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Breather, Soliton and Rational Solutions for the (2 + 1)-Dimensional Hirota Equation*

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Abstract By virtue of Hirota's bilinear method and Kadomtsev-Petviashvili hierarchy reduction technique, the general breather, soliton and rational solutions in the $(2+1)$-dimensional Hirota equation are constructed. These solutions are expressed in terms of Gram determinants and Schur polynomials. The Nth-order breather and soliton solutions contain $2N$ free complex parameters, while Nth-order rational ones possess N free complex parameters. By utilizing the Hermitian matrices, the range of free parameters is determined such that it ensures the regularity of these breather and soliton solutions. For the rational solutions, their non-singularity is proved and the parity-time-symmetric condition is derived. Furthermore, the rich dynamic patterns of breather, soliton and rational solutions are established by various choices of free parameters.

Keywords Hirota's bilinear method, Kadomtsev-Petviashvili hierarchy reduction technique, $(2+1)$-Dimensional Hirota equation
2020 MR Subject Classification 37K40, 35Q53

1 Introduction

In the 1840s, Scott Russell reported the solitary waves on the water surface (see [1]). Henceforth, solitary waves were studied extensively in mathematics and physics (see [2–4]). They retained shapes and velocities after collisions. In 1979, several breathers were investigated. Especially, Ma breathers and Akhmediev breathers were spatially and temporally periodic, respectively (see [5–6]). The Akhmediev-Kuznetsov-Ma breathers were spatiotemporally periodic (see [7]). Recently, rogue waves have also attracted considerable attention. They are localized waves in both space and time, described as large and spontaneous waves in oceanography (see [8]). Rogue waves have also been derived in experiments (see [9]). Similarly, lumps were just localized waves in space for higher-dimensional integrable systems (see [10]). These phenomena were observed in various fields such as optics (see [11–14]), plasma physics (see [15]) finance

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(see [16]), etc.

Up to now, the soliton, breather, rogue wave, and lump solutions have been established in nonlinear evolution equations by using various methods, including the inverse scattering technique (see [17]), Darboux transformation (see [18]), algebraic geometry approach (see [19]), Hirota's bilinear method (see [20]), etc. In recent years, based on Hirota's bilinear method, the Kadomtsev-Petviashvili hierarchy reduction technique was explored to construct the soliton, breather, rogue wave, and lump solutions of several nonlinear integrable equations. In 2012, Yang and Ohta proposed the Kadomtsev-Petviashvili hierarchy reduction technique to derive the general rogue waves of the nonlinear Schrödinger equation (NLSE for short) (see [21]). By virtue of this technique, general rogue waves of other bilinear integrable equations were also discovered, including Davey-Stewartson I equation (see [22]), Davey-Stewartson II equation (see [23]) and Mel'nikov equation (see [24]). Subsequently, Chen et al. (2018) tried to extend this technique to derive the higher-order rogue waves of the long-short wave resonance equation and Schrödinger-Boussinesq equation (see [25–26]). However, the iterated relation is too complex to derive the explicit expression of higher-order rogue waves. Up to 2020, Yang et al. introduced an innovative differential operator to solve the above problem, and then they successfully established the general rogue waves of many bilinear integrable equations, including Boussinesq equation (see [27]), generalized derivative Schrödinger equation (see [28]), the three-wave resonance interaction equation (see [29]) and the massive Thirring model (see [30]). These solutions were expressed elegantly and concisely in terms of Gram determinants. We note that the solutions of these equations were derived from first- or second-order equations in the Kadomtsev-Petviashvili hierarchy. In this work, we would like to extend this technique to $(2+1)$-dimensional Hirota equation from the third-order equations in the Kadomtsev-Petviashvili hierarchy.

The $(2+1)$-dimensional Hirota equation has the following form

$$\begin{aligned} &\mathrm{i}\Phi_t + \Phi_{xy} + \mathrm{i}\beta\Phi_{xxx} - \Phi\Psi + 6\mathrm{i}\beta|\Phi|^2\Phi_x = 0, \\ &\Psi_x + 2(|\Phi|^2)_y = 0, \end{aligned} \tag{1.1}$$

where Φ is a complex function and Ψ is a real function with respect to space variables x, y and time t, and β is a real nonzero parameter representing the strength of higher-order linear and nonlinear effects. (1.1) was derived from $(1+1)$-dimensional Hirota equation by employing an asymptotically exact reduction method (see [31]). If $y = x$, (1.1) will be transformed into the $(1+1)$-dimensional Hirota equation describing the wave propagation of ultrashort light pulses in optical fibers (see [32–33]). The general higher-order rogue waves of $(1+1)$-dimensional Hirota equation have been discovered by utilizing Hirota's bilinear method (see [34]). For (1.1), using the bilinear transformation method, they obtained general higher-order rogue waves (see [35]). However, these results did not include lump solutions, and these researchers did not analyze breather and soliton ones. This work is to center around general breather, soliton and rational solutions of (1.1) by using Hirota's bilinear method and Kadomtsev-Petviashvili hierarchy reduction technique. We focus on the range of free parameters to satisfy the regularity of breather and soliton solutions, and demonstrate that obtained rational solutions are nonsingular. The parity-time-symmetric condition for rational solutions is also discussed. Furthermore, we display the various dynamic patterns of breather, soliton and rational solutions to (1.1). In particular, the superpositions of the breather and soliton solutions exhibit interesting dynamic patterns. We also discover that these rational solutions have both rogue wave and lump solutions in the framework of $(2+1)$-dimensional Hirota equation (1.1).

The structure of this paper is organized as follows: In Section 2, the explicit expressions of the general breather, soliton and rational solutions for (1.1) are presented in terms of the Gram determinant. In Section 3, the derivation of these solutions is demonstrated with the aid of two types of Gram determinants. Furthermore, the regularity and symmetry of obtained solutions are discussed. In Section 4, the dynamic behaviors of fundamental breather, soliton and rational solutions are investigated analytically and graphically. Moreover, the superpositions of the breather and soliton solutions are explored. Finally, the main results of the paper are summarized in Section 5.

2 Breather, Soliton and Rational Solutions of (2 + 1)-Dimensional Hirota Equation

The general breather and soliton solutions in (1.1) are given by the following theorem.

Theorem 2.1 *The* $(2+1)$*-dimensional Hirota equation* (1.1) *possesses the* Nth*-order breather and soliton solutions as follows*

$$\Phi = \frac{\sigma_1}{\sigma_0}, \quad \Psi = -2(\ln \sigma_0)_{xy}. \tag{2.1}$$

Here,

$$\sigma_n = \det_{1\leq u,v\leq N}(m_{u,v}^{(n)}), \tag{2.2}$$

whose matrix elements are defined as

$$\begin{aligned} m_{u,v}^{(n)} = {} & \frac{1}{p_{u1}+\overline{p}_{v1}}\Big(-\frac{p_{u1}}{\overline{p}_{v1}}\Big)^n \mathrm{e}^{\xi_u+\overline{\xi}_v} + \frac{1}{p_{u1}+\overline{p}_{v2}}\Big(-\frac{p_{u1}}{\overline{p}_{v2}}\Big)^n \mathrm{e}^{\xi_u} \\ & + \frac{1}{p_{u2}+\overline{p}_{v1}}\Big(-\frac{p_{u2}}{\overline{p}_{v1}}\Big)^n \mathrm{e}^{\overline{\xi}_v} + \frac{1}{p_{u2}+\overline{p}_{v2}}\Big(-\frac{p_{u2}}{\overline{p}_{v2}}\Big)^n, \end{aligned}$$

where

$$\xi_u = (p_{u1} - p_{u2})(x - [4(p_{u1} + p_{u2})^2 - 4p_{u1}p_{u2} + 6]\beta t) \\ + 3\mathrm{i}(p_{u1}^2 + p_{u2}^2)\beta y + \xi_u^0$$

with the "$-$" representing complex conjugation. In addition, ξ_u^0 *are arbitrary complex constants and the complex free parameters* p_{u1}, p_{u2} *satisfy the following constraints*

$$p_{u1} \cdot p_{u2} = 1, \quad u = 1, 2, \cdots, N. \tag{2.3}$$

The derivation of this theorem will appear in Section 3. Note that since there are forms $\frac{1}{p_{u\widetilde{m}}+\overline{p}_{v\widetilde{n}}}$ $(u,\ v = 1, 2, \cdots, N,\ \widetilde{m},\ \widetilde{n} = 1, 2)$ in Theorem 2.1, the $p_{u\widetilde{m}}$ and $p_{v\widetilde{n}}$ should satisfy that the values of $p_{u\widetilde{m}} + \overline{p}_{v\widetilde{n}}$ are nonzero. From the parameter relations (2.3), these Nth-order breather and soliton solutions (2.1) contain $2N$ free complex parameters $p_{u1},\ \xi_u^0\ (u = 1, 2, \cdots, N)$. With the help of these free parameters, we can establish various superposition patterns of the breather and soliton solutions. We note that the solutions (2.1) are soliton types when p_{u1} are real while the solutions (2.1) are breather solutions when p_{u1} are non-real. It is well known that breather solutions could become rational solutions in the limit case, which can contain rogue waves or lump solutions. For the $(2+1)$-dimensional Hirota equation (1.1), our general rational solutions are represented by Theorem 2.2 as follows.

Theorem 2.2 *The* $(2+1)$*-dimensional Hirota equation* (1.1) *has the* Nth*-order rational solutions*

$$\Phi = \frac{\tau_1}{\tau_0}, \quad \Psi = -2(\ln \tau_0)_{xy}, \tag{2.4}$$

here,

$$\tau_n = \det_{1 \le u, v \le N}(\widehat{m}^{(n)}_{2u-1,2v-1}),$$

whose matrix elements are defined as

$$\widehat{m}^{(n)}_{u,v} = \frac{(p\partial_p + \xi' + n)^u}{u!} \frac{(q\partial_q + \eta' - n)^v}{v!} \frac{(p+1)(q+1)}{2(p+q)}\Bigg|_{p=q=1},$$

where

$$\xi' = px + 6\mathrm{i}\beta p^2 y - 6\beta(p + 2p^3)t + \sum_{r=0}^{\infty} \widehat{a}_{2r+1} \ln^{2r}(p),$$

$$\eta' = qx - 6\mathrm{i}\beta q^2 y - 6\beta(q + 2q^3)t + \sum_{r=0}^{\infty} \overline{\widehat{a}}_{2r+1} \ln^{2r}(q)$$

and $\widehat{a}_{2r+1}$ $(r = 0, 1, 2, \cdots, N)$ *are free complex parameters.*

The proof of Theorem 2.2 will also be presented in Section 3. In contrast to the previous works [35], our results have different matrix elements, which will give rise to new types of solutions. Specifically, the rational solutions in [35] are nonlocalized in the planes (x,y) and (y,t), that is, these rational solutions are normal algebraic solitons. However, our rational solutions will exhibit lump solutions in the plane (x,y) and rogue wave solutions in the plane (y,t). Interestingly, these results are different from other bilinear integrable systems, such as the Davey-Stewartson equation (see [22–23]), the Mel'nikov equation (see [24]), the $(2+1)$-dimensional dispersive long wave system (see [36]), etc.

3 Derivation of the Breather, Soliton and Rational Solutions to the (2+1)-Dimensional Hirota Equation

In this section, we focus on constructing the breather, soliton and rational solutions of (1.1) by using Hirota's bilinear method and Kadomtsev-Petviashvili hierarchy reduction technique. Firstly, using the independent variable transformation (see [35])

$$\Phi=\frac{g}{f},\quad \Psi=-2(\ln f)_{xy}, \tag{3.1}$$

where f is a real function and g is a complex function with respect to variables x,y and t, (1.1) is transformed into the bilinear forms

$$(D_x^2+2)f\cdot f=2g\overline{g}, \tag{3.2}$$

$$(\mathrm{i}\beta D_x^3+D_xD_y+6\mathrm{i}\beta D_x+\mathrm{i}D_t)g\cdot f=0, \tag{3.3}$$

where D is the Hirota's bilinear differential operator defined by

$$\begin{aligned}&P(D_x,D_y,D_t)[g(x,y,t)\cdot f(x,y,t)]\\ \equiv\ &P(\partial_x-\partial_x',\partial_y-\partial_y',\partial_t-\partial_t')g(x,y,t)f(x',y',t')|_{x'=x,\,y'=y,\,t'=t}.\end{aligned}$$

The solutions of bilinear equations (3.2)–(3.3) can be obtained based on the following lemma.

Lemma 3.1 (see [34]) *If functions $\widetilde{m}_{u,v}^{(n)}$, $\widetilde{\varphi}_u^{(n)}$ and $\widetilde{\psi}_v^{(n)}$ of variables (x_{-1},x_1,x_2,x_3) satisfy the following differential and difference relations*

$$\begin{aligned}\partial_{x_1}\widetilde{m}_{u,v}^{(n)}&=\widetilde{\varphi}_u^{(n)}\widetilde{\psi}_v^{(n)},\\ \partial_{x_2}\widetilde{m}_{u,v}^{(n)}&=(\partial_{x_1}\widetilde{\varphi}_u^{(n)})\widetilde{\psi}_v^{(n)}-\widetilde{\varphi}_u^{(n)}(\partial_{x_1}\widetilde{\psi}_v^{(n)}),\\ \partial_{x_3}\widetilde{m}_{u,v}^{(n)}&=\partial_{x_1}^2\widetilde{\varphi}_u^{(n)}\widetilde{\psi}_v^{(n)}-\partial_{x_1}\widetilde{\varphi}_u^{(n)}\partial_{x_1}\widetilde{\psi}_v^{(n)}+\widetilde{\varphi}_u^{(n)}\partial_{x_1}^2\widetilde{\psi}_v^{(n)},\\ \partial_{x_{-1}}\widetilde{m}_{u,v}^{(n)}&=-\widetilde{\varphi}_u^{(n-1)}\widetilde{\psi}_v^{(n+1)},\quad \widetilde{m}_{u,v}^{(n+1)}=\widetilde{m}_{u,v}^{(n)}+\widetilde{\varphi}_u^{(n)}\widetilde{\psi}_v^{(n+1)}\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
\partial_{x_1}\widetilde{\varphi}_u^{(n)} &= \widetilde{\varphi}_u^{(n+1)}, \quad \partial_{x_1}\widetilde{\psi}_v^{(n)} = -\widetilde{\psi}_v^{(n-1)},\\
\partial_{x_2}\widetilde{\varphi}_u^{(n)} &= \partial_{x_1}^2\widetilde{\varphi}_u^{(n)}, \quad \partial_{x_2}\widetilde{\psi}_v^{(n)} = -\partial_{x_1}^2\widetilde{\psi}_v^{(n)},\\
\partial_{x_3}\widetilde{\varphi}_u^{(n)} &= \partial_{x_1}^3\widetilde{\varphi}_u^{(n)}, \quad \partial_{x_3}\widetilde{\psi}_v^{(n)} = \partial_{x_1}^3\widetilde{\psi}_v^{(n)},\\
\partial_{x_{-1}}\widetilde{\varphi}_u^{(n)} &= \widetilde{\varphi}_u^{(n-1)}, \quad \partial_{x_{-1}}\widetilde{\psi}_v^{(n)} = -\widetilde{\psi}_v^{(n+1)},
\end{aligned}
\tag{3.5}
$$

then the function

$$
\widetilde{\tau}_n = \det_{1\le u,v\le N}(\widetilde{m}_{i_u,j_v}^{(n)}) \tag{3.6}
$$

satisfies the higher-dimensional bilinear equations in the Kadomtsev-Petviashvili hierarchy

$$
(D_{x_1}D_{x_{-1}} - 2)\widetilde{\tau}_n\cdot\widetilde{\tau}_n + 2\widetilde{\tau}_{n+1}\cdot\widetilde{\tau}_{n-1} = 0, \tag{3.7}
$$

$$
(D_{x_1}^3 + 3D_{x_1}D_{x_2} - 4D_{x_3})\widetilde{\tau}_{n+1}\cdot\widetilde{\tau}_n = 0. \tag{3.8}
$$

Here, $(i_1, i_2, \cdots, i_N)$ *and* $(j_1, j_2, \cdots, j_N)$ *are arbitrary sequences of indices.*

Next, we restrict $\widetilde{\tau}_n$ to satisfy the dimensional reduction and conjugate conditions

$$
(\partial_{x_{-1}} + \partial_{x_1})\widetilde{\tau}_n = C\widetilde{\tau}_n, \tag{3.9}
$$

$$
\widetilde{\tau}_{-n} = \overline{\widetilde{\tau}}_n, \tag{3.10}
$$

where C is an arbitrary constant. Substituting (3.9) into (3.7), we obtain

$$
(D_{x_1}^2 + 2)\widetilde{\tau}_n\cdot\widetilde{\tau}_n - 2\widetilde{\tau}_{n+1}\cdot\widetilde{\tau}_{n-1} = 0. \tag{3.11}
$$

From (3.10), we can define

$$
f = \widetilde{\tau}_0, \quad g = \widetilde{\tau}_1, \quad \overline{g} = \widetilde{\tau}_{-1}.
$$

Then, substituting the above formulas into (3.11) and (3.8), we obtain

$$
(D_{x_1}^2 + 2)f\cdot f = 2g\overline{g}, \tag{3.12}
$$

$$
(D_{x_1}^3 + 3D_{x_1}D_{x_2} - 4D_{x_3})g\cdot f = 0. \tag{3.13}
$$

Subsequently, through the coordinate transformation

$$
x_1 = x - 6\beta t, \quad x_2 = 3\mathrm{i}\beta y, \quad x_3 = -4\beta t, \tag{3.14}
$$

the bilinear equations (3.12)–(3.13) become the bilinear equations (3.2)–(3.3), respectively. Finally, using variable transformation (3.1), we derive the solutions of (1.1).

Proof of Theorem 2.1 In order to construct the breather and soliton solutions of the $(2+1)$-dimensional Hirota equation (1.1) in Theorem 2.1, we introduce $\widetilde{m}_{u,v}^{(n)}$, $\widetilde{\psi}_u^{(n)}$ and $\widetilde{\varphi}_v^{(n)}$ as follows (see [37])

$$\begin{aligned}\widetilde{m}_{u,v}^{(n)} &= \int \widetilde{\varphi}_u^{(n)}\widetilde{\psi}_v^{(n)}\,\mathrm{d}x_1 = \sum_{\widetilde{m},\widetilde{n}=1}^{2}\frac{1}{p_{u\widetilde{m}}+q_{v\widetilde{n}}}\Big(-\frac{p_{u\widetilde{m}}}{q_{v\widetilde{n}}}\Big)^n \mathrm{e}^{\xi_{u\widetilde{m}}+\eta_{v\widetilde{n}}},\\ \widetilde{\varphi}_u^{(n)} &= p_{u1}^n\mathrm{e}^{\xi_{u1}}+p_{u2}^n\mathrm{e}^{\xi_{u2}},\quad \widetilde{\psi}_v^{(n)} = (-q_{v1})^{-n}\mathrm{e}^{\eta_{v1}}+(-q_{v2})^{-n}\mathrm{e}^{\eta_{v2}},\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}\xi_{u\widetilde{m}} &= \frac{1}{p_{u\widetilde{m}}}x_{-1}+p_{u\widetilde{m}}x_1+p_{u\widetilde{m}}^2x_2+p_{u\widetilde{m}}^3x_3+\xi_{u\widetilde{m}}^0,\\ \eta_{v\widetilde{n}} &= \frac{1}{q_{v\widetilde{n}}}x_{-1}+q_{v\widetilde{n}}x_1-q_{v\widetilde{n}}^2x_2+q_{v\widetilde{n}}^3x_3+\eta_{v\widetilde{n}}^0.\end{aligned}$$

Here, $p_{u\widetilde{m}}$, $q_{v\widetilde{n}}$, $\xi_{u\widetilde{m}}^0$ and $\eta_{v\widetilde{n}}^0$ are arbitrary complex parameters. We select sequences $(i_1,i_2,\cdots,i_N)$ and $(j_1,j_2,\cdots,j_N)$ as natural sequence $(1,2,3,\cdots,N)$, and rewrite τ function (3.6) as

$$\widetilde{\tau}_n = \det_{1\le u,v\le N}(\widetilde{m}_{u,v}^{(n)}) = \prod_{u=1}^{N}\mathrm{e}^{\xi_{u2}+\eta_{u2}}\sigma_n,\quad \sigma_n = \det_{1\le u,v\le N}(m_{u,v}^{(n)}),\tag{3.16}$$

where

$$\begin{aligned}m_{u,v}^{(n)} &= \mathrm{e}^{\xi_{u1}-\xi_{u2}+\eta_{v1}-\eta_{v2}}F_n(p_{u1},q_{v1})+\mathrm{e}^{\xi_{u1}-\xi_{u2}}F_n(p_{u1},q_{v2})\\ &\quad+\mathrm{e}^{\eta_{v1}-\eta_{v2}}F_n(p_{u2},q_{v1})+F_n(p_{u2},q_{v2})\end{aligned}$$

with

$$\begin{aligned}F_n(p,q) &= \frac{1}{p+q}\Big(-\frac{p}{q}\Big)^n,\\ \xi_{u1}-\xi_{u2} &= (p_{u1}-p_{u2})x_1-(p_{u1}^2-p_{u2}^2)x_2+(p_{u1}^3-p_{u2}^3)x_3\\ &\quad+\Big(\frac{1}{p_{u1}}-\frac{1}{p_{u2}}\Big)x_{-1}+\xi_{u1}^0-\xi_{u2}^0,\\ \eta_{u1}-\eta_{u2} &= (q_{u1}-q_{u2})x_1-(q_{u1}^2-q_{u2}^2)x_2+(q_{u1}^3-q_{u2}^3)x_3\\ &\quad+\Big(\frac{1}{q_{u1}}-\frac{1}{q_{u2}}\Big)x_{-1}+\eta_{v1}^0-\eta_{v2}^0.\end{aligned}$$

Note that

$$\begin{aligned}(\partial_{x_1}+\partial_{x_{-1}})m_{u,v}^{(n)} &= [[G(p_{u1},p_{u2})+G(q_{v1},q_{v2})]F_n(p_{u1},q_{v1})\mathrm{e}^{\xi_u+\eta_v}\\ &\quad+G(p_{u1},p_{u2})F_n(p_{u1},q_{v2})\mathrm{e}^{\xi_u}+G(q_{v1},q_{v2})F_n(p_{u2},q_{v1})\mathrm{e}^{\eta_v}],\end{aligned}$$

where

$$\xi_u = \xi_{u1}-\xi_{u2},\quad \eta_u = \eta_{u1}-\eta_{u2},$$

$$G(p,q)=(p-q)+\frac{1}{p}-\frac{1}{q}=(p-q)\Big(1-\frac{1}{pq}\Big).$$

Thus, by taking

$$p_{u1}\cdot p_{u2}=1,\quad q_{u1}\cdot q_{u2}=1,$$

we can obtain

$$\partial_{x_1}\sigma_n=\sum_{u,v=1}^{N}\Delta_{u,v}\partial_{x_1}m_{u,v}^{(n)}=-\sum_{u,v=1}^{N}\Delta_{u,v}\partial_{x_{-1}}m_{u,v}^{(n)}=-\partial_{x_{-1}}\sigma_n,$$

where $\Delta_{u,v}$ denotes the (u,v)-cofactor of the matrix $(m_{u,v}^{(n)})$. Therefore, the dimensional reduction condition (3.9) is satisfied. From the above form together with the gauge freedom of $\widetilde{\tau}_n$ in (3.6), it yields

$$\begin{aligned}&(D_{x_1}^2+2)\sigma_n\cdot\sigma_n=2\sigma_{n+1}\cdot\sigma_{n-1},\\&(D_{x_1}^3+3D_{x_1}D_{x_2}-4D_{x_3})\sigma_{n+1}\cdot\sigma_n=0.\end{aligned}$$

Next, setting

$$q_{u\widetilde{m}}=\overline{p}_{u\widetilde{m}},\quad \eta_{u\widetilde{m}}^0=\overline{\xi}_{u\widetilde{m}}^0,\quad u=1,2,\cdots,N,\ \widetilde{m}=1,2,$$

we have $\xi_{u\widetilde{m}}=\overline{\eta}_{u\widetilde{m}}$ under the coordinate transformations (3.14), which implies

$$\begin{aligned}m_{u,v}^{(-n)}&=\frac{1}{p_{u1}+\overline{p}_{v1}}\Big(-\frac{\overline{p}_{v1}}{p_{u1}}\Big)^n e^{\xi_{u1}-\xi_{u2}+\overline{\xi}_{v1}-\overline{\xi}_{v2}}+\frac{1}{p_{u2}+\overline{p}_{v2}}\Big(-\frac{\overline{p}_{v2}}{p_{u2}}\Big)^n\\&\quad+\frac{1}{p_{u1}+\overline{p}_{v2}}\Big(-\frac{\overline{p}_{v2}}{p_{u1}}\Big)^n e^{\xi_{u1}-\xi_{v2}}+\frac{1}{p_{u2}+\overline{p}_{v1}}\Big(-\frac{\overline{p}_{v1}}{p_{u2}}\Big)^n e^{\overline{\xi}_{u1}-\overline{\xi}_{u2}}\\&=\frac{1}{\overline{p}_{v1}+p_{u1}}\Big(-\frac{\overline{p}_{v1}}{p_{u1}}\Big)^n e^{\xi_{u1}-\xi_{u2}+\overline{\xi}_{v1}-\overline{\xi}_{v2}}+\frac{1}{\overline{p}_{v2}+p_{u2}}\Big(-\frac{\overline{p}_{v2}}{p_{u2}}\Big)^n\\&\quad+\frac{1}{\overline{p}_{v2}+p_{u1}}\Big(-\frac{\overline{p}_{v2}}{p_{u1}}\Big)^n e^{\xi_{u1}-\xi_{v2}}++\frac{1}{\overline{p}_{v1}+p_{u2}}\Big(-\frac{\overline{p}_{v1}}{p_{u2}}\Big)^n e^{\overline{\xi}_{v1}-\overline{\xi}_{u2}}\\&=\overline{m}_{v,u}^{(n)},\end{aligned}$$

i.e.,

$$\sigma_n=\overline{\sigma}_{-n}.$$

Finally, defining

$$\sigma_0=f,\quad \sigma_1=g,\quad \sigma_{-1}=\overline{g},$$

together with coordinate transformations (3.14), we arrive at the bilinear equations (3.2)–(3.3) of (1.1). Since the bilinear equations (3.2)–(3.3) do not contain derivation of the variable x_{-1}, we set $x_{-1}=0$ for convenience. Thus, this completes the proof of Theorem 2.1.

Proof of Theorem 2.2 In order to derive the rational solutions of the $(2+1)$-dimensional Hirota equation (1.1) in Theorem 2.2, we select $\widetilde{m}_{u,v}^{(n)}$, $\widetilde{\psi}_u^{(n)}$ and $\widetilde{\varphi}_v^{(n)}$ as follows

$$\begin{aligned} \widetilde{m}_{u,v}^{(n)} &= \int \widetilde{\psi}_u^{(n)} \widetilde{\varphi}_v^{(n)}\,\mathrm{d}x_1 = \frac{(p\partial_p)^u}{u!}\frac{(q\partial_q)^v}{v!}\frac{(p+1)(q+1)}{2(p+q)}\Big(-\frac{p}{q}\Big)^n \mathrm{e}^{\xi+\eta}, \\ \widetilde{\psi}_u^{(n)} &= \frac{(p\partial_p)^u}{u!}\frac{p^n(p+1)}{2}\mathrm{e}^{\xi}, \quad \widetilde{\varphi}_v^{(n)} = \frac{(q\partial_q)^v}{v!}(-q)^{-n}(q+1)\mathrm{e}^{\eta}, \end{aligned} \tag{3.17}$$

where

$$\xi = \frac{1}{p}x_{-1} + px_1 + p^2x_2 + p^3x_3 + \sum_{r=1}^{\infty}\frac{\widehat{a}_r}{r}\ln^r(p),$$
$$\eta = \frac{1}{q}x_{-1} + qx_1 - q^2x_2 + q^3x_3 + \sum_{r=1}^{\infty}\frac{\widehat{b}_r}{r}\ln^r(q).$$

Here, $p, q, \widehat{a}_r$ and $\widehat{b}_r$ are arbitrary complex constants. Particularly, setting the sequences $(i_1, i_2, \cdots, i_N)$ and $(j_1, j_2, \cdots, j_N)$ as odd sequence $(1, 3, 5, \cdots, 2N-1)$ and taking $p = q = 1$, the determinant (see [21])

$$\tau_n = \det_{1\le u,v\le N}(\widehat{m}_{2u-1,2v-1}^{(n)}|_{p=q=1})$$

satisfies the dimension reduction condition

$$(\partial_{x_{-1}} + \partial_{x_1})\tau_n = 4N\tau_n.$$

Furthermore, let free parameters

$$\widehat{a}_r = \overline{\widehat{b}}_r, \quad r = 1, 2, \cdots, N.$$

Then, under the coordinate transformation

$$x_1 = x - 6\beta t, \quad x_2 = 3\mathrm{i}\beta y, \quad x_3 = -4\beta t,$$

one can readily find that

$$\xi|_{p=q=1} = \overline{\eta}|_{p=q=1}, \quad \widehat{m}_{u,v}^{(n)}|_{p=q=1} = \overline{\widehat{m}}_{v,u}^{(-n)}|_{p=q=1},$$

i.e.,

$$\tau_n = \overline{\tau}_{-n}.$$

Thus, we can define

$$\tau_0 = f, \quad \tau_1 = g, \quad \tau_{-1} = \overline{g}.$$

Substituting the above results into bilinear equations (3.7)–(3.8) together with coordinate transformations (3.14), we also arrive at the bilinear equations (3.2)–(3.3) of (1.1). Similar to the proof of Theorem 2.1, we set $x_{-1}=0$. Finally, using the operator relations

$$p\partial_p p^n \mathrm{e}^{\xi} = p^n \mathrm{e}^{\xi}[p\partial_p + \xi' + n],$$
$$q\partial_q (-q)^{-n} \mathrm{e}^{\eta} = (-q)^{-n} \mathrm{e}^{\eta}[q\partial_q + \eta' - n],$$

where

$$\xi' = p(x-6\beta t) + 6\mathrm{i}p^2\beta y - 12p^3\beta t + \sum_{r=0}^{\infty} \widehat{a}_{r+1} \ln^r(p), \tag{3.18}$$
$$\eta' = q(x-6\beta t) - 6\mathrm{i}q^2\beta y - 12q^3\beta t + \sum_{r=0}^{\infty} \overline{\widehat{a}}_{r+1} \ln^r(q), \tag{3.19}$$

we have the rational solutions of (1.1),

$$\Phi = \frac{\tau_1}{\tau_0}, \quad \Psi = -2(\ln \tau_0)_{xy}. \tag{3.20}$$

Here

$$\tau_n = \det_{1\le u,v\le N} (\widehat{m}^{(n)}_{2u-1,2v-1}),$$

whose matrix elements are defined as

$$\widehat{m}^{(n)}_{u,v} = \Big(-\frac{q}{p}\Big)^n \mathrm{e}^{\xi+\eta} \frac{(p\partial_p + \xi' + n)^u}{u!} \frac{(q\partial_q + \eta' - n)^v}{v!} \frac{(p+1)(q+1)}{2(p+q)}\Big|_{p=q=1},$$

where

$$\xi' = p(x-6\beta t) + 6\mathrm{i}\beta p^2 y - 12p^3\beta t + \sum_{r=0}^{\infty} \widehat{a}_{r+1} \ln^r(p),$$
$$\eta' = q(x-6\beta t) - 6\mathrm{i}\beta q^2 y - 12q^3\beta t + \sum_{r=0}^{\infty} \overline{\widehat{a}}_{r+1} \ln^r(q).$$

Due to the gauge freedom of $\widetilde{\tau}_n$ function, we remove the formula $\big(-\frac{q}{p}\big)^n \mathrm{e}^{\xi+\eta}$. In the end, similar to NLSE (see [21]), we may also take $\widehat{a}_2 = \widehat{a}_4 = \cdots = \widehat{a}_{2N-2} = 0$. Thus, this completes the proof of Theorem 2.2.

Discussions of Regularity Next, we discuss the regularity of the obtained breather, soliton and rational solutions of (1.1). From (3.15), we find that $\widetilde{\tau}_0$ can be given by the determinant of a Hermite matrix

$$M_1 = (\widetilde{m}^{(0)}_{u,v})^N_{u,v=1}, \quad \widetilde{m}^{(0)}_{u,v} = \int (\mathrm{e}^{\xi_{u1}} + \mathrm{e}^{\xi_{u2}})(\mathrm{e}^{\overline{\xi}_{v1}} + \mathrm{e}^{\widetilde{\xi}_{v2}})\mathrm{d}x_1.$$

For any zero vector $\mathbf{s}=(s_1,s_2,\cdots,s_N)^{\mathrm{T}}$ and $a_{u1}>0$ ($u=1,2,\cdots,N$, a_{u1} is real part of p_{u1}), we have

$$\begin{aligned}\mathbf{s}^{\dagger}M_1\mathbf{s}&=\sum_{u,v=1}^{N}\overline{s}_u\widetilde{m}_{u,v}^{(0)}s_v=\int_{-\infty}^{x_1}\sum_{u,v=1}^{N}\overline{s}_us_v(\mathrm{e}^{\xi_{u1}}+\mathrm{e}^{\xi_{u2}})(\mathrm{e}^{\overline{\xi}_{v1}}+\mathrm{e}^{\widetilde{\xi}_{v2}})\,\mathrm{d}x_1\\&=\int_{-\infty}^{x_1}\Big|\sum_{u=1}^{N}s_u(\mathrm{e}^{\xi_{u1}}+\mathrm{e}^{\xi_{u2}})\Big|^2\,\mathrm{d}x_1>0,\end{aligned}$$

where the symbol "†" indicates the conjugate transpose. It is shown that the Hermite matrix M_1 is positive definite, hence $\widetilde{\tau}_0>0$. From (3.16), we have $\sigma_0>0$. Similarly, when $a_{u1}<0$ for $u=1,2,\cdots,N$, it leads to $\sigma_0<0$. Thus, we have the following theorem.

Theorem 3.1 *Assuming the free complex parameters* $p_{u1}=a_u+\mathrm{i}b_u$*, for* $u=1,2,\cdots,N$*, when real parameters* $a_u>0$ *or* $a_u<0$ *for all* $u\geq 1$*, the breather and soliton solutions* (2.1) *of* (1.1) *in Theorem* 2.1 *are nonsingular.*

Furthermore, from (3.17), we find that τ_0 can also be given by the determinant of a Hermite matrix

$$\begin{aligned}M_2&=(\widehat{m}_{2u-1,2v-1}^{(0)})_{u,v=1}^{N},\\\widehat{m}_{u,v}^{(0)}&=\int_{-\infty}^{x_1}\frac{(p\partial_p)^u}{u!}\frac{(q\partial_q)^v}{v!}\frac{(p+1)(q+1)}{2}\mathrm{e}^{\xi+\overline{\xi}}\Big|_{p=q=1}\mathrm{d}x_1.\end{aligned}$$

Thus, it leads to

$$\begin{aligned}\mathbf{s}^{\dagger}M_2\mathbf{s}&=\sum_{u,v=1}^{N}\overline{s}_us_v\frac{(p\partial_p)^{2u-1}}{(2u-1)!}\frac{(q\partial_q)^{2v-1}}{(2v-1)!}\frac{(p+1)(q+1)}{2}\int_{-\infty}^{x_1}\mathrm{e}^{\xi+\overline{\xi}}\Big|_{p=q=1}\mathrm{d}x_1\\&=\int_{-\infty}^{x_1}\Big|\sum_{u=1}^{N}s_u\frac{(p\partial_p)^{2u-1}}{(2u-1)!}\frac{\sqrt{2}(p+1)}{2}\mathrm{e}^{\xi}\Big|_{p=1}\Big|^2\mathrm{d}x_1>0.\end{aligned}$$

It implies that the Hermite matrix M_2 is positive, i.e., $\tau_0>0$. Therefore, the rational solutions in Theorem 2.2 are nonsingular.

Finally, we analyze the parity-time-symmetric condition of rational solutions (2.4) for (1.1) in Theorem 2.2. Our derivation is based on the rational solutions expressed in terms of the Schur polynomial defined by

$$\sum_{r=0}^{\infty}S_r(\mathbf{x})\kappa^r=\exp\Big(\sum_{r=1}^{\infty}x_r\kappa^r\Big),\tag{3.21}$$

where $\mathbf{x}=(x_1,x_2,\cdots)$. Applying the simplification technique (see [21]), we have the following Nth-order rational solutions of (1.1) from Theorem 2.2.

Lemma 3.2 (1.1) *has* Nth-*order rational solutions*

$$\Phi=\frac{\widehat{\sigma}_1}{\widehat{\sigma}_0},\quad\Psi=-2(\ln\widehat{\sigma}_0)_{xy}.$$

Here

$$\widehat{\sigma}_n = \det_{1\le u,v\le N}(\widehat{m}^{(n)}_{2u-1,2v-1}), \tag{3.22}$$

whose elements are

$$\widehat{m}^{(n)}_{u,v} = \sum_{j=0}^{min(u,v)} \frac{1}{4^j} S_{u-j}(\mathbf{x}^+(n)+j\mathbf{s})S_{v-j}(\mathbf{x}^-(n)+j\mathbf{s}),$$

where vectors $\mathbf{x}^{\pm}(n) = (x_1^{\pm}(n), x_2^{\pm}(n), \cdots)$ *are defined as*

$$\begin{cases} x_1^+(n) = x + 6\mathrm{i}\beta(y+3\mathrm{i}t) + n + \widehat{a}_1, \\ x_1^-(n) = x - 6\mathrm{i}\beta(y-3\mathrm{i}t) - n + \overline{\widehat{a}}_1, \end{cases}$$

and for all $r \ge 1$*,*

$$\begin{cases} x_{2r}^+(n) = x_{2r}^-(n) = 0, \\ x_{2r+1}^+(n) = \dfrac{1}{(2r+1)!}(x + 6\mathrm{i}\beta[2^{2r}y + \mathrm{i}(1+2\cdot 3^{2r})t]) + \widehat{a}_{2r+1}, \\ x_{2r+1}^-(n) = \dfrac{1}{(2r+1)!}(x - 6\mathrm{i}\beta[2^{2r}y - \mathrm{i}(1+2\cdot 3^{2r})t]) + \overline{\widehat{a}}_{2r+1}. \end{cases}$$

In addition, the components s_r *of vector* $\mathbf{s} = (s_1, s_2, \cdots)$ *are defined as*

$$\sum_{r=1}^{\infty} s_r \kappa^r = \ln\left[\frac{2}{\kappa}\tanh\left(\frac{\kappa}{2}\right)\right], \tag{3.23}$$

and $\widehat{a}_{2r+1}$ $(r = 0, 1, 2, \cdots)$ *are free complex parameters.*

By taking $a_{2r+1} = 0$ for all $r \ge 0$ in Lemma 3.2, we have

$$\overline{x}_r^{\pm}(-x,-t) = -x_r^{\mp}(x,t), \quad r \ge 1.$$

Because $s_1 = s_3 = \cdots = s_{\mathrm{odd}} = 0$ from (3.23), we obtain

$$\widehat{S}_u(\mathbf{x}^{\pm}(n)+j\mathbf{s}) = S_u(-x_1^{\mp}, js_2, -x_3^{\mp}, js_4, \cdots),$$

where $\widehat{S}_u(x,t) = \overline{S}_u(-x,-t)$. According to the definition of Schur polynomial (3.21), one can readily obtain

$$\widehat{S}_u(\mathbf{x}^{\pm}(n)+j\mathbf{s}) = (-1)^u \sum_{r=0}^{\lfloor \frac{u}{2} \rfloor} S_{u-2r}(\mathbf{x}^{\mp}(n)+j\mathbf{s}).$$

Finally, substituting the above formula into (3.22) and using simple row manipulations, we find

$$\overline{\Phi}(-x,-y,-t) = \Phi(x,y,t), \quad \Psi(-x,-y,-t) = \Psi(x,y,t).$$

Thus, we have the following theorem.

Theorem 3.2 *When free complex parameters* $\widehat{a}_{2r+1} = 0$ $(r = 0, 1, 2, \cdots)$ *in Theorem* 2.2*, the rational solutions* (2.4) *of* (1.1) *are parity-time-symmetric.*

4 Dynamic Analysis

In this section, we delineate the dynamic behaviors of the general breather, soliton and rational solutions of (1.1).

4.1 First-order breather and soliton solutions

By setting $N = 1$ and $p_{11} = a_1 + \mathrm{i}b_1$ in Theorem 2.1, we derive the first-order breather and soliton solutions of (1.1),

$$\begin{aligned} &\Phi_1 = -1 - \frac{g_1}{f_1}, \quad \Psi_1 = -2(\ln f_1)_{xy}, \\ &g_1 = \frac{a_1 + \mathrm{i}b_1}{a_1^2 + b_1^2} \mathrm{e}^{\bar{\xi}_1}(\mathrm{e}^{\xi_1} + 1) + (a_1 - \mathrm{i}b_1)(\mathrm{e}^{\xi_1} + 1), \\ &f_1 = \frac{1}{2a_1}(\mathrm{e}^{2\xi_{11}} + a_1^2 + b_1^2) + \frac{2a_1}{a_1^2 + b_1^2 + 1} \cos(\xi_{12}) \mathrm{e}^{\xi_{11}}, \end{aligned} \tag{4.1}$$

where $\xi_1 = \xi_{11} + \mathrm{i}\xi_{12}$ and

$$\begin{aligned} &\xi_{11} = \alpha_1 x - \alpha_3 y - \alpha_5 t + \Re(\xi_1^0), \\ &\xi_{12} = \alpha_2 x + \alpha_4 y - \alpha_6 t + \Im(\xi_1^0), \\ &\alpha_1 = a_1\Big(1 - \frac{1}{a_1^2 + b_1^2}\Big), \quad \alpha_3 = 6a_1 b_1 \beta\Big(1 + \frac{1}{(a_1^2 + b_1^2)^2}\Big), \\ &\alpha_2 = b_1\Big(1 + \frac{1}{a_1^2 + b_1^2}\Big), \quad \alpha_4 = 3\beta(a_1^2 - b_1^2)\Big(1 - \frac{1}{(a_1^2 + b_1^2)^2}\Big), \\ &\alpha_5 = 4a_1\beta(a_1^2 - 3b_1^2)\Big(1 - \frac{1}{(a_1^2 + b_1^2)^3}\Big) + 6a_1\beta\Big(1 - \frac{1}{a_1^2 + b_1^2}\Big), \\ &a_6 = 4b_1\beta(3a_1^2 - b_1^2)\Big(1 + \frac{1}{(a_1^2 + b_1^2)^3}\Big) + 6b_1\beta\Big(1 + \frac{1}{a_1^2 + b_1^2}\Big). \end{aligned}$$

Here, $\Re(\xi_1^0)$ and $\Im(\xi_1^0)$ represent the real part and imaginary ones of ξ_1^0, respectively. It is shown that when $\alpha_2, \alpha_4, \alpha_6 \neq 0$, the solutions Φ_1, Ψ_1 exhibit periodicity in both x, y and t, with periods $\frac{2\pi}{\alpha_2}$, $\frac{2\pi}{\alpha_4}$ and $\frac{2\pi}{\alpha_6}$, respectively. The family of the first-order breather and soliton solutions (4.1) for (1.1) contains three parameters a_1, b_1 and ξ_1^0. Hence, there are various shapes of breather and soliton solutions. Firstly, we consider the first-order breather solutions in the plane (x, y). Taking $t = 0$ for convenience, we obtain such breather solutions as depicted in Figure 1. As illustrated in Figure 1, the first-order breather solutions in the plane (x, y) propagate along the straight line $\alpha_1 x - \alpha_3 y = 0$ and admit bidirectional travel. Furthermore, by fixing $y = 0$, the "normal" waves are observed when α_2 is small and α_1 is comparatively large (see Figure 2(a)). Conversely, spindle-like waves emerge when α_1 is small and α_2 is significantly larger than α_1 (see Figure 2(b)). From Figure 1, it is noted that the structures of Φ and Ψ are similar; therefore, we omit the discussions of the real function Ψ in subsequent analysis.

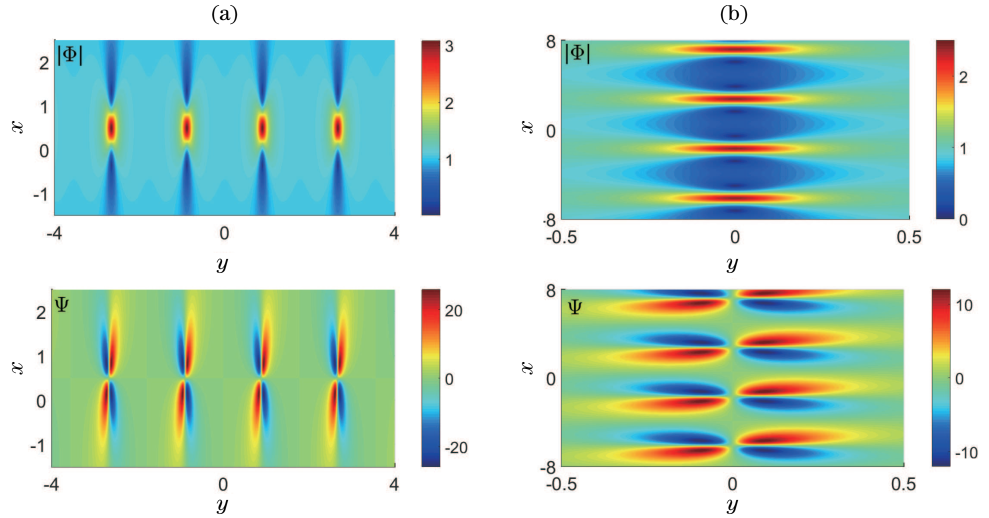

Figure 1 The first-order breather solutions (4.1) of (1.1) with parameters $\beta = 1$, $\xi_1^0 = 0$ and $t = 0$. (a) The breather solutions $|\Phi|$ (top) and Ψ (bottom) are periodic along the y-axis with $p_{11} = \sqrt{2}$. (b) The breather solutions $|\Phi|$ (top) and Ψ (bottom) are periodic along the x-axis with $p_{11} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$.

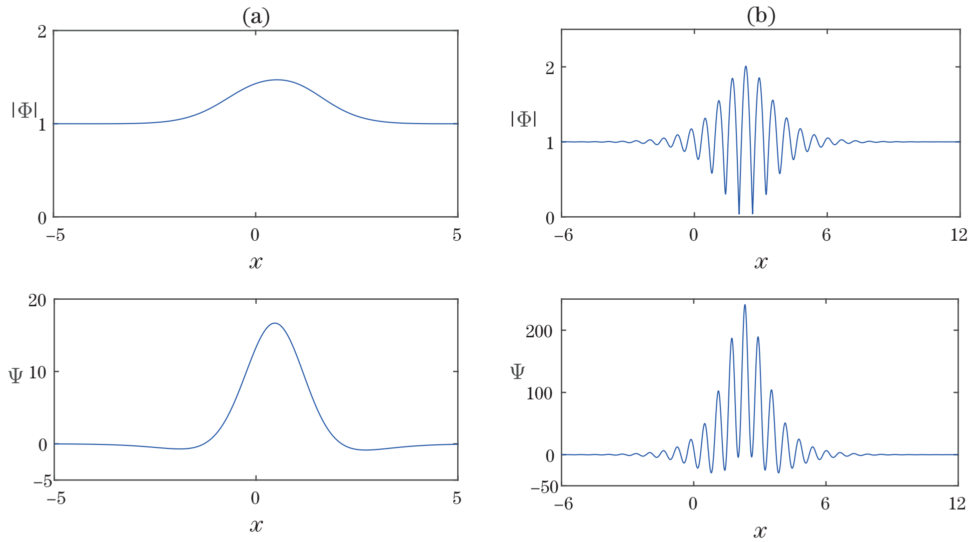

Figure 2 Shape of breather solutions given by (4.1) at $y = t = 0$. (a) A normal shape of $|\Phi|$ (top) and Ψ (bottom) with parameters $\beta = 1$, $p_{11} = 2 + \frac{1}{2}\mathrm{i}$, $\xi_1^0 = 0$. (b) The spindle shape of $|\Phi|$ (top) and Ψ (bottom) with parameters the same as (a) except $p_1 = 1 + 10\mathrm{i}$.

Next, we explore the first-order breather solutions in the space-time plane (x, t). Setting

$$y = 0, \quad \beta = -\frac{1}{12}, \quad a_1 = b_1 = \frac{\sqrt{2}}{2}, \quad \xi_1^0 = 0, \tag{4.2}$$

we obtain the first-order breather solutions of (1.1) in plane (x, t) depicted in Figure 3(c). As shown in Figure 3(c), this solution is periodic in both x and t with oscillations moving along the line $x = t$. Interestingly, by selecting the parameters

$$y = 0, \quad \beta = 1, \quad a_1 = b_1 = \frac{\sqrt{2}}{2}, \quad \xi_1^0 = 0, \tag{4.3}$$

we obtain a first-order line breather solution. Unlike the normal breather solutions discussed previously, the line breather solutions maintain a stable peak height, resembling soliton solutions on the periodic background (see Figure 3(b)).

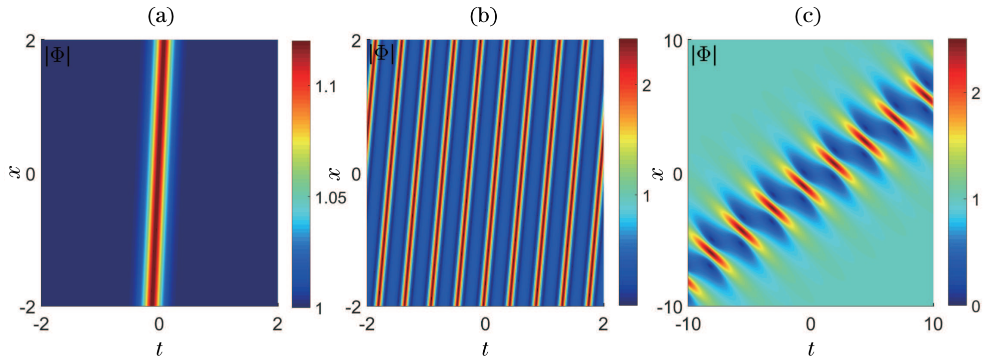

Figure 3 The first-order breather and soliton solutions Φ of (1.1) with parameters $\xi_1^0 = 0$ and $y = 0$. (a) A soliton solution with $\beta = 1$, $p_{11} = \sqrt{2}$. (b) A line breather solution with $\beta = 1$, $p_{11} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$. (c) A breather solution with $\beta = -\frac{1}{12}$, $p_{11} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$.

Furthermore, if $\alpha_2 = \alpha_6 = 0$, i.e., $b_1 = 0$ in (4.1), we can obtain the first-order soliton solutions of (1.1) in the space-time plane (x, t). The soliton solution is illustrated in Figure 3(a) with parameters $\beta = 1, p_{11} = \sqrt{2}$, $\xi_{11}^0 = 0$ and $y = 0$. As shown in Figure 3(a), the soliton solutions of (1.1) possess

$$|\Phi|^2_{Amp} = \left[\left(1 + \frac{(a_1^2-1)^2\cos(\theta)}{|a_1|(a_1^2+1) + 4a_1^2\cos(\theta)}\right)^2 + \left(\frac{(a_1^4-1)^2\sin(\theta)}{|a_1|(a_1^2+1) + 4a_1^2\cos(\theta)}\right)^2\right],$$

where $\theta = 3\beta\big(a_1^2 - \frac{1}{a_1^2}\big)y + \Im(\xi_1^0)$, and top trajectory

$$x(t) = \left(a_1^2 + \frac{1}{a_1^2} + 10\right)\beta t + \frac{a_1(\Re(\xi_1^0) - \ln|a_1|)}{a_1^2 - 1}.$$

4.2 Higher-order breather and soliton solutions

In order to obtain the second-order breather and soliton solutions, we set $N = 2$ in Theorem 2.1. In this case, we have

$$\Phi_2 = \frac{g_2}{f_2}, \quad \Psi_2 = -2(\ln f_2)_{xy},$$
$$g_2 = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,2}^{(1)} \\ m_{2,1}^{(1)} & m_{2,2}^{(1)} \end{vmatrix}, \quad f_2 = \begin{vmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} \\ m_{2,1}^{(0)} & m_{2,2}^{(0)} \end{vmatrix}, \tag{4.4}$$

where the functions $m^{(1)}_{u,v}$, $m^{(0)}_{u,v}$ $(u, v = 1, 2)$ are given in (2.2). Because the explicit expressions of f_2, g_2 are very cumbersome, we omit them. It is well known that the second-order breathers describe the interactions between two first-order breathers. To illustrate characteristics of superpositions of breathers in (1.1), we plot Figure 4(a), were the parameters $\beta = 1$, $p_{11} = 2$, $p_{21} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$, $\xi^0_1 = \xi^0_2 = 0$ and $t = 0$.

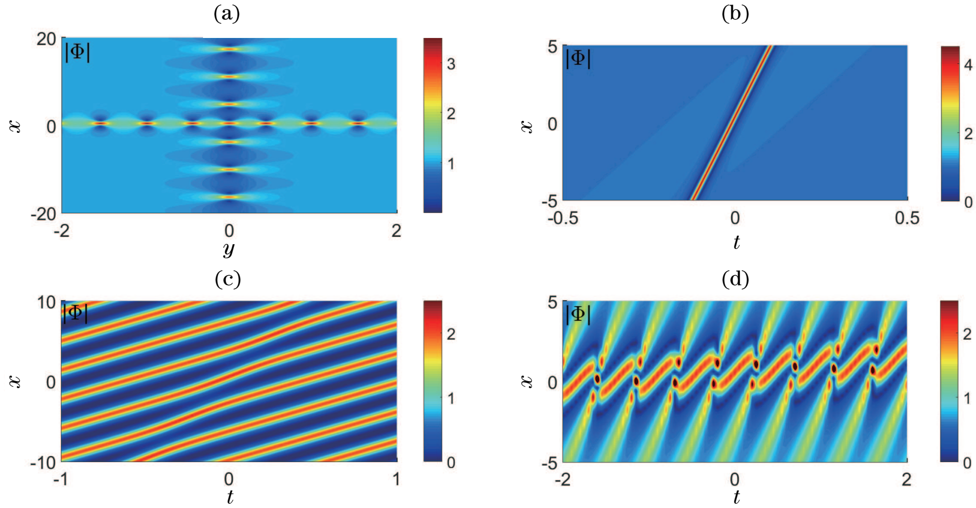

Figure 4 The superpositions of breathers and solitons with the parameters $\beta = 1$ and $\xi^0_1 = \xi^0_2 = 0$. (a) A superposition of two breathers with $p_{11} = 2$, $p_{21} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$ and $t = 0$. (b) A superposition of two solitons with $p_{11} = \sqrt{2}$, $p_{21} = 2\sqrt{2}$ and $y = 0$. (c) A superposition of one soliton and one breather with $p_{11} = \sqrt{2}$, $p_{21} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$ and $y = 0$. (d) A superposition of two breathers with $p_{21} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$, $p_{11} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$ and $y = 0$.

Furthermore, we can also construct the second-order breather and soliton solutions in the space-time plane (x, t) by selecting appropriate values of free parameters. We plot three kinds of superpositions of the breather and soliton solutions in Figure 4. Figure 4(b) describes the interaction between two first-order solitons. We can see clearly that the shapes and velocities of solitons are maintained after collision except for some phase shifts, indicating that this collision is elastic. Similarly, we may obtain the superposition of one breather and one soliton (see Figure 4(c)) and of the two breathers (see Figure 4(d)). By setting $N \geq 3$ in Theorem 3.1, one can readily derive Nth-order breather and soliton solutions for (1.1). Generally, the Nth-order breather and soliton solutions describe the superpositions between u $(u = 1, 2, \cdots, N)$ first-order solitons and N-u first-order breathers. Due to their complexity, explicit expressions are not provided here. This discussion focuses on the case of $N = 3$. Results for different choices of free parameters, such as third-order breather and soliton solutions, are illustrated in Figures 5–6. As shown in Figure 5(a), three first-order breathers can interact in pairs or collide simultaneously. Figure 5(b) demonstrates how the free parameters ξ^0_u can be adjusted to manipulate the collision points between first-order breathers, which can be useful to construct various dynamic patterns. In addition, we display the four superposition patterns of breather and soliton solutions in Figure 6.

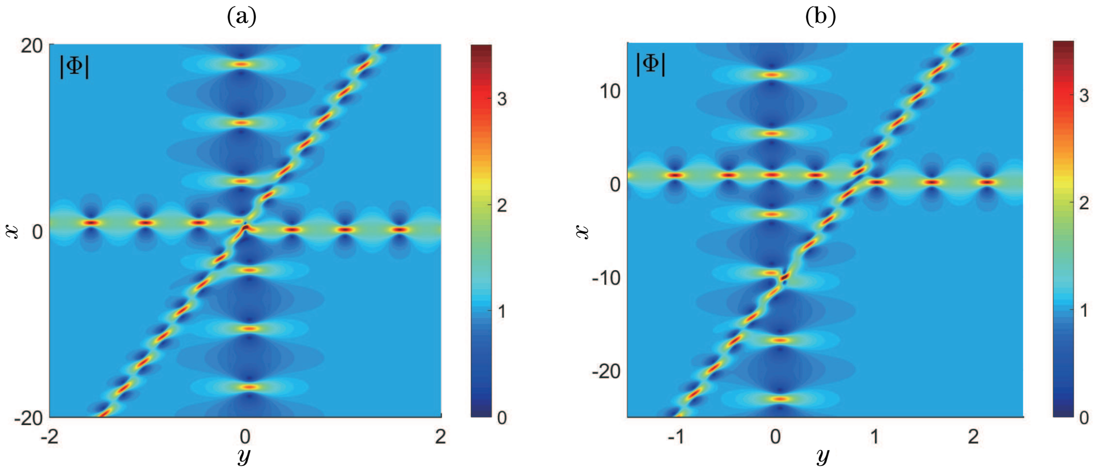

Figure 5 The third-order breather solutions Φ of (1.1) for the parameters $\beta = 1$, $t = 0$, $p_{11} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$, $p_{21} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$, $p_{31} = 2 + 2\mathrm{i}$, $\xi_1^0 = \xi_2^0 = 0$ with (a) $\xi_3^0 = 0$ and (b) $\xi_3^0 = 50$.

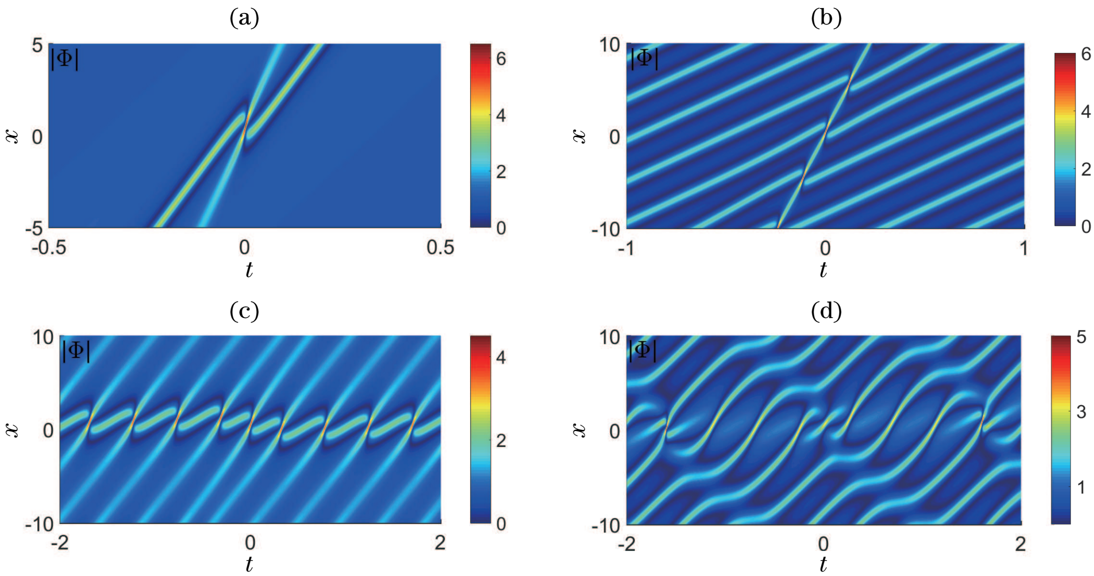

Figure 6 The superpositions of breathers and solitons with the parameters $\beta = 1$, $\xi_1^0 = \xi_2^0 = \xi_3^0 = 0$ and $y = 0$. (a) A superposition of three solitons with $p_{11} = \sqrt{2}$, $p_{21} = 2$, $p_{31} = 2\sqrt{2}$. (b) A superposition of two solitons and one breather with $p_{11} = \sqrt{2}$, $p_{21} = 2\sqrt{2}$, $p_{31} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$. (c) A superposition of one soliton and two breathers with $p_{11} = \sqrt{2}$, $p_{21} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$, $p_{31} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$. (d) A superposition of three breathers with $p_{11} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$, $p_{21} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$, $p_{31} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$.

4.3 The rational solutions of (2 + 1)-dimensional Hirota equation

Taking $N = 1$ in Theorem 2.2, we obtain the first-order rational solutions

$$\begin{aligned}
&\Phi_{1r} = 1 - \frac{G}{F}, \quad \Psi_{1r} = \frac{H}{F^2},\\
&G = 12\mathrm{i}\beta y + \widehat{a}_1 - \overline{\widehat{a}}_1 + 1,\\
&H = 8(6\mathrm{i}\beta y + \widehat{a}_1 - \overline{\widehat{a}}_1)(x - 18t + \widehat{a}_1 + \overline{\widehat{a}}_1),\\
&F = (x + 6\mathrm{i}\beta(y + 3\mathrm{i}t) + \widehat{a}_1)(x - 6\mathrm{i}\beta(y - 3\mathrm{i}t) + \overline{\widehat{a}}_1) + \frac{1}{4}.
\end{aligned} \tag{4.5}$$

From this formula, we see that regardless of the arbitrary value chosen for the time variable t, the first-order rational solutions (4.5) exhibit structures similar to the Peregrine soliton of NLSE (see [11]). These are identified as lump solutions of (1.1). In order to understand the dynamic behaviors of these lumps, we plot the first-order lump solution in Figure 7.

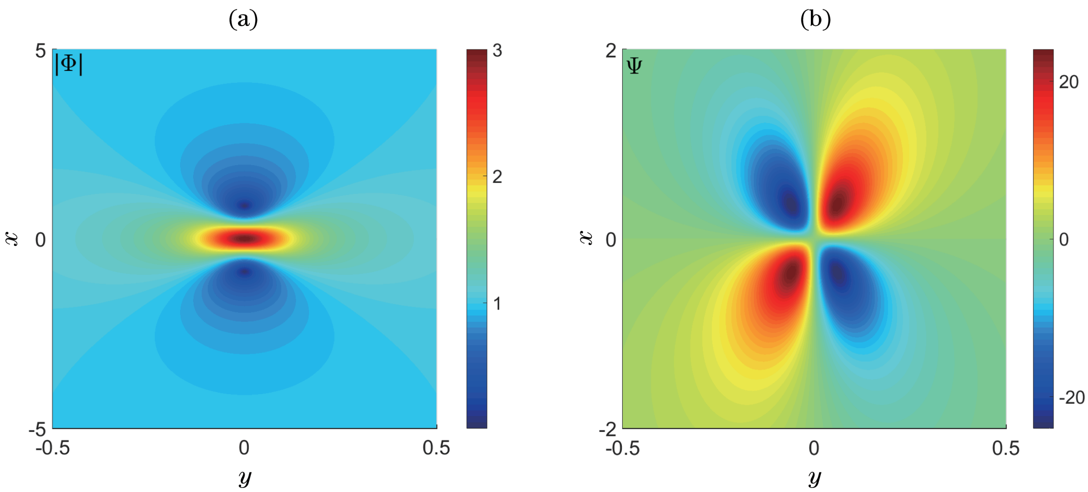

Figure 7 The first-order rational solutions Φ and Ψ of (1.1) with parameters $\beta = 1$, $\widehat{a}_1 = 0$ and $t = 0$.

Next, fixing $t = 0$, we find that the intensities Φ_{1r} and ψ_{1r} tend to the constant backgrounds 1 and 0 as $x, y \to \infty$. $|\Phi_{1r}|^2$ has one local maximum 3 at (x_1, y_1) and two local minima 0 at points $(x_2^{\pm}, y_2^{\pm})$. Meanwhile, Ψ_{1r} also has two local maximum $24|\beta|$ at $(x_3^{\pm}, y_3^{\pm})$ and two local minima $-24|\beta|$ at points $(x_4^{\pm}, y_4^{\pm})$. Here, the four critical points are defined by

$$\begin{aligned}
&(x_1, y_1) = \Big(\frac{1}{2}, 0\Big), \quad (x_2^{\pm}, y_2^{\pm}) = \Big(\frac{1 \pm \sqrt{3}}{2}, 0\Big),\\
&(x_3^{\pm}, y_3^{\pm}) = \Big(\frac{2 \pm \sqrt{2}}{4}, \frac{\pm\sqrt{2}}{24\beta}\Big), \quad (x_4^{\pm}, y_4^{\pm}) = \Big(\frac{2 \mp \sqrt{2}}{4}, \frac{\pm\sqrt{2}}{24\beta}\Big).
\end{aligned}$$

Therefore, for the first-order lump solutions Φ_{1r} and Ψ_{1r}, it gives rise to

$$W_1 = \sqrt{3}, \quad W_2 = \frac{\sqrt{2(108 + 72\sqrt{2})\beta^2 + 2}}{12|\beta|},$$

where W_1 and W_2 are wave widths of $|\Phi_{1r}|$ and Ψ_{1r}, respectively. Therefore, the parameter β does not effect the amplitude of lumps Φ_{1r}, but influences the duration of Peregrine solitons Φ_{1r} and Ψ_{1r}. Furthermore, we plot other patterns of the first-order rational solution at the different planes, which are displayed in Figures 8–9. Similarly, by setting $N=2$, $\beta=1$ and $\widehat{a}_1=\widehat{a}_3=0$ in Theorem 2.2, we plot the second-order rational solution in Figure 10. Figure 10(a) displays a lump whose maximum amplitude is 5 and whose structure is similar to second-order rogue waves of NLSE. As shown in Figure 10(b), the second-order rational solutions belong to the type of rogue waves of modified Korteweg-de Vries equation in the plane (x,t) (see [36]). (1.1) also possesses rogue waves, as shown in Figure 10(c).

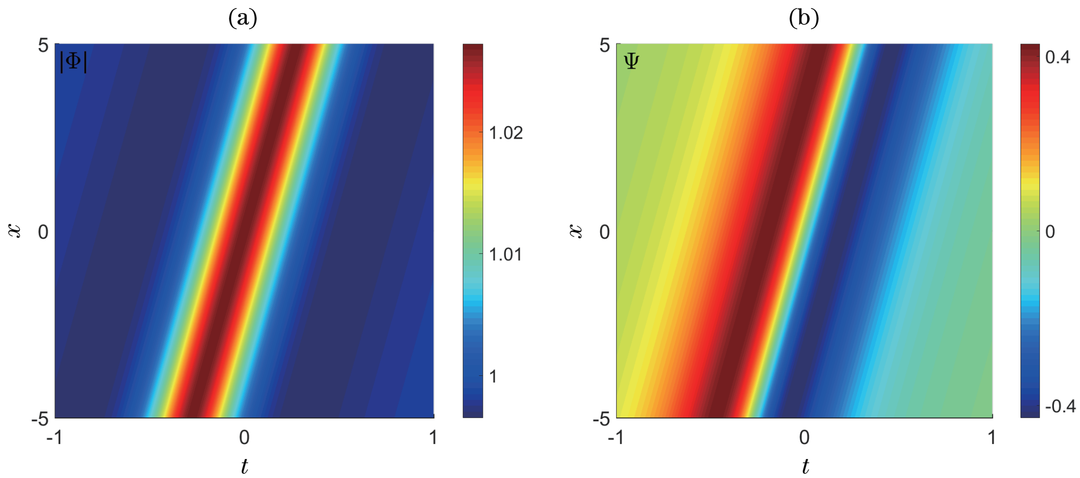

Figure 8 The first-order rational solutions Φ and Ψ of (1.1) with parameters $\beta=1$, $\widehat{a}_1=0$ and $y=1$.

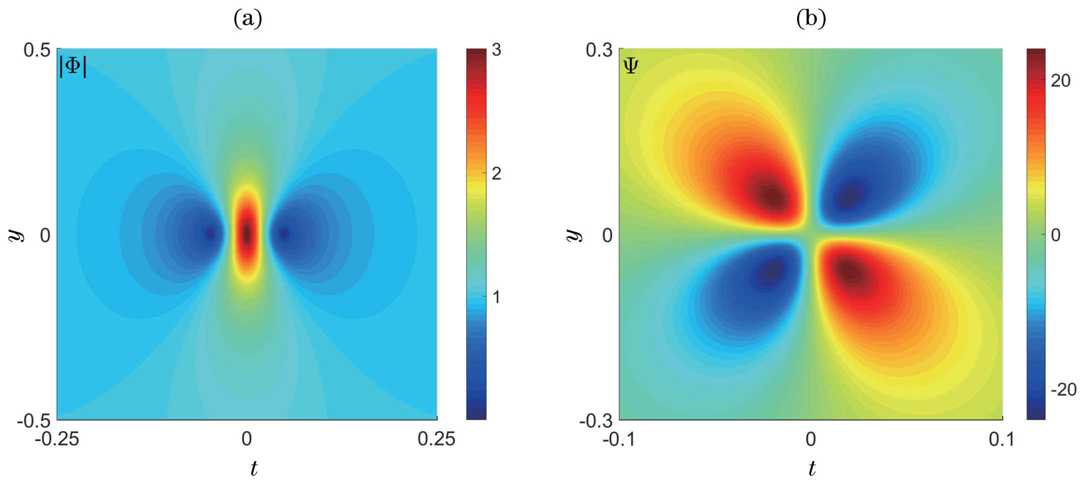

Figure 9 The first-order rational solutions Φ and Ψ of (1.1) with parameters $\beta=1$, $\widehat{a}_1=0$ and $x=0$.

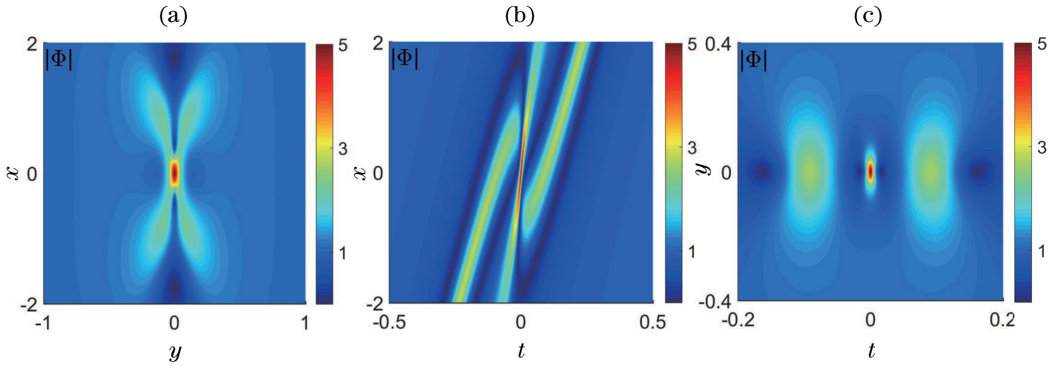

Figure 10 Three patterns of second-order rational solutions Φ of (1.1) for the parameters $\beta = 1$, $\widehat{a}_1 = 0$ and $\widehat{a}_3 = 0$. (a) $t = 0$, (b) $y = 1$, (c) $x = 0$.

5 Conclusions

In this paper, utilizing Hirota's bilinear method and Kadomtsev-Petviashvili hierarchy reduction technique, we have constructed the general breather, soliton and rational solutions of (1.1). These solutions are explicitly presented in terms of Gram determinant. We discuss their regularity and symmetry. The superpositions of the breathers and solitons of (1.1) are established by different choices of free parameters. Furthermore, we analyze dynamic behaviors of fundamental rational solutions and obtain three kinds of dynamic patterns in (1.1). These results demonstrate that Hirota's bilinear method and Kadomtsev-Petviashvili hierarchy reduction technique effectively investigate the breather, soliton and rational solutions of nonlinear integrable equations. Utilizing this method to study higher-order integrable systems may be an interesting topic in the future.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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