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Symplectic Mean Curvature Flow in $\mathbb{CP}^2$ with Normal Curvature Pinched*

Xiang LI[1] Jun SUN[2]

Abstract In this paper, the authors show that the symplectic mean curvature flow in $\mathbb{CP}^2$ with normal curvature pinched exists for a long time and converges to a holomorphic curve.

Keywords Symplectic mean curvature flow, Holomorphic curve
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1 Introduction

Let M be a Kähler surface, ω be the Kähler form on M and J be a complex structure compatible with ω. For vector fields U, V on M, the Riemannian metric $\overline{g}$ on M is defined by

$$\overline{g}(U,V)=\omega(U,JV).$$

For a compact oriented real surface Σ which is smoothly immersed in M, the Kähler angle α of Σ was defined by [4],

$$\omega|_{\Sigma}=\cos\alpha \mathrm{d}\mu_{\Sigma}, \tag{1.1}$$

where $\mathrm{d}\mu_{\Sigma}$ is the area element of the induced metric on Σ. We say that Σ is a symplectic surface if $\cos\alpha>0$ and Σ is a holomorphic curve if $\cos\alpha\equiv 1$.

The existence of holomorphic curves is a fundamental problem in differential geometry. By Wirtinger's inequality, holomorphic curves are always area-minimizing in its homological class, thus must be symplectic stable minimal surface. On the other hand, Wolfson [11] showed that any symplectic minimal surface in Kähler-Einstein surface with nonnegative scalar curvature must be holomorphic curve. Hence, it is natural to consider the existence problem for symplectic minimal surfaces. One important idea is to use the mean curvature flow, which is the negative gradient flow for the area functional. The other way is to use variational method (see [6]).

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[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China.
E-mail: lihsiang@mail.ustc.edu.cn
[2]Corresponding author. School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail: sunjun@whu.edu.cn
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Fortunately, Chen-Li [2] and Wang [10] independently proved that "symplectic" property is preserved by the mean curvature flow, in which case we call it "symplectic mean curvature flow (SMCF for short)". The symplectic mean curvature flow exists globally and converges at infinity in graphic cases (see [3]). Han-Li [5] proved that, in a Kähler-Einstein surface with positive scalar curvature, if the initial surface is sufficiently close to a holomorphic curve, then the symplectic mean curvature flow exists globally and converges to a holomorphic curve at infinity. In the space $\mathbb{CP}^2$ with constant holomorphic sectional curvature $k > 0$, Han-Li-Yang [8] proved that if the Kähler angle of the initial surface has a certain lower bound and satisfies certain pinching estimate, then the symplectic mean curvature flow exists for a long time and converges to a holomorphic curve.

Furthermore, Chen-Li [2] and Wang [10] proved that there is no finite time Type I singularity for symplectic mean curvature flow. Therefore, it is important to study Type II singularities for the symplectic mean curvature flow, which are always eternal solutions. An important type of eternal solutions to the mean curvature flow is translating solitons. There are many rigidity results on symplectic translating solitons. For instance, together with Han, the second author (see [9]) showed that any symplectic translating soliton with nonpositive normal curvature cannot arise as blow up limit of symplectic mean curvature flow. On the other hand, for general blow up flow, we (see [7]) proved that any eternal mean curvature flow which is normally flat cannot arise as blow up limit for symplectic mean curvature flow. So we are interested in the symplectic mean curvature flow with normal curvature pinched. Recently, Baker-Nguyen [1] studied codimension two surfaces pinched by normal curvature evolving by mean curvature flow, they proved that codimension two surfaces satisfying a nonlinear curvature condition depending on normal curvature smoothly evolve by mean curvature flow to round points.

In this paper, we mainly study the symplectic mean curvature flow in space $\mathbb{CP}^2$. We use common notations, such as H for mean curvature, A for the second fundamental form and $K^\perp$ for normal curvature. We show that the symplectic mean curvature flow with normal curvature pinched exists for a long time and converges to a holomorphic curve, i.e., the following theorem.

Theorem 1.1 *Suppose Σ is a symplectic surface in $\mathbb{CP}^2$ with constant holomorphic sectional curvature $k > 0$. Taking $\mu \in [0, 1]$, assume that*

$$|A|^2 + 2\mu\gamma|K^\perp| \le \lambda|H|^2 + \frac{160\lambda + 4 + 2\mu - 6\mu\cos^2\alpha}{40\lambda + 1}\frac{2\lambda - 1}{4\lambda}k,$$

where $\gamma = \frac{1}{40\lambda+1}$ and

$$\cos\alpha \ge \max\{S_1(\lambda, \mu), S_2(\lambda, \mu), S_3(\lambda, \mu)\}$$

holds on the initial surface for any $\frac{1}{2} < \lambda \le \frac{2}{3} - \frac{1}{12}\mu$, then it remains true along the symplectic mean curvature flow. Furthermore, under this assumption, the symplectic mean curvature flow exists for a long time and converges to a holomorphic curve.

Remark 1.1 Here we need to point out that $S_1(\lambda,\mu)$, $S_2(\lambda,\mu)$ and $S_3(\lambda,\mu)$ are polynomials with respect to λ and μ, in the following forms:

$$S_1(\mu,\lambda) \doteq \sqrt{\frac{11-4\lambda}{12(\lambda+1)}},$$
$$S_2(\mu,\lambda) \doteq \sqrt{\frac{472\lambda-320\lambda^2+12+6\mu}{6(160\lambda^2+4\lambda+3\mu)}}$$

and

$$S_3(\mu,\lambda) \doteq \sqrt{\frac{400\lambda^2(7\lambda-3)(2\lambda-1)(40\lambda+1)^2+\mu(J_2(\mu,\lambda)-J_1(\mu,\lambda))}{1200\lambda^3(2\lambda-1)(40\lambda+1)^2+\mu J_2(\mu,\lambda)}},$$

where

$$\begin{aligned}J_1(\mu,\lambda) &= 4\lambda(2\lambda-1)(40\lambda+1)[(30-40\lambda)(40\lambda+1)-(40\lambda+\mu)(40\lambda+1)\\ &\quad -(30-40-\mu)(40\lambda+\mu)]\end{aligned}$$

and

$$J_2(\mu,\lambda) = \lambda(40\lambda+1)[240\lambda^2(40\lambda+1)-6(2\lambda-1)(30-40\lambda-\mu)(40\lambda+\mu)].$$

And we know by numerical calculation that

$$\frac{1}{3} < \max\{S_1(\lambda,\mu), S_2(\lambda,\mu), S_3(\lambda,\mu)\} \le 1.$$

Remark 1.2 $S_1(\lambda,\mu)$, $S_2(\lambda,\mu)$ and $S_3(\lambda,\mu)$ look very complicated in terms of how they are expressed, and that is because we are thinking about normal curvature pinched condition. This condition is valuable because submanifolds with non-flat normal bundle are more general.

Remark 1.3 When $\mu=0$, the inequality (3.7) does not exist and then (3.12)–(3.14) are meaningless. At this time, the pinching condition reduces to

$$|A|^2 \le \lambda|H|^2 + \frac{2\lambda-1}{\lambda}k$$

and

$$\cos\alpha \ge S_3(0,\lambda) = \sqrt{\frac{7\lambda-3}{3\lambda}},$$

where $\frac{1}{2} < \lambda \le \frac{2}{3}$. This is exactly the assumptions of the main result of Han-Li-Yang [8].

Note that the holomorphic sectional curvature of $\mathbb{CP}^2$ is $k>0$. Using a similar method, we can consider the case of flat torus $\mathbb{T}^4$. We have the following result.

Theorem 1.2 *Suppose Σ is a symplectic surface in the flat torus $\mathbb{T}^4$. Assume that $|A|^2 + 2\mu\gamma|K^\perp| \le \lambda|H|^2$, where $\mu\in[0,1]$, $\gamma=\frac{1}{40\lambda+1}$ and $\cos\alpha \ge \delta$ ($0<\delta\le 1$ is a constant) holds on the initial surface for any $\frac{1}{2}<\lambda\le\frac{2}{3}$, then it remains true along the symplectic mean curvature flow. Furthermore, under this assumption, the symplectic mean curvature flow exists for a long time and converges to a holomorphic curve.*

Unfortunately, we do not know whether the symplectic curvature flow has long-time existence and convergence in manifolds with negative curvature.

Throughout this paper we will adopt the following ranges of indices:

$$\begin{aligned} A, B, \cdots &= 1, \cdots, 4, \\ \alpha, \beta, \gamma, \cdots &= 3, 4, \\ i, j, k, \cdots &= 1, 2. \end{aligned}$$

2 Preliminaries

In this section, we adhere to the notation of [8]. Now suppose M is a Kähler surface with constant holomorphic sectional curvature k, then from [12, Theorems 2.1 and 2.3], we have the following results.

Lemma 2.1 *M has a curvature tensor of the form*

$$K_{kjih} = -\frac{k}{4}[(g_{kh}g_{ji} - g_{jh}g_{ki}) + (J_{kh}J_{ji} - J_{jh}J_{ki}) - 2J_{kj}J_{ih}].$$

Thus M is symmetric. Furthermore, M is Einstein

$$K_{ji} = \frac{3}{2}k\overline{g}_{ij}.$$

Suppose that Σ is a submanifold in a Riemannian manifold M, we choose an orthonormal basis $\{e_i\}$ for $T\Sigma$ and $\{e_\alpha\}$ for $N\Sigma$. Recall the evolution equation for the second fundamental form h_{ij}^α and $|A|^2$ along the mean curvature flow (see [2, 10]).

Lemma 2.2 *For a mean curvature flow $F : \Sigma \times [0, t_0) \to M$, the second fundamental form h_{ij}^α satisfies the following equation*

$$\begin{aligned} \frac{\partial}{\partial t}h_{ij}^\alpha = {} & \Delta h_{ij}^\alpha + (\overline{\nabla}_{\partial_k}K)_{\alpha ijk} + (\overline{\nabla}_{\partial_j}K)_{\alpha kik} \\ & - 2K_{lijk}h_{lk}^\alpha + 2K_{\alpha\beta jk}h_{ik}^\beta + 2K_{\alpha\beta ik}h_{jk}^\beta \\ & - K_{lkik}h_{lj}^\alpha - K_{lkjk}h_{il}^\alpha + K_{\alpha k\beta k}h_{ij}^\beta \\ & - H^\beta(h_{ik}^\beta h_{jk}^\alpha + h_{jk}^\beta h_{ik}^\alpha) + h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta \\ & - 2h_{im}^\beta h_{mk}^\alpha h_{kj}^\beta + h_{mj}^\alpha h_{mk}^\beta h_{ik}^\beta \\ & + h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta + h_{ij}^\beta \langle e_\beta, \overline{\nabla}_H e_\alpha \rangle, \end{aligned} \tag{2.1}$$

where K_{ABCD} is the curvature tensor of M and $\overline{\nabla}$ is the covariant derivative of M. Therefore

$$\begin{aligned} \frac{\partial}{\partial t}|A|^2 = {} & \Delta|A|^2 - 2|\nabla A|^2 + [(\overline{\nabla}_{\partial_k}K)_{\alpha ijk} + (\overline{\nabla}_{\partial_j}K)_{\alpha kik}]h_{ij}^\alpha \\ & - 4K_{lijk}h_{lk}^\alpha h_{ij}^\alpha + 8K_{\alpha\beta jk}h_{ik}^\beta h_{ij}^\alpha - 4K_{lkik}h_{lj}^\alpha h_{ij}^\alpha + 2K_{\alpha k\beta k}h_{ij}^\beta h_{ij}^\alpha \\ & + 2\sum_{\alpha,\beta,i,j}\Big(\sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)\Big)^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big)^2. \end{aligned}$$

Corollary 2.1 *Along the mean curvature flow, the length of the mean curvature vector satisfies*

$$\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2K_{\alpha k\beta k}H^\alpha H^\beta + 2\sum_{i,j}\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2.$$

Using Lemmas 2.1–2.2 and Corollary 2.1, Han-Li-Yang [8] computed the evolution equation of the length of the second fundamental form and the length of the mean curvature vector in $\mathbb{CP}^2$ as follows.

Corollary 2.2 *For a mean curvature flow $F : \Sigma^2 \times [0, t_0) \to \mathbb{CP}^2$, the length of the second fundamental form and the length of the mean curvature vector satisfy*

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2 - 2|\nabla A|^2 - k|A|^2 - \frac{k}{2}(3\cos^2\alpha + 1)|A|^2 \\ &\quad + k(3\cos^2\alpha + 1)|H|^2 - 2k(3\cos^2\alpha - 1)|\overline{\nabla} J_{\Sigma_t}|^2 \\ &\quad + 2\sum_{\alpha,\beta,i,j}\Big(\sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)\Big)^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big)^2. \end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 &= \Delta|H|^2 - 2|\nabla H|^2 + 3k|H|^2 - \frac{k}{2}(3\cos^2\alpha + 1)|H|^2 \\ &\quad + 2\sum_{i,j}\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2. \end{aligned} \tag{2.3}$$

Suppose that M is a compact Kähler surface. Let Σ be a smooth surface in M. The Kähler angle of Σ in M is defined by (1.1). Recall the evolution equation of $\cos\alpha$ (see [2]).

Lemma 2.3 *Along the symplectic mean curvature flow, $\cos\alpha$ satisfies*

$$\frac{\partial}{\partial t}\cos\alpha = \Delta\cos\alpha + |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + \mathrm{Ric}(Je_1, e_2)\sin^2\alpha, \tag{2.4}$$

where $|\overline{\nabla} J_{\Sigma_t}|^2 = |h_{1k}^3 - h_{2k}^4|^2 + |h_{2k}^3 + h_{1k}^4|^2$, $\{e_1, e_2, e_3, e_4\}$ is any orthonormal basis for TM such that $\{e_1, e_2\}$ is the basis for $T\Sigma$ and $\{e_3, e_4\}$ is the basis for $N\Sigma$.

It is easy to see that $|\overline{\nabla} J_{\Sigma_t}|^2$ is independent of the choice of the frame and only depends on the orientation of the frame.

By Ricci equation, we have

$$R_{ij\alpha\beta} - K_{ij\alpha\beta} = h_{ip}^\alpha h_{jp}^\beta - h_{jp}^\alpha h_{ip}^\beta.$$

Now let Σ be a surface in a Kähler surface M. Then the normal curvature is

$$K^\perp := R_{1234} - K_{1234} = h_{1p}^3 h_{2p}^4 - h_{2p}^3 h_{1p}^4. \tag{2.5}$$

We will choose a special frame for M. Actually, at the point where $|H| \neq 0$, we will choose local orthonormal normal frame $\{\nu_3, \nu_4\}$ with $\nu_3 = \frac{H}{|H|}$. It is also possible to choose the tangent frame $\{e_1, e_2\}$ to diagonalize A_1, where A_1 is the second fundamental form corresponding to ν_3. Therefore, there exist functions a, b, c so that the second fundamental form can be expressed as

$$A = \begin{pmatrix} \frac{|H|}{2} + a & 0 \\ 0 & \frac{|H|}{2} - a \end{pmatrix} \nu_3 + \begin{pmatrix} b & c \\ c & -b \end{pmatrix} \nu_4. \tag{2.6}$$

Then we have

$$h_{11}^3 = \frac{|H|}{2} + a, \quad h_{22}^3 = \frac{|H|}{2} - a, \quad h_{12}^3 = h_{21}^3 = 0, \quad h_{11}^4 = b, h_{22}^4 = -b, \quad h_{12}^4 = h_{21}^4 = c.$$

The mean curvature vector is given by $H = H^3\nu_3 + H^4\nu_4$ with $H^3 = |H|$ and $H^4 = 0$.

In this local frame, we see by direct computation that

$$K^\perp = h_{1p}^3 h_{2p}^4 - h_{2p}^3 h_{1p}^4 = h_{11}^3 h_{21}^4 + h_{12}^3 h_{22}^4 - h_{21}^3 h_{11}^4 - h_{22}^3 h_{12}^4 = 2ac, \tag{2.7}$$

$$|\mathring{A}|^2 = 2(a^2 + b^2 + c^2), \tag{2.8}$$

$$|A|^2 = \frac{|H|^2}{2} + 2(a^2 + b^2 + c^2) \tag{2.9}$$

and

$$|\overline{\nabla} J_{\Sigma_t}|^2 = |h_{1k}^3 - h_{2k}^4|^2 + |h_{2k}^3 + h_{1k}^4|^2 = \frac{1}{2}|H|^2 + 2[b^2 + (a-c)^2] = |A|^2 - 2K^\perp. \tag{2.10}$$

Here, $\mathring{A}$ denotes the trace-free part of the second fundamental form.

Next, we would like to compute the evolution equation for the normal curvature.

Lemma 2.4 *For a mean curvature flow $F : \Sigma^2 \times [0, t_0) \to \mathbb{CP}^2$, the normal curvature satisfies the equation*

$$\begin{aligned} \frac{\partial}{\partial t} K^\perp = {} & \Delta K^\perp - 2\nabla_{\mathrm{evol}} K^\perp + |\mathring{A}|^2 k (3\cos^2\alpha - 1) + \frac{1}{2} k K^\perp (1 - 15\cos^2\alpha) \\ & + K^\perp (|A|^2 + 2|\mathring{A}|^2). \end{aligned}$$

Proof From (2.5), we calculate

$$\begin{aligned} \Big(\frac{\partial}{\partial t} - \Delta\Big) K^\perp = {} & -2(\nabla_q h_{1p}^3 \nabla_q h_{2p}^4 - \nabla_q h_{2p}^3 \nabla_q h_{1p}^4) + h_{2p}^4 \Big(\frac{\partial}{\partial t} - \Delta\Big) h_{1p}^3 \\ & + h_{1p}^3 \Big(\frac{\partial}{\partial t} - \Delta\Big) h_{2p}^4 - h_{1p}^4 \Big(\frac{\partial}{\partial t} - \Delta\Big) h_{2p}^3 - h_{2p}^3 \Big(\frac{\partial}{\partial t} - \Delta\Big) h_{1p}^4. \end{aligned}$$

We denote

$$\nabla_{\mathrm{evol}} K^\perp := \nabla_q h_{1p}^3 \nabla_q h_{2p}^4 - \nabla_q h_{2p}^3 \nabla_q h_{1p}^4. \tag{2.11}$$

Notice that $\mathbb{CP}^2$ is locally symmetric. Using (2.1) and our choice of frame, we compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) K^\perp = {} & -2\nabla_{\mathrm{evol}} K^\perp \\ & + h_{2p}^4 [-2K_{l1pk} h_{lk}^3 + 2K_{3\beta pk} h_{1k}^\beta + 2K_{3\beta 1k} h_{pk}^\beta - K_{lk1k} h_{lp}^3 - K_{lkpk} h_{1l}^3 \end{aligned}$$

$$
\begin{aligned}
&+ K_{3k\beta k}h^{\beta}_{1p} - H^{\beta}(h^{\beta}_{1k}h^3_{pk} + h^{\beta}_{pk}h^3_{1k}) + h^3_{1m}h^{\beta}_{mk}h^{\beta}_{kp} - 2h^{\beta}_{1m}h^3_{mk}h^{\beta}_{kp} \\
&+ h^{\beta}_{1k}h^{\beta}_{km}h^3_{mp} + h^3_{km}h^{\beta}_{mk}h^{\beta}_{1p} + h^{\beta}_{1p}\langle e_\beta, \overline{\nabla}_H e_3\rangle] \\
&+ h^3_{1p}[-2K_{l2pk}h^4_{lk} + 2K_{4\beta pk}h^{\beta}_{2k} + 2K_{4\beta 2k}h^{\beta}_{pk} - K_{lk2k}h^4_{lp} - K_{lkpk}h^4_{2l} \\
&+ K_{4k\beta k}h^{\beta}_{2p} - H^{\beta}(h^{\beta}_{2k}h^4_{pk} + h^{\beta}_{pk}h^4_{2k}) + h^4_{2m}h^{\beta}_{mk}h^{\beta}_{kp} - 2h^{\beta}_{2m}h^4_{mk}h^{\beta}_{kp} \\
&+ h^{\beta}_{2k}h^{\beta}_{km}h^4_{mp} + h^4_{km}h^{\beta}_{mk}h^{\beta}_{2p} + h^{\beta}_{2p}\langle e_\beta, \overline{\nabla}_H e_4\rangle] \\
&- h^4_{1p}[-2K_{l2pk}h^3_{lk} + 2K_{3\beta pk}h^{\beta}_{2k} + 2K_{3\beta 2k}h^{\beta}_{pk} - K_{lk2k}h^3_{lp} - K_{lkpk}h^3_{2l} \\
&+ K_{3k\beta k}h^{\beta}_{2p} - H^{\beta}(h^{\beta}_{2k}h^3_{pk} + h^{\beta}_{pk}h^3_{2k}) + h^3_{2m}h^{\beta}_{mk}h^{\beta}_{kp} - 2h^{\beta}_{2m}h^3_{mk}h^{\beta}_{kp} \\
&+ h^{\beta}_{2k}h^{\beta}_{km}h^3_{mp} + h^3_{km}h^{\beta}_{mk}h^{\beta}_{2p} + h^{\beta}_{2p}\langle e_\beta, \overline{\nabla}_H e_3\rangle] \\
&- h^3_{2p}[-2K_{l1pk}h^4_{lk} + 2K_{4\beta pk}h^{\beta}_{1k} + 2K_{4\beta 1k}h^{\beta}_{pk} - K_{lk1k}h^4_{lp} - K_{lkpk}h^4_{1l} \\
&+ K_{4k\beta k}h^{\beta}_{1p} - H^{\beta}(h^{\beta}_{1k}h^4_{pk} + h^{\beta}_{pk}h^4_{1k}) + h^4_{1m}h^{\beta}_{mk}h^{\beta}_{kp} - 2h^{\beta}_{1m}h^4_{mk}h^{\beta}_{kp} \\
&+ h^{\beta}_{1k}h^{\beta}_{km}h^4_{mp} + h^4_{km}h^{\beta}_{mk}h^{\beta}_{1p} + h^{\beta}_{1p}\langle e_\beta, \overline{\nabla}_H e_4\rangle] \\
= &-2\nabla_{\mathrm{evol}}K^{\perp} \\
&+ [-2K_{2112}h^3_{22}h^4_{21} + 2K_{34pk}h^4_{1k}h^4_{2p} + 2K_{3412}h^4_{p2}h^4_{2p} - K_{1212}h^3_{11}h^4_{21} \\
&- K_{1212}h^3_{11}h^4_{21} + K_{3k\beta k}h^{\beta}_{1p}h^4_{2p} - |H|(h^3_{11}h^3_{11}h^4_{21} + h^3_{11}h^3_{11}h^4_{21}) \\
&+ h^3_{11}h^{\beta}_{1k}h^{\beta}_{kp}h^4_{2p} - 2h^{\beta}_{1m}h^3_{mk}h^{\beta}_{kp}h^4_{2p} + h^{\beta}_{1k}h^{\beta}_{km}h^3_{mp}h^4_{2p} \\
&+ h^3_{km}h^{\beta}_{mk}h^{\beta}_{1p}h^4_{2p} + h^4_{2p}h^4_{1p}\langle e_4, \overline{\nabla}_H e_3\rangle] \\
&+ [-2K_{1212}h^4_{12}h^3_{11} + 2K_{4312}h^3_{22}h^3_{11} + 2K_{4321}h^3_{11}h^3_{11} - K_{2121}h^4_{21}h^3_{11} \\
&- K_{1212}h^4_{21}h^3_{11} + K_{4k\beta k}h^{\beta}_{21}h^3_{11} - |H|(h^3_{22}h^4_{12}h^3_{11} + h^3_{11}h^4_{21}h^3_{11}) \\
&+ h^4_{2m}h^{\beta}_{mk}h^{\beta}_{k1}h^3_{11} - 2h^{\beta}_{2m}h^4_{mk}h^{\beta}_{k1}h^3_{11} + h^{\beta}_{2k}h^{\beta}_{km}h^4_{m1}h^3_{11} \\
&+ h^4_{km}h^{\beta}_{mk}h^{\beta}_{21}h^3_{11}] \\
&- [-2K_{1221}h^3_{11}h^4_{12} + 2K_{34pk}h^4_{2k}h^4_{1p} + 2K_{3421}h^4_{p1}h^4_{1p} - K_{2121}h^3_{22}h^4_{12} \\
&- K_{2121}h^3_{22}h^4_{12} + K_{3k\beta k}h^{\beta}_{2p}h^4_{1p} - |H|(h^3_{22}h^3_{22}h^4_{12} + h^3_{22}h^3_{22}h^4_{12}) \\
&+ h^3_{22}h^{\beta}_{2k}h^{\beta}_{kp}h^4_{1p} - 2h^{\beta}_{2m}h^3_{mk}h^{\beta}_{kp}h^4_{1p} + h^{\beta}_{2k}h^{\beta}_{km}h^3_{mp}h^4_{1p} + h^3_{km}h^{\beta}_{mk}h^{\beta}_{2p}h^4_{1p} \\
&+ h^4_{1p}h^4_{2p}\langle e_4, \overline{\nabla}_H e_3\rangle] \\
&- [-2K_{2121}h^4_{21}h^3_{22} + 2K_{4321}h^3_{11}h^3_{22} + 2K_{4312}h^3_{22}h^3_{22} - K_{1212}h^4_{12}h^3_{22} \\
&- K_{2121}h^4_{12}h^3_{22} + K_{4k\beta k}h^{\beta}_{12}h^3_{22} - |H|(h^3_{11}h^4_{21}h^3_{22} + h^3_{22}h^4_{12}h^3_{22}) \\
&+ h^4_{1m}h^{\beta}_{mk}h^{\beta}_{k2}h^3_{22} - 2h^{\beta}_{1m}h^4_{mk}h^{\beta}_{k2}h^3_{22} + h^{\beta}_{1k}h^{\beta}_{km}h^4_{m2}h^3_{22} \\
&+ h^4_{km}h^{\beta}_{mk}h^{\beta}_{12}h^3_{22}] \\
= &-2\nabla_{\mathrm{evol}}K^{\perp} - 8K_{1212}(h^3_{11} - h^3_{22})h^4_{12} + (K_{3k3k} + K_{4k4k})(h^3_{11} - h^3_{22})h^4_{12} \\
&+ 2K_{1234}[|A|^2 + 2(h^4_{12})^2 - 2h^4_{11}h^4_{22} - 2h^3_{11}h^3_{22}] \\
&+ [2|A|^2 + 2(h^3_{11})^2 + +2(h^3_{22})^2 + 2(h^4_{12})^2 - 2h^4_{11}h^4_{22} \\
&- 3|H|^2](h^3_{11} - h^3_{22})h^4_{12}.
\end{aligned}
$$

It is known that

$$K_{1212} = K_{3434} = \frac{k}{4}(3\cos^2\alpha + 1), \quad K_{1234} = \frac{k}{4}(3\cos^2\alpha - 1), \quad K_{ij} = \frac{3}{2}k\overline{g}_{ij}.$$

Therefore, we have

$$\begin{aligned}
& -8K_{1212}(h^3_{11} - h^3_{22})h^4_{12} + (K_{3k3k} + K_{4k4k})(h^3_{11} - h^3_{22})h^4_{12} \\
& + 2K_{1234}[|A|^2 + 2(h^4_{12})^2 - 2h^4_{11}h^4_{22} - 2h^3_{11}h^3_{22}] \\
= {} & [K_{33} + K_{44} - 2K_{3434} - 8K_{1212}](h^3_{11} - h^3_{22})h^4_{12} \\
& + 2K_{1234}\Big[|A|^2 + 2c^2 + 2b^2 - \frac{1}{2}|H|^2 + 2a^2\Big] \\
= {} & 2ac\Big[3k - 10 \cdot \frac{k}{4}(3\cos^2\alpha + 1)\Big] + 4|\mathring{A}|^2 \cdot \frac{k}{4}(3\cos^2\alpha - 1) \\
= {} & |\mathring{A}|^2 k(3\cos^2\alpha - 1) + ack(1 - 15\cos^2\alpha).
\end{aligned}$$

Corollary 2.3 *For a mean curvature flow $F : \Sigma^2 \times [0, t_0) \to \mathbb{CP}^2$, the length of the normal curvature satisfies the inequality*

$$\begin{aligned}
\frac{\partial}{\partial t}|K^\perp| \le {} & \Delta|K^\perp| + 2|\nabla_{\mathrm{evol}}K^\perp| + |\mathring{A}|^2 k|3\cos^2\alpha - 1| \\
& + \frac{1}{2}k|K^\perp|(1 - 15\cos^2\alpha) + |K^\perp|(|A|^2 + 2|\mathring{A}|^2).
\end{aligned} \tag{2.12}$$

Proof At the point $|K^\perp| \ne 0$, we notice that

$$\begin{aligned}
\Delta|K^\perp| &= \Delta\langle K^\perp, K^\perp\rangle^{\frac{1}{2}} = g^{ij}\nabla_i\nabla_j\langle K^\perp, K^\perp\rangle^{\frac{1}{2}} \\
&= g^{ij}\nabla_i \frac{\langle K^\perp, \nabla_j K^\perp\rangle}{\langle K^\perp, K^\perp\rangle^{\frac{1}{2}}} \\
&= \frac{|\nabla K^\perp|^2}{\langle K^\perp, K^\perp\rangle^{\frac{1}{2}}} + \frac{\langle K^\perp, \Delta K^\perp\rangle}{\langle K^\perp, K^\perp\rangle^{\frac{1}{2}}} - \frac{|\nabla K^\perp|^2}{\langle K^\perp, K^\perp\rangle^{\frac{1}{2}}} \\
&= \frac{\langle K^\perp, \Delta K^\perp\rangle}{\langle K^\perp, K^\perp\rangle^{\frac{1}{2}}}.
\end{aligned}$$

Consequently, we infer that

$$\begin{aligned}
\frac{\partial}{\partial t}|K^\perp| &= \frac{\partial}{\partial t}\langle K^\perp, K^\perp\rangle^{\frac{1}{2}} = \frac{\langle \frac{\partial}{\partial t}K^\perp, K^\perp\rangle}{\langle K^\perp, K^\perp\rangle^{\frac{1}{2}}} \\
&= \Delta|K^\perp| - 2\frac{K^\perp}{|K^\perp|}\nabla_{\mathrm{evol}}K^\perp + \frac{K^\perp}{|K^\perp|}|\mathring{A}|^2 k(3\cos^2\alpha - 1) \\
&\quad + \frac{1}{2}k|K^\perp|(1 - 15\cos^2\alpha) + |K^\perp|(|A|^2 + 2|\mathring{A}|^2) \\
&\le \Delta|K^\perp| + 2|\nabla_{\mathrm{evol}}K^\perp| + |\mathring{A}|^2 k|3\cos^2\alpha - 1| \\
&\quad + \frac{1}{2}k|K^\perp|(1 - 15\cos^2\alpha) + |K^\perp|(|A|^2 + 2|\mathring{A}|^2).
\end{aligned}$$

At the point $|K^\perp| = 0$, we can compute $\frac{\partial}{\partial t}\sqrt{|K^\perp|^2 + \varepsilon}$, then we can take $\varepsilon \to 0$ and we will end up with the same result.

3 Pinching Estimate

In order to prove the long-time existence of symplectic mean curvature flow, we need an important pinching condition.

The following lemma is necessary when we estimate the gradient term in the evolution equation.

Lemma 3.1 (1) *For any $\eta > 0$, we have the inequality*

$$|\nabla A|^2 \geq \Big(\frac{3}{4} - \eta\Big)|\nabla H|^2 - \Big(\frac{1}{4\eta} - 1\Big)|w|^2, \tag{3.1}$$

where $w_i^\alpha = \sum\limits_l K_{\alpha lil}$, $|w^\alpha|^2 = \sum\limits_i |w_i^\alpha|^2$ *and* $|w|^2 = \sum\limits_\alpha |w^\alpha|^2 = \frac{9k^2}{8}\cos^2\alpha\sin^2\alpha$.

(2) $|\nabla A|^2 \geq 2|\nabla_{\rm evol}K^\perp| \geq 2\nabla_{\rm evol}K^\perp$, *if* $n = 2$.

Proof For (1), see [8, Lemma 3.1].

Using Cauchy inequality, we can easily get (2).

Lemma 3.2 *Suppose Σ is a symplectic surface in $\mathbb{CP}^2$ with constant holomorphic sectional curvature $k > 0$. Taking $\mu \in [0, 1]$, assume that*

$$|A|^2 + 2\mu\gamma|K^\perp| \leq \lambda|H|^2 + \frac{160\lambda + 4 + 2\mu - 6\mu\cos^2\alpha}{40\lambda + 1}\frac{2\lambda - 1}{4\lambda}k,$$

where $\gamma = \frac{1}{40\lambda+1}$ *and*

$$\cos\alpha \geq \max\{S_1(\lambda,\mu), S_2(\lambda,\mu), S_3(\lambda,\mu)\}$$

holds on the initial surface for any $\frac{1}{2} < \lambda \leq \frac{2}{3} - \frac{1}{12}\mu$, then it remains true along the symplectic mean curvature flow. Here $S_1(\lambda,\mu)$, $S_2(\lambda,\mu)$ and $S_3(\lambda,\mu)$ are defined in Remark 1.1.

Proof From Lemma 2.1 and Lemma 2.3, we know that

$$\frac{\partial}{\partial t}\cos\alpha = \Delta\cos\alpha + |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + \frac{3k}{2}\sin^2\alpha\cos\alpha.$$

Thus at any time t,

$$\cos\alpha \geq \max\{S_1(\lambda,\mu), S_2(\lambda,\mu), S_3(\lambda,\mu)\},$$

if it holds on the initial surface.

Let us first define $Q = |A|^2 + 2\mu\gamma|K^\perp| - \lambda|H|^2 - dk$. Using (2.2)–(2.3) and (2.12), we can compute

$$\begin{aligned}&\frac{\partial}{\partial t}Q\\ &\leq \Delta Q - 2(|\nabla A|^2 - 2\mu\gamma|\nabla_{\rm evol}K^\perp| - \lambda|\nabla H|^2) - \frac{k}{2}(3\cos^2\alpha + 1)Q - \frac{dk^2}{2}(3\cos^2\alpha + 1)\\ &\quad - k|A|^2 + 2k|H|^2 + 2\mu\gamma|\mathring{A}|^2k(3\cos^2\alpha - 1) + \mu\gamma k|K^\perp|(1 - 15\cos^2\alpha) - 3\lambda k|H|^2\end{aligned}$$

$$
\begin{aligned}
&+\mu\gamma k|K^\perp|(3\cos^2\alpha+1)+2\mu\gamma|K^\perp|(|A|^2+2|\mathring{A}|^2)-4k(3\cos^2\alpha-1)[b^2+(a-c)^2]\\
&+2\sum_{\alpha,\beta,i,j}\Big(\sum_k(h^\alpha_{ik}h^\beta_{jk}-h^\alpha_{jk}h^\beta_{ik})\Big)^2+2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2-2\lambda\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2\\
\le{}&\Delta Q+\Big(-2+2\mu\gamma+\frac{2\lambda}{\frac34-\eta}\Big)|\nabla A|^2+\frac{2\lambda}{\frac34-\eta}\frac{1-4\eta}{4\eta}\frac{9k^2}{8}\cos^2\alpha\sin^2\alpha-\frac{k}{2}(3\cos^2\alpha+1)Q\\
&-\frac{dk^2}{2}(3\cos^2\alpha+1)-k|A|^2+(2-3\lambda)k|H|^2+2\mu\gamma|\mathring{A}|^2k(3\cos^2\alpha-1)\\
&+\mu\gamma k|K^\perp|(1-15\cos^2\alpha)+\mu\gamma k|K^\perp|(3\cos^2\alpha+1)+2\mu\gamma|K^\perp|(|A|^2+2|\mathring{A}|^2)\\
&-4k(3\cos^2\alpha-1)[b^2+(a-c)^2]\\
&+2\sum_{\alpha,\beta,i,j}\Big(\sum_k(h^\alpha_{ik}h^\beta_{jk}-h^\alpha_{jk}h^\beta_{ik})\Big)^2+2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2-2\lambda\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2\\
\le{}&\Delta Q+\Big(-2+2\mu\gamma+\frac{2\lambda}{\frac34-\eta}\Big)|\nabla A|^2+\frac{2\lambda}{\frac34-\eta}\frac{1-4\eta}{4\eta}\frac{9k^2}{8}\cos^2\alpha\sin^2\alpha-\frac{k}{2}(3\cos^2\alpha+1)Q\\
&-\frac{dk^2}{2}(3\cos^2\alpha+1)-k|A|^2+(2-3\lambda)k|H|^2+2\mu\gamma|\mathring{A}|^2k(3\cos^2\alpha-1)\\
&+\mu\gamma k|K^\perp|(1-15\cos^2\alpha)+\mu\gamma k|K^\perp|(3\cos^2\alpha+1)+2\mu\gamma|K^\perp|(|A|^2+2|\mathring{A}|^2)\\
&+2|\mathring{h}_3|^4+2|\mathring{h}_4|^4+(2-2\lambda)|\mathring{h}_3|^2|H|^2-\frac{2\lambda-1}{2}|H|^4+4|\mathring{h}_3|^2|\mathring{h}_4|^2+16a^2c^2. \qquad (3.2)
\end{aligned}
$$

The first inequality in (3.2) used (2.10) and

$$\cos\alpha\ge\max\{S_1(\lambda,\mu),S_2(\lambda,\mu),S_3(\lambda,\mu)\}\ge\frac13,$$

the second inequality used Lemma 3.1, and the third inequality used

$$\cos\alpha\ge\max\{S_1(\lambda,\mu),S_2(\lambda,\mu),S_3(\lambda,\mu)\}\ge\frac13$$

and the fact that

$$
\begin{aligned}
&2\sum_{\alpha,\beta,i,j}\Big(\sum_k(h^\alpha_{ik}h^\beta_{jk}-h^\alpha_{jk}h^\beta_{ik})\Big)^2+2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2-2\lambda\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2\\
&=2|\mathring{h}_3|^4+2|\mathring{h}_4|^4+(2-2\lambda)|\mathring{h}_3|^2|H|^2-\frac{2\lambda-1}{2}|H|^4+4|\mathring{h}_3|^2|\mathring{h}_4|^2+16a^2c^2
\end{aligned}
$$

computed by using (2.6).

Since $|H|^2=\frac{2}{2\lambda-1}(|\mathring{h}_3|^2+|\mathring{h}_4|^2+2\mu\gamma|K^\perp|-Q-dk)$, substituting it into the above inequality, we obtain that

$$
\begin{aligned}
\frac{\partial}{\partial t}Q\le{}&\Delta Q-\frac{2}{2\lambda-1}Q^2+\Big[-\frac{k}{2}(3\cos^2\alpha+1)+3k-\frac{2\mu\gamma}{2\lambda-1}|K^\perp|-\frac{2(2-2\lambda)}{2\lambda-1}|\mathring{h}_3|^2\\
&+\frac{2}{2\lambda-1}|\mathring{h}_3|^2+\frac{2}{2\lambda-1}|\mathring{h}_4|^2+\frac{4\mu\gamma}{2\lambda-1}|K^\perp|-\frac{2}{2\lambda-1}dk+\frac{2}{2\lambda-1}|\mathring{h}_3|^2\\
&+\frac{2}{2\lambda-1}|\mathring{h}_4|^2+\frac{4\mu\gamma}{2\lambda-1}|K^\perp|-\frac{2}{2\lambda-1}dk\Big]Q+\frac{2\lambda}{\frac34-\eta}\frac{1-4\eta}{4\eta}\frac{9k^2}{8}\cos^2\alpha\sin^2\alpha\\
&-\frac{3dk^2}{2}\cos^2\alpha+6\mu\gamma|\mathring{A}|^2k\cos^2\alpha-12\mu\gamma|K^\perp|k\cos^2\alpha-\frac{dk^2}{2}+3dk^2
\end{aligned}
$$

$$
\begin{aligned}
&- 4k(|\mathring{h}_3|^2 + |\mathring{h}_4|^2) - 6\mu\gamma|K^\perp|k - 2\mu\gamma(|\mathring{h}_3|^2 + |\mathring{h}_4|^2)k + \mu\gamma|K^\perp|k + \mu\gamma|K^\perp|k \\
&+ 6\mu\gamma|K^\perp|(|\mathring{h}_3|^2 + |\mathring{h}_4|^2) + \frac{2}{2\lambda-1}\mu\gamma|K^\perp|(|\mathring{h}_3|^2 + |\mathring{h}_4|^2 + 2\mu\gamma|K^\perp| - dk) + 2|\mathring{h}_3|^4 \\
&+ 2|\mathring{h}_4|^4 + \frac{2(2-2\lambda)}{2\lambda-1}|\mathring{h}_3|^2(|\mathring{h}_3|^2 + |\mathring{h}_4|^2 + 2\mu\gamma|K^\perp| - dk) + 4|\mathring{h}_3|^2|\mathring{h}_4|^2 + 16a^2c^2 \\
&- \frac{2}{2\lambda-1}|\mathring{h}_3|^4 - \frac{4}{2\lambda-1}|\mathring{h}_3|^2|\mathring{h}_4|^2 - \frac{4}{2\lambda-1}2\mu\gamma|\mathring{h}_3|^2|K^\perp| + \frac{4}{2\lambda-1}dk|\mathring{h}_3|^2 \\
&- \frac{2}{2\lambda-1}|\mathring{h}_4|^4 - \frac{4}{2\lambda-1}2\mu\gamma|\mathring{h}_4|^2|K^\perp| + \frac{4}{2\lambda-1}dk|\mathring{h}_4|^2 - \frac{8}{2\lambda-1}\mu^2\gamma^2|K^\perp|^2 \\
&+ \frac{4}{2\lambda-1}2\mu\gamma dk|K^\perp| - \frac{2}{2\lambda-1}d^2k^2,
\end{aligned}
$$

where we assume that

$$0 \le \mu\gamma \le 1 - \frac{\lambda}{\frac{3}{4}-\eta}.$$

For convenience, let us define a new function

$$
\begin{aligned}
P = &-\frac{2}{2\lambda-1}Q^2 + \frac{2\lambda}{\frac{3}{4}-\eta}\frac{1-4\eta}{4\eta}\frac{9k^2}{8}\cos^2\alpha\sin^2\alpha - \frac{3dk^2}{2}\cos^2\alpha + 6\mu\gamma|\mathring{A}|^2k\cos^2\alpha \\
&- 12\mu\gamma|K^\perp|k\cos^2\alpha - \frac{dk^2}{2} + 3dk^2 - 4k(|\mathring{h}_3|^2 + |\mathring{h}_4|^2) - 6\mu\gamma|K^\perp|k + \mu\gamma|K^\perp|k \\
&- 2\mu\gamma(|\mathring{h}_3|^2 + |\mathring{h}_4|^2)k + \mu\gamma|K^\perp|k + \frac{2}{2\lambda-1}\mu\gamma|K^\perp|(|\mathring{h}_3|^2 + |\mathring{h}_4|^2 + 2\mu\gamma|K^\perp| - dk) \\
&+ 6\mu\gamma|K^\perp|(|\mathring{h}_3|^2 + |\mathring{h}_4|^2) + 2|\mathring{h}_3|^4 + \frac{4}{2\lambda-1}2\mu\gamma dk|K^\perp| - \frac{2}{2\lambda-1}d^2k^2 \\
&+ 2|\mathring{h}_4|^4 + \frac{2(2-2\lambda)}{2\lambda-1}|\mathring{h}_3|^2(|\mathring{h}_3|^2 + |\mathring{h}_4|^2 + 2\mu\gamma|K^\perp| - dk) + 4|\mathring{h}_3|^2|\mathring{h}_4|^2 + 16a^2c^2 \\
&- \frac{2}{2\lambda-1}|\mathring{h}_3|^4 - \frac{4}{2\lambda-1}|\mathring{h}_3|^2|\mathring{h}_4|^2 - \frac{4}{2\lambda-1}2\mu\gamma|\mathring{h}_3|^2|K^\perp| + \frac{4}{2\lambda-1}dk|\mathring{h}_3|^2 \\
&- \frac{2}{2\lambda-1}|\mathring{h}_4|^4 - \frac{4}{2\lambda-1}2\mu\gamma|\mathring{h}_4|^2|K^\perp| + \frac{4}{2\lambda-1}dk|\mathring{h}_4|^2 - \frac{8}{2\lambda-1}\mu^2\gamma^2|K^\perp|^2 \\
= &-\frac{2}{2\lambda-1}Q^2 + \frac{2\lambda}{\frac{3}{4}-\eta}\frac{1-4\eta}{4\eta}\frac{9k^2}{8}\cos^2\alpha\sin^2\alpha - \frac{3dk^2}{2}\cos^2\alpha - 12\mu\gamma|K^\perp|k\cos^2\alpha \\
&+ \frac{5}{2}dk^2 - 4\mu\gamma|K^\perp|k + \Big(-4 - 2\mu\gamma + \frac{4d\lambda}{2\lambda-1} + 6\mu\gamma\cos^2\alpha\Big)k|\mathring{h}_3|^2 \\
&+ \Big(2 - \frac{2}{2\lambda-1}\Big)\mu\gamma|K^\perp||\mathring{h}_3|^2 + \Big(4 - \frac{4\lambda}{2\lambda-1}\Big)|\mathring{h}_3|^2|\mathring{h}_4|^2 + \Big(2 - \frac{2}{2\lambda-1}\Big)(|\mathring{h}_4|^2 - \frac{1}{2}dk)^2 \\
&+ \Big(-4 - 2\mu\gamma + \frac{4d\lambda}{2\lambda-1} + 6\mu\gamma\cos^2\alpha\Big)k|\mathring{h}_4|^2 + \Big(6 - \frac{6}{2\lambda-1}\Big)\mu\gamma|\mathring{h}_4|^2|K^\perp| \\
&- \frac{4}{2\lambda-1}\mu^2\gamma^2|K^\perp|^2 + \frac{6}{2\lambda-1}\mu\gamma|K^\perp|dk - \frac{\lambda+1}{2\lambda-1}d^2k^2 + 16a^2c^2.
\end{aligned}
$$

In order to apply the maximum principle for parabolic equation, our goal is to show

$$P \le 0. \tag{3.3}$$

We estimate P as follows

$$P \le -\frac{2}{2\lambda-1}Q^2 + \frac{2\lambda}{\frac{3}{4}-\eta}\frac{1-4\eta}{4\eta}\frac{9k^2}{8}\cos^2\alpha\sin^2\alpha - \frac{3dk^2}{2}\cos^2\alpha - 12\mu\gamma|K^\perp|k\cos^2\alpha$$

$$
\begin{aligned}
&+\frac{5}{2}dk^2-4\mu\gamma|K^\perp|k+\Big(-4-2\mu\gamma+\frac{4d\lambda}{2\lambda-1}+6\mu\gamma\cos^2\alpha\Big)k|\mathring{h}_3|^2\\
&+\Big(2-\frac{2}{2\lambda-1}\Big)\mu\gamma|K^\perp||\mathring{h}_3|^2+\Big(8-\frac{4\lambda}{2\lambda-1}\Big)|\mathring{h}_3|^2|\mathring{h}_4|^2+\Big(2-\frac{2}{2\lambda-1}\Big)\Big(|\mathring{h}_4|^2-\frac{1}{2}dk\Big)^2\\
&+\Big(-4-2\mu\gamma+\frac{4d\lambda}{2\lambda-1}+6\mu\gamma\cos^2\alpha\Big)k|\mathring{h}_4|^2+\Big(6-\frac{6}{2\lambda-1}\Big)\mu\gamma|\mathring{h}_4|^2|K^\perp|\\
&-\frac{4}{2\lambda-1}\mu^2\gamma^2|K^\perp|^2+\frac{6}{2\lambda-1}\mu\gamma|K^\perp|dk-\frac{\lambda+1}{2\lambda-1}d^2k^2\\
\le&-\frac{2}{2\lambda-1}Q^2+\Big(\frac{2\lambda}{\frac{3}{4}-\eta}\frac{1-4\eta}{4\eta}\frac{9}{8}\sin^2\alpha-\frac{3d}{2}\Big)k^2\cos^2\alpha+\Big(2-\frac{2}{2\lambda-1}\Big)\mu\gamma|K^\perp||\mathring{h}_3|^2\\
&+\Big(\frac{6}{2\lambda-1}d-12\cos^2\alpha-4\Big)\mu\gamma|K^\perp|k+\Big(2-\frac{2}{2\lambda-1}\Big)\Big(|\mathring{h}_4|^2-\frac{1}{2}dk\Big)^2\\
&+\Big(-4-2\mu\gamma+\frac{4d\lambda}{2\lambda-1}+6\mu\gamma\cos^2\alpha\Big)k|\mathring{h}_3|^2+\Big(6-\frac{6}{2\lambda-1}\Big)\mu\gamma|\mathring{h}_4|^2|K^\perp|\\
&+\Big(-4-2\mu\gamma+\frac{4d\lambda}{2\lambda-1}+6\mu\gamma\cos^2\alpha\Big)k|\mathring{h}_4|^2+\Big(8-\frac{4\lambda}{2\lambda-1}\Big)|\mathring{h}_3|^2|\mathring{h}_4|^2\\
&+\Big(\frac{5}{2}-\frac{\lambda+1}{2\lambda-1}d\Big)dk^2,
\end{aligned}\tag{3.4}
$$

where we used the fact that

$$16a^2c^2\le 4|\mathring{h}_3|^2|\mathring{h}_4|^2.$$

We have assumed that

$$0\le\mu\gamma\le 1-\frac{\lambda}{\frac{3}{4}-\eta}\Rightarrow 0<\eta\le\frac{3}{4}-\lambda,$$

so we can take

$$
\begin{aligned}
\eta&=\frac{3}{4}\Big(1-\frac{1}{30}\mu\Big)-\lambda,\\
\gamma&=\frac{1}{40\lambda+1}.
\end{aligned}
$$

And then we want the coefficients on the right-hand side of (3.4) to be less than or equal to 0, i.e.,

$$\frac{2\lambda}{\frac{3}{4}-\eta}\frac{1-4\eta}{4\eta}\frac{9}{8}\sin^2\alpha-\frac{3d}{2}\le 0,\tag{3.5}$$

$$6-\frac{6}{2\lambda-1}\le 0,\quad 2-\frac{2}{2\lambda-1}\le 0,\quad 8-\frac{4\lambda}{2\lambda-1}\le 0,\tag{3.6}$$

$$\frac{6}{2\lambda-1}d-4-12\cos^2\alpha\le 0,\tag{3.7}$$

$$-4-2\mu\gamma+\frac{4\lambda}{2\lambda-1}d+6\mu\gamma\cos^2\alpha\le 0\tag{3.8}$$

and

$$\frac{5}{2}-\frac{\lambda+1}{2\lambda-1}d\le 0.\tag{3.9}$$

From (3.6), we can obtain

$$\frac{1}{2} < \lambda \leq \frac{2}{3}.$$

From (3.7)–(3.9), we can obtain

$$\begin{aligned} &\frac{5(2\lambda-1)}{2(\lambda+1)} \leq d \leq \frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{4\lambda}, \\ &\frac{5(2\lambda-1)}{2(\lambda+1)} \leq d \leq \frac{(4+12\cos^2\alpha)(2\lambda-1)}{6}. \end{aligned} \tag{3.10}$$

We need to find out the right conditions to ensure that (3.10) is reasonable. That is

$$\begin{aligned} &\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{4\lambda} - \frac{5(2\lambda-1)}{2(\lambda+1)} \geq 0 \\ \Leftrightarrow\ &2(\lambda+1)(160\lambda+4+2\mu-6\mu\cos^2\alpha) - 5(160\lambda^2+4\lambda) \geq 0 \\ \Leftrightarrow\ &-480\lambda^2 + (308+4\mu-12\mu\cos^2\alpha)\lambda + 8 + 4\mu - 12\mu\cos^2\alpha \geq 0 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\frac{(4+12\cos^2\alpha)(2\lambda-1)}{6} - \frac{5(2\lambda-1)}{2(\lambda+1)} \geq 0 \\ \Leftrightarrow\ &2(4+12\cos^2\alpha)(\lambda+1) - 30 \geq 0 \\ \Leftrightarrow\ &\cos^2\alpha \geq \frac{11-4\lambda}{12(\lambda+1)} \\ \Leftrightarrow\ &\cos\alpha \geq \sqrt{\frac{11-4\lambda}{12(\lambda+1)}} \doteq S_1(\mu,\lambda). \end{aligned} \tag{3.12}$$

We have to deal with (3.11) carefully. Let us define

$$f(\lambda) = -480\lambda^2 + (308+4\mu-12\mu\cos^2\alpha)\lambda + 8 + 4\mu - 12\mu\cos^2\alpha,$$

then the two roots of the equation $f(\lambda) = 0$ are

$$\begin{aligned} \lambda_1 &= \frac{12\mu\cos^2\alpha - 4\mu - 308 + \sqrt{(308+4\mu-12\mu\cos^2\alpha)^2 - 4(-480)(8+4\mu-12\mu\cos^2\alpha)}}{-960} \\ &< \frac{1}{2} \end{aligned}$$

and

$$\lambda_2 = \frac{12\mu\cos^2\alpha - 4\mu - 308 - \sqrt{(308+4\mu-12\mu\cos^2\alpha)^2 - 4(-480)(8+4\mu-12\mu\cos^2\alpha)}}{-960}.$$

Noting that λ_2 is monotonically decreasing with respect to $\cos^2\alpha$, we take $\cos\alpha = 1$ to obtain $\lambda_2 \geq \frac{2}{3} - \frac{1}{12}\mu$. What we need to note is that the $\cos^2\alpha$ here is only a parameter, not our final $\cos^2\alpha$ range. Therefore, (3.10) is reasonable when λ and $\cos\alpha$ satisfy the following two conditions:

$$\frac{1}{2} < \lambda \leq \frac{2}{3} - \frac{1}{12}\mu$$

and

$$\cos\alpha \geq \sqrt{\frac{11-4\lambda}{12(\lambda+1)}}.$$

Furthermore, we notice that when

$$\cos\alpha \geq \sqrt{\frac{472\lambda - 320\lambda^2 + 12 + 6\mu}{6(160\lambda^2 + 4\lambda + 3\mu)}} \doteq S_2(\mu,\lambda), \tag{3.13}$$

we have

$$\frac{(4+12\cos^2\alpha)(2\lambda-1)}{6} - \frac{160\lambda + 4 + 2\mu - 6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{4\lambda} \geq 0. \tag{3.14}$$

Then we can take $d = \frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{4\lambda}$ in (3.5), thus

$$\begin{aligned}\cos\alpha &\geq \sqrt{1 - \frac{400\lambda^2(3-4\lambda)(2\lambda-1)(40\lambda+1)^2 + \mu J_1(\mu,\lambda)}{1200\lambda^3(2\lambda-1)(40\lambda+1)^2 + \mu J_2(\mu,\lambda)}} \\ &= \sqrt{\frac{400\lambda^2(7\lambda-3)(2\lambda-1)(40\lambda+1)^2 + \mu(J_2(\mu,\lambda) - J_1(\mu,\lambda))}{1200\lambda^3(2\lambda-1)(40\lambda+1)^2 + \mu J_2(\mu,\lambda)}} \doteq S_3(\mu,\lambda),\end{aligned} \tag{3.15}$$

where $J_1(\mu,\lambda)$ and $J_2(\mu,\lambda)$ are polynomials with respect to μ and λ, in the following forms:

$$\begin{aligned}J_1(\mu,\lambda) = 4\lambda(2\lambda-1)(40\lambda+1)&[(30-40\lambda)(40\lambda+1) \\ &- (40\lambda+\mu)(40\lambda+1) - (30-40-\mu)(40\lambda+\mu)]\end{aligned}$$

and

$$J_2(\mu,\lambda) = \lambda(40\lambda+1)[240\lambda^2(40\lambda+1) - 6(2\lambda-1)(30-40\lambda-\mu)(40\lambda+\mu)].$$

From (3.12)–(3.13) and (3.15), we can get that Kähler angle α satisfies

$$\cos\alpha \geq \max\{S_1(\mu,\lambda), S_2(\mu,\lambda), S_3(\mu,\lambda)\}.$$

Finally, under assumptions

$$\frac{1}{2} < \lambda \leq \frac{2}{3} - \frac{1}{12}\mu$$

and

$$\cos\alpha \geq \max\{S_1(\mu,\lambda), S_2(\mu,\lambda), S_3(\mu,\lambda)\},$$

we show that (3.3) is correct.

At the point $|H| = 0$, we use the following inequality (see [8]):

$$2\sum_{\alpha,\beta,i,j}\Big(\sum_k (h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik})\Big)^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\Big)^2 \leq 3|A|^2.$$

The proof of case of $|H| = 0$ is equal to that given earlier for case of $|H| \neq 0$ and so is omitted.

Applying the maximum principle for parabolic equation to

$$\frac{\partial}{\partial t} Q \leq \Delta Q + CQ,$$

we know that

$$Q \leq 0$$

along the flow, if it is true on initial surface for $\frac{1}{2} < \lambda \leq \frac{2}{3} - \frac{1}{12}\mu$ and

$$\cos\alpha \geq \max\{S_1(\mu,\lambda), S_2(\mu,\lambda), S_3(\mu,\lambda)\}.$$

This completes the proof of Lemma 3.2.

4 Long Time Existence and Convergence

In this section we prove the long time existence and convergence of the symplectic mean curvature flow.

For convenience, we consider the case of $\mu = 1$, and the other cases are similar to $\mu = 1$.

Theorem 4.1 *When $\mu = 1$, under the assumption of Lemma 3.2 and the initial surface satisfies $\cos\alpha \geq \sqrt{1-\delta}$, where*

$$\sqrt{1-\delta} = \max\big\{S_1(\lambda,1), S_2(\lambda,1), S_3(\lambda,1)\big\},$$

then the symplectic mean curvature flow exists for long time. Here $S_1(\lambda,\mu)$, $S_2(\lambda,\mu)$ and $S_3(\lambda,\mu)$ are defined in Remark 1.1.

Proof Suppose f is a positive increasing function which will be determined later. Now we compute the evolution equation of $|H|^2 f\big(\frac{1}{\cos\alpha}\big)$,

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\Big(|H|^2 f\Big(\frac{1}{\cos\alpha}\Big)\Big) = \Big(\frac{\partial}{\partial t} - \Delta\Big)|H|^2 f\Big(\frac{1}{\cos\alpha}\Big) + |H|^2\Big(\frac{\partial}{\partial t} - \Delta\Big) f\Big(\frac{1}{\cos\alpha}\Big) - 2\nabla|H|^2 \cdot \nabla f\Big(\frac{1}{\cos\alpha}\Big).$$

It follows that

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)\cos\alpha &= |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + \frac{3}{2}k\sin^2\alpha\cos\alpha \\ &\geq |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha.\end{aligned}$$

And, we also have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)|H|^2 &\leq -2|\nabla H|^2 + \Big(1 + \frac{3\delta}{2}\Big)k|H|^2 + 2|H|^2|A|^2 \\ &\leq -2|\nabla H|^2 + \Big(1 + \frac{3\delta}{2}\Big)k|H|^2 + 2|H|^2\Big(\lambda|H|^2 - \frac{2}{40\lambda+1}|K^\perp|\end{aligned}$$

$$+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{4\lambda}k\Big)$$
$$=-2|\nabla H|^2+\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k|H|^2+2\lambda|H|^4.$$

Putting the above inequality into the evolution equation of $|H|^2f\big(\frac{1}{\cos\alpha}\big)$, we get that

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(|H|^2f\Big(\frac{1}{\cos\alpha}\Big)\Big)\\
&\le f\Big(-2|\nabla H|^2+\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k|H|^2+2\lambda|H|^4\Big)\\
&\quad-|H|^2\Big(f'\frac{|\overline{\nabla}J_{\Sigma_t}|^2}{\cos\alpha}+2f'\frac{|\nabla\cos\alpha|^2}{\cos^3\alpha}+f''\frac{|\nabla\cos\alpha|^2}{\cos^4\alpha}\Big)\\
&\quad-2\frac{\nabla(f|H|^2)-|H|^2\nabla f}{f}\nabla f\Big(\frac{1}{\cos\alpha}\Big)\\
&=|H|^2f\Big(2\lambda|H|^2-2\frac{|\nabla H|^2}{|H|^2}-\frac{f'}{f}\frac{|\overline{\nabla}J_{\Sigma_t}|^2}{\cos\alpha}\Big)-2|H|^2\frac{\nabla(f|H|^2)}{f|H|^2}\nabla f\Big(\frac{1}{\cos\alpha}\Big)\\
&\quad+|H|^2\Big(-f''+2\frac{(f')^2}{f}-2f'\cos\alpha\Big)\frac{|\nabla\cos\alpha|^2}{\cos^4\alpha}\\
&\quad+\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k|H|^2f.
\end{aligned}$$

Set $\phi=f|H|^2$. At the point where $\phi\neq0$, it is easy to see that

$$\nabla\phi=f\nabla|H|^2+|H|^2\nabla f=f\nabla|H|^2-|H|^2f'\frac{\nabla\cos\alpha}{\cos^2\alpha},$$

i.e.,

$$\frac{\nabla\cos\alpha}{\cos^2\alpha}=\frac{f}{f'}\Big(\frac{\nabla|H|^2}{|H|^2}-\frac{\nabla\phi}{\phi}\Big).$$

Then we have inequality as follows:

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)\phi\\
&\le\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k\phi+\phi\Big(2\lambda|H|^2-2\frac{|\nabla H|^2}{|H|^2}-\frac{f'}{f}\frac{|\overline{\nabla}J_{\Sigma_t}|^2}{\cos\alpha}\Big)\\
&\quad+2|H|^2f'\frac{\nabla\phi}{\phi}\frac{\nabla\cos\alpha}{\cos^2\alpha}+\frac{\phi f}{(f')^2}\Big(-f''+2\frac{(f')^2}{f}-2f'\cos\alpha\Big)\Big(\frac{|\nabla|H|^2|^2}{|H|^4}\\
&\quad-2\frac{\nabla|H|^2}{|H|^2}\frac{\nabla\phi}{\phi}+\frac{|\nabla\phi|^2}{\phi^2}\Big)\\
&\le\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k\phi+\phi\Big(-\frac{f'}{2f}\frac{|H|^2}{\cos\alpha}+2\lambda|H|^2\Big)\\
&\quad+\phi\Big(-2\frac{|\nabla H|^2}{|H|^2}-4\frac{ff''}{(f')^2}\frac{|\nabla|H||^2}{|H|^2}+8\frac{|\nabla|H||^2}{|H|^2}-8\frac{f}{f'}\cos\alpha\frac{|\nabla|H||^2}{|H|^2}\Big)\\
&\quad+\frac{\phi f}{(f')^2}\Big(-f''+2\frac{(f')^2}{f}-2f'\cos\alpha\Big)\Big(\frac{|\nabla|H|^2|^2}{|H|^4}-2\frac{\nabla|H|^2}{|H|^2}\frac{\nabla\phi}{\phi}\Big)+2|H|^2f'\frac{\nabla\phi}{\phi}\frac{\nabla\cos\alpha}{\cos^2\alpha}\\
&=\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k\phi+\phi\Big(-\frac{f'}{2f}\frac{1}{\cos\alpha}+2\lambda\Big)|H|^2
\end{aligned}$$

$$
\begin{aligned}
&+\phi\Big(2-\frac{ff''}{(f')^2}-2\frac{f}{f'}\cos\alpha\Big)\Big(-2\frac{\nabla|H|^2}{|H|^2}\frac{\nabla\phi}{\phi}+\frac{|\nabla\phi|^2}{\phi^2}\Big)\\
&+\phi\Big(6-4\frac{ff''}{(f')^2}-8\frac{f}{f'}\cos\alpha\Big)\frac{|\nabla|H||^2}{|H|^2}+2|H|^2f'\frac{\nabla\phi}{\phi}\frac{\nabla\cos\alpha}{\cos^2\alpha}.
\end{aligned}
$$

Setting $\frac{f}{f'}=g$, we choose g such that for $x\in\big[1,\frac{1}{\sqrt{1-\delta}}\big]$,

$$
\begin{aligned}
\frac{x}{g}&\ge 4\lambda,\\
-4g'+\frac{8g}{x}-2&=0.
\end{aligned}
$$

Let $g(x)=xp(x)$; then $p(x)$ needs to satisfy

$$
\begin{aligned}
0<p(x)&\le\frac{1}{4\lambda},\\
-2xp'&=1-2p.
\end{aligned}
$$

We choose $p(x)=\frac{1}{2}-qx$ by solving the last equation, where q will be defined later. It reduces to solve the inequality

$$
0<\frac{1}{2}-qx\le\frac{1}{4\lambda},\quad x\in\Big[1,\frac{1}{\sqrt{1-\delta}}\Big],
$$

i.e.,

$$
\Big(\frac{1}{2}-\frac{1}{4\lambda}\Big)\frac{1}{x}\le q\le\frac{1}{2x},\quad x\in\Big[1,\frac{1}{\sqrt{1-\delta}}\Big].
$$

Thus if $\sqrt{1-\delta}>1-\frac{1}{2\lambda}$, we can choose $q=\frac{1}{2}-\frac{1}{4\lambda}$. Then

$$
g=x\Big(\frac{1}{2}-qx\Big)=\frac{x}{2}-\Big(\frac{1}{2}-\frac{1}{4\lambda}\Big)x^2
$$

and

$$
f(x)=\frac{(1-2q)^2x^2}{(1-2qx)^2}=\frac{x^2}{(2\lambda-(2\lambda-1)x)^2},\quad x\in\Big[1,\frac{1}{\sqrt{1-\delta}}\Big].
$$

It is evident that for $x\in\big[1,\frac{1}{\sqrt{1-\delta}}\big]$,

$$
1\le f(x)\le\frac{1}{(2\lambda\sqrt{1-\delta}-(2\lambda-1))^2}.
$$

So, we have

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)\phi\le{}&\Big(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}\Big)k\phi+2|H|^2f'\frac{\nabla\phi}{\phi}\frac{\nabla\cos\alpha}{\cos^2\alpha}\\
&+\phi\Big(2-\frac{ff''}{(f')^2}-2\frac{f}{f'}\cos\alpha\Big)\Big(-2\frac{\nabla|H|^2}{|H|^2}\frac{\nabla\phi}{\phi}+\frac{|\nabla\phi|^2}{\phi^2}\Big).
\end{aligned}
$$

Applying the maximum principle, we get that

$$
|H|^2\le|H|^2f\Big(\frac{1}{\cos\alpha}\Big)\le\mathrm{e}^{(1+\frac{3\delta}{2}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda})kt}|H|^2(0)f\Big(\frac{1}{\cos\alpha}\Big)(0).
$$

So pinching inequality implies

$$
|A|^2\le C_0\mathrm{e}^{C_1t}+\frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{2\lambda}k,
$$

where C_0 depends only on $\max\limits_{\Sigma_0}|H|^2$ and λ. Therefore the flow exists for all time. The proof is completed.

Theorem 4.2 *Under the assumption of Lemma* 3.2, *the symplectic mean curvature flow converges to a holomorphic curve.*

Proof We can rewrite the evolution equation of $\cos\alpha$

$$\frac{\partial}{\partial t}\cos\alpha = \Delta\cos\alpha + |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + \frac{3k}{2}\sin^2\alpha\cos\alpha$$

as

$$\begin{aligned}\frac{\partial}{\partial t}\sin^2\Big(\frac{\alpha}{2}\Big) &= -\Delta\cos\alpha + |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha - 3k\sin^2\Big(\frac{\alpha}{2}\Big)\cos^2\Big(\frac{\alpha}{2}\Big)\cos\alpha\\ &\le -c\sin^2\Big(\frac{\alpha}{2}\Big),\end{aligned}$$

where $c>0$ depends only on k and the lower bound of $\cos\alpha$. Applying the maximum principle, we get that $\sin^2(\frac{\alpha}{2})\le \mathrm{e}^{-ct}$. From [5, Proposition 2.1], we know that

$$\int_{\Sigma_t}\frac{\sin^2\alpha}{\cos\alpha}\mathrm{d}\mu_t \le C_0\mathrm{e}^{-Kt},$$

where C_0 is a constant which depends only on the initial surface, i.e., $C_0=\int_{\Sigma_0}\frac{\sin^2\alpha}{\cos\alpha}\mathrm{d}\mu_0$, and K is scalar curvature of $\mathbb{CP}^2$. As $t\to+\infty$, $C_0\mathrm{e}^{-Kt}$ is sufficiently small. So we can apply [5, Theorem 2.5] to obtain that the mean curvature flow with the initial surface Σ_0 exists globally and it converges to a holomorphic curve. The proof of the theorem is completed.

Thus we can summarize what we have proved as the following theorem.

Theorem 4.3 *Suppose* Σ *is a symplectic surface in* $\mathbb{CP}^2$ *with constant holomorphic sectional curvature* $k>0$. *Taking* $\mu\in[0,1]$, *assume that*

$$|A|^2 + 2\mu\gamma|K^\perp| \le \lambda|H|^2 + \frac{160\lambda+4+2\mu-6\mu\cos^2\alpha}{40\lambda+1}\frac{2\lambda-1}{4\lambda}k,$$

where $\gamma=\frac{1}{40\lambda+1}$ *and*

$$\cos\alpha \ge \max\{S_1(\lambda,\mu), S_2(\lambda,\mu), S_3(\lambda,\mu)\}$$

holds on the initial surface for any $\frac12<\lambda\le\frac23-\frac{1}{12}\mu$, *then it remains true along the symplectic mean curvature flow. Furthermore, under this assumption, the symplectic mean curvature flow exists for a long time and converges to a holomorphic curve. Here* $S_1(\lambda,\mu)$, $S_2(\lambda,\mu)$ *and* $S_3(\lambda,\mu)$ *are defined in Remark* 1.1.

5 A Special Case

When our objective manifold is a flat torus $\mathbb{T}^4$, using a similar method, the above Theorem 1.1 can be greatly simplified.

Lemma 5.1 *If a solution* $F:\Sigma\times[0,t_0)\to\mathbb{T}^4$ *of SMCF satisfies* $|A|^2+2\mu\gamma|K^\perp|\le\lambda|H|^2$, *where* $\mu\in[0,1]$, $\gamma=\frac{1}{40\lambda+1}$ *and* $\frac12<\lambda\le\frac23$, *then this remains true for all* $0\le t<t_0$.

Proof In the space $\mathbb{T}^4$, we can get the following formula:

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2\sum_{\alpha,\beta,i,j}\Big(\sum_k(h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik})\Big)^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2,$$
$$\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2$$

and

$$\frac{\partial}{\partial t}|K^\perp| \le \Delta|K^\perp| + 2|\nabla_{\mathrm{evol}}K^\perp| + |K^\perp|(|A|^2 + 2|\mathring{A}|^2).$$

Using the same argument as in the proof of Lemma 3.2, we can easily carry out the proof of this Lemma.

Since the space $\mathbb{T}^4$ is flat, the formulas (3.5), (3.7)–(3.9) are meaningless. So we just have to set Σ_0 to be symplectic, i.e., $\cos\alpha \ge \delta$ ($0 < \delta \le 1$ is a constant).

Theorem 5.1 *Under the assumption of Lemma* 5.1 *and the condition that the initial surface satisfies* $\cos\alpha \ge \delta$, *where* $0 < \delta \le 1$ *is a constant, the symplectic mean curvature flow exists for a long time and converges to a minimal surface, which is a holomorphic curve with respect to some compatible complex structure on the flat torus.*

Proof Notice that the space $\mathbb{T}^4$ is flat, so we just have to set Σ_0 to be symplectic, i.e., $\cos\alpha \ge \delta$, where $0 < \delta \le 1$ is a constant. The proof of this theorem can be proved by the same method as employed in the last section. Finally, we can get the following inequality

$$|H|^2 \le |H|^2 f\Big(\frac{1}{\cos\alpha}\Big) \le |H|^2(0) f\Big(\frac{1}{\cos\alpha}\Big)(0).$$

So pinching inequality implies $|A|^2 \le |H|^2(0)f\big(\frac{1}{\cos\alpha}\big)(0) < +\infty$. This implies that the mean curvature flow exists for a long time and converges to a minimal surface at infinity. It is known that a symplectic minimal surface in a Calabi-Yau surface is holomorphic with respect to some compatible complex structure. The proof of the theorem is completed.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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