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Skew Constacyclic Codes over a Family of Finite Rings and Their Applications to LCD and Quantum Codes

Abdullah DERTLI[1]

Abstract This paper studies skew constacyclic codes over a family of finite rings denoted by B_k to obtain quantum codes over the fields $\mathbb{F}_{p^r}$ and to construct Euclidean LCD skew constacyclic codes. The author investigates the structural properties of skew constacyclic codes over B_k using a decomposition approach, and also finds necessary and sufficient conditions for skew constacyclic codes that contain their duals. Finally, the author gives some examples of quantum codes obtained via the construction and LCD codes.

Keywords Quantum codes, Skew constacyclic codes, LCD codes, Gray map
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1 Introduction

In the classical computer and digital platform, the classical error-correcting codes are developed in order to transmit information and to correct mistakes which occur in information. Recently, instead of classical computers, the quantum computers are considered. Moreover, quantum computers are known to be able to solve certain problems faster than classical computers can. With the expected arrival of quantum computers which work with respect to quantum mechanics basics in the near future, research into quantum information theory has intensified significantly.

Quantum computers outrun the classical computers in their ability to solve complex problems. While the problem of factorizing a number into its primes is easily achievable for small numbers, it takes months for larger numbers even with the best computers. It is believed that a quantum computer can overcome the same problem within a few minutes if properly implemented. Also, while no efficient algorithm is known for the integer factorization problem for classical computers, an efficient (polynomial time) algorithm is known for quantum computers.

Quantum error-correcting codes (QECCs for short) are used in quantum computing and communication to correct errors that occur during the transmission in a noisy channel and to protect quantum information from decoherence. The application of error-correcting codes by quantum computers can be labeled as one of the pivotal reasons for this efficiency.

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[1]Department of Mathematics, Ondokuz Mayıs University, Samsun, 55139, Turkey.
E-mail: abdullah.dertli@gmail.com

Effective techniques for quantum error-correction were first developed by Shor and independently by Steane. They discovered quantum error-correcting codes (see [23–24]). Calderbank et al. presented a way of constructing quantum codes from classical codes (see [7]). Later, Ketkar et al. generalized these results to a non-binary case (see [16]). Lately, quantum codes are studied in [2, 10, 13–14, 19].

In classical coding theory, cyclic codes play a prominent role due to their algebraic structures. There are many useful generalizations of cyclic codes. One important generalization of cyclic codes that has received a lot of attention in recent years is the class of skew cyclic codes. Boucher et al. [3] introduced skew cyclic codes as a generalization of cyclic codes using the skew polynomial ring $F[x,\theta]$, where F is a finite field and θ is a non-trivial automorphism over F. Later, many researchers investigated skew codes over various finite rings (see [1, 4–5, 26]).

A linear complementary dual code (called LCD) is defined as a linear code C whose dual code $C^{\perp}$ satisfies $C \cap C^{\perp} = \{0\}$. LCD codes were introduced by Massey [21]. Yang and Massey classified cyclic LCD codes over finite fields (see [27]). These codes have gained serious attention due to their recent successful application in cryptography and are used in communications systems, data storage and consumer electronics. LCD codes over F_2 play an important role in implementations against side channel attacks (SCA for short, which consists in passively recording some leakage, and this is the source of information to retrieve the key) and fault injection attacks (FIA for short, which consists in actively perturbing the computation so as to obtain exploitable differences at the output) (see [8]). Tzeng and Hartmann proved that the minimum distance of a class of LCD codes is greater than that given by the BCH bound (see [25]). Sendrier showed that LCD codes meet the asymptotic Gilbert-Varshamov bound using properties of the hull dimension spectrum of linear codes (see [22]). Dougherty et al. gave a linear programming bound on the largest size of an LCD code of given length and minimum distance (see [11]). Recently, LCD codes are studied in [6, 12, 17–18, 20].

Motivated by the previous works, we study quantum codes that are obtained from skew constacyclic codes and Euclidean LCD skew constacyclic codes over an infinite family of the finite rings denoted by B_k.

2 Preliminaries

In [15], Irwansyah et al. introduced the family of finite rings B_k. We summarize some of the relevant results from [15] in this section. The infinite family of the finite rings B_k is a generalization of the family of finite rings A_k (see [9]).

Let F_{p^r} be the finite field of order p^r for a prime p and a positive integer r. The family of the finite rings B_k is defined as

$$B_k := F_{p^r}[v_1, v_2, \cdots, v_k]/\langle v_i^2 - v_i, v_i v_j - v_j v_i \rangle$$

for all $i, j = 1, 2, \cdots, k$. We also define $B_0 = F_{p^r}$. If $k = 1$, then $B_1 = F_{p^r} + v_1 F_{p^r}$, where $v_1^2 = v_1$; if $k = 2$, then $B_2 = F_{p^r} + v_1 F_{p^r} + v_2 F_{p^r} + v_1 v_2 F_{p^r}$, where $v_1^2 = v_1,\ v_2^2 = v_2,\ v_1 v_2 = v_2 v_1$. The rings in this family are finite commutative rings with cardinality $(p^r)^{2^k}$ and with characteristic p.

Let $B \subseteq \{1, 2, \cdots, k\}$ and $v_B = \prod_{i \in B} v_i$. In particular $v_\varnothing = 1$. Each element of B_k is of the form $\sum_{B \in P_k} \alpha_B v_B$, where $\alpha_B \in F_{p^r}$, and P_k is the power set of $\{1, 2, \cdots, k\}$. For $A, B \subseteq \{1, 2, \cdots, k\}$ we have that $v_A v_B = v_{A \cup B}$ which gives that $\sum_{B \in P_k} \alpha_B v_B \cdot \sum_{C \in P_k} \beta_C v_C = \sum_{D \in P_k} \big(\sum_{B \cup C = D} \alpha_B \beta_C \big) v_D$. For more on the ring B_k we refer the reader to [15].

A code C of length n over B_k is a subset of B_k^n. A linear code C of length n over B_k is a B_k-submodule of B_k^n. An element $c = (c_0, c_1, \cdots, c_{n-1}) \in C$ is called a codeword. Let λ_k be a unit in B_k. A linear code C of length n over B_k is said to be λ_k-constacyclic code if C is invariant under the constacyclic shift operator $\upsilon_{\lambda_k} : B_k^n \to B_k^n$ defined by $\upsilon_{\lambda_k}(c_0, c_1, \cdots, c_{n-1}) = (\lambda_k c_{n-1}, c_0, \cdots, c_{n-2})$. Note that the constacyclic code is a cyclic code for $\lambda_k = 1$ and the negacyclic code for $\lambda_k = -1$. By identifying each codeword $c = (c_0, c_1, \cdots, c_{n-1}) \in B_k^n$ with a polynomial $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$ in $B_k[x]/\langle x^n - \lambda_k \rangle$, we see that a linear code C is a λ_k -constacyclic code of length n over B_k if and only if it is an ideal of the ring $B_k[x]/\langle x^n - \lambda_k \rangle$.

Let $x = (x_0, x_1, \cdots, x_{n-1})$ and $y = (y_0, y_1, \cdots, y_{n-1})$ be two elements of B_k^n. Then the Euclidean inner product of x and y is defined as $x \cdot y = x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1}$. The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in B_k^n \mid x \cdot y = 0, \forall y \in C\}$. A code C is called self-orthogonal if $C \subseteq C^\perp$ and self dual if $C = C^\perp$. The reciprocal of a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is defined as $f^*(x) = x^{\deg(f(x))} f(x^{-1})$. A polynomial $f(x)$ is called self-reciprocal if $f(x) = f^*(x)$.

The skew reciprocal polynomial of $g(x) = \sum_{i=0}^{k} g_i x^i$ of degree k is $g^*(x) = \sum_{i=0}^{k} \theta^i (g_{k-i}) x^i$, where θ is a non-trivial automorphism. If g_0 does not cancel, the left monic skew reciprocal polynomial of g is $g^\natural(x) = \frac{1}{\theta^k(g_0)} g^*(x)$. If a skew polynomial is equal to its left monic skew reciprocal polynomial, then it is called self-reciprocal (see [6]).

We define

$$A_1 \oplus A_2 \oplus \cdots \oplus A_{2^k} = \{a_1 + a_2 + \cdots + a_{2^k} : a_i \in A_i, i = 1, 2, \cdots, 2^k\},$$
$$A_1 \otimes A_2 \otimes \cdots \otimes A_{2^k} = \{(a_1, a_2, \cdots, a_{2^k}) : a_i \in A_i, i = 1, 2, \cdots, 2^k\}.$$

Let

$$e_{v_\varnothing} = 1 + (-1)^{|B|} \sum_{B \in P_k} v_B$$

and

$$e_{v_i} = v_i + (-1)^{|B|+1} \sum_{\substack{i \in B \in P_k \\ |B| \geq 2}} v_B$$

for $i = 1, 2, \cdots, k$. The total number of e_{v_i}'s is $\binom{k}{1} = k$,

$$e_{v_i v_j} = \underset{i<j}{v_i v_j} + (-1)^{|B|+2} \sum_{\substack{i,j \in B \in P_k \\ |B| \geq 3}} v_B$$

for $i, j = 1, 2, \cdots, k$. The total number of $e_{v_i v_j}$'s is $\binom{k}{2}$,

$$e_{v_i v_j v_s} = \underset{i<j<s}{v_i v_j v_s} + (-1)^{|B|+3} \sum_{\substack{i,j,s \in B \in P_k \\ |B| \geq 4}} v_B$$

for $i, j, s = 1, 2, \cdots, k$. The total number of $e_{v_i v_j v_s}$'s is $\binom{k}{3}$,

$$\vdots$$

$$e_{v_1 v_2 \cdots v_k} = v_1 v_2 \cdots v_k.$$

The number of $e_{v_1 v_2 \cdots v_k}$ is $\binom{k}{k} = 1$.

Then we have $\sum\limits_{B \in P_k} e_{v_B} = 1$, $(e_{v_B})^2 = e_{v_B}$ and $e_{v_B} e_{v_A} = 0$ if $A \neq B$ for any $A \subseteq \{1, 2, \cdots, k\}$, $B \subseteq \{1, 2, \cdots, k\}$. Hence $B_k = \bigoplus\limits_{B \in P_k} e_{v_B} B_k \cong \bigoplus\limits_{B \in P_k} e_{v_B} F_{p^r}$. Thus, we know that every element of B_k can be uniquely expressed as $z = \sum\limits_{B \in P_k} a_{v_B} e_{v_B}$, where $a_{v_B} \in F_{p^r}$.

Example 2.1 Let $k = 3$. Then $B_3 = F_{p^r} + v_1 F_{p^r} + v_2 F_{p^r} + v_3 F_{p^r} + v_1 v_2 F_{p^r} + v_1 v_3 F_{p^r} + v_2 v_3 F_{p^r} + v_1 v_2 v_3 F_{p^r}$. We have

$$\begin{aligned}
e_{v_\varnothing} &= e_1 = 1 - v_1 - v_2 - v_3 + v_1 v_2 + v_1 v_3 + v_2 v_3 - v_1 v_2 v_3, \\
e_{v_1} &= v_1 - v_1 v_2 - v_1 v_3 + v_1 v_2 v_3, \\
e_{v_2} &= v_2 - v_1 v_2 - v_2 v_3 + v_1 v_2 v_3, \\
e_{v_3} &= v_3 - v_1 v_3 - v_2 v_3 + v_1 v_2 v_3, \\
e_{v_1 v_2} &= v_1 v_2 - v_1 v_2 v_3, \\
e_{v_1 v_3} &= v_1 v_3 - v_1 v_2 v_3, \\
e_{v_2 v_3} &= v_2 v_3 - v_1 v_2 v_3, \\
e_{v_1 v_2 v_3} &= v_1 v_2 v_3.
\end{aligned}$$

Hence, $B_k = e_1 F_{p^r} \oplus e_{v_1} F_{p^r} \oplus e_{v_2} F_{p^r} \oplus e_{v_3} F_{p^r} \oplus e_{v_1 v_2} F_{p^r} \oplus e_{v_1 v_3} F_{p^r} \oplus e_{v_2 v_3} F_{p^r} \oplus e_{v_1 v_2 v_3} F_{p^r}$.

Let $\Omega \in \mathrm{Aut}(F_{p^r})$. We define a non-trivial automorphism which is different from [15],

$$\Delta_k : B_k \to B_k$$

by $\Delta_k\big(\sum\limits_{B \in P_k} \alpha_B v_B\big) = \sum\limits_{B \in P_k} \Omega(\alpha_B) v_B$.

The set of polynomials

$$B_k[x, \Delta_k] = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in B_k, n \in \mathbb{N}\}$$

is the skew polynomial ring over B_k with the usual addition of polynomials and the non-commutative multiplication given by

$$(ax^i)(bx^j) = a\Delta_k^i(b)x^{i+j}$$

and extended to all polynomials with distributivity.

Definition 2.1 *Let Δ_k be a non-trivial automorphism over B_k and λ_k be a unit in B_k. C is called skew λ_k-constacyclic code of length n over B_k if the following conditions hold:*

(i) *C is a B_k-submodule of B_k^n,*

(ii) *if $s = (s_0, s_1, \cdots, s_{n-1}) \in C$, then $\Delta_{\lambda_k}(s) = (\Delta_k(\lambda_k s_{n-1}), \Delta_k(s_0), \cdots, \Delta_k(s_{n-2})) \in C$.*

As the ring $B_k[x, \Delta_k]$ is non-commutative, its ideal $\langle x^n - \lambda_k \rangle$ is two sided only if n is even. So if n is even, then the set $B_{k_{\Delta_k}, n} = B_k[x, \Delta_k]/\langle x^n - \lambda_k \rangle$ is a residue class ring. For an arbitrary n, $B_{k_{\Delta_k}, n}$ is a left $B_k[x, \Delta_k]$-module.

Theorem 2.1 *A skew λ_k-constacyclic code of length n over B_k is defined as a left $B_k[x, \Delta_k]$-submodule of $B_k[x, \Delta_k]/\langle x^n - \lambda_k \rangle$.*

Theorem 2.2 *Let $C = \langle f(x) \rangle$ be a left $B_k[x, \Delta_k]$-submodule of $B_k[x, \Delta_k]/\langle x^n - \lambda_k \rangle$. Then $f(x)$ is a right divisor of $x^n - \lambda_k$, where $f(x)$ is a monic polynomial of minimum degree in C.*

Note that a skew λ_k-constacyclic code is a skew cyclic code for $\lambda_k = 1$ and a skew negacyclic code for $\lambda_k = -1$.

The Gray map Ψ_k is

$$\begin{aligned} &\Psi_k : B_k \to F_{p^r}^{2^k} \\ &z = \sum_{B \in P_k} a_{v_B} e_{v_B} \mapsto \Psi_k(z) = \Upsilon, \end{aligned}$$

where

$$\begin{aligned} \Upsilon = \Big(&\sum_{B=\varnothing} a_{v_B}, \sum_{B \subseteq \{1\}} a_{v_B}, \sum_{B \subseteq \{2\}} a_{v_B}, \cdots, \sum_{B \subseteq \{k\}} a_{v_B}, \sum_{B \subseteq \{1,2\}} a_{v_B}, \sum_{B \subseteq \{1,3\}} a_{v_B}, \cdots, \\ &\sum_{\substack{B \subseteq \{i,j\} \\ i<j}} a_{v_B}, \sum_{B \subseteq \{1,2,3\}} a_{v_B}, \cdots, \sum_{\substack{B \subseteq \{i,j,s\} \\ i<j<s}} a_{v_B}, \cdots, \sum_{B \subseteq \{1,2,\cdots,k\}} a_{v_B} \Big), \end{aligned}$$

$a_{v_B} \in F_{p^r}$ for $i, j, s = 1, 2, \cdots, k$.

The Gray map Ψ_k can be extended from B_k^n to $F_{p^r}^{2^k n}$.

Example 2.2 Let $k = 3$. Then

$$\begin{aligned} &\Psi_3 : B_3 \to F_{p^r}^{8} \\ &z = \sum_{B \in P_3} a_{v_B} e_{v_B} \mapsto \Psi_3(z) = \Upsilon, \end{aligned}$$

where

$$\Upsilon = (a_1, a_1 + a_{v_1}, a_1 + a_{v_2}, a_1 + a_{v_3}, a_1 + a_{v_1} + a_{v_2} + a_{v_1v_2}, a_1 + a_{v_1} + a_{v_3} + a_{v_1v_3},$$
$$a_1 + a_{v_2} + a_{v_3} + a_{v_2v_3}, a_1 + a_{v_1} + a_{v_2} + a_{v_3} + a_{v_1v_2} + a_{v_1v_3} + a_{v_2v_3} + a_{v_1v_2v_3}).$$

For any $x = \sum_{B \in P_k} \alpha_B v_B \in B_k$, let the Lee weight be defined as $w_L(x) = w_H(\Psi_k(x))$, where w_H is the Hamming weight. The Lee weight of a vector $a = (a_1, \cdots, a_n) \in B_k$ is defined as $w_L(a) = \sum_{i=1}^{n} w_L(a_i)$. For any elements $a_1, a_2 \in C$, the Lee distance is given by $d_L(a_1, a_2) = w_L(a_1 - a_2)$. The minimum Lee distance of C is defined as $d_L = d_L(C) = \min\{d_L(c, \acute{c}) : \forall c, \acute{c} \in C, c \neq \acute{c}\}$.

Theorem 2.3 *The Gray map Ψ_k is a linear and distance preserving map.*

Let C be a linear code of length n over B_k. We define

$$C_{v_\varnothing} = C_1 = \Big\{\mathbf{a}_{v_\varnothing} : \exists \underset{B\neq\varnothing}{\mathbf{a}_{v_B}} \in F_{p^r}^n, \sum_{B\in P_k} \mathbf{a}_{v_B} e_{v_B} \in C\Big\},$$
$$C_{v_1} = \Big\{\mathbf{a}_{v_1} : \exists \underset{B\neq\{1\}}{\mathbf{a}_{v_B}} \in F_{p^r}^n, \sum_{B\in P_k} \mathbf{a}_{v_B} e_{v_B} \in C\Big\},$$
$$C_{v_2} = \Big\{\mathbf{a}_{v_2} : \exists \underset{B\neq\{2\}}{\mathbf{a}_{v_B}} \in F_{p^r}^n, \sum_{B\in P_k} \mathbf{a}_{v_B} e_{v_B} \in C\Big\},$$
$$\vdots$$
$$C_{v_k} = \Big\{\mathbf{a}_{v_k} : \exists \underset{B\neq\{k\}}{\mathbf{a}_{v_B}} \in F_{p^r}^n, \sum_{B\in P_k} \mathbf{a}_{v_B} e_{v_B} \in C\Big\},$$
$$C_{v_1v_2} = \Big\{\mathbf{a}_{v_1v_2} : \exists \underset{B\neq\{1,2\}}{\mathbf{a}_{v_B}} \in F_{p^r}^n, \sum_{B\in P_k} \mathbf{a}_{v_B} e_{v_B} \in C\Big\},$$
$$\vdots$$
$$C_{v_1v_2\cdots v_k} = \Big\{\mathbf{a}_{v_1v_2\cdots v_k} : \exists \underset{B\neq\{1,2,\cdots,k\}}{\mathbf{a}_{v_B}} \in F_{p^r}^n, \sum_{B\in P_k} \mathbf{a}_{v_B} e_{v_B} \in C\Big\}.$$

The number of C_{v_B} is 2^k. Then C_{v_B} is a linear code of length n over F_{p^r} and the linear code C of length n over B_k can be expressed uniquely as

$$C = \bigoplus_{B\in P_k} e_{v_B} C_{v_B}$$

such that $d_L(C) = \min\{d_H(C_{v_B})\}$ and $|C| = \prod_{B\in P_k} |C_{v_B}|$.

Theorem 2.4 *Let $C = \bigoplus_{B\in P_k} e_{v_B} C_{v_B}$ be a linear code of length n over B_k. Then the dual $C^\perp = \bigoplus_{B\in P_k} e_{v_B} C_{v_B}^\perp$ is also a linear code of length n over B_k.*

Let G be a generator matrix of C over B_k. If G_{v_B} is a generator matrix of C_{v_B}, then a generator matrix of C is $G = [e_{v_B} G_{v_B}]$ and a generator matrix of $\Psi_k(C)$ is $[\Psi_k(e_{v_B} G_{v_B})]$.

Theorem 2.5 *If C is an (n, M, d_L) linear code over B_k, then $\Psi_k(C)$ is a $(2^k n, M, d_H)$ linear code over F_{p^r}, where $d_L = d_H$.*

Proof By Theorem 2.3, Ψ_k is a linear and distance preserving map. Hence $d_L = d_H$. Since Ψ_k is a bijection, $|C| = |\Psi_k(C)| = (p^r)^{2^k}$. Also, the set $\Psi_k(C)$ is a code of length $2^k n$ over F_{p^r}. So, $\Psi_k(C)$ is a linear $(2^k n, M, d_H)$ code over F_{p^r}.

Theorem 2.6 *Let C be a code over B_k. Then C is a self-orthogonal code over B_k if and only if C_{v_B} is a self-orthogonal code over F_{p^r}.*

Proof Let C be a self-orthogonal code over B_k and $\mathbf{x} = \sum\limits_{B \in P_k} \mathbf{a}_{v_B} e_{v_B} \in C$, where $\mathbf{a}_{v_B} \in C_{v_B}$. Since C is a self-orthogonal code,

$$\begin{aligned}\mathbf{x} \cdot \mathbf{x} &= \Big(\sum_{B \in P_k} \mathbf{a}_{v_B} e_{v_B}\Big)\Big(\sum_{B \in P_k} \mathbf{a}_{v_B} e_{v_B}\Big) \\ &= \sum_{B \in P_k} \mathbf{a}_{v_B}^2 e_{v_B} = 0.\end{aligned}$$

We get $\mathbf{a}_{v_B}^2 = 0$. Hence $\mathbf{a}_{v_B} \in C_{v_B}^{\perp}$ implying C_{v_B} is a self-orthogonal code over F_{p^r}.

The other direction is obvious by the expression of C.

Theorem 2.7 *Let C be a linear code C of length n over B_k. Then $\Psi_k(C^{\perp}) = \Psi_k(C)^{\perp}$. Moreover, if C is a self-dual code, then so is $\Psi_k(C)$.*

Proof This can be proved similarly to [10].

Theorem 2.8 *Let C be a linear code C of length n over B_k. Then $\Psi_k(C) = \bigotimes\limits_{B \in P_k} C_{v_B}$ and $|\Psi_k(C)| = \prod\limits_{B \in P_k} |C_{v_B}|$.*

Proof This can be proved similarly to [10].

3 Skew Constacyclic Codes over B_k

Let $\lambda_k \in B_k$ be a unit. Then

$$\lambda_k = \sum_{B \in P_k} \lambda_{v_B} v_B,$$

where λ_{v_B} is a unit in F_{p^r}. Since

$$\lambda_k e_{v_B} = e_{v_B}\Big(\sum_{B \in P_k} \lambda_{v_B}\Big),$$

we have

$$\begin{aligned}\lambda_k &= \lambda_k\Big(\sum_{B \in P_k} e_{v_B}\Big) \\ &= \sum_{B \in P_k} \lambda_{v_B} e_{v_B}.\end{aligned}$$

Theorem 3.1 *Let $\lambda_k \in B_k$. Then λ_k is a unit in B_k if and only if $\sum\limits_{B\in P_k} \lambda_{v_B}$ is a unit in $F^*_{p^r}$ for $B \subseteq \{1, 2, \cdots, k\}$.*

Proof By the Chinese Remainder Theorem, λ_k is a unit in B_k if and only if $\sum\limits_{B\in P_k} \lambda_{v_B}$ is a unit in $F^*_{p^r}$.

Theorem 3.2 *Let $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ be a linear code of length n over B_k. Then C is a skew λ_k-constacyclic code over B_k if and only if every C_{v_B} is a skew λ_{v_B}-constacyclic code over F_{p^r}.*

Proof Let $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ be a skew λ_k-constacyclic code of length n over B_k and $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in C$, where $a_i = \sum\limits_{B\in P_k} e_{v_B} x^i_{v_B}$ and $x^i_{v_B} \in F_{p^r}$ for $i = 1, 2, \cdots, n$. Then $\mathbf{x}_{v_B} = (x^1_{v_B}, x^2_{v_B}, \cdots, x^n_{v_B}) \in C_{v_B}$, $x^i_{v_B} \in F_{p^r}$ for $i = 1, 2, \cdots, n$. Since C is a skew λ_k-constacyclic code, $\Delta_{\lambda_k}(a) \in C$. We have $\lambda_k e_{v_B} = e_{v_B}\big(\sum\limits_{B\in P_k} \lambda_{v_B}\big)$ and Δ_{λ_k} fixes $v_1, v_2, \cdots, v_k$. Then $\Delta_{\lambda_k}\big(\sum\limits_{B\in P_k} \lambda_{v_B}\big) = \sum\limits_{B\in P_k} \lambda_{v_B}$. We have

$$\Delta_{\lambda_k}(\lambda_k a_n) = \sum_{B\in P_k} e_{v_B} \Omega\Big(\sum_{B\in P_k} \lambda_{v_B} x^n_{v_B}\Big).$$

Hence, $\Omega\big(\sum\limits_{B\in P_k} \lambda_{v_B} x^i_{v_B}\big) \in C_{v_B}$ for $i = 1, 2, \cdots, n$. So, C_{v_B} is a skew λ_{v_B}-constacyclic code over F_{p^r}.

The converse can be shown similarly.

Theorem 3.3 *Let $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ be a skew λ_k-constacyclic code of length n over B_k. Then*

$$C = \langle e_1 f_1, e_{v_1} f_{v_1}, \cdots, e_{v_k} f_{v_k}, e_{v_1 v_2} f_{v_1 v_2}, \cdots, e_{v_1 v_2 \cdots v_k} f_{v_1 v_2 \cdots v_k} \rangle$$

and $|C| = (p^r)^{2^k n - \deg\left(\sum\limits_{B\in P_k} f_{v_B}\right)}$, where f_{v_B} is a generator polynomial of C_{v_B}.

Proposition 3.1 *Suppose C is a skew λ_k-constacyclic code of length n over B_k. Then there is a unique polynomial $f(x)$ such that $C = \langle f(x) \rangle$ and $f(x) \mid x^n - \lambda_k$, where $f(x) = \sum\limits_{B\in P_k} e_{v_B} f_{v_B}$.*

Proposition 3.2 *Suppose C is a skew λ_k-constacyclic code of length n over B_k, then $C^\perp = \bigoplus\limits_{B\in P_k} e_{v_B} C^\perp_{v_B}$ is a skew λ_k^{-1}-constacyclic code of length n over B_k and all $C^\perp_{v_B}$ are skew $\big(\sum\limits_{B\in P_k} \lambda_{v_B}\big)^{-1}$-constacyclic codes of length n over F_{p^r}.*

Proposition 3.3 *If $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ is a skew λ_k-constacyclic code of length n over B_k, then*

$$C^\perp = \Big\langle \sum_{B\in P_k} e_{v_B} h^*_{v_B} \Big\rangle$$

and $|C^{\perp}| = (p^r)^{\deg\left(\sum\limits_{B\in P_k} f_{v_B}\right)}$, *where* $x^n - \lambda_k = h_{v_B} f_{v_B}$ *and* $h^*_{v_B}$ *is the skew reciprocal polynomial of* h_{v_B}.

4 Construction of LCD Skew Constacyclic Codes over B_k

Definition 4.1 *The hull of the linear code* C *over* F_q *is defined to be* $\mathrm{Hull}(C) = C \cap C^{\perp}$. *When* $\mathrm{Hull}(C) = \{\mathbf{0}\}$, *the code* C *is called an LCD code.*

It is clear that, $\mathrm{Hull}(C)$ is a linear code.

Theorem 4.1 (see [6]) *Consider* F_q *a finite field,* θ *an automorphism of* F_q *of order* μ, $R = F_q[x, \theta]$, n *in* N^* *and* $\lambda \in \{1, -1\}$. *Consider a* (θ, λ)*-constacyclic code* C *with length* n, *skew generator polynomial* g *and consider* h *in* R *such that* $\Theta^n(h).g = x^n - \lambda$. C *is a Euclidean LCD code if and only if* $\mathrm{gcrd}(g, h^{\natural}) = 1$, *where* $h^{\natural}(x) = \frac{1}{\theta^k(h_0)} h^*(x)$.

Theorem 4.2 *Let* $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ *be an LCD code over* B_k *if and only if every* C_{v_B} *is an LCD code over* F_{p^r}.

Proof A linear code $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ has dual code $C^{\perp} = \bigoplus\limits_{B\in P_k} e_{v_B} C^{\perp}_{v_B}$. We have $\mathrm{Hull}(C) = C \cap C^{\perp} = \bigoplus\limits_{B\in P_k} e_{v_B}(C_{v_B} \cap C^{\perp}_{v_B})$. $\mathrm{Hull}(C) = \{\mathbf{0}\}$ if and only if $C_{v_B} \cap C^{\perp}_{v_B} = \{\mathbf{0}\}$.

By Theorem 4.1, we can obtain the following theorem.

Theorem 4.3 *Let* $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ *be a skew* λ_k*-constacyclic code over* B_k *with length* n. C *is an LCD code if and only if* $\mathrm{gcrd}(f_{v_B}(x), h^{\natural}_{v_B}(x)) = 1$.

Lemma 4.1 *Let* C *be a linear code over* B_k *with length* n. *Then* $\Psi_k(C \cap C^{\perp}) = \Psi_k(C) \cap \Psi_k(C)^{\perp}$.

Theorem 4.4 *If* $C = \bigoplus\limits_{B\in P_k} e_{v_B} C_{v_B}$ *is an LCD code of length* n *over* B_k, *then* $\Psi_k(C)$ *is an LCD code of length* $2^k n$ *over* F_{p^r}.

Proof Let C be an LCD code. Then $C \cap C^{\perp} = \{\mathbf{0}\}$, so $\Psi_k(C \cap C^{\perp}) = \{\mathbf{0}\}$. From Lemma 4.1, $\Psi_k(C) \cap \Psi_k(C)^{\perp} = \{\mathbf{0}\}$. Therefore, $\Psi_k(C)$ is an LCD code.

Conversely, let $\Psi_k(C)$ be an LCD code. Then $\Psi_k(C) \cap \Psi_k(C)^{\perp} = \{\mathbf{0}\}$. From Lemma 4.1, we have $\Psi_k(C \cap C^{\perp}) = \{\mathbf{0}\}$. Since Ψ_k is injective, $C \cap C^{\perp} = \{\mathbf{0}\}$. Hence, C is an LCD code.

Remark 4.1 Let C be an $[n, k, d]$ linear code. If it attains the Singleton bound, i.e., $d = n - k + 1$, it is called a maximum distance separable code, or MDS code.

Example 4.1 Let $k = 3, F_9 = F_3[\xi]$ with $\xi^2 = \xi + 1$ and $\Omega(\varrho) = \varrho^3$ for any $\varrho \in F_9$. Let

$n = 4$. We have

$$x^4 + 1 = (x^2 + \xi^3 x + 1)(x^2 + \xi^7 x + 1) \in F_9[x, \Omega],$$
$$x^4 - 1 = (x + 1)(x + 2)(x + \xi)(x + \xi^7) \in F_9[x, \Omega].$$

If $g_1(x) = g_{v_1}(x) = g_{v_2}(x) = g_{v_3}(x) = x + \xi^7$ and $g_{v_1v_2}(x) = g_{v_1v_3}(x) = g_{v_2v_3}(x) = g_{v_1v_2v_3}(x) = x^2 + \xi^7 x + 1$, then $C_1 = C_{v_1} = C_{v_2} = C_{v_3} = \langle x + \xi^7 \rangle$ is a skew cyclic code of length 4 over F_9 and $C_{v_1v_2} = C_{v_1v_3} = C_{v_2v_3} = C_{v_1v_2v_3} = \langle x^2 + \xi^7 x + 1 \rangle$ is a skew negacyclic code of length 4 over F_9. The skew reciprocal polynomial of $x^3 + \xi x^2 + 2x + \xi^5$ is $x^3 + \xi^5 x^2 + 2x + \xi$. The skew reciprocal polynomial of $x^2 + \xi^3 x + 1$ is $x^2 + \xi x + 1$. By Theorem 4.1, $C_1 = C_{v_1} = C_{v_2} = C_{v_3}$ is a Euclidean LCD MDS code with parameters $[4, 3, 2]$ and $C_{v_1v_2} = C_{v_1v_3} = C_{v_2v_3} = C_{v_1v_2v_3}$ is a Euclidean LCD MDS code with parameters $[4, 2, 3]$. Hence the code C is an LCD code over B_3 with length 4 and $\Psi_3(C)$ is an LCD code with parameters $[32, 20, 2]$.

It can be generalized for a suitable k.

Example 4.2 Let $k = 2, F_9 = F_3[\xi]$ with $\xi^2 = \xi + 1$ and $\Omega(\varrho) = \varrho^3$ for any $\varrho \in F_9$. Let $n = 6$. We have

$$\begin{aligned} x^6 + 1 &= (x + \xi^7)^3 (x + \xi)^3 \in F_9[x, \Omega] \\ &= (x^3 + \xi^2 x^2 - x + \xi^5)(x^3 + \xi^2 x^2 - x + \xi^3), \\ x^6 - 1 &= (x + \xi^2)^2 (x + \xi^6)^2 (x + 1)(x + 2) \in F_9[x, \Omega] \\ &= (x^3 + \xi x^2 + x + 1)(x^3 + \xi^7 x^2 + x + \xi^4). \end{aligned}$$

If $g_1(x) = g_{v_1}(x) = x^3 + \xi^7 x^2 + x + \xi^4$ and $g_{v_2}(x) = g_{v_1v_2}(x) = x^3 + \xi^2 x^2 - x + \xi^3$, then $C_1 = C_{v_1}$ is a skew cyclic code of length 6 over F_9 and $C_{v_2} = C_{v_1v_2}$ is a skew negacyclic code of length 6 over F_9. The skew reciprocal polynomial of $x^3 + \xi x^2 + x + 1$ is $x^3 + \xi x^2 + x + 1$. The skew reciprocal polynomial of $x^3 + \xi^2 x^2 - x + \xi^5$ is $x^3 + \xi^5 x^2 + \xi^7 x + \xi$. By Theorem 4.1, $C_1 = C_{v_1}$ is a Euclidean LCD MDS code with parameters $[6, 3, 4]$ and $C_{v_2} = C_{v_1v_2}$ is a Euclidean LCD MDS code with parameters $[6, 3, 4]$. Hence the code C is an LCD code over B_2 with length 6 and $\Psi_2(C)$ is an LCD code with parameters $[24, 12, 4]$.

It can be generalized for a suitable k.

5 Quantum Codes from Skew Constacyclic Codes over B_k

Definition 5.1 *A q-ary quantum code is a q^t dimensional subspace of the Hilbert space $\mathbb{C}^{q^n}$. A quantum code with length n, dimension t and minimum distance d over F_q is denoted by $[[n, t, d]]_q$.*

In the sequel, we will construct quantum codes from dual containing skew constacyclic codes over B_k.

Lemma 5.1 *Let C be an $[n,t,d]$ linear code over F_q. If $C^{\perp} \subseteq C$, then there exists a quantum code of type $[[n, 2t-n, \geq d]]$ over F_q (see [16]).*

Lemma 5.2 *Let C be a skew constacyclic code of length n over F_q with $\lambda \in F_q^*$. If $C^{\perp} \subseteq C$, then $\lambda = \lambda^{-1}$ (see [26]).*

Lemma 5.3 *Let $C = \langle f(x) \rangle$ be a skew λ-constacyclic code of length n over F_q such that the order of automorphism Ω divides n, where $\lambda = \mp 1$. Then $C^{\perp} \subseteq C$ if and only if $h^*(x)h(x)$ is divisible by $x^n - \lambda$ on the right (see [16]).*

Theorem 5.1 *Let $C = \left\langle \sum\limits_{B \in P_k} e_{v_B} f_{v_B} \right\rangle$ be a skew λ_k-constacyclic code of length n over B_k such that the order of automorphism Δ_k divides n and $\sum\limits_{B \in P_k} \lambda_{v_B} = \mp 1$. Then $C^{\perp} \subseteq C$ if and only if $h_{v_B}^* h_{v_B}$ is divisible by $x^n - \sum\limits_{B \in P_k} \lambda_{v_B}$ on the right.*

Proof Let $C = \left\langle \sum\limits_{B \in P_k} e_{v_B} f_{v_B} \right\rangle$ be a skew λ_k-constacyclic code of length n over B_k, where $f_{v_B} \in C_{v_B}$. If $h_{v_B}^* h_{v_B}$ is divisible by $x^n - \sum\limits_{B \in P_k} \lambda_{v_B}$ on the right, then by Lemma 5.3, we can get $C_{v_B}^{\perp} \subseteq C_{v_B}$, which implies that $e_{v_B} C_{v_B}^{\perp} \subseteq e_{v_B} C_{v_B}$. Therefore, $\bigoplus\limits_{B \in P_k} e_{v_B} C_{v_B}^{\perp} \subseteq \bigoplus\limits_{B \in P_k} e_{v_B} C_{v_B}$. So, $C^{\perp} \subseteq C$.

Conversely, if $C^{\perp} \subseteq C$, then $\bigoplus\limits_{B \in P_k} e_{v_B} C_{v_B}^{\perp} \subseteq \bigoplus\limits_{B \in P_k} e_{v_B} C_{v_B}$. Hence, $e_{v_B} C_{v_B}^{\perp} = e_{v_B} C^{\perp} \subseteq e_{v_B} C_{v_B} = e_{v_B} C$. That is $C_{v_B}^{\perp} \subseteq C_{v_B}$. By Lemma 5.3, $h_{v_B}^* h_{v_B}$ is divisible by $x^n - \sum\limits_{B \in P_k} \lambda_{v_B}$ on the right.

By Lemma 5.1 and Theorems 2.6, 5.1, we can obtain quantum codes from skew λ_k-constacyclic codes over B_k.

Theorem 5.2 *Let $C = \bigoplus\limits_{B \in P_k} e_{v_B} C_{v_B}$ be a skew λ_k-constacyclic code of length n over B_k such that the order of automorphism Δ_k divides n. If $C^{\perp} \subseteq C$, then there exists a quantum error-correcting code with parameters $[[2^k n, 2t - 2^k n, d_L]]$, where d_L denotes the minimum Lee distance of C and t is the dimension of the code $\Psi_k(C)$.*

Remark 5.1 Let C be a quantum $[[n,k,d]]$ code. If it attains the (quantum) Singleton bound, i.e., $2d = n - k + 2$, it is called a maximum distance separable code or quantum MDS code.

Example 5.1 Let $k = 3, F_9 = F_3[\xi]$ with $\xi^2 = \xi + 1$ and $\Omega(\varrho) = \varrho^3$ for any $\varrho \in F_9$. Let $n = 6$. We have

$$x^6 + 1 = (x + \xi^7)^3 (x + \xi)^3 \in F_9[x, \Omega],$$
$$x^6 - 1 = (x + \xi^2)^2 (x + \xi^6)^2 (x + 1)(x + 2) \in F_9[x, \Omega].$$

If $f_1(x) = f_{v_1}(x) = f_{v_2}(x) = f_{v_3}(x) = x^2 + \xi^6 x + 1$ and $f_{v_1 v_2}(x) = f_{v_1 v_3}(x) = f_{v_2 v_3}(x) = f_{v_1 v_2 v_3}(x) = x + \xi$, then $C_1 = C_{v_1} = C_{v_2} = C_{v_3} = \langle x^2 + \xi^6 x + 1 \rangle$ is a skew cyclic code of length 6 over F_9 and $C_{v_1v_2} = C_{v_1v_3} = C_{v_2v_3} = C_{v_1v_2v_3} = \langle x+\xi \rangle$ is a skew negacyclic code of length 6 over F_9. Hence the code $C = \langle \sum_{B\in P_3} e_{v_B} f_{v_B} \rangle$ is a skew $(1+v_1v_2+v_1v_3+v_2v_3+v_1v_2v_3)$-constacyclic code over B_3 and $\Psi_3(C)$ has parameters $[48, 36, 3]$.

Since

$$\begin{aligned} h_1(x) = h_{v_1}(x) = h_{v_2}(x) = h_{v_3}(x) &= x^4 + \xi^2 x^3 + \xi^6 x + 2, \\ h_1^*(x) = h_{v_1}^*(x) = h_{v_2}^*(x) = h_{v_3}^*(x) &= x^4 + \xi^6 x^3 + \xi^2 x + 2, \\ h_{v_1v_2}(x) = h_{v_1v_3}(x) = h_{v_2v_3}(x) = h_{v_1v_2v_3}(x) &= x^5 + \xi^7 x^4 + 2x^3 + \xi^3 x^2 + x + \xi^7, \\ h_{v_1v_2}^*(x) = h_{v_1v_3}^*(x) = h_{v_2v_3}^*(x) = h_{v_1v_2v_3}^*(x) &= x^5 + \xi^3 x^4 + 2x^3 + \xi^7 x^2 + x + \xi^3, \end{aligned}$$

we have $h_{v_B}^*(x)h_{v_B}(x)$ is divisible by $x^6 \pm 1$ on the right. Therefore, $C^\perp \subseteq C$. Hence, by Theorem 5.2, we obtain a quantum code with parameters $[[48, 24, 3]]$.

Similarly, we obtain a quantum code with parameters $[[3 \cdot 2^{k+1}, 3 \cdot 2^k, 3]]$ for a suitable k.

Example 5.2 Let $F_9 = F_3[\xi]$ with $\xi^2 = \xi + 1$ and $\Omega(\varrho) = \varrho^3$ for any $\varrho \in F_9$. Let $n = 8$. We have

$$x^8 - 1 = (x^2 - \xi^6)(x^2 + \xi^6 x + \xi^3)(x - \xi^5)(x^2 - \xi^3 x + 1)(x + \xi^6) \in F_9[x, \Omega].$$

If $f_{v_B}(x) = x + \xi^6$, then $C_{v_B} = \langle x + \xi^6 \rangle$ is a skew cyclic code of length 8 over F_9 with parameters $[8, 7, 2]$. Since $h_{v_B}^*(x)h_{v_B}(x)$ is divisible by $x^8 - 1$ on the right, $C_{v_B}^\perp \subseteq C_{v_B}$. Hence we obtain a quantum MDS code with parameters $[[8, 6, 2]]$ over F_9.

Let $\lambda_k = 1$. The code $C = \langle \sum_{B\in P_k} e_{v_B}(x + \xi^6) \rangle$ is a skew λ_k-constacyclic code over B_k and $[2^{k+3}, 2^{k+3} - 2^k, 2]$ are the parameters of $\Psi_k(C)$. Since $h_{v_B}^*(x)h_{v_B}(x)$ is divisible by $x^8 - 1$ on the right, $C^\perp \subseteq C$. Hence, by Theorem 5.2, we obtain a quantum code with parameters $[[2^{k+3}, 2^{k+3} - 2^{k+1}, 2]]$ for a suitable k.

Example 5.3 Let $F_9 = F_3[\xi]$ with $\xi^2 = \xi + 1$ and $\Omega(\varrho) = \varrho^3$ for any $\varrho \in F_9$. Let $n = 20, \lambda_k = 1$. If $f_{v_B}(x) = x + \xi^2$, then $C_{v_B} = \langle x + \xi^2 \rangle$ is a skew cyclic code of length 20 over F_9. Hence the code $C = \langle \sum_{B\in P_k} e_{v_B}(x + \xi^2) \rangle$ is a skew λ_k-constacyclic code over B_k and $[5 \cdot 2^{k+2}, 5 \cdot 2^{k+2} - 2^k, 4]$ are the parameters of $\Psi_k(C)$. Since $h_{v_B}^*(x)h_{v_B}(x)$ is divisible by $x^{20} - 1$ on the right, $C^\perp \subseteq C$. Hence, by Theorem 5.2, we obtain a quantum code with parameters $[[5 \cdot 2^{k+2}, 9 \cdot 2^{k+1}, 4]]$ for a suitable k.

Example 5.4 Let $F_{25} = F_5[\xi]$ with $\xi^2 = \xi + 3$ and $\Omega(\varrho) = \varrho^5$ for any $\varrho \in F_{25}$. Let $n = 12, \lambda_k = -1$. If $f_{v_B}(x) = x^5 + \xi x^4 + \xi^8 x^3 + \xi^{21} x^2 + \xi^{21} x + \xi^{11}$, then $C_{v_B} = \langle f_{v_B}(x) \rangle$ is a skew negacyclic code of length 12 over F_{25}. Hence the code $C = \langle \sum_{B\in P_k} e_{v_B} f_{v_B}(x) \rangle$ is a skew λ_k-constacyclic code over B_k and $[3 \cdot 2^{k+2}, 2^{k+3} - 2^k, 4]$ are the parameters of $\Psi_k(C)$. Since $h_{v_B}^*(x)h_{v_B}(x)$ is divisible by $x^{12} - 1$ on the right, $C^\perp \subseteq C$. Hence, by Theorem 5.2, we obtain a quantum code with parameters $[[3 \cdot 2^{k+2}, 2^{k+1}, 4]]$ for a suitable k.

Example 5.5 Let $F_9 = F_3[\xi]$ with $\xi^2 = \xi + 1$ and $\Omega(\varrho) = \varrho^3$ for any $\varrho \in F_9$. Let $n = 18$, $f_1(x) = f_{v_1}(x) = \cdots = f_{v_k}(x) = x^8 + \xi^4 x^7 - \xi x^6 + x^5 - \xi^6 x^3 + \xi^5 x^2 - \xi^2 x - \xi^2$ and other $f_{v_B}(x) = x^6 - \xi^2 x^5 + \xi^3 x^4 - \xi^6 x^3 + \xi^7 x^2 - \xi^2 x + \xi^6$, then $C_1 = C_{v_1} = C_{v_2} = \cdots = C_{v_k} = \langle x^8 + \xi^4 x^7 - \xi x^6 + x^5 - \xi^6 x^3 + \xi^5 x^2 - \xi^2 x - \xi^2 \rangle$ is a skew cyclic code of length 18 over F_9 and other $C_{v_B} = \langle f_{v_B}(x) \rangle$ is a skew negacyclic code of length 18 over F_9. Hence the code $C = \left\langle \sum_{B \in P_k} e_{v_B} f_{v_B} \right\rangle$ is a skew λ_k-constacyclic code over B_k and $[9 \cdot 2^{k+1}, 11 \cdot 2^k, 6]$ are the parameters of $\Psi_k(C)$. Since $h^*_{v_B}(x) h_{v_B}(x)$ is divisible by $x^{18} \pm 1$ on the right, $C^\perp \subseteq C$. Hence, by Theorem 5.2, we obtain a quantum code with parameters $[[9 \cdot 2^{k+1}, 2^{k+2}, 6]]$ for a suitable k.

6 Conclusion

In this paper, by using the skew constacyclic codes over the family of finite rings B_k, the parameters of quantum codes and LCD codes were obtained, and some computations were made. In the future, the asymmetric quantum codes can be obtained from skew constacyclic codes, and entanglement-assisted quantum error-correcting codes can be obtained from LCD skew constacyclic codes over the family of finite rings B_k. Also, codes with better parameters can be found.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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