^{h(z)})=T\Big(r,a_0(z)-a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big)\Big)+O(\log r)=S(r,\mathrm{e}^{h(z)}),$$

a contradiction. So, $\frac{c_2}{c_1}-1\neq 0$.

If $\frac{c_3}{c_1}-1=0$, then (3.27) shows

$$\Big(\frac{c_2}{c_1}-1\Big)H(z-1)\mathrm{e}^{h(z-1)}=a_0(z)-a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big). \tag{3.28}$$

By (3.23) and (3.28), we have

$$\begin{aligned}&H(z+1)\mathrm{e}^{h(z+1)}-a_1(z)H(z)\mathrm{e}^{h(z)}\\ &=\frac{c_2}{c_2-c_1}\Big(a_0(z)-a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big)\Big).\end{aligned} \tag{3.29}$$

Applying Lemma 2.7 to (3.29), there exist constants $c_4$ and $c_5$, at least one of them is not zero, such that

$$c_4H(z+1)\mathrm{e}^{h(z+1)}-c_5a_1(z)H(z)\mathrm{e}^{h(z)}=0. \tag{3.30}$$

By (3.30), $c_4 = 0$ if and only if $c_5 = 0$. So, $c_4c_5 \neq 0$. By (3.30), we have

$$\mathrm{e}^{h(z+1)-h(z)} = \frac{c_5 a_1(z)H(z)}{c_4 H(z+1)}. \tag{3.31}$$

If $h(z+1) - h(z)$ is not a constant, then (3.31) shows

$$T(r, \mathrm{e}^{h(z+1)-h(z)}) = O(\log r),$$

a contradiction.

So, $h(z+1) - h(z)$ is a constant, say $h(z+1) - h(z) = d$. Hence, $h(z-1) = h(z) - d$, together with (3.28), we have

$$\Big(\frac{c_2}{c_1} - 1\Big)\mathrm{e}^{-d} H(z-1)\mathrm{e}^{h(z)} = a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big).$$

Combining this with (3.24),

$$T(r, \mathrm{e}^{h(z)}) = T(r, h'(z)) + O(\log r) = S(r, \mathrm{e}^{h(z)}),$$

a contradiction. So, $\frac{c_3}{c_1} - 1 \neq 0$.

Applying Lemma 2.7 to (3.27), there exist constants $c_6$ and $c_7$, at least one of them is not zero, such that

$$c_6\Big(\frac{c_2}{c_1} - 1\Big)H(z-1)\mathrm{e}^{h(z-1)} + c_7\Big(\frac{c_3}{c_1} - 1\Big)a_1(z)H(z)\mathrm{e}^{h(z)} = 0. \tag{3.32}$$

By $\frac{c_2}{c_1} - 1 \neq 0, \frac{c_3}{c_1} - 1 \neq 0$, in (3.32), if $c_6 = 0$, then $c_7 = 0$, and vice versa. So, $c_6 \neq 0$, $c_7 \neq 0$. By (3.32), we have

$$H(z-1)\mathrm{e}^{h(z-1)} = -\frac{c_7\Big(\dfrac{c_3}{c_1} - 1\Big)}{c_6\Big(\dfrac{c_2}{c_1} - 1\Big)} a_1(z)H(z)\mathrm{e}^{h(z)}.$$

Substituting the last equality into (3.27), we have

$$\Big(\frac{c_3}{c_1} - 1\Big)\Big(1 - \frac{c_7}{c_6}\Big)a_1(z)H(z)\mathrm{e}^{h(z)} = a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big). \tag{3.33}$$

By $a_0(z) - a(z)\big(\frac{H'(z)}{H(z)} + h'(z)\big) \not\equiv 0$, $1 - \frac{c_7}{c_6} \neq 0$ holds in (3.33). However, (3.24) and (3.33) show that $T(r, \mathrm{e}^{h(z)}) = S(r, \mathrm{e}^{h(z)})$, a contradiction.

**Case 2** $c_2 \neq 0$.

Using a same method similar to Case 1, we also obtain a contradiction.

**Case 3** $c_3 \neq 0$.

We see from (3.25) that at least one of $c_1$ and $c_2$ is not zero. If not, $c_1 = c_2 = 0$, then (3.25) can be written as $c_3 a_1(z)H(z)\mathrm{e}^{h(z)} = 0$, a contradiction. By Cases 1–2, we may obtain a contradiction.

Thus, from above three cases, we have

$$a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big) \equiv 0,$$

and so $h(z)$ is a polynomial.

(3.23) shows

$$H(z+1)e^{h(z+1)-h(z)+h(z)-h(z-1)} - a_1(z)H(z)e^{h(z)-h(z-1)} - H(z-1) = 0. \tag{3.34}$$

If $\deg h(z) \geq 2$, then $\deg(h(z) - h(z-1)) \geq 1$. Set $y(z) = e^{h(z)-h(z-1)}$, then (3.34) can be written as

$$H(z+1)y(z+1)y(z) - a_1(z)H(z)y(z) - H(z-1) = 0.$$

Set

$$P(z, y) = H(z+1)y(z+1)y(z) - a_1(z)H(z)y(z) - H(z-1) \equiv 0.$$

Obviously, $P(z, 0) = -H(z-1) \not\equiv 0$. By Lemma 2.3, we have

$$m\Big(r, \frac{1}{y(z)}\Big) = S(r, y),$$

which yields

$$N\Big(r, \frac{1}{e^{h(z)-h(z-1)}}\Big) = T(r, e^{h(z)-h(z-1)}) + S(r, e^{h(z)-h(z-1)}),$$

a contradiction.

Thus, $\deg h(z) = 1$ and $h'(z) = d$ is a nonzero constant, and so

$$\frac{a_0(z)}{a(z)} \equiv \frac{H'(z)}{H(z)} + d.$$

By this and (3.29), we may assume $w(z) = H(z)e^{dz}$. Substituting these into (2.7), we have

$$e^d H(z+1)e^{dz} - e^{-d}H(z-1)e^{dz} - a_1(z)H(z)e^{dz} \equiv 0,$$

which yields

$$a_1(z) = e^d \frac{H(z+1)}{H(z)} - e^{-d}\frac{H(z-1)}{H(z)}.$$

### 3.3 $\deg_w P \geq 2$, $\deg_w Q = 0$

Theorem 3.3 also shows if $\deg_w P \geq 2$, then all transcendental meromorphic solutions $w$ of (3.21) with rational coefficients satisfy $\sigma_2(w) \geq 1$. So, we only need to solve the existence problem of transcendental entire solutions of (3.21), and we obtain the following theorem.

**Theorem 3.7** *Let $a(z)$ be a nonzero rational function, $P(z, w)$ be a polynomial in $w$ with rational coefficients and* $\deg_w P \geq 2$, *then* (3.21) *has no transcendental entire solution.*

**Proof** Denote

$$P(z, w) = a_p(z)w^p(z) + a_{p-1}(z)w^{p-1}(z) + \cdots + a_1(z)w(z) + a_0(z), \tag{3.35}$$

where $p(\geq 2)$ is a positive integer, $a_p(z)(\not\equiv 0)$, $a_{p-1}(z), \cdots, a_1(z), a_0(z)$ are rational functions.

Suppose $w(z)$ is a transcendental entire solution of (3.21), then $w(z)$ must have finitely many zeros. Otherwise, $w(z+1)-w(z-1)-P(z,w)$ has finitely many poles, while $a(z)\frac{w'(z)}{w(z)}$ has infinitely many poles, it is a contradiction. So, $w(z)$ has the form

$$w(z) = H(z)\mathrm{e}^{h(z)}, \tag{3.36}$$

where $H(z)$ is a nonzero polynomial, and $h(z)$ is a nonconstant entire function.

Substituting (3.36) into (3.21), we have

$$\begin{aligned} &H(z+1)\mathrm{e}^{h(z+1)} - H(z-1)\mathrm{e}^{h(z-1)} + a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big) \\ &= a_p(z)H^p(z)\mathrm{e}^{ph(z)} + \cdots + a_1(z)H(z)\mathrm{e}^{h(z)} + a_0(z) \end{aligned} \tag{3.37}$$

or

$$H(z+1)\mathrm{e}^{h(z+1)} - H(z-1)\mathrm{e}^{h(z-1)} - \sum_{j=1}^{p} a_j(z)H^j(z)\mathrm{e}^{jh(z)} = a_0^*(z), \tag{3.38}$$

where

$$a_0^*(z) = a_0(z) - a(z)\Big(\frac{H'(z)}{H(z)} + h'(z)\Big). \tag{3.39}$$

Obviously,

$$T(r, a_0^*(z)) = T(r, h'(z)) + O(\log r) = S(r, \mathrm{e}^{h(z)}). \tag{3.40}$$

**Case 1** Suppose that none of $h(z+1)-h(z-1)$, $h(z+1)-jh(z)$, $h(z-1)-jh(z)$, $j=1,\cdots,p$ are constants.

Applying Lemma 2.7 to (3.38), there exist constants $d_1, \cdots, d_{p+2}$, at least one of them is not zero, such that

$$d_{p+2}H(z+1)\mathrm{e}^{h(z+1)} - d_{p+1}H(z-1)\mathrm{e}^{h(z-1)} - \sum_{j=1}^{p} d_j a_j(z)H^j(z)\mathrm{e}^{jh(z)} = 0. \tag{3.41}$$

Set

$$\begin{cases} f_j(z) = a_j(z)H^j(z)\mathrm{e}^{jh(z)}, \quad j = 1, 2, \cdots, p, \\ f_{p+2}(z) = H(z+1)\mathrm{e}^{h(z+1)}, \quad f_{p+1}(z) = H(z-1)\mathrm{e}^{h(z-1)}. \end{cases} \tag{3.42}$$

Thus, (3.41) can be rewritten as

$$d_{p+2}f_{p+2}(z) - d_{p+1}f_{p+1}(z) - d_p f_p(z) - \cdots - d_1 f_1(z) = 0. \tag{3.43}$$

Obviously,

$$\frac{f_j(z)}{f_k(z)} = \frac{a_j(z)}{a_k(z)}H^{j-k}(z)\mathrm{e}^{(j-k)h(z)}, \quad 1 \leq j < k \leq p,$$

$$\frac{f_j(z)}{f_{p+2}(z)} = \frac{a_j(z)H^j(z)}{H(z+1)}\mathrm{e}^{jh(z)-h(z+1)},$$
$$\frac{f_j(z)}{f_{p+1}(z)} = \frac{a_j(z)H^j(z)}{H(z-1)}\mathrm{e}^{jh(z)-h(z-1)}, \quad j=1,2,\cdots,p,$$
$$\frac{f_{p+1(z)}}{f_{p+2}(z)} = \frac{H(z-1)}{H(z+1)}\mathrm{e}^{h(z-1)-h(z+1)}$$

are transcendental, and so

$$\sum_{j=1}^{p+2}\Big(N(r,f_j(z))+N\Big(r,\frac{1}{f_j(z)}\Big)\Big) = O(\log r) = o(\tau(r)), \tag{3.44}$$

where $\tau(r) = \min\limits_{1\le j<k\le p+2}\big\{T\big(r,\frac{f_j}{f_k}\big)\big\}$.

Applying Lemma 2.8 to (3.43), we have $d_j = 0 (j=1,2,\cdots,p+2)$, a contradiction.

**Case 2** Suppose that $h(z+1)-h(z)=d$ is a constant. Then

$$h(z+1)=h(z)+d, \quad h(z-1)=h(z)-d.$$

Substituting this into (3.37), we have

$$(\mathrm{e}^d H(z+1)-\mathrm{e}^{-d}H(z-1))\mathrm{e}^{h(z)} + a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big) = \sum_{j=0}^{p} a_j(z)H^j(z)\mathrm{e}^{jh(z)}. \tag{3.45}$$

Applying Valiron Monho'ko Lemma to (3.45), we obtain

$$\begin{aligned} pT(r,\mathrm{e}^{h(z)}) &= T\Big(r,\sum_{j=0}^{p} a_j(z)H^j(z)\mathrm{e}^{jh(z)}\Big) + S(r,\mathrm{e}^{h(z)}) \\ &\le T(r,\mathrm{e}^{h(z)}) + S(r,\mathrm{e}^{h(z)}), \end{aligned}$$

which yields $p\le 1$, contradicting $p\ge 2$.

**Case 3** Suppose that $h(z+1)-j_0h(z)$ is a constant, say $h(z+1)-j_0h(z)=d$, for some $2\le j_0\le p,\ j_0\in\mathbb{N}$. Then

$$h(z+1)=j_0h(z)+d, \quad h(z-1)=\frac{1}{j_0}h(z)-\frac{d}{j_0}.$$

Substituting this into (3.37), we have

$$\begin{aligned} &\mathrm{e}^d H(z+1)\mathrm{e}^{j_0h(z)} - \mathrm{e}^{-\frac{d}{j_0}}H(z-1)\mathrm{e}^{\frac{1}{j_0}h(z)} + a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big) \\ =&\sum_{j=0}^{p} a_j(z)H^j(z)\mathrm{e}^{jh(z)} \end{aligned}$$

or

$$-\mathrm{e}^{-\frac{d}{j_0}}H(z-1)\mathrm{e}^{\frac{1}{j_0}h(z)} = \sum_{j=0}^{p} a_j(z)H^j(z)\mathrm{e}^{jh(z)} - \mathrm{e}^d H(z+1)\mathrm{e}^{j_0h(z)} - a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big).$$

Thus

$$T(r,\mathrm{e}^{h(z)})=T(r,(\mathrm{e}^{-\frac{d}{j_0}}H(z-1)\mathrm{e}^{\frac{1}{j_0}h(z)})^{j_0})+O(\log r)$$
$$=j_0T(r,\mathrm{e}^{-\frac{d}{j_0}}H(z-1)\mathrm{e}^{\frac{1}{j_0}h(z)})+O(\log r)$$
$$=nj_0T(r,\mathrm{e}^{h(z)})+S(r,\mathrm{e}^{h(z)}), \tag{3.46}$$

where $0\le n\le p,\ n\in\mathbb{N}$.

By (3.46), $nj_0=1$. Thus, $j_0=n=1$, contradicting $j_0\ge 2$.

**Case 4** Suppose that $h(z-1)-h(z)=-d$ is a constant. Clearly, $h(z+1)-h(z)=d$ is also a constant. By Case 2, we obtain a contradiction.

**Case 5** Suppose that $h(z-1)-j_1h(z)=d$ is a constant for some $2\le j_1\le p,\ j_1\in\mathbb{N}$. Then

$$h(z+1)=\frac{1}{j_1}h(z)-\frac{d}{j_1},\quad h(z-1)=j_1h(z)+d.$$

Substituting this into (3.37), we have

$$\mathrm{e}^{-\frac{d}{j_1}}H(z+1)\mathrm{e}^{\frac{1}{j_1}h(z)}-\mathrm{e}^{d}H(z-1)\mathrm{e}^{j_1h(z)}+a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big)=\sum_{j=0}^{p}a_j(z)H^j(z)\mathrm{e}^{jh(z)}$$

or

$$\mathrm{e}^{-\frac{d}{j_1}}H(z+1)\mathrm{e}^{\frac{1}{j_1}h(z)}=\sum_{j=0}^{p}a_j(z)H^j(z)\mathrm{e}^{jh(z)}+\mathrm{e}^{d}H(z-1)\mathrm{e}^{j_1h(z)}-a(z)\Big(\frac{H'(z)}{H(z)}+h'(z)\Big).$$

Thus

$$T(r,\mathrm{e}^{h(z)})=T(r,(\mathrm{e}^{-\frac{d}{j_1}}H(z+1)\mathrm{e}^{\frac{1}{j_1}h(z)})^{j_1})+O(\log r)$$
$$=j_1T(r,\mathrm{e}^{-\frac{d}{j_1}}H(z+1)\mathrm{e}^{\frac{1}{j_1}h(z)})+O(\log r)$$
$$=nj_1T(r,\mathrm{e}^{h(z)})+S(r,\mathrm{e}^{h(z)}), \tag{3.47}$$

where $0\le n\le p,\ n\in\mathbb{N}$.

By (3.47), $nj_1=1$. Thus, $j_1=n=1$, contradicting $j_1\ge 2$.

**Case 6** Suppose that $h(z+1)-h(z-1)=d$ is a constant.

Substituting this into (3.38), we have

$$(\mathrm{e}^{d}H(z+1)-H(z-1))\mathrm{e}^{h(z-1)}-\sum_{j=1}^{p}a_j(z)H^j(z)\mathrm{e}^{jh(z)}=a_0^*(z). \tag{3.48}$$

If $\mathrm{e}^{d}H(z+1)-H(z-1)\equiv 0$, then (3.38) can be written as

$$-\sum_{j=1}^{p}a_j(z)H^j(z)\mathrm{e}^{jh(z)}=a_0^*(z).$$

Together with (3.39), we obtain

$$pT(r,\mathrm{e}^{h(z)})=T\Big(r,\sum_{j=1}^{p}a_j(z)H^j(z)\mathrm{e}^{jh(z)}\Big)+S(r,\mathrm{e}^{h(z)})$$

$$= T(r, a_0^*(z)) + S(r, \mathrm{e}^{h(z)})$$
$$= S(r, \mathrm{e}^{h(z)}),$$

which yields $p = 0$, a contradiction.

If $\mathrm{e}^d H(z+1) - H(z-1) \not\equiv 0$, by (3.39), applying Lemma 2.7 to (3.48), there exist constants $c_j (j = 1, 2, \cdots, p+1)$, at least one of them is not zero, such that

$$c_{p+1}(\mathrm{e}^d H(z+1) - H(z-1))\mathrm{e}^{h(z-1)} - \sum_{j=1}^{p} c_j a_j(z) H^j(z) \mathrm{e}^{jh(z)} = 0. \tag{3.49}$$

If there exists some $j_1, 1 \le j_1 \le p$ such that $h(z-1) - j_1 h(z)$ is a constant, then by Cases 4–5, we obtain a contradiction.

Thus, $h(z-1) - jh(z)(j = 1, 2, \cdots, p)$ is not constant. Set

$$f_{p+1}(z) = (\mathrm{e}^d H(z+1) - H(z-1))\mathrm{e}^{h(z-1)}, \quad f_j(z) = a_j(z) H^j(z) \mathrm{e}^{jh(z)}, \quad j = 1, \cdots, p.$$

(3.49) can be written as

$$c_{p+1} f_{p+1}(z) - c_p f_p(z) - \cdots - c_1 f_1(z) = 0. \tag{3.50}$$

For $1 \le j < k \le p,\ i = 1, 2, \cdots, p$, we obtain

$$\frac{f_j(z)}{f_k(z)} = \frac{a_j(z)}{a_k(z)} H^{j-k}(z) \mathrm{e}^{(j-k)h(z)},$$
$$\frac{f_i(z)}{f_{p+1}(z)} = \frac{a_i(z) H^i(z)}{\mathrm{e}^d H(z+1) - H(z-1)} \mathrm{e}^{ih(z) - h(z-1)}$$

are transcendental, and so

$$\sum_{j=1}^{p+1} \Big( N(r, f_j(z)) + N\Big(r, \frac{1}{f_j(z)}\Big)\Big) = O(\log r) = o(\tau(r)), \tag{3.51}$$

where $\tau(r) = \min\limits_{1 \le j < k \le p+1} \big\{T\big(r, \frac{f_j}{f_k}\big)\big\}$.

Applying Lemma 2.8 to (3.50), we have $c_j = 0\ (j = 1, 2, \cdots, p+1)$, a contradiction.

In conclusion, (3.21) has no transcendental entire solution.

### 3.4 $\deg_w Q \ge 1$

Theorem 1.1 shows that if $\deg_w P \ge \deg_w Q + 2$ or $\deg_w Q \ge 2,\ \deg_w P = 0$, then all transcendental meromorphic solutions $w$ of (1.1) with rational coefficients satisfy $\sigma_2(w) \ge 1$. We further obtain the following theorem.

**Theorem 3.8** *Let $a(z)$ be a nonzero rational function, $P(z, w)$ and $Q(z, w)$ be prime polynomials in $w$ with rational coefficients. If $\deg_w Q(z, w) \ge 1$ and $Q(z, 0) \not\equiv 0$, then (1.1) has no transcendental entire solution.*

**Proof** Denote

$$\begin{cases} P(z,w) = a_p(z)w^p(z) + \cdots + a_1(z)w(z) + a_0(z), \\ Q(z,w) = b_q(z)w^q(z) + \cdots + b_1(z)w(z) + b_0(z), \end{cases} \tag{3.52}$$

where $a_p(z)(\not\equiv 0), \cdots, a_0(z), b_q(z)(\not\equiv 0), \cdots, b_0(z)$ are rational functions.

Suppose $w(z)$ is a transcendental entire solution of (1.1), then $w(z)$ must have finitely many zeros. Otherwise, suppose $w(z)$ has infinitely many zeros, the zeros of $w(z)$ are simple poles $a(z)\frac{w'(z)}{w(z)}$, except finitely many, so $a(z)\frac{w'(z)}{w(z)}$ has infinitely many poles. By $Q(z,0) = b_0(z) \not\equiv 0$, the zeros of $w(z)$ are not poles of

$$w(z+1) - w(z-1) - \frac{a_p(z)w^p(z) + \cdots + a_1(z)w(z) + a_0(z)}{b_q(z)w^q(z) + \cdots + b_1(z)w(z) + b_0(z)},$$

except finitely many. It is a contradiction. So, $w(z)$ has the form

$$w(z) = H(z)\mathrm{e}^{h(z)}, \tag{3.53}$$

where $H(z)$ is a nonzero polynomial, and $h(z)$ is a nonconstant entire function.

**Case 1** $p \le q$. Denote

$$c(z) = \frac{P(z,w)}{Q(z,w)} = w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)}. \tag{3.54}$$

By $p \le q$, $a_p(z)b_q(z)b_0(z) \not\equiv 0$, (3.53) and Lemma 2.10, we have

$$m(r, c(z)) = m\Big(r, \frac{P(z,w)}{Q(z,w)}\Big) = S(r, \mathrm{e}^{h(z)}) = S(r,w). \tag{3.55}$$

By (3.54), we see the poles of $c(z)$ come from the poles of $a(z)$, zeros of $w(z)$, both have the finite number. Thus

$$N(r, c(z)) = O(\log r) = S(r,w). \tag{3.56}$$

By (3.55)–(3.56), we have

$$T(r, c(z)) = m(r, c(z)) + N(r, c(z)) = S(r,w).$$

So, $c(z)$ is a nonzero small function of $w(z)$.

By (3.52) and (3.54), we have

$$a_p(z)w^p(z) + \cdots + a_1(z)w(z) + a_0(z) = c(z)(b_q(z)w^q(z) + \cdots + b_1(z)w(z) + b_0(z)),$$

thus,

$$p = q, \quad a_j(z) \equiv c(z)b_j(z), \quad j = 0, \cdots, p,$$

and so

$$P(z,w) \equiv c(z)Q(z,w),$$

which contradicts the fact that $P(z,w)$, $Q(z,w)$ are prime polynomials in $w$.

**Case 2** $p>q$.

By $p>q$, we have

$$P(z,w)=P_1(z,w)Q(z,w)+P_2(z,w),$$

where $P_i(z,w)$ $(i=1,2)$ are polynomials in $w$ with rational coefficients and $\deg_w P_i=p_i$ $(i=1,2)$ such that $p_1=p-q\geq 1, 1\leq p_2<q$.

Thus, (1.1) can be written as

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=\frac{P_1(z,w)Q(z,w)+P_2(z,w)}{Q(z,w)}=P_1(z,w)+\frac{P_2(z,w)}{Q(z,w)}.$$

Denote

$$c^*(z)=\frac{P_2(z,w)}{Q(z,w)}=w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}-P_1(z,w). \tag{3.57}$$

By $p_2<q$, $Q(z,0)\not\equiv 0$, (3.53) and Lemma 2.10, we have

$$m(r,c^*(z))=S(r,w). \tag{3.58}$$

By (3.57), the poles of $c^*(z)$ come from the poles of $a(z)$, zeros of $w(z)$ and the poles of coefficients of $P_1(z,w)$. Since $w(z)$ has finitely many zeros, $a(z)$ and the coefficients of $P_1(z,w)$ are rational functions. So,

$$N(r,c^*(z))=O(\log r)=S(r,w). \tag{3.59}$$

By (3.58)–(3.59), we have

$$T(r,c^*(z))=m(r,c^*(z))+N(r,c^*(z))=S(r,w),$$

thus, $c^*(z)$ is a nonzero small function of $w(z)$.

By (3.57), we obtain

$$\begin{aligned}p_2T(r,w)&=T(r,P_2(z,w))+S(r,w)\\&=T(r,c^*(z)Q(z,w))+S(r,w)\\&=qT(r,w)+S(r,w).\end{aligned}$$

Hence, $p_2=q$, contradicting $p_2<q$.

**Remark 3.1** The condition "$Q(z,0)\not\equiv 0$" in Theorem 3.8 is to ensure that meromorphic solutions of (1.1) have finitely many zeros.

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## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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