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$\mathbb{Z}_N$-Graded Toda Lattices*

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Abstract The $\mathbb{Z}_N$-graded Toda lattices are introduced and investigated under both infinite and periodic boundary conditions. Initially, a hierarchy of integrable $\mathbb{Z}_N$-graded Toda lattices is constructed using the technique of discrete zero curvature equations under infinite boundary conditions. The integrability of these lattices is demonstrated through their bi-Hamiltonian structures. Subsequently, particular emphasis is placed on the study of the $\mathbb{Z}_N$-graded Toda lattice, the first nontrivial lattice in the hierarchy. It is discovered that this lattice can be represented in a Newtonian form with an exponential potential in the Flaschka-Manakov variables. Furthermore, the periodic $\mathbb{Z}_N$-graded Toda lattice is identified as either a periodic Toda lattice or a set of independent periodic Toda lattices sharing the same periodicity. Finally, the complete integrability of the periodic $\mathbb{Z}_N$-graded Toda lattice as a Hamiltonian system in the Liouville sense is established.

Keywords $\mathbb{Z}_N$-Graded Toda lattice, Zero curvature representation, Bi-Hamiltonian structure, Integrable Hamiltonian system
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1 Introduction

The Toda lattice is a well-known integrable system that describes the dynamics of a one-dimensional chain of particles with exponential interaction between nearest neighbors. It has been extensively studied in various fields of mathematics and physics (see [1–4]). The Newtonian equation of motion of the Toda lattice is

$$\ddot{q}(n,t) = \exp\{q(n+1,t) - q(n,t)\} - \exp\{q(n,t) - q(n-1,t)\}, \tag{1.1}$$

where $q(n,t)$ represents the position of the particle at site n and time t, and $\dot{q}(n,t)$ denotes its time derivative. The lattice may be subjected to an infinite boundary condition with $-\infty < n < \infty$ or a τ-periodic boundary condition with $1 \le n \le \tau$ and $q(n+\tau,t) = q(n,t)$ for $n \in \mathbb{Z}$, leading to both infinite and finite lattices.

The introduction of Flaschka-Manakov variables (see [5–6]), represented by $v(n,t) = \exp(q(n,t) - q(n-1,t))$ and $p(n,t) = -\dot{q}(n,t)$, allows for an evolutionary form of the equation of motion for the Toda lattice:

$$\begin{cases} \dot{p}(n,t) = v(n,t) - v(n+1,t), \\ \dot{v}(n,t) = v(n,t)p(n-1,t) - p(n,t)v(n,t), \quad n \in \mathbb{Z}. \end{cases} \tag{1.2}$$

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This formulation has indeed led to the discovery of profound mathematical structures inherent in the Toda lattice. As a result, the study of the Toda lattice has established connections with various branches of mathematics, including Lie groups, Lie algebras, random matrices, numerical computation, differential geometry, and algebraic geometry (see [7–10]).

Over the past sixty years, the Toda lattice has been generalized and extended in various directions. For example, the two-dimensional and three-dimensional Toda lattices introduce multiple continuous or discrete spatial variables (see [11–14]). The Bogoyavlensky-Toda lattices extend the Toda lattice associated with the root system of type A_n to other simple Lie algebras (see [15]). The full Kostant-Toda lattice extends the Toda lattice to generic symmetric matrices (see [16–18]). In particular, Hu introduced two-dimensional signed Toda equations and showed how the Darboux transformation method can be used to explicitly construct periodic Laplace sequences of surfaces (see [19]).

In this paper, we propose and study $\mathbb{Z}_N$-graded Toda lattices. Under the infinite boundary condition, we introduce a spectral problem involving a $2N\times 2N$ matrix with $\mathbb{Z}_N$-graded matrix blocks and utilize the technique of discrete zero curvature equations to construct a hierarchy of $\mathbb{Z}_N$-graded Toda lattices. By employing the trace identity approach developed by Tu [20], we establish the bi-Hamiltonian formula for the $\mathbb{Z}_N$-graded Toda lattices. Focusing on the $\mathbb{Z}_N$-graded Toda lattice ($\mathbb{Z}_N$TL for short), we demonstrate that the $\mathbb{Z}_N$TL can be expressed in a Newtonian form with an exponential potential in the Flaschka-Manakov variables. Additionally, we show that the periodic $\mathbb{Z}_N$TL is either a periodic Toda lattice or a set of independent periodic-Toda lattices sharing the same periodicity. This analysis confirms the complete integrability of the periodic $\mathbb{Z}_N$TL as a Hamiltonian system in the Liouville sense.

2 The $\mathbb{Z}_N$-Graded Toda Lattices and Their Bi-Hamiltonian Structures

Let Ω be the $N\times N$ permutation matrix given by

$$(\Omega)_{i,j}=\delta_{j-i,1}+\delta_{i-j,N-1}.$$

By direct computation, we have the following lemma.

Lemma 2.1 *For a fixed integer r $(1\le r\le N-1)$, the following results hold:*

(1)

$$(\Omega^r)_{i,j}=\delta_{j-i,r}+\delta_{i-j,N-r},\quad \Omega^N=I_N,$$

where I_N is the $N\times N$ identity matrix.

(2) $\Omega^{-r}=\Omega^{N-r}$*, and specifically,* $\Omega^{-1}=\Omega^{N-1}=\Omega^{\mathrm{T}}$*, where* T *denotes the transpose of the matrix.*

(3) *If* $A=\mathrm{diag}(a_1,\cdots,a_N)$ *is a diagonal matrix, then*

$$\mathrm{Ad}_{\Omega^k}A\triangleq\Omega^kA\Omega^{-k}=\mathrm{diag}(a_{k+1},\cdots,a_N,a_1,\cdots,a_k)$$

is still a diagonal matrix. We refer to Ad_Ω *as the adjoint action operator.*

Following [21], a matrix A of size $N\times N$ given by

$$A=\mathrm{diag}(a_1,\cdots,a_N)\Omega^r,\quad 0\le r\le N-1$$

is said to have level r, denoted as $\text{lev}(A) = r$. In the ring R of the $N \times N$ matrices and the set R_k of all $N \times N$ matrices of level k, we have a direct sum decomposition given by

$$R = \bigoplus_{k \in \mathbb{Z}_N} R_k, \quad R_k R_l \subseteq R_{k+l}.$$

Consequently, the ring R forms a $\mathbb{Z}_N$-graded ring, where the levels $k \in \mathbb{Z}_N$ define the grading. $j \in \mathbb{Z}_N$ specifies that j-indices are interpreted as being modulo N. For convenience, we usually use $1 \leq j \leq N$ instead of $0 \leq j \leq N-1$.

Consider the $2N \times 2N$ matrix discrete spectral problem

$$S\Phi = U(P, V, \lambda)\Phi, \quad U(P, V, \lambda) = \begin{pmatrix} 0 & \Omega^{-r} \\ -V\Omega^{-r} & \lambda\Omega^{-r} - P\Omega^{-r} \end{pmatrix}. \tag{2.1}$$

Where λ is a spectral parameter, $\Phi = (\phi_1(n,t), \cdots, \phi_{2N}(n,t))^{\mathrm{T}}$ is a column vector, and S is the space shift operator defined as: $Sf(n,t) = f(n+1,t)$, $S^{-1}f(n,t) = f(n-1,t)$ for any (matrix) function $f(n,t)$. Moreover, matrices $P = \text{diag}(p_1(n,t), \cdots, p_N(n,t))$ and $V = \text{diag}(v_1(n,t), \cdots, v_N(n,t))$ are diagonal matrices and consequently matrices Ω^r, $V\Omega^r$ and $P\Omega^r$ are all of level r.

From the stationary discrete zero-curvature equation

$$S(\Gamma)U = U\Gamma, \quad \Gamma = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{2.2}$$

we arrive at

$$\begin{cases} C = -\Omega^r S(B) V \Omega^{-r}, \\ \lambda S(B) = (SB)P - \Omega^{-r} A \Omega^r - S(A), \\ \lambda \Omega^{-r} C = S(A) V \Omega^{-r} + V\Omega^{-r} A + P\Omega^{-r} C, \\ \lambda(\Omega^r S(A) \Omega^{-r} - A) = \Omega^r V \Omega^{-r} B + \Omega^r S(C) \Omega^{-r} + \Omega^r S(A) P \Omega^{-r} - \Omega^r P \Omega^{-r} A, \end{cases} \tag{2.3}$$

where $A = A(n,t)$, $B = B(n,t)$, $C = C(n,t)$ are undetermined diagonal matrices of lattice functions.

Upon setting

$$\begin{aligned} A &= \sum_{j=0}^{\infty} A_j \lambda^{-j} \equiv \sum_{j=0}^{\infty} \text{diag}(A_{j,1}(,n,t), \cdots, A_{j,N}(n,t))\lambda^{-j}, \\ B &= \sum_{j=0}^{\infty} B_j \lambda^{-j} \equiv \sum_{j=0}^{\infty} \text{diag}(B_{j,1}(n,t), \cdots, B_{j,N}(n,t))\lambda^{-j}, \\ C &= \sum_{j=0}^{\infty} C_j \lambda^{-j} \equiv \sum_{j=0}^{\infty} \text{diag}(C_{j,1}(n,t), \cdots, C_{j,N}(n,t))\lambda^{-j} \end{aligned}$$

in (2.3), we obtain the following relations

$$\begin{cases} C_j = -\Omega^r S(B_j) V \Omega^{-r}, \\ S(B_{j+1}) = S(B_j)P - \Omega^r A_j \Omega^{-r} - S(A_j), \\ C_{j+1} = \Omega^r S(A_j) V \Omega^{-r} + \Omega^r V \Omega^{-r} A_j + \Omega^r P \Omega^{-r} C_j, \\ \Omega^r S(A_{j+1}) \Omega^{-r} - A_{j+1} = \Omega^r S(C_j) \Omega^{-r} \\ \quad + \Omega^r S(A_j) P \Omega^{-r} + \Omega^r V \Omega^{-r} B_j - \Omega^r P \Omega^{-r} A_j, \quad j \geq 1. \end{cases} \tag{2.4}$$

Upon taking initial data:

$$A_0 = \frac{1}{2} I_N, \quad A_1 = B_0 = 0, \quad B_1 = -I_N,$$

we can uniquely determine the lattice functions A_j, B_j, C_j $(j \geq 0)$ by requiring

$$C_0|_{P=V=0} = C_1|_{P=V=0} = 0, \quad A_j|_{P=V=0} = B_j|_{P=V=0} = C_j|_{P=V=0} = 0, \quad j \geq 2.$$

In particular, we have

$$A_2 = \Omega^r V \Omega^{-r}, \quad B_2 = -S^{-1}(P), \tag{2.5}$$

$$C_0 = 0, \quad C_1 = \Omega^r V \Omega^{-r}, \quad C_2 = \Omega^r P V \Omega^{-r}. \tag{2.6}$$

Define

$$M^{(m)} = (\lambda^m \Gamma)_+ + \begin{pmatrix} B_{m+1} & 0 \\ 0 & 0 \end{pmatrix}, \quad m \geq 1,$$

where $(\lambda^m \Gamma)_+$ means to take polynomial part of $(\lambda^m \Gamma)$ with respect to λ. Then the discrete zero-curvature equation

$$U_{t_m} = S(M^{(m)})U - UM^{(m)}, \quad m \geq 1 \tag{2.7}$$

gives rise to the $\mathbb{Z}_N$-graded Toda lattice hierarchy

$$\begin{cases} P_{t_m} = \Omega^{-r} A_{m+1} \Omega^r - S(A_{m+1}), \\ V_{t_m} = S(B_{m+1})V - V\Omega^{-r} B_{m+1} \Omega^r, \quad m \geq 1. \end{cases} \tag{2.8}$$

In particular, the m-th lattice equation in the hierarchy takes the form

$$\begin{cases} p_{j,t_m}(n,t) = A_{m+1,j-r}(n,t) - A_{m+1,j}(n+1,t), \\ v_{j,t_m}(n,t) = v_j(n,t)(B_{m+1,j}(n+1,t) - B_{m+1,j-r}(n,t)), \quad j \in \mathbb{Z}_N, \end{cases} \tag{2.9}$$

where $j \in \mathbb{Z}_N$ indicates that $1 \leq j \leq N$ and all component j-indices are understood to be modulo N:

$$p_{j+N}(n) = p_j(n), \quad v_{j+N}(n) = v_j(n), \quad 1 \leq j \leq N. \tag{2.10}$$

It is worth noting that we adopt $p_N(n,t)$ and $v_N(n,t)$ instead of $p_0(n,t)$ and $v_0(n,t)$, respectively.

To sum up, we have the following theorem.

Theorem 2.1 *The m-th lattice equation* (2.9) *of the* $\mathbb{Z}_N$TL *hierarchy has zero-curvature representation* (2.7).

In order to establish the bi-Hamiltonian structure for each lattice equation in the $\mathbb{Z}_N$TL hierarchy, we take into account the trace identity (see [20]),

$$\begin{pmatrix} \dfrac{\delta}{\delta p_k} \\ \dfrac{\delta}{\delta v_k} \end{pmatrix} \sum_{n \in \mathbb{Z}} \mathrm{tr}\Big(\Lambda \frac{\partial U}{\partial \lambda}\Big) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} \mathrm{tr}\Big(\dfrac{\partial U}{\partial p_k} \Lambda\Big) \\ \mathrm{tr}\Big(\dfrac{\partial U}{\partial v_k} \Lambda\Big) \end{pmatrix}, \quad \Lambda = \Gamma U^{-1}, \tag{2.11}$$

where γ is an undetermined integer.

It is easy to see that

$$U^{-1}=\begin{pmatrix}\Omega^r V^{-1}(\lambda I_N-P) & -\Omega^r V^{-1}\\ \Omega^r & 0\end{pmatrix}.$$

Thus

$$\Lambda=\Gamma U^{-1}=\begin{pmatrix}A\Omega^r V^{-1}(\lambda I_N-P)+B\Omega^r & -A\Omega^r V^{-1}\\ C\Omega^r V^{-1}(\lambda I_N-P)-A\Omega^r & -C\Omega^r V^{-1}\end{pmatrix}.$$

Moreover we have

$$\mathrm{tr}\Big(\Lambda\frac{\partial U}{\partial\lambda}\Big)=-\mathrm{tr}(CV^{-1})=\mathrm{tr}S(B), \tag{2.12}$$

$$\mathrm{tr}\Big(\frac{\partial U}{\partial p_k}\Lambda\Big)=\sum_{l=0}^{\infty}\frac{1}{v_k}C_{l,k-r}(n,t)\lambda^{-l}, \tag{2.13}$$

$$\mathrm{tr}\Big(\frac{\partial U}{\partial v_k}\Lambda\Big)=\sum_{l=0}^{\infty}\frac{1}{v_k}A_{l,k-r}(n,t)\lambda^{-l}. \tag{2.14}$$

Inserting these expressions to (2.11) and making use of (2.4), we arrive at

$$\begin{pmatrix}\dfrac{\delta}{\delta p_k}\\[2mm] \dfrac{\delta}{\delta v_k}\end{pmatrix}\sum_{n\in\mathbb{Z}}\mathrm{tr}S(B_{l+1})=(\gamma-l)\begin{pmatrix}-B_{l,k}(n+1,t)\\[2mm] \dfrac{1}{v_k}A_{l,k-r}(n,t)\end{pmatrix}.$$

To fix the constant γ, we set $l=1$ in the above equation to get

$$(-1,0)^{\mathrm{T}}=(\gamma-1)(1,0)^{\mathrm{T}}.$$

Hence $\gamma=0$ and thus

$$\begin{pmatrix}\dfrac{\delta H_m}{\delta p_k}\\[2mm] \dfrac{\delta H_m}{\delta v_k}\end{pmatrix}=\begin{pmatrix}-B_{m,k}(n+1,t)\\[2mm] \dfrac{1}{v_k}A_{m,k-r}(n,t)\end{pmatrix}$$

with

$$H_m=-\frac{1}{m}\sum_{n\in\mathbb{Z}}\mathrm{tr}B_{m+1}(n,t).$$

Here we have used the identity: $\sum\limits_{n\in\mathbb{Z}}\mathrm{tr}B_{m+1}(n+1,t)=\sum\limits_{n\in\mathbb{Z}}\mathrm{tr}B_{m+1}(n,t)$.

We notice that (2.9) can be written as

$$\begin{pmatrix}P_{t_m}\\ V_{t_m}\end{pmatrix}=\mathcal{J}\begin{pmatrix}-S(B_{m+1})\\ \Omega^{-r}A_{m+1}\Omega^r V^{-1}\end{pmatrix}, \tag{2.15}$$

where

$$\mathcal{J}=\begin{pmatrix}0 & V-Ad_{\Omega^r}SV\\ -V+VAd_{\Omega^{-r}}S^{-1} & 0\end{pmatrix}.$$

To express (2.9) in a Hamiltonian form, we introduce an N-dimensional column vector $\mathbb{1}_N = (1, \cdots, 1)^{\mathrm{T}}$ and a $2N$-dimensional vector G_m denoted as

$$G_m = \begin{pmatrix} -S(B_m) \\ \Omega^{-r} A_m \Omega^r V^{-1} \end{pmatrix} \mathbb{1}_N.$$

It is easy to see that

$$\vec{u} \equiv (p_1, \cdots, p_N, v_1, \cdots, v_N)^{\mathrm{T}} = \begin{pmatrix} P \\ V \end{pmatrix} \mathbb{1}_N$$

and

$$\Omega^{-r} \mathbb{1}_N = \Omega^r \mathbb{1}_N = \mathbb{1}_N.$$

With these notation, (2.9) or (2.15) can be written as a Hamiltonian form

$$\vec{u}_{t_m} = JG_{m+1} = J \frac{\delta H_{m+1}}{\delta \vec{u}},$$

where

$$J = \begin{pmatrix} 0 & V - \Omega^r SV \\ -V + VS^{-1}\Omega^{-r} & 0 \end{pmatrix}$$

is a Hamiltonian operator.

Furthermore, by using recursion relation (2.4), we solve equation

$$KG_m = JG_{m+1} \tag{2.16}$$

to give rise to another Hamiltonian operator

$$K = \begin{pmatrix} VS^{-1}\Omega^{-r} - \Omega^r SV & VP - P\Omega^r SV \\ -VP + V\Omega^r S^{-1} P & -VS\Omega^r V + V\Omega^r S^{-1} V \end{pmatrix}. \tag{2.17}$$

We have the following theorem.

Theorem 2.2 *The* m-th *lattice equation* (2.9) *of the* $\mathbb{Z}_N$TL *hierarchy can be written as a bi-Hamiltonian form*

$$\vec{u}_{t_m} = J \frac{\delta H_{m+1}}{\delta \vec{u}} = K \frac{\delta H_m}{\delta \vec{u}}.$$

3 The $\mathbb{Z}_N$-Graded Toda Lattice and Its Periodic Reductions

3.1 The $\mathbb{Z}_N$-graded Toda lattice and the periodic Toda lattice

In the following we focus on the $\mathbb{Z}_N$-graded Toda lattice given by

$$\begin{cases} p_{j,t}(n,t) = v_j(n,t) - v_{j+r}(n+1,t), \\ v_{j,t}(n,t) = v_j(n,t)(p_{j-r}(n-1,t) - p_j(n,t)), \quad j \in \mathbb{Z}_N, \ n \in \mathbb{Z}, \end{cases} \tag{3.1}$$

where $t = t_1$. This is the first non-trivial lattice equation in the $\mathbb{Z}_N$TL hierarchy.

Similar to the Flaschka-Manakov variables, we define

$$p_j(n,t) = -\frac{\mathrm{d}q_j(n,t)}{\mathrm{d}t}, \quad v_j(n,t) = \mathrm{e}^{q_j(n,t) - q_{j-r}(n-1,t)}, \quad 1 \leq j \leq N.$$

Then the $\mathbb{Z}_N$TL (3.1) can be written as a Newtonian form:

$$\frac{\mathrm{d}^2 q_j(n,t)}{\mathrm{d}t^2} = \mathrm{e}^{q_{j+r}(n+1,t)-q_j(n,t)} - \mathrm{e}^{q_j(n,t)-q_{j-r}(n-1,t)}, \quad j \in \mathbb{Z}_N,\ n \in \mathbb{Z}. \tag{3.2}$$

(3.2) shows that the $\mathbb{Z}_N$TL is an infinite chain that describes the motion of particles with exponential interaction. Specifically, the dynamics of a particle located at the (n, j) position are influenced solely by particles at the $(n-1, j-r)$ position and $(n+1, j+r)$ position.

When $N = 1$ (which corresponds to $r = 1$ and $\Omega = 1$), the Toda lattice (1.2) is recovered. Under the τ-periodic boundary condition given by

$$a_{\tau+s} = a_s, \quad b_{\tau+s} = b_s, \quad 1 \le s \le \tau,$$

where τ is a positive integer and $a_s = p(s,t),\ b_s = v(s,t)$, we obtain the τ-periodic Toda lattice

$$\begin{cases} \dot{a}_s = b_s - b_{s+1}, \\ \dot{b}_s = b_s(a_{s-1} - a_s), \quad s = 1, 2, \cdots, \tau, \end{cases} \tag{3.3}$$

where $a_0 = a_\tau,\ b_{\tau+1} = b_1$.

This system can be expressed in a Hamiltonian form as follows:

$$\dot{a}_s = \{a_s, H\}, \quad \dot{b}_s = \{b_s, H\},$$

where the Hamiltonian is given by

$$H = \sum_{s=1}^{\tau} \left(\frac{1}{2} a_s^2 + b_s \right)$$

and Poisson bracket is defined as

$$\{a_i, a_j\} = \{b_i, b_j\} = 0, \quad \{a_i, b_j\} = b_i \delta_{ij} - b_{i+1} \delta_{i+1,j}, \quad 1 \le i, j \le \tau.$$

Here δ_{ij} is Kronecker delta which is 1 if $i = j$, and 0 otherwise. It is well-known that τ-periodic Toda lattice (3.3) is a completely integrable Hamiltonian system in the Liouville sense (see [22]).

3.2 The periodic $\mathbb{Z}_N$-graded Toda lattice

The τ-periodic boundary condition of the $\mathbb{Z}_N$TL (3.1) is defined as

$$p_j(s+\tau, t) = p_j(s,t), \quad v_j(s+\tau, t) = v_j(s,t), \quad 1 \le j \le N \tag{3.4}$$

for any integer s $(1 \le s \le \tau)$.

As a result, the τ-periodic $\mathbb{Z}_N$TL (3.5) can be expressed as follows:

$$\begin{cases} \dot{p}_j(s,t) = v_j(s,t) - v_{j+r}(s+1,t), \\ \dot{v}_j(s,t) = v_j(s,t)(p_{j-r}(s-1,t) - p_j(s,t)), \quad j \in \mathbb{Z}_N,\ s \in \mathbb{Z}_\tau. \end{cases} \tag{3.5}$$

It is evident that the τ-periodic $\mathbb{Z}_N$TL (3.1) is a system of $2N\tau$ first-order differential equations. This system can be viewed as the Toda lattice defined on the torus $\mathbb{Z}_N \times \mathbb{Z}\tau$.

In the following, we perform a detailed analysis of the τ-periodic $\mathbb{Z}_N$TL (3.5) in three different cases. In this context, the notation (a, b) stands for the greatest common divisor of the integers a and b.

Case I $(N, \tau) = 1$ and $(N, r) = 1$.

We first establish the following lemma.

Lemma 3.1 *Let N and $r\,(1 \le r < N)$ be coprime numbers, satisfying $(N, r) = 1$. Then, in the modulo N sense, the sequence*

$$kr+1, \tau r+kr+1, \cdots, (j-1)\tau r+kr+1, \cdots, (N-1)\tau r+kr+1$$

is a permutation of the sequence: $1, 2, \cdots, N$, *where k $(1 \le k \le \tau-1)$ is a given number.*

Proof It suffices to show that $((i-1)\tau r+kr+1) \pmod N$ is not equal to $((j-1)\tau r+kr+1) \pmod N$ for $1 \le i < j \le N$. Let $N_i = ((i-1)\tau r+kr+1) \pmod N$. Then, $N_j - N_i = (j-i)\tau r \pmod N$. Therefore, $N_j - N_i = 0$ implies that $N \mid (j-i)\tau r$. Since $(N, r) = 1$ and $(N, \tau) = 1$, $N_j - N_i = 0$ is equivalent to $N \mid (j-i)$. However, $1 \le i < j \le N$ implies that $(j-i) < N$, and thus $N \nmid (j-i)$. Consequently, $N_i \ne N_j$ if $i < j$. The lemma follows.

We state the theorem.

Theorem 3.1 *Assuming that τ, r and N are coprime, satisfying $(N, \tau) = 1$ and $(N, r) = 1$, the τ-periodic $\mathbb{Z}_N$TL (3.5) is equivalent to an $(N\tau)$-periodic Toda lattice and thus a completely integrable Hamiltonian system in the Liouville sense.*

Proof To prove the theorem, we rearrange the variables $\{p_j(s), v_j(s) \mid 1 \le j \le N, 1 \le s \le \tau\}$ into $\{a_j = p_{(j-1)r+1}(j,t), b_j = v_{(j-1)r+1}(j,t) \mid 1 \le j \le N\tau\}$ such that a_j's and b_j's form an $(N\tau)$-periodic Toda lattice.

Using the τ-periodic condition, we can establish the following identity:

$$\begin{pmatrix} p_1(1) & p_{r+1}(2) & \cdots & p_{(\tau-1)r+1}(\tau) \\ p_{\tau r+1}(\tau+1) & p_{(\tau+1)r+1}(\tau+2) & \cdots & p_{(2\tau-1)r+1}(2\tau) \\ \cdots & \cdots & \cdots & \cdots \\ p_{(N-1)\tau r+1}((N-1)\tau+1) & p_{(N-1)\tau r+2}((N-1)\tau+2) & \cdots & p_{(N\tau-1)r+1}(N\tau) \end{pmatrix}$$
$$= \begin{pmatrix} p_1(1) & p_{r+1}(2) & \cdots & p_{(\tau-1)r+1}(\tau) \\ p_{\tau r+1}(1) & p_{(\tau+1)r+1}(2) & \cdots & p_{(2\tau-1)r+1}(\tau) \\ \cdots & \cdots & \cdots & \cdots \\ p_{(N-1)\tau r+1}(1) & p_{(N-1)\tau r+r+1}(2) & \cdots & p_{(N\tau-1)r+1}(\tau) \end{pmatrix}.$$

The Lemma 3.1 implies that the entries in the k-th column of the matrix are a permutation of the variables: $p_1(k), p_2(k), \cdots, p_N(k)$. Therefore, we obtain the following equation:

$$\{p_j(s), 1 \le j \le N, 1 \le s \le \tau\} = \{p_{(j-1)r+1}(j), 1 \le j \le \tau N\}. \tag{3.6}$$

Similarly, we have

$$\{v_j(s), 1 \le j \le N, 1 \le s \le \tau\} = \{v_{(j-1)r+1}(j), 1 \le j \le \tau N\}. \tag{3.7}$$

Define the variables as follows:

$$a_j = p_{(j-1)r+1}(j,t), \quad b_j = v_{(j-1)r+1}(j,t), \quad 1 \le j \le \tau N. \tag{3.8}$$

By directly checking we find that for $1 \le j \le (N\tau)$, the dynamics of a_j and b_j are

$$\begin{cases} \dot{a}_j = b_j - b_{j+1}, \\ \dot{b}_j = b_j(a_{j-1} - a_j), \quad j \in \mathbb{Z}_{N\tau}. \end{cases}$$

These equations show that the τ-periodic $\mathbb{Z}_N$TL is indeed equivalent to an $(N\tau)$-periodic Toda lattice. This completes the proof of the theorem.

Case II $(N,\tau)=d$ and $(N,r)=1$.

Let d be the greatest common divisor of N and τ, satisfying $1<d\le N$ and $1<d\le\tau$. We can further define positive integers α and β as follows:

$$N=\alpha d,\quad \tau=\beta d.$$

It follows that $(\alpha,\beta)=(\alpha,r)=(d,r)=1$ from $(N,\tau)=d$ and $(N,r)=1$.

Lemma 3.2 *Let $(N,\tau)=d$ and $(N,r)=1$. In the modulo N sense, the sequence*

$$kr+s, \tau r+kr+s, \cdots, (j-1)\tau r+kr+s, \cdots, (N-1)\tau r+kr+s$$

consists of α distinct numbers, where s $(1\le s\le N)$ and k $(1\le k\le \tau-1)$ are two given numbers.

Proof Let $m_j=((j-1)\tau r+kr+s)\pmod N$. Then, $m_j-m_i=(j-i)\tau r\pmod N$. Therefore, if $m_j-m_i=0$, it implies that $N\mid(j-i)\tau r$. Due to $(N,r)=1$ and $(N,\tau)=d$, we have $\alpha\mid(j-i)$. However, this contradicts the condition $1\le i<j\le\alpha$. Thus, the lemma is proven.

Theorem 3.2 *Assuming that $(N,\tau)=d$ and $(\alpha,r)=1$, the τ-periodic $\mathbb{Z}_N$TL (3.5) is a set of d independent βN-periodic Toda lattices.*

Proof By using the periodic boundary conditions we can observe that the variable group

$$\{p_{(j-1)r+1}(j), v_{(j-1)r+1}(j)\mid 1\le j\le N\tau\}$$

only consists of $2\beta N$ variables: $p_1(1),\cdots,p_{(\beta N-1)r+1)}(\beta N)$, instead of $2\tau N$ variables. Since $\tau N=d\beta N$, we take d groups variables,

$$\{p_{(j-1)r+s}(j), v_{(j-1)r+s}(j)\mid 1\le j\le \beta N\},\quad s=1,2,\cdots,d. \tag{3.9}$$

The s-th $(1\le s\le d)$ variable group can be expressed as

$$\{p_s(1),p_{r+s}(2),\cdots,p_{(\beta N-1)r+s}(\beta N), v_s(1), v_{r+s}(2),\cdots,v_{(\beta N-1)r+s}(\beta N)\}. \tag{3.10}$$

Again, using the τ-periodic condition, we can establish the following identity:

$$\begin{aligned}&\begin{pmatrix} p_s(1) & p_{r+s}(2) & \cdots & p_{(\tau-1)r+s}(\tau)\\ p_{\tau r+s}(\tau+1) & p_{(\tau+1)r+s}(\tau+2) & \cdots & p_{(2\tau-1)r+s}(2\tau)\\ \cdots & \cdots & \cdots & \cdots\\ p_{(\alpha-1)\tau r+s}((\alpha-1)\tau+1) & p_{(\alpha-1)\tau r+r+s}((\alpha-1)\tau+2) & \cdots & p_{(\tau\alpha-1)r+s}(\alpha\tau)\end{pmatrix}\\ =&\begin{pmatrix} p_s(1) & p_{r+s}(2) & \cdots & p_{(\tau-1)r+s}(\tau)\\ p_{\tau r+s}(1) & p_{(\tau+1)r+s}(2) & \cdots & p_{(2\tau-1)r+s}(\tau)\\ \cdots & \cdots & \cdots & \cdots\\ p_{(\alpha-1)\tau r+s}(1) & p_{(\alpha-1)\tau r+r+s}(2) & \cdots & p_{(\tau\alpha-1)r+s}(\tau)\end{pmatrix}.\end{aligned} \tag{3.11}$$

Based on Lemma 3.2, we can conclude that the above matrix, and consequently, the s-th variables group, contains (βN) distinct variables $p_j(s)$. Moreover, from the first column of (3.11), it is clear that the d groups contain all the elements $p_1(k), 1 \le k \le \tau$, as s ranges from 1 to d. This pattern holds for all $p_j(k), 1 \le j \le N, 1 \le k \le \tau$.

Furthermore, by setting

$$a_j^{(s)} = p_{(j-1)r+s}(j), \quad b_j^{(s)} = v_{(j-1)r+s}(j), \quad 1 \le j \le \beta N,$$

we obtain the following equations:

$$\begin{cases} \dot{a}_j^{(s)} = b_j^{(s)} - b_{j+1}^{(s)}, \\ \dot{b}_j^{(s)} = b_j(a_{j-1}^{(s)} - a_j^{(s)}), \quad j \in \mathbb{Z}_{\beta N}. \end{cases}$$

These equations show that the motions of each group of variables form a closed system, precisely the βN-periodic Toda lattice. Consequently, the whole τ-periodic $\mathbb{Z}_N$TL (3.5) consists of d groups of βN-periodic Toda lattices. In other words, the τ-periodic $\mathbb{Z}_N$TL (3.5) is essentially a collection of d independent βN-periodic Toda lattices. This completes the proof of the theorem.

Case III $(N, \tau) = d$ and $(\alpha, r) = \theta$.

In this general case, we set

$$N = \alpha d, \quad \tau = \beta d, \quad \alpha = \xi\theta, \quad r = \rho\theta, \quad d > 1,\ \theta > 1,$$

where $\alpha, \beta, d, \xi, \rho$ and θ are positive integers. We have the following relationships:

$$(\alpha, \beta) = 1, \quad (\xi, \rho) = 1.$$

Additionally, we set $g = \frac{\beta N}{\theta}$. It is evident that $g = \frac{\beta N}{\theta} = \xi\tau$.

The following lemma holds.

Lemma 3.3 *Let $(N, \tau) = d$ and $(\alpha, r) = 1$. Then, in the modulo N sense, the sequence*

$$kr + s, \tau r + kr + s, \cdots, (j-1)\tau r + kr + s, \cdots, (\xi - 1)\tau r + kr + s$$

consists of ξ distinct numbers, where s $(1 \le s \le N)$ and k $(1 \le k \le \tau - 1)$ are two given numbers.

Proof Let $s_j = ((j-1)\tau r + kr + s) \pmod N$. Then, $s_j - s_i = (j-i)\tau r \pmod N$. Therefore, if $s_j - s_i = 0$, it implies that $N \mid (j-i)\tau r$, namely, $\alpha d \mid (j-i)\beta d r$. Since $(\alpha, \beta) = 1$, we have $\alpha \mid (j-i)r$, namely, $\xi \mid (j-i)\rho$. Finally, since $(\xi, \rho) = 1$, we conclude that $\xi \mid (j-i)$. However, the condition $1 \le i < j \le \xi$ conflicts with $\xi \mid (j-i)$. Hence, we have completed the proof of the lemma.

We have the following theorem.

Theorem 3.3 *Assuming that $(N, \tau) = d$, $(\alpha, r) = \theta$ and $d > 1, \theta > 1$, the τ-periodic $\mathbb{Z}_N$TL (3.5) is a set of θd independent g-periodic Toda lattices.*

Proof In this case, for a given s the s-th variable group (3.10) contains only $2g$ variables, instead of $2\beta N$ variables. Therefore, we need to consider θd groups of variables with the pattern $p_{(j-1)r+s}(j)$ and $v_{(j-1)r+s}(j)$ as follows

$$\{p_s(1), p_{r+s}(2), \cdots, p_{(g-1)r+s}(g), v_s(1), v_{r+s}(2), \cdots, v_{(g-1)r+s}(g)\}, \quad 1 \le s \le \theta d. \qquad (3.12)$$

Once again, utilizing the τ-periodic condition, we can establish the following identity:

$$\begin{pmatrix} p_s(1) & p_{r+s}(2) & \cdots & p_{(\tau-1)r+s}(\tau) \\ p_{\tau r+s}(\tau+1) & p_{(\tau+1)r+s}(\tau+2) & \cdots & p_{(2\tau-1)r+s}(2\tau) \\ \cdots & \cdots & \cdots & \cdots \\ p_{(\xi-1)\tau r+s}((\xi-1)\tau+1) & p_{(\xi-1)\tau r+r+s}((\xi-1)\tau+2) & \cdots & p_{(\xi\tau-1)r+s}(\xi\tau) \end{pmatrix}$$

$$= \begin{pmatrix} p_s(1) & p_{r+s}(2) & \cdots & p_{(\tau-1)r+s}(\tau) \\ p_{\tau r+s}(1) & p_{(\tau+1)r+s}(2) & \cdots & p_{(2\tau-1)r+s}(\tau) \\ \cdots & \cdots & \cdots & \cdots \\ p_{(\xi-1)\tau r+s}(1) & p_{(\xi-1)\tau r+r+s}(2) & \cdots & p_{(\tau\xi-1)r+s}(\tau) \end{pmatrix}. \tag{3.13}$$

Analogous to the analysis conducted for Case II, we can infer that the dynamics of each group of variables in Case III is equivalent to that of a g-periodic Toda lattice. Therefore, the τ-periodic $\mathbb{Z}_N$TL (3.5) can be decoupled to θd independent g-periodic Toda lattices. Thus, we have completed the proof of Theorem 3.3.

Furthermore, based on the fact that the union of the independent integrable Hamiltonian systems remains integrable, we conclude that all the τ-periodic $\mathbb{Z}_N$ TL systems in the above three cases are integrable systems. In summary, we have the following theorem.

Theorem 3.4 *The periodic* $\mathbb{Z}_N$TL *system* (3.5) *is either a periodic Toda lattice or a set of independent periodic-Toda lattices with the same periodicity. Hence, the periodic* $\mathbb{Z}_N$TL *system* (3.5) *is a completely integrable Hamiltonian system in the Liouville sense.*

Remark 3.1 We have constructed the $\mathbb{Z}_N$ Toda lattices and established their bi-Hamiltonian structure. When considering the infinite boundary condition, the $\mathbb{Z}_N$-graded lattice equations can be viewed as $(1+2)$-dimensional integrable lattices. These lattices involve one continuous variable, denoted as $t \in \mathbb{R}$, one infinite discrete variable, denoted as $n \in \mathbb{Z}$, and one N-periodic variable, denoted as j. Under periodic boundary conditions, it may seem that these lattice equations are bi-periodic. However, we have demonstrated that the $\mathbb{Z}_N$TL is either a periodic-Toda lattice or a set of independent periodic-Toda lattices with the same periodicity. This work is a continuation of our prior research as presented in [23–24].

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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