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The Asymptotic Stability of Dirac Solitons in the Massive Thirring Model*

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Abstract In this paper, the authors employ the $\overline{\partial}$-steepest descent method and Bäcklund transformation to investigate the asymptotic stability of Dirac solitons in the context of the massive Thirring model (MTM for short) system. They formulate the solution to the Cauchy problem for the MTM system in terms of the solution to a Riemann-Hilbert (RH for short) problem. This RH problem is decomposed into two components: A pure radiation solution and a soliton solution. As a direct outcome of this decomposition, they establish the asymptotic stability of Dirac solitons within the MTM system.

Keywords Massive Thirring model, Riemann-Hilbert problem, $\overline{\partial}$-Steepest descent method, Bäcklund transformation, Asymptotic stability
2020 MR Subject Classification 17B40, 17B50

1 Introduction

This paper is devoted to the asymptotic stability of Dirac solitons in the massive Thirring model (MTM for short) system

$$\begin{cases} \mathrm{i}(u_t + u_x) + v + u|v|^2 = 0, \\ \mathrm{i}(v_t - v_x) + u + v|u|^2 = 0. \end{cases} \tag{1.1}$$

Given initial data $(u(0,x), v(0,x)) = (u_0, v_0) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. Here, $H^2(\mathbb{R})$ is the standard Sobolev space defined as

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f', f'' \in L^2(\mathbb{R})\}.$$

Additionally, $H^{1,1}(\mathbb{R})$ is defined as

$$H^{1,1}(\mathbb{R}) = \{f \in L^{2,1}(\mathbb{R}) : f' \in L^{2,1}(\mathbb{R})\}$$

with $\|f(x)\|^p_{L^{p,s}(\mathbb{R})} = \int_{\mathbb{R}} (1+x^2)^{\frac{ps}{2}} |f(x)|^p \mathrm{d}x$. The MTM system, developed by Thirring [1], describes a nonlinear Dirac equation that is invariant under relativistic transformations in one-dimensional space. The Gross-Neveu model (see [2]), also known as the massive Soler

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model when formulated in three-dimensional space (see [3]), provides another example of a relativistically invariant one-dimensional Dirac equation.

Recent research has focused on studying various mathematical properties of the MTM system. The Lax pair and new exact solutions for the MTM system were constructed (see [4]). It is known that the Cauchy problem for the MTM system is locally well-posed in the function space $H^s(\mathbb{R})$ for $s > 0$ (see [5]). Furthermore, it has been established that the Cauchy problem is globally well-posed for $s > \frac{1}{2}$ (see [6]). Recent works have also demonstrated the global well-posedness of the MTM system in the function space $L^2(\mathbb{R})$ (see [7–8]). The soliton solutions of the MTM system were obtained using the inverse scattering method (IST for short) (see [9]). Moreover, research has also addressed the $\overline{\partial}$-problem related to the class of rapidly decaying potentials of the MTM system (see [10]). Recently, the $\overline{\partial}$-steepest descent method has been used to study the long time behavior of the MTM system for the weighted Sobolev initial data $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ (see [11]) and $H^{2,2}(\mathbb{R})$ (see [12]), respectively. The L^2-orbital stability of solitons for the MTM system was established using the Bäcklund transformation (see [13]).

In this study, we establish the asymptotic stability of Dirac solitons in the MTM system (1.1) by employing the $\overline{\partial}$-steepest descent method and the Bäcklund transformation. The explicit one-soliton solutions are presented as [13],

$$u_\lambda(t,x) = \mathrm{i}\gamma^{-1} \sin\alpha \operatorname{sech}\Big(f(x-vt) - \mathrm{i}\frac{\alpha}{2}\Big)\mathrm{e}^{-\mathrm{i}\beta(t+vx)}, \tag{1.2}$$

$$v_\lambda(t,x) = -\mathrm{i}\gamma \sin\alpha \operatorname{sech}\Big(f(x-vt) + \mathrm{i}\frac{\alpha}{2}\Big)\mathrm{e}^{-\mathrm{i}\beta(t+vx)}, \tag{1.3}$$

the discrete spectral $\lambda = \gamma \mathrm{e}^{\frac{\mathrm{i}\alpha}{2}}$ is associated with

$$f = \frac{\gamma^2+\gamma^{-2}}{2}\sin\alpha, \quad \beta = \frac{\gamma^2+\gamma^{-2}}{2}\cos\alpha, \quad v = \frac{\gamma^{-2}-\gamma^2}{\gamma^{-2}+\gamma^2}.$$

We now state our main result as follows.

Theorem 1.1 *Consider the Dirac solitons $(u_{\lambda_0}, v_{\lambda_0})$ for the MTM system given by* (1.2)–(1.3)*. Then, there exist positive constants $\epsilon_0(\lambda_0), C(\lambda_0), T(\lambda_0)$ such that if the initial data $(u_0, v_0) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ satisfy*

$$\varepsilon = (\|u_0 - u_{\lambda_0}(0,x)\| + \|v_0 - v_{\lambda_0}(0,x)\|)_{H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R})} \le \varepsilon_0(\lambda_0), \tag{1.4}$$

there exists λ such that

$$|\lambda - \lambda_0| \le C(\lambda_0)\varepsilon, \tag{1.5}$$

and for all $t \ge T(\lambda_0)$ and $\left|\frac{x}{t}\right| < 1$,

$$\|u(t,x) - u_\lambda(t,x)\|_{L^\infty(\mathbb{R})} + \|v(t,x) - v_\lambda(t,x)\|_{L^\infty(\mathbb{R})} < C(\lambda_0)\varepsilon t^{-\frac{1}{2}}. \tag{1.6}$$

The paper is structured as follows. In Section 2, we discuss the IST for the spectral problem (2.1), and also provide an equivalent formulation of the RH problem associated with it. In Section 3, we analyze the asymptotic for the pure radiation solution using the $\overline{\partial}$-steepest descent method. In Section 4, we prove the asymptotic stability of solitons, as formulated in Theorem 1.1, utilizing the Bäcklund transformation.

2 Inverse Scattering Transform

In this section, we consider the IST for the MTM system based on its Lax pair.

2.1 Analyticity and symmetry of Jost functions

The MTM system can be integrated associated with Lax pair:

$$\psi_x - \frac{\mathrm{i}}{4}\Big(\lambda^2 - \frac{1}{\lambda^2}\Big)\sigma_3\psi = L\psi, \tag{2.1}$$

$$\psi_t - \frac{\mathrm{i}}{4}\Big(\lambda^2 + \frac{1}{\lambda^2}\Big)\sigma_3\psi = A\psi, \tag{2.2}$$

where

$$L = \frac{\mathrm{i}}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{\mathrm{i}\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{\mathrm{i}}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix},$$

$$A = -\frac{\mathrm{i}}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{\mathrm{i}\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{\mathrm{i}}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix},$$

and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be shown that (2.1) admits

$$\psi^{\pm}(\lambda) \sim \mathrm{e}^{\frac{\mathrm{i}x\sigma_3(\lambda^2-\lambda^{-2})}{4}}, \quad x \to \pm\infty.$$

Further making the transformation

$$\varphi^{\pm}(\lambda) = \psi^{\pm}(\lambda)\mathrm{e}^{-\frac{\mathrm{i}x\sigma_3(\lambda^2-\lambda^{-2})}{4}}, \tag{2.3}$$

then $\varphi^{\pm}(\lambda)$ solves the spectral problem

$$\varphi^{\pm}_x(\lambda) - \frac{\mathrm{i}}{4}\Big(\lambda^2 - \frac{1}{\lambda^2}\Big)[\sigma_3, \varphi^{\pm}(\lambda)] = L\varphi^{\pm}(\lambda) \tag{2.4}$$

with asymptotic condition

$$\varphi^{\pm}(\lambda) \sim I, \quad x \to \pm\infty.$$

It follows from (2.4), we acquire

$$\varphi^{-}(\lambda) = I + \int_{-\infty}^{x} \mathrm{e}^{\frac{\mathrm{i}(x-y)\widehat{\sigma}_3(\lambda^2-\lambda^{-2})}{4}} L\varphi^{-}(\lambda)\mathrm{d}y, \tag{2.5}$$

$$\varphi^{+}(\lambda) = I - \int_{x}^{+\infty} \mathrm{e}^{\frac{\mathrm{i}(x-y)\widehat{\sigma}_3(\lambda^2-\lambda^{-2})}{4}} L\varphi^{+}(\lambda)\mathrm{d}y, \tag{2.6}$$

where $\mathrm{e}^{\widehat{\sigma}_3}L := \mathrm{e}^{\sigma_3}L\mathrm{e}^{-\sigma_3}$ for a 2×2 matrix L. Denote $\varphi^{\pm}(\lambda) = [\varphi^{\pm}_1(\lambda), \varphi^{\pm}_2(\lambda)]$, where $\varphi^{\pm}_j(\lambda)$ represents the jth column of $\varphi^{\pm}(\lambda)$.

Proposition 2.1 *Let* $(u_0, v_0) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. *Then, the Volterra integral equations* (2.5)–(2.6) *admit unique solutions* $\varphi^+(\lambda)$ *and* $\varphi^-(\lambda)$, *respectively. Moreover, the functions* $\varphi_1^+(\lambda)$ *and* $\varphi_2^-(\lambda)$ *can be analytically continued into the domain* $\Omega^+ = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda^2 > 0\}$, *while* $\varphi_1^-(\lambda)$ *and* $\varphi_2^+(\lambda)$ *can be analytically continued into the domain* $\Omega^- = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda^2 < 0\}$. *Here,* $\mathrm{Im}\, z$ *denotes the imaginary part of the complex number* z, *while* $\mathrm{Re}\, z$ *represents its real component* (see [14]).

By symmetry of (2.1), we have

$$\varphi^{\pm}(\lambda) = \sigma_3 \varphi^{\pm}(-\lambda)\sigma_3, \quad \varphi^{\pm}(\lambda) = \sigma_2 \overline{\varphi^{\pm}(\overline{\lambda})}\sigma_2. \tag{2.7}$$

Moreover there exists a continuous matrix function $S(\lambda)$ such that

$$\varphi^-(\lambda) = \varphi^+(\lambda) \mathrm{e}^{\frac{\mathrm{i}x\widehat{\sigma}_3(\lambda^2-\lambda^{-2})}{4}} S(\lambda), \quad \lambda \in \Sigma, \tag{2.8}$$

with $\Sigma = \{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Im}\,\lambda^2 = 0\}$ and

$$S(\lambda) = \begin{pmatrix} \alpha(\lambda) & -\overline{\beta(\overline{\lambda})} \\ \beta(\lambda) & \overline{\alpha(\overline{\lambda})} \end{pmatrix}, \tag{2.9}$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ can be expressed by Wronskian determinant

$$\alpha(\lambda) = Wr(\varphi_1^-(\lambda), \varphi_2^+(\lambda)), \tag{2.10}$$

$$\beta(\lambda) = \mathrm{e}^{\frac{\mathrm{i}x(\lambda^2-\lambda^{-2})}{2}} Wr(\varphi_1^+(\lambda), \varphi_1^-(\lambda)), \tag{2.11}$$

we have the following symmetry relations:

$$\alpha(-\lambda) = \alpha(\lambda), \quad \beta(-\lambda) = -\beta(\lambda).$$

Additionally, it is established from (2.10) that the function $\alpha(\lambda)$ is analytic in $\mathbb{C}^-$. Let $\lambda_k, k = 1, \cdots, N$ be the zeros of $\alpha(\lambda)$, it follows that there exists a constant $\widetilde{C}_k$ such that

$$\varphi_1^-(\lambda_k) = \widetilde{C}_k \varphi_2^+(\lambda_k).$$

Subsequently, we define norming constant $C_k = \frac{\widetilde{C}_k}{\alpha'(\lambda_k)}$.

Assumption 2.1 There exists an open dense set $\mathcal{G} \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ such that, for $(u_0, v_0) \in \mathcal{G}$, the function $\alpha(\lambda)$ has no zeros on $\mathbb{R} \cup \mathrm{i}\mathbb{R}$. We denote by $\mathcal{G}_n$ the open subset of $\mathcal{G}$ satisfying N-solitons are contained in $\mathcal{G}_n$ and it turns out also neighborhoods of N-solitons belong to $\mathcal{G}_n$. In particular, the number of zeros does not vary in time and the set $\mathcal{G}$ are invariant under the MTM system (see [15–16]).

2.2 Jost functions for $\lambda \to 0$

Define the transformation matrix by [11],

$$\Psi(\lambda) = T(u; \lambda)\psi(\lambda), \quad \lambda \neq 0, \tag{2.12}$$

where

$$T(u;\lambda) = \begin{pmatrix} 1 & 0 \\ u(x) & \lambda^{-1} \end{pmatrix}. \tag{2.13}$$

Setting $z = \lambda^2$, then $\Psi(z)$ satisfies

$$\Psi_x(z) - \frac{\mathrm{i}}{4}\Big(z - \frac{1}{z}\Big)\sigma_3\Psi(z) = \mathcal{L}\Psi(z) \tag{2.14}$$

with

$$\mathcal{L} = Q_1(u,v) + zQ_2(u,v), \tag{2.15}$$

where

$$Q_1(u,v) = \begin{pmatrix} -\frac{\mathrm{i}}{4}(|u|^2+|v|^2) & \frac{\mathrm{i}}{2}\overline{u} \\ u_x - \frac{\mathrm{i}}{2}u|v|^2 - \frac{\mathrm{i}}{2}v & \frac{\mathrm{i}}{4}(|u|^2+|v|^2) \end{pmatrix}$$

and

$$Q_2(u,v) = \frac{\mathrm{i}}{2}\begin{pmatrix} u\overline{v} & -\overline{v} \\ u + u^2\overline{v} & -u\overline{v} \end{pmatrix}.$$

Consider the Jost function $\Psi(z)$ satisfying

$$\Psi^{\pm}(z) \sim \mathrm{e}^{\frac{\mathrm{i}x\sigma_3(z-z^{-1})}{4}}, \quad x \to \pm\infty.$$

Letting

$$M^{\pm}(z) = \Psi^{\pm}(z)\mathrm{e}^{-\frac{\mathrm{i}x\sigma_3(z-z^{-1})}{4}},$$

then we have

$$\lim_{x\to\pm\infty} M_1^{\pm}(z) = \lim_{x\to\pm\infty} T(u;\lambda)\varphi_1^{\pm}(\lambda) = e_1,$$
$$\lim_{x\to\pm\infty} M_2^{\pm}(z) = \lim_{x\to\pm\infty} \lambda T(u;\lambda)\varphi_2^{\pm}(\lambda) = e_2,$$

where $M_j^{\pm}(z)$ represents the j-th column of $M^{\pm}(z)$ and e_j denotes the j-th column of the identity matrix, and they satisfy the Volterra integral equations

$$M_1^{\pm}(z) = e_1 + \int_{\pm\infty}^{x} \mathrm{diag}(0, \mathrm{e}^{-\frac{\mathrm{i}}{2}(z-z^{-1})(x-y)})\mathcal{L}M_1^{\pm}(z)\mathrm{d}y, \tag{2.16}$$

$$M_2^{\pm}(z) = e_2 + \int_{\pm\infty}^{x} \mathrm{diag}(\mathrm{e}^{\frac{\mathrm{i}}{2}(z-z^{-1})(x-y)}, 0)\mathcal{L}M_2^{\pm}(z)\mathrm{d}y. \tag{2.17}$$

There exists $S_1(z)$ such that

$$M^-(z) = M^+(z)S_1(z), \quad z \in \mathbb{R}\setminus\{0\}, \tag{2.18}$$

where

$$S_1(z)=\mathrm{e}^{\frac{\mathrm{i}x\widehat{\sigma}_3(z-z^{-1})}{4}}\begin{pmatrix} a(z) & -\overline{b(\overline{z})}\\ zb(z) & \overline{a(\overline{z})}\end{pmatrix}$$

and

$$a(z)=Wr(M_1^-(z),M_2^+(z)), \tag{2.19}$$

$$\overline{b(z)}=\mathrm{e}^{-\frac{\mathrm{i}x(z-z^{-1})}{2}}Wr(M_2^+(z),M_2^-(z)). \tag{2.20}$$

2.3 Jost functions for $|\lambda|\to\infty$

Define the transformation by [11],

$$\widehat{\Psi}(z)=\widehat{T}(v;\lambda)\psi(\lambda), \tag{2.21}$$

where $z=\lambda^2$ and

$$\widehat{T}(v;\lambda)=\begin{pmatrix} 1 & 0\\ v(x) & \lambda\end{pmatrix}. \tag{2.22}$$

Then $\widehat{\Psi}(z)$ satisfies

$$\widehat{\Psi}_x(z)-\frac{\mathrm{i}}{4}\Big(z-\frac{1}{z}\Big)\sigma_3\widehat{\Psi}(z)=\widehat{\mathcal{L}}\Psi(z), \tag{2.23}$$

where

$$\widehat{\mathcal{L}}=\widehat{Q}_1(u,v)+\frac{1}{z}\widehat{Q}_2(u,v) \tag{2.24}$$

with

$$\widehat{Q}_1(u,v)=\begin{pmatrix} \frac{\mathrm{i}}{4}(|u|^2+|v|^2) & -\frac{\mathrm{i}}{2}\overline{v}\\ v_x+\frac{\mathrm{i}}{2}|u|^2v+\frac{\mathrm{i}}{2}u & -\frac{\mathrm{i}}{4}(|u|^2+|v|^2)\end{pmatrix},$$

$$\widehat{Q}_2(u,v)=-\frac{\mathrm{i}}{2}\begin{pmatrix} \overline{u}v & -\overline{u}\\ v+\overline{u}v^2 & -\overline{u}v\end{pmatrix}.$$

Consider the Jost solution of (2.23) with asymptotics

$$\widehat{\Psi}^{\pm}(z)\sim\mathrm{e}^{\frac{\mathrm{i}x\sigma_3(z-z^{-1})}{4}},\quad x\to\pm\infty,$$

making the transformation

$$N^{\pm}(z)=\widehat{\Psi}^{\pm}(z)\mathrm{e}^{-\frac{\mathrm{i}x\sigma_3(z-z^{-1})}{4}},$$

then we have

$$\lim_{x\to\pm\infty}N_1^{\pm}(z)=\lim_{x\to\pm\infty}\widehat{T}(u;\lambda)\varphi_1^{\pm}(\lambda)=e_1,$$

$$\lim_{x\to\pm\infty} N_2^{\pm}(z) = \lim_{x\to\pm\infty} \lambda^{-1}\widehat{T}(u;\lambda)\varphi_2^{\pm}(\lambda) = e_2,$$

where $N_j^{\pm}(z)$ represents the j-th column of $N^{\pm}(z)$, and they satisfy the Volterra integral equations

$$N_1^{\pm}(z) = e_1 + \int_{\pm\infty}^{x} \mathrm{diag}(0, \mathrm{e}^{-\frac{\mathrm{i}}{2}(z-z^{-1})(x-y)})\widehat{\mathcal{L}}N_1^{\pm}(z)\mathrm{d}y, \tag{2.25}$$

$$N_2^{\pm}(z) = e_2 + \int_{\pm\infty}^{x} \mathrm{diag}(\mathrm{e}^{\frac{\mathrm{i}}{2}(z-z^{-1})(x-y)}, 0)\widehat{\mathcal{L}}N_2^{\pm}(z)\mathrm{d}y. \tag{2.26}$$

Proposition 2.2 *The functions $M_1^{\pm}(z)$ and $M_2^{\mp}(z)$ are analytic within the upper/lower half-planes $\mathbb{C}^{\pm}$ for z, and they are also continuous on $\mathbb{C}^{\pm}\cup\mathbb{R}$. Similarly, the functions $N_1^{\pm}(z)$ and $N_2^{\mp}(z)$ are analytic in $\mathbb{C}^{\pm}$ for z, and they are continuous on $\mathbb{C}^{\pm}\cup\mathbb{R}$* (see [11]).

Again, there exists $S_2(z)$ such that

$$N^{-}(z) = N^{+}(z)S_2(z), \quad z\in\mathbb{R}, \tag{2.27}$$

where

$$S_2(z) = \mathrm{e}^{\frac{\mathrm{i}x\widehat{\sigma}_3(z-z^{-1})}{4}} \begin{pmatrix} \widehat{a}(z) & -\overline{\widehat{b}(\overline{z})} \\ z\widehat{b}(z) & \overline{\widehat{a}(\overline{z})} \end{pmatrix}.$$

It follows from (2.27) that

$$\widehat{a}(z) = Wr(N_1^{-}(z), N_2^{+}(z)), \tag{2.28}$$

$$\overline{\widehat{b}(z)} = \mathrm{e}^{-\frac{\mathrm{i}x(z-z^{-1})}{2}} Wr(N_2^{+}(z), N_2^{-}(z)). \tag{2.29}$$

We can observe the following relations

$$\alpha(\lambda) = a(z) = \widehat{a}(z), \quad \lambda\beta(\lambda) = b(z) = \lambda^2\widehat{b}(z). \tag{2.30}$$

For $z\in\mathbb{R}\setminus\{0\}$, we define the reflection coefficient as

$$r(z) = \frac{b(z)}{a(z)}, \quad \widehat{r}(z) = \frac{\widehat{b}(z)}{a(z)}. \tag{2.31}$$

Drawing from (2.30), we can also deduce that $a(z)$ is analytically continuous in $\mathbb{C}^{-}$. Let

$$\mathcal{Z} = \{z_k \mid z_k = \lambda_k^2, k = 1, \cdots, N\}$$

be the set of zeros of $a(z)$ in $\mathbb{C}^{-}$. Then there exist constants $\widetilde{c}_k$ and $\widetilde{w}_k$ such that

$$M_1^{-}(z_k) = \widetilde{c}_k M_2^{+}(z_k), \quad N_1^{-}(z_k) = \widetilde{w}_j N_2^{+}(z_k),$$

where the norming constants are $c_k = \frac{\widetilde{c}_k}{a'(z_k)}$ and $w_k = \frac{\widetilde{w}_j}{a'(z_k)}$, respectively.

2.4 The RH problem for the potential (u, v)

In this section, we will present two RH problems. The first problem aims to reconstruct the component u using $M^{\pm}(z)$ as $z \to 0$, while the second problem deals with $N^{\pm}(z)$ and aims to reconstruct the component v as $|z| \to \infty$. It is important to note that both components satisfy the MTM system (1.1).

Define

$$\begin{cases} P_+(z) = \begin{pmatrix} \dfrac{M_1^-(z)}{a(z)} & M_2^+(z) \end{pmatrix}, & z \in \mathbb{C}^+, \\ P_-(z) = \begin{pmatrix} M_1^+(z) & \dfrac{M_2^-(z)}{\overline{a(\bar{z})}} \end{pmatrix}, & z \in \mathbb{C}^- \end{cases} \tag{2.32}$$

and

$$\begin{cases} \widehat{P}_+(z) = \begin{pmatrix} N_1^+(z) & \dfrac{N_2^-(z)}{\overline{\widehat{a}(z)}} \end{pmatrix} & z \in \mathbb{C}^+, \\ \widehat{P}_-(z) = \begin{pmatrix} \dfrac{N_1^-(z)}{\widehat{a}(z)} & N_2^+(z) \end{pmatrix}, & z \in \mathbb{C}^-. \end{cases} \tag{2.33}$$

With asymptotic limits as [14],

$$P_{\pm}(z) \to \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{4}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{4}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} \end{pmatrix} = P^{\infty}(x), \quad z \to 0,$$

$$\widehat{P}_{\pm}(z) \to \begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{4}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} & 0 \\ 0 & \mathrm{e}^{\frac{\mathrm{i}}{4}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} \end{pmatrix} = \widehat{P}^{\infty}(x), \quad |z| \to \infty.$$

Define

$$\mathcal{M}_{\pm}(z) = P^{\infty}(x)^{-1} P_{\pm}(z), \quad z \in \mathbb{C}^{\pm},$$
$$\mathcal{N}_{\pm}(z) = \widehat{P}^{\infty}(x)^{-1} \widehat{P}_{\pm}(z), \quad z \in \mathbb{C}^{\pm}.$$

Then we obtain the following two RH problems.

RH Problem 2.1 Find an analytic function $\mathcal{M}(z) : \mathbb{C} \setminus \mathbb{R} \to SL_2(\mathbb{C})$ with the following properties.

(1) $\mathcal{M}(z) = I + \mathcal{O}(z)$ as $z \to 0$.

(2) For each $z \in \mathbb{R}$, the boundary values $\mathcal{M}_{\pm}(z)$ satisfy the jump relation

$$\mathcal{M}_+(z) = \mathcal{M}_-(z) V_1(z), \tag{2.34}$$

where

$$V_1(z) = \begin{pmatrix} 1 + z|r(z)|^2 & \overline{r(z)}\mathrm{e}^{\mathrm{i}\Theta(z)t)} \\ zr(z)\mathrm{e}^{-\mathrm{i}\Theta(z)t} & 1 \end{pmatrix}$$

and

$$\Theta(z) = \frac{1}{2}(z - z^{-1})\frac{x}{t} + \frac{1}{2}(z + z^{-1}). \tag{2.35}$$

(3) $\mathcal{M}(z)$ has simple poles at each $z_k \in \mathcal{Z}$ and $\overline{z}_k \in \overline{\mathcal{Z}}$ at which

$$\operatorname*{Res}_{z=z_k} \mathcal{M}(z) = \lim_{z\to z_k} \mathcal{M}(z) \begin{pmatrix} 0 & 0 \\ c_k z_k \mathrm{e}^{-\mathrm{i}\Theta(z_k)t} & 0 \end{pmatrix},$$
$$\operatorname*{Res}_{z=\overline{z}_k} \mathcal{M}(z) = \lim_{z\to \overline{z}_k} \mathcal{M}(z) \begin{pmatrix} 0 & -\overline{c}_k \mathrm{e}^{\mathrm{i}\Theta(\overline{z}_k)t} \\ 0 & 0 \end{pmatrix}.$$

RH Problem 2.2 Find an analytic function $\mathcal{N}(k) : \mathbb{C} \setminus \mathbb{R} \to SL_2(\mathbb{C})$ with the following properties.

(1) $\mathcal{N}(z) = I + \mathcal{O}(z^{-1})$ as $|z| \to \infty$.

(2) For each $z \in \mathbb{R}$, the boundary values $\mathcal{N}_\pm(z)$ satisfy the jump relation

$$\mathcal{N}_+(z) = \mathcal{N}_-(z)V_2(z), \tag{2.36}$$

where

$$V_2(z) = \begin{pmatrix} 1 & -\overline{\widehat{r}(z)}\mathrm{e}^{\mathrm{i}\Theta(z)t} \\ -z\widehat{r}(z)\mathrm{e}^{-\mathrm{i}\Theta(z)t} & 1 + z|\widehat{r}(z)|^2 \end{pmatrix}.$$

(3) $\mathcal{N}(z)$ has simple poles at each $z_k \in \mathcal{Z}$ and $\overline{z}_k \in \overline{\mathcal{Z}}$ at which

$$\operatorname*{Res}_{z=z_k} \mathcal{N}(z) = \lim_{z\to z_k} \mathcal{N}(z) \begin{pmatrix} 0 & -w_k \mathrm{e}^{\mathrm{i}\Theta(z_k)t} \\ 0 & 0 \end{pmatrix},$$
$$\operatorname*{Res}_{z=\overline{z}_k} \mathcal{N}(z) = \lim_{z\to \overline{z}_k} \mathcal{N}(z) \begin{pmatrix} 0 & 0 \\ \overline{w}_k \overline{z}_k \mathrm{e}^{-\mathrm{i}\Theta(z_k)t} & 0 \end{pmatrix}.$$

The solutions $(u(t,x), v(t,x))$ to the MTM system (1.1) are connected to the solutions of the RH problems 2.1 and 2.2 through the reconstruction formulas:

$$\overline{u(t,x)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} = \lim_{z\to 0} z^{-1}[\mathcal{M}(z)]_{12},$$
$$\overline{v(t,x)}\mathrm{e}^{\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} = \lim_{|z|\to \infty} z[\mathcal{N}(z)]_{12}.$$

Proposition 2.3 *Suppose* $(u(t,x), v(t,x)) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$*, there exists a bi-Lipschitz map*

$$(u(t,x), v(t,x)) \mapsto (r(z), \widehat{r}(z)) \in H^{1,1}(\mathbb{R}) \cap L^{2,-2}(\mathbb{R}),$$

where $L^{2,-2}(\mathbb{R}) = \dot{L}^{2,-2}(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $\|f(x)\|^2_{\dot{L}^{2,-2}(\mathbb{R})} = \int_{\mathbb{R}} |x|^{-2}|f(x)|^2\mathrm{d}x.$

Proof The proof can be founded in [11, 14].

3 Dispersion for Pure Radiation Solutions

In this section, we will focus on the MTM system (1.1) in the soliton region with $\left|\frac{x}{t}\right| < 1$, which is of interest from a physical perspective.

Theorem 3.1 *Let* $(u_0, v_0) \in \mathcal{G}_0 \cap H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. *Then there exist constant* $C(u_0, v_0) > 0$ *and* $T(u_0, v_0) > 0$ *such that for all* $t \geq T(u_0, v_0)$ *and* $\left|\frac{x}{t}\right| < 1$, *the solution of MTM system* (1.1) *satisfies*

$$\|u(t,x)\|_{L^\infty(\mathbb{R})} + \|v(t,x)\|_{L^\infty(\mathbb{R})} \leq C(u_0, v_0) t^{-\frac{1}{2}}. \tag{3.1}$$

In the rest of Section 3 we prove Theorem 3.1.

3.1 New coordinates

For $\left|\frac{x}{t}\right| < 1$, the phase function in (2.35) can be simplified as following:

$$\Theta(z)t = \tau\theta(k), \tag{3.2}$$

where

$$\tau = \sqrt{t^2 - x^2}, \quad k = \frac{z}{\mu}, \quad \mu = \frac{t-x}{\tau} \tag{3.3}$$

and $\theta(k) = \frac{1}{2}\big(k + \frac{1}{k}\big)$ with stationary points at $\xi = \pm 1$.

We consider the RH problem 3.1, which is related to the RH problems 2.1–2.2 as discussed in Lemma 3.1.

RH Problem 3.1 Find an analytic function $M(k) : \mathbb{C} \setminus \mathbb{R} \to SL_2(\mathbb{C})$ with the following properties.

(1) $M(k) = I + (k^{-1})$ as $k \to \infty$.

(1) For each $k \in \mathbb{R}$, the boundary values $M_\pm(k)$ satisfy the jump relation

$$M_+(k) = M_-(k)V(k),$$

where

$$V(k) = \begin{pmatrix} 1 + \rho(k)\check{\rho}(k) & \rho(k)e^{-i\tau\theta(k)} \\ \check{\rho}(k)e^{i\tau\theta(k)} & 1 \end{pmatrix}. \tag{3.4}$$

Lemma 3.1 *RH problem* 2.1 *with* $\mathcal{G}_0$ *and RH problem* 3.1 *are equivalent for the following choice if* $\rho(k)$ *and* $\check{\rho}(k)$ *in* (3.4):

$$\rho(k) = \overline{r(k\mu)}e^{2i\tau\theta(k)}, \quad \check{\rho}(k) = k\mu r(k\mu)e^{-2i\tau\theta(k)}. \tag{3.5}$$

RH problem 2.2 *with* $\mathcal{G}_0$ *and RH problem* 3.1 *are equivalent for the following choice if* $\rho(k)$ *and* $\check{\rho}(k)$ *in* (3.4):

$$\rho(k) = -\overline{\widehat{r}}(k\mu)d_-(z)d_+(z)e^{2i\tau\theta(k)}, \quad \check{\rho}(k) = -\frac{k\mu \cdot \widehat{r}(k\mu)e^{-2i\tau\theta(k)}}{d_-(z)d_+(z)}, \tag{3.6}$$

where

$$\begin{cases} d(z) = \exp\Big\{ -\dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_{-\infty}^{\infty} \dfrac{\log(1+s|r(s)|^2)}{s-z} \mathrm{d}s \Big\}, & z \in \mathbb{C} \setminus \mathbb{R}, \\ d_\pm(z) = \lim\limits_{\varepsilon \to 0} d(z \pm \mathrm{i}\varepsilon), & z \in \mathbb{R}. \end{cases} \tag{3.7}$$

Consider the following scalar RH problem.

RH Problem 3.2 Find a scalar function $\delta(k)$ analytic for $k \in \mathbb{C} \setminus (-1,1)$ with the following properties.

- $\delta(k) = 1 + \mathcal{O}(k^{-1})$ as $|k| \to \infty$.
- For each $k \in \mathbb{R}$, the boundary values $\delta_\pm(k)$ satisfy the jump relation

$$\delta_+(k) = \begin{cases} \delta_-(k), & k \in \mathbb{R} \setminus (-1,1), \\ \delta_-(k)(1+\rho(k)\breve{\rho}(k)), & k \in (-1,1). \end{cases}$$

It can be shown that the RH problem 3.2 has a unique solution

$$\delta(k) = \exp\Big\{ \mathrm{i} \int_{-1}^{1} \frac{\nu(s)}{s-k} \mathrm{d}s \Big\}, \tag{3.8}$$

where

$$\nu(k) = -\frac{1}{2\pi} \log(1+\rho(k)\breve{\rho}(k))$$

and the logarithm is principally branched along $(-1,1)$.

Proposition 3.1 *The function $\delta(k)$ has the following properties.*

(1) *The function $\delta(k)$ satisfies the estimate*

$$\mathrm{e}^{-\frac{\|\nu(k)\|_{L^\infty}}{2}} \le |\delta(k)| \le \mathrm{e}^{\frac{\|\nu(k)\|_{L^\infty}}{2}}. \tag{3.9}$$

(2) *Along any ray of $L = \pm 1 + \mathrm{e}^{\mathrm{i}\phi}\mathbb{R}^+$ with $0 < \phi < \pi$, $C(\rho, \breve{\rho}) > 0$, we have*

$$|\delta(k) - \delta_0(k)(k \pm 1)^{\mathrm{i}\nu(\pm 1)}| \le C(\rho, \breve{\rho})|k \pm 1|^{\frac{1}{2}}. \tag{3.10}$$

Here $\delta_0(k) = \mathrm{e}^{\mathrm{i}\beta(\pm 1, \pm 1)}$ and

$$\beta(k, \pm 1) = -\nu(\pm 1) \log\Big(k \mp \frac{5}{6}\Big) + \int_{-1}^{1} \frac{\nu(s) - \chi(s)\nu(\pm 1)}{s-k} \mathrm{d}s, \tag{3.11}$$

where $\chi(k)$ is the characteristic function of the interval

$$I = \Big\{ k \mid k \in \Big(-1, -\frac{5}{6}\Big) \text{ or } k \in \Big(\frac{5}{6}, 1\Big) \Big\}. \tag{3.12}$$

Proof For part (1), we use the fact that $\|\nu(k)\|_{H^{1,1}(\mathbb{R})} \le \frac{1}{2\pi}\|\rho\breve{\rho}\|_{L^1}$ to establish the bound. For part (2), we write the function $\delta(k)$ as follows:

$$\delta(k) = \exp\Big(\mathrm{i} \int_I \frac{\nu(s)}{s-k} \mathrm{d}s + \mathrm{i} \int_{-1}^{1} \frac{\nu(s) - \chi(s)\nu(\pm 1)}{s-k} \mathrm{d}s \Big)$$

$$= (k \pm 1)^{\mathrm{i}\nu(\pm 1)} \exp(\mathrm{i}\beta(k, \pm 1)).$$

Using the fact that

$$|(k \pm 1)^{\mathrm{i}\nu(\pm 1)}| \leq \mathrm{e}^{-\pi\nu(\pm 1)} = \sqrt{1 + \rho(k)\breve{\rho}(k)},$$

we deduce that

$$|\beta(k, \pm 1) - \beta(\pm 1, \pm 1)| \leq C(\rho, \breve{\rho})|k \pm 1|^{\frac{1}{2}}.$$

Define

$$M^{(1)}(k) = M(k)\delta(k)^{-\sigma_3}, \tag{3.13}$$

then $M^{(1)}(k)$ satisfies the following RH problem.

RH Problem 3.3 Find an analytic function $M^{(1)}(k) : \mathbb{C} \setminus \mathbb{R} \to SL_2(\mathbb{C})$ with the following properties.

(1) $M^{(1)}(k) = I + \mathcal{O}(k^{-1})$ as $k \to \infty$.

(2) For each $k \in \mathbb{R}$, the boundary values $M_{\pm}^{(1)}(k)$ satisfy the jump relation

$$M_{+}^{(1)}(k) = M_{-}^{(1)}(k) V^{(1)}(k), \tag{3.14}$$

where

$$V^{(1)}(k) = \begin{cases} \begin{pmatrix} 1 & \rho(k)\delta^2(k)\mathrm{e}^{-\mathrm{i}\tau\theta(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \breve{\rho}(k)\delta(k)^{-2}\mathrm{e}^{\mathrm{i}\tau\theta(k)} & 1 \end{pmatrix}, & |k| \geq 1, \\ \begin{pmatrix} 1 & 0 \\ \dfrac{\breve{\rho}(k)\delta_-^{-2}(k)}{1 + \rho(k)\breve{\rho}(k)}(k)\mathrm{e}^{\mathrm{i}\tau\theta(k)} & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\rho(k)\delta_+^2(k)}{1 + \rho(k)\breve{\rho}(k)}\mathrm{e}^{-\mathrm{i}\tau\theta(k)} \\ 0 & 1 \end{pmatrix}, & |k| < 1. \end{cases}$$

3.2 $\overline{\partial}$-extensions of jump factorization

In this section, our objective is to obtain factorizations of the jump matrix that allow for continuous extension beyond the real axis $\mathbb{R}$ according to the decay and growth associated with $\theta(k)$ (see Figure 1). To achieve this, we define a new contour $\Sigma^{(2)} = \bigcup_{j=1}^{9} \Sigma_j$ (see Figure 2).

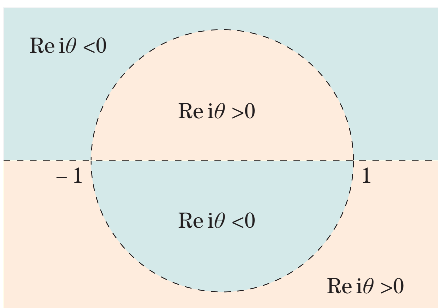

Figure 1 Signature table for Re iθ.

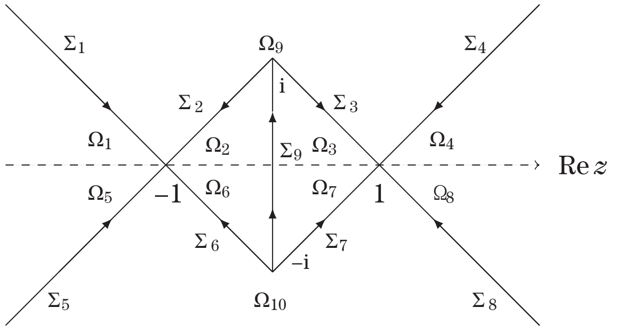

Figure 2 The contours Σ_k and regions $\Omega_k, k = 1, 2, \cdots, 10$.

Following this contour definition, we construct the necessary extension functions.

Lemma 3.2 *It is possible to define functions* $R_j(k) : \overline{\Omega}_j \to \mathbb{C}\,(k = 1, 2, \cdots, 8)$ *with boundary values satisfying*

$$R_j(k)|_{j=1,4} = \begin{cases} -\breve{\rho}(k)\delta^{-2}(k), & k \in I_j, \\ -\breve{\rho}(\xi)\delta^{-2}(\xi)\mathrm{e}^{-2\chi(\xi)}(k-\xi)^{-2\mathrm{i}\nu}, & k \in \Sigma_j, \end{cases}$$

$$R_j(k)|_{j=2,3} = \begin{cases} -\dfrac{\rho(k)}{1+\rho(k)\breve{\rho}(k)}\delta_+^2(k), & k \in I_j, \\ -\dfrac{\rho(\xi)}{1+\rho(\xi)\breve{\rho}(\xi)}\delta_+^2(\xi)\mathrm{e}^{2\chi(\xi)}(k-\xi)^{2\mathrm{i}\nu}, & k \in \Sigma_j, \end{cases}$$

$$R_j(k)|_{j=5,8} = \begin{cases} \rho(k)\delta^2(k), & k \in I_j, \\ \rho(\xi)\mathrm{e}^{2\chi(\xi)}\delta^2(\xi)(k-\xi)^{2\mathrm{i}\nu}, & k \in \Sigma_j, \end{cases}$$

$$R_j(k)|_{j=6,7} = \begin{cases} \dfrac{\breve{\rho}(k)}{1+\rho(k)\breve{\rho}(k)}\delta_-^{-2}(k), & k \in I_j, \\ \dfrac{\breve{\rho}(\xi)}{1+\rho(\xi)\breve{\rho}(\xi)}\delta_-^{-2}(\xi)\mathrm{e}^{-2\chi(\xi)}(k-\xi)^{-2\mathrm{i}\nu}, & k \in \Sigma_j, \end{cases}$$

where

$$\xi = \begin{cases} -1, & j = 1, 2, 5, 6, \\ 1, & j = 3, 4, 7, 8, \end{cases} \tag{3.15}$$

and we define I_j *as the projection of* Σ_j *onto* $\mathbb{R}$. *Then, for a fixed constant* $c > 0$ *such that*

$$|\overline{\partial} R_j(k)| \le c|k \pm 1|^{-\frac{1}{2}} + c(|\rho'(\mathrm{Re}\,k)| + |\breve{\rho}'(\mathrm{Re}\,k)|).$$

Proof Let us consider the case when $j = 4$. Writing $k - 1 = s\mathrm{e}^{\mathrm{i}\phi}$, we define the function $R_4(k)$ as follows:

$$f_4(k) = \breve{\rho}(1)\mathrm{e}^{-2\chi(1)}\delta^2(k), \tag{3.16}$$

and let

$$R_4(k) = [f_4(k) + (\breve{\rho}(\mathrm{Re}k) - f_4(k)\mathcal{K}(\phi)]\delta^{-2}(k),$$

where $\mathcal{K}(\phi)$ is a smooth function on $\left(0,\frac{\pi}{4}\right)$ with

$$\mathcal{K}(\phi)=\begin{cases}1, & \phi\in\left[0,\dfrac{\pi}{12}\right],\\[2ex] 0, & \phi\in\left[\dfrac{\pi}{6},\dfrac{\pi}{4}\right].\end{cases}$$

The function $R_4(k)$ takes the same values on both Σ_4 and $(1,\infty)$. Since

$$\overline{\partial}=\frac{1}{2}\Big(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\Big)=\frac{1}{2}\mathrm{e}^{\mathrm{i}\phi}\Big(\frac{\partial}{\partial s}+\frac{\mathrm{i}}{s}\frac{\partial}{\partial\phi}\Big),$$

we calculate

$$\overline{\partial}R_4(k)=\frac{1}{2}\check{\rho}'(\operatorname{Re}k)\mathcal{K}(\phi)\delta^{-2}(k)-(\check{\rho}(\operatorname{Re}k)-f_4(k))\delta^{-2}\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}}{|k-1|}\mathcal{K}'(\phi),$$

where the first term is bounded by (3.9). For the second term, we write

$$|\check{\rho}(\operatorname{Re}k)-f_4(k)|\le|\check{\rho}(\operatorname{Re}k)-\check{\rho}(1)|+|\check{\rho}(1)-f_4(k)|,$$

and apply the Cauchy-Schwarz inequality to bound each term as follows:

$$|\check{\rho}(\operatorname{Re}k)-\check{\rho}(1)|\le\Big|\int_1^{\operatorname{Re}k}\check{\rho}'(s)\mathrm{d}s\Big|\le\|\check{\rho}\|_{H^1(\mathbb{R})}|k-1|^{\frac{1}{2}}$$

and

$$\begin{aligned}|\check{\rho}(1)-f_4(k)|&\le|\check{\rho}(1)|(1+|\check{\rho}(1)|^2)|\delta^2(1)-\delta_0^2(1)(k-1)^{2\mathrm{i}\nu}|\\&\le C(\check{\rho})\|\check{\rho}\|_{H^1(\mathbb{R})}|k-1|^{-\frac{1}{2}}.\end{aligned}$$

The last estimate employs Proposition 3.1, and the result is immediately obtained.

By using $R_j(k),\ j=1,\cdots,8$, we define the function

$$\mathcal{R}^{(2)}(k)=\begin{cases}\begin{pmatrix}1 & 0\\ R_j(k)\mathrm{e}^{\mathrm{i}\tau\theta(k)} & 1\end{pmatrix}, & k\in\Omega_{j|_{j=1,4,6,7}},\\[2ex] \begin{pmatrix}0 & R_j(k)\mathrm{e}^{-\mathrm{i}\tau\theta(k)}\\ 0 & 0\end{pmatrix}, & k\in\Omega_{j|_{j=2,3,5,8}},\\[2ex] I, & k\in\Omega_9\cup\Omega_{10}.\end{cases}\tag{3.17}$$

We now introduce another matrix-valued function

$$M^{(2)}(k)=M^{(1)}(k)\mathcal{R}^{(2)}(k),\tag{3.18}$$

which converts the RH problem 3.3 into a mixed $\overline{\partial}$-RH problem 3.4.

RH Problem 3.4 Find a meromorphic function $M^{(2)}(k):\mathbb{C}\setminus\Sigma^{(2)}\to SL_2(\mathbb{C})$ with the following properties.

(1) $M^{(2)}(k)=I+\mathcal{O}(k^{-1})$ as $k\to\infty$.

(1) For each $k \in \Sigma^{(2)}$, the boundary values $M_{\pm}^{(2)}(k)$ satisfy the jump relation

$$M_{+}^{(2)}(k) = M_{-}^{(2)}(k)V^{(2)}(k), \tag{3.19}$$

where

$$V^{(2)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ R_j(k)\mathrm{e}^{\mathrm{i}\tau\theta(k)} & 1 \end{pmatrix}, \quad k \in \Sigma_{j|_{j=1,4,6,7}}, & \begin{pmatrix} 1 & \Delta R_{32}(k)\mathrm{e}^{-\mathrm{i}\tau\theta(k)} \\ 0 & 1 \end{pmatrix}, \quad k \in (0, i), \\ \begin{pmatrix} 1 & R_j(k)\mathrm{e}^{-\mathrm{i}\tau\theta(k)} \\ 0 & 1 \end{pmatrix}, \quad k \in \Sigma_{j|_{j=2,3,5,8}}, & \begin{pmatrix} 1 & 0 \\ \Delta R_{67}(k)\mathrm{e}^{\mathrm{i}\tau\theta(k)} & 1 \end{pmatrix}, \quad k \in (-i, 0). \end{cases}$$

where $\Delta R_{32}(k) = R_3(k) - R_2(k)$ and $\Delta R_{67}(k) = R_6(k) - R_7(k)$.

(3) For $z \in \mathbb{C} \setminus \Sigma^{(2)}$ we have $\overline{\partial} M^{(2)}(k) = M^{(2)}(k)\overline{\partial}\mathcal{R}^{(2)}$, where

$$\overline{\partial}\mathcal{R}^{(2)} = \begin{cases} \begin{pmatrix} 0 & 0 \\ \overline{\partial} R_j(k)\mathrm{e}^{\mathrm{i}\tau\theta(k)} & 0 \end{pmatrix}, & k \in \Omega_{j=1,4,6,7}, \\ \begin{pmatrix} 0 & \overline{\partial} R_j(k)\mathrm{e}^{-\mathrm{i}\tau\theta(k)} \\ 0 & 0 \end{pmatrix}, & k \in \Omega_{j=2,3,5,8}, \\ 0, & k \in \Omega_9 \cup \Omega_{10}. \end{cases} \tag{3.20}$$

3.3 Analysis of pure RH problem

In this section, we can factorize $M^{(2)}(k)$ as follows:

$$M^{(2)}(k) = M^{(3)}(k)M^{rhp}(k). \tag{3.21}$$

Here, $M^{rhp}(k)$ corresponds to the solution of the RH problem 3.4 for $M^{(2)}(k)$ by dropping the $\overline{\partial}$ component, while $M^{(3)}(k)$ represents the solution of the pure $\overline{\partial}$-RH problem 3.6.

The matrix $M^{rhp}(k)$ is meromorphic away from the contour $\Sigma^{(2)}$ on which its boundary values satisfy the jump relation (3.19). However, at any distance from the phase points ξ, the jump is uniformly near identity. Using (3.19) and the definition $\theta(k)$, there exists a constant $c > 0$ satisfying

$$\|V^{(2)} - I\|_{L^\infty(\Sigma^{(2)})} = \mathcal{O}(\mathrm{e}^{-ct}), \tag{3.22}$$

which is exponentially small in $\Sigma^{(2)}$. Based on this estimation, we will construct the solution $M^{rhp}(k) = (1 + \mathcal{O}(\mathrm{e}^{-ct}))M^{lc}(k)$, where $M^{lc}(k)$ is constructed below.

RH Problem 3.5 Find a meromorphic function $M^{lc}(k) : \mathbb{C} \setminus \Sigma^{(2)} \to SL_2(\mathbb{C})$ with the following properties.

(1) $M^{lc}(k) = I + \mathcal{O}(k^{-1}), \quad |k| \to \infty$.

(2) For each $k \in \bigcup_{j=1}^{8} \Sigma_j$, the boundary values $M_{\pm}^{lc}(k)$ satisfy the jump relation

$$M_{+}^{lc}(k) = M_{-}^{lc}(k)V^{(2)}(k).$$

By using the standard method (see [17]), it can shown that

Proposition 3.2 *The solution to the RH problem* 3.5 *admits the following expansion*

$$
\begin{aligned}
M^{lc}(k) = I &+ \frac{1}{\sqrt{2\tau}(k-1)}\begin{pmatrix} 0 & -\mathrm{i}\beta_{12} \\ \mathrm{i}\beta_{21} & 0 \end{pmatrix} \\
&+ \frac{1}{\sqrt{2\tau}(k+1)}\begin{pmatrix} 0 & -\mathrm{i}\overline{\beta}_{12} \\ \mathrm{i}\overline{\beta}_{21} & 0 \end{pmatrix} + \mathcal{O}(k^{-2}),
\end{aligned}
\tag{3.23}
$$

where

$$
\beta_{12} = \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4\mathrm{e}^{-\pi\nu/2}}}{\varrho\Gamma(-\mathrm{i}\nu)}, \quad \beta_{21} = \frac{\nu}{\beta_{12}}, \tag{3.24}
$$

and $\varrho = \check{\rho}(1)\mathrm{e}^{\mathrm{i}t}\delta^{-2}(1)\mathrm{e}^{2\mathrm{i}\nu(1)\log\sqrt{2t}}$ *and* $\Gamma(k)$ *is a Gamma function.*

3.4 Analysis of remaining $\overline{\partial}$-problem

RH Problem 3.6 Find a function $M^{(3)}(k): \mathbb{C} \to SL_2(\mathbb{C})$ with the following properties.
(1) $M^{(3)}(k) = I + \mathcal{O}(k^{-1})$ as $|k| \to \infty$.
(2) For $z \in \mathbb{C}$, we have

$$
\overline{\partial} M^{(3)}(k) = M^{(3)}(k)W(k), \tag{3.25}
$$

where

$$
W(k) = M^{rhp}(k)\overline{\partial}\mathcal{R}^{(2)}(k)M^{rhp}(k)^{-1}, \tag{3.26}
$$

and $\overline{\partial}\mathcal{R}^{(2)}(k)$ is defined by (3.20).

A matrix-valued function $M^{(3)}(k)$ that is both bounded and continuous is equivalent to solving a Fredholm-type integral equation (see [18]):

$$
M^{(3)}(k) = I + \frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W(s)}{s-k}\mathrm{d}A(s), \tag{3.27}
$$

where $\mathrm{d}A$ denotes the Lebesgue measure on $\mathbb{C}$, and s refers to a complex variable.

(3.27) can be written as

$$
(I - \mathcal{S})M^{(3)}(k) = I, \tag{3.28}
$$

where $\mathcal{S}$ is the solid Cauchy operator

$$
\mathcal{S}(f)(k) = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(s)W(s)}{s-k}\mathrm{d}A(s). \tag{3.29}
$$

Proposition 3.3 demonstrates that as $t \to +\infty$, the operator $\mathcal{S}$ has a small norm.

Proposition 3.3 *There exists a constant* $C(\rho, \check{\rho})$ *such that for all* $\tau > 0$, *the operator* (3.29) *satisfies the inequality*

$$
\|\mathcal{S}\|_{L^\infty \to L^\infty} \le C(\rho, \check{\rho})\tau^{-\frac{1}{4}}. \tag{3.30}
$$

Proof We consider the case in the region $k \in \Omega_3$, with the case for other regions following similarly. Let $s-1=x+\mathrm{i}y$ and $k-1=u+\mathrm{i}v$, and let $f \in L^\infty(\Omega_3)$. Then, from (3.20) and (3.25), it follows that

$$\begin{aligned}|S(f)(k)| &\le \iint_{\Omega_3} \frac{|f(s)M^{rhp}(s)\overline{\partial}\mathcal{R}(s)M^{rhp}(s)^{-1}|}{|s-k|}\mathrm{d}A(s)\\ &\le \|f\|_{L^\infty(\Omega_3)}\|M^{rhp}\|_{L^\infty(\Omega_3)}\|M^{rhp^{-1}}\|_{L^\infty(\Omega_3)}\iint_{\Omega_3}\frac{|\overline{\partial}R_4(s)||\mathrm{e}^{\mathrm{i}\tau\theta(s)}|}{|s-k|}\mathrm{d}A(s)\\ &\le C(I_1+I_2), \end{aligned} \tag{3.31}$$

where I_1, I_2 are defined by

$$\begin{aligned}I_1 &= \iint_{\Omega_3}\frac{(|\rho'(\operatorname{Re} s)|+|\check{\rho}'(\operatorname{Re} s)|)|\mathrm{e}^{\mathrm{i}\tau\theta(s)}|}{|s-k|}\mathrm{d}A(s),\\ I_2 &= \iint_{\Omega_3}\frac{|s-1|^{-\frac{1}{2}}|\mathrm{e}^{\mathrm{i}\tau\theta(s)}|}{|s-k|}\mathrm{d}A(s).\end{aligned}$$

We use the parametrization $k=a+(1-a)\mathrm{i}\in\Sigma_3$ with $0\le a\le 1$. A simple computation shows that

$$\operatorname{Im}(\theta(a+(1-a)\mathrm{i})) = \frac{-a(a-1)^2}{a^2+(1-a)^2} \le -a(a-1)^2 \le 0.$$

Define

$$I(a)=\begin{cases} -\dfrac{1}{2}a^2, & 0\le a\le \dfrac{1}{2},\\[2mm] -\dfrac{1}{2}(a-1)^2, & \dfrac{1}{2}\le a\le 1,\end{cases} \tag{3.32}$$

we have $\operatorname{Im}(\theta(a+(1-a)\mathrm{i}))\le I(a)\le 0$, and then

$$\int_{\Omega_3}|\mathrm{e}^{-\mathrm{i}\tau\theta(k)}|\mathrm{d}k \le \int_0^1 \mathrm{e}^{\tau I(a)}\mathrm{d}a \le C\tau^{-\frac{1}{2}}. \tag{3.33}$$

Let us now begin with the estimate of I_1, we acquire

$$\begin{aligned}I_1 &= \int_0^\infty\int_y^\infty \frac{(|\rho'(s+1)|+|\check{\rho}'(s+1)|)|\mathrm{e}^{\mathrm{i}\tau\theta(1+x+\mathrm{i}y)}|}{\sqrt{(x-u)^2+(y-v)^2}}\mathrm{d}x\mathrm{d}y\\ &\le c(\|\rho'\|_{L^2(\mathbb{R})}+\|\check{\rho}'\|_{L^2(\mathbb{R})})\int_0^\infty \mathrm{e}^{-\tau I(y)}|y-v|^{-\frac{1}{2}}\mathrm{d}y \lesssim \tau^{-\frac{1}{4}}.\end{aligned}$$

For I_2, we have

$$I_2 \le \int_0^\infty \mathrm{e}^{-\tau I(y)}\Big\|\frac{1}{(x^2+y^2)^{\frac{1}{4}}}\Big\|_{L_x^2(y,\infty)}\Big\|\frac{1}{\sqrt{(x-u)^2+(y-v)^2}}\Big\|_{L_x^2((y,\infty))}\mathrm{d}y.$$

A direct computation shows that $\|(x^2+y^2)^{-\frac{1}{4}}\|_{L_x^2((y,\infty))}$ does not depend on $y>0$. Thus, we can replicate the arguments above to obtain $I_3 \lesssim \tau^{-\frac{1}{4}}$, which proves (3.31). This suffices to prove Proposition 3.3.

In order to determine the potential $(u(t,x),v(t,x))$, we must first identify the asymptotic properties of the coefficient corresponding to the k^{-1} term in the Laurent series expansion of

$M^{(3)}(k)$ as $|k|\to\infty$. This coefficient is characterized by an integral expression that emerges from the expansion

$$M^{(3)}(k)=I+\frac{M_1^{(3)}}{k}+\frac{1}{\pi}\iint_{\mathbb{C}}\frac{sM^{(3)}(s)W(s)}{z(s-z)}\mathrm{d}A(s), \tag{3.34}$$

where

$$M_1^{(3)}=\frac{1}{\pi}\iint_{\mathbb{C}}M^{(3)}(s)W(s)\mathrm{d}A(s).$$

Proposition 3.4 *For all $\tau>0$, there exists a constant $C(\rho,\breve{\rho})$ such that*

$$|M_1^{(3)}|\le C(\rho,\breve{\rho})\tau^{-\frac{3}{4}}. \tag{3.35}$$

Proof Let us first consider the case of $\Omega_3\cup\Omega_4$ (see Figure 3).

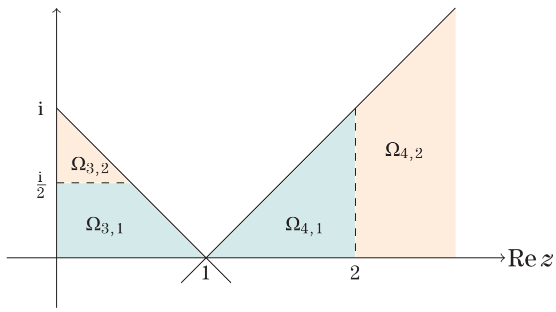

Figure 3 Decomposition of Ω_3 and Ω_4.

$$\begin{aligned}|M_1^{(3)}|&\le\frac{1}{\pi}\iint_{\Omega_4}|M^{(3)}(s)M^{rhp}(s)\overline{\partial}\mathcal{R}^{(2)}(s)M^{rhp}(s)^{-1}|\mathrm{d}A(s)\\&\le\frac{1}{\pi}\|M^{(3)}\|_{L^\infty(\Omega_4)}\|M^{rhp}\|_{L^\infty(\Omega_4)}\|M^{rhp^{-1}}\|_{L^\infty(\Omega_4)}\iint_{\Omega_4}|\overline{\partial}R_4||\mathrm{e}^{\mathrm{i}\tau\theta(s)}|\mathrm{d}A(s)\\&\le C(I_4+I_5),\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}I_4&=\iint_{\Omega_4}(|\rho'(\mathrm{Re}s)|+|\breve{\rho}'(\mathrm{Re}s)|)|\mathrm{e}^{\mathrm{i}\tau\theta(s)}|\mathrm{d}A(s),\\I_5&=\iint_{\Omega_4}|s-1|^{-\frac{1}{2}}|\mathrm{e}^{\mathrm{i}\tau\theta(s)}|\mathrm{d}A(s).\end{aligned}$$

We write $s-1=x+\mathrm{i}y\in\Omega_4$, then we can obtain

$$|\mathrm{e}^{-\mathrm{i}\tau\theta(s)}|\le\begin{cases}\mathrm{e}^{-\tau xy}, & s\in\Omega_{4,1},\\ \mathrm{e}^{-\tau y}, & s\in\Omega_{4,2}.\end{cases} \tag{3.37}$$

Next, we present the computation for I_4 as

$$I_4\le\int_0^\infty\int_y^\infty(|\breve{\rho}'(x+1)|+|\rho'(x+1)|)\mathrm{e}^{-\tau xy}\mathrm{d}x\mathrm{d}y$$

$$+\int_0^\infty\int_y^\infty(|\check{\rho}'(x+1)|+|\rho'(x+1)|)\mathrm{e}^{-\tau y}\mathrm{d}x\mathrm{d}y$$
$$\le c(\|\rho'\|_{H^1(\mathbb{R})}+\|\check{\rho}'\|_{H^1(\mathbb{R})})\Big(\frac{1}{\tau^{\frac34}}+\frac{1}{\tau}\Big). \tag{3.38}$$

Recalling the bounds of I_4 from Proposition 3.3, we can similarly bound I_5. Then we acquire

$$\Big|\iint_{\Omega_4}M^{(3)}(s)W(s)\mathrm{d}A(s)\Big|\le c\tau^{-\frac34}.$$

Let us now estimate $\iint_{\Omega_3}M^{(3)}(s)W(s)\mathrm{d}A(s)$. We have that for any $y\in\big(0,\frac12\big)$ and any $x\in(-1,-y)$, it is the case that $|\mathrm{e}^{-\mathrm{i}\tau\theta(s)}|\le\mathrm{e}^{-\frac{\tau xy}{2}}$ for $s\in\Omega_{3,1}$. It follows that

$$\int_{\Omega_{3,1}}|\rho'(\operatorname{Re}s)||\mathrm{e}^{-\mathrm{i}\tau\theta(s)}|\mathrm{d}A(s)\le\int_0^1\int_{-1}^{-y}|\rho'(x+1)||\mathrm{e}^{-\frac{\tau xy}{2}}|\mathrm{d}x\mathrm{d}y$$
$$\le\|\rho'\|_{L^2(\mathbb{R})}\int_0^\infty\|\mathrm{e}^{-\frac{\tau xy}{2}}\|_{L^2_x(y,\infty)}\mathrm{d}y\le C(\rho,\check{\rho})\tau^{-\frac34}.$$

To bound the integral over $\Omega_{3,2}$, we write $s-\mathrm{i}=x+\mathrm{i}y$. For $s\in\Omega_{3,2}$, we have $|\mathrm{e}^{\mathrm{i}\tau\theta(s)}|\le\mathrm{e}^{-\frac{\tau y}{2}}$, it follows that

$$\int_{\Omega_{3,2}}|\rho'(\operatorname{Re}s)||\mathrm{e}^{-\mathrm{i}\tau\theta(s)}|\mathrm{d}A(s)\le\int_{-1}^0\int_0^{-y}|\rho'(x)||\mathrm{e}^{-\frac{\tau y}{2}}|\mathrm{d}x\mathrm{d}y$$
$$\le\|\rho'\|_{L^1(\mathbb{R})}\int_0^\infty\mathrm{e}^{-\frac{\tau y}{2}}\mathrm{d}y\le C(\rho,\check{\rho})\tau^{-1}. \tag{3.39}$$

Since other regions Ω_j can be considered in a similar way, we can conclude the proof.

4 Asymptotic Stability of Solitons

In this section, without loss of generality, we assume $\mu=1$ to establish the asymptotic stability of the one-soliton solution.

4.1 The Bäcklund transformation

In this subsection, we construct a map such that

$$\mathcal{G}_1\times\mathbb{C}_-\ni\{(u_0,v_0),z_1\}\mapsto(\widetilde{u}_0,\widetilde{v}_0)\in\mathcal{G}_0 \tag{4.1}$$

via the transformation

$$\widetilde{r}(z)=r(z)\frac{z-\overline{z}_1}{z-z_1}. \tag{4.2}$$

By definition (4.2), if $r(z)\in H^{1,1}(\mathbb{R})\cap L^{2,-2}(\mathbb{R})$, then $\widetilde{r}(z)\in H^{1,1}(\mathbb{R})\cap L^{2,-2}(\mathbb{R})$. Consequently, there exists a constant $C>0$ such that

$$\|\widetilde{r}(z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,-2}(\mathbb{R})}\le C\|r(z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,-2}(\mathbb{R})}.$$

We construct Bäcklund transformation for MTM system as follows.

Proposition 4.1 *The solutions* (u,v) *and* $(\widetilde{u},\widetilde{v})$ *to the MTM system in* (1.1) *are connected by the Bäcklund transformation*

$$\widetilde{u}(t,x)=u(t,x)+\mathbf{B}_u, \tag{4.3}$$

$$\widetilde{v}(t,x)=v(t,x)+\mathbf{B}_v, \tag{4.4}$$

where

$$\mathbf{B}_u=2\mathrm{i}\frac{\operatorname{Im} z_1\overline{\eta}_1\eta_2}{|\eta_2|^2+z|\eta_1|^2}\mathrm{e}^{\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y}, \tag{4.5}$$

$$\mathbf{B}_v=-2\mathrm{i}\frac{\operatorname{Im} z_1\overline{\gamma}_1\gamma_2}{|\gamma_2|^2+z|\gamma_1|^2}\mathrm{e}^{\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} \tag{4.6}$$

and

$$(\eta_1,\eta_2)^{\mathrm{T}}=2\mathrm{i}c_1\operatorname{Im} z_1\mathrm{e}^{\mathrm{i}\Theta(z_1)t}\mathcal{M}(z_1)e_2, \tag{4.7}$$

$$(\gamma_1,\gamma_2)^{\mathrm{T}}=-2\mathrm{i}w_1\operatorname{Im} z_1\mathrm{e}^{-\mathrm{i}\Theta(z_1)t}\mathcal{N}(z_1)e_1, \tag{4.8}$$

where phase function $\Theta(z)$ *is defined in* (2.35).

Proof We will explore the Bäcklund transformation for $\mathbf{B}_u$ and a similar approach for $\mathbf{B}_v$. Assuming that $H=(h_1,h_2)$ is a determined invertible matrix solution of the spectral problem (2.14). Our objective is to eliminate a simple pole in the first column at $z_1\in\mathbb{C}^-$, as well as in the second column at $\overline{z}_1\in\mathbb{C}^+$ within the RH problem 2.1. Further define a transformation

$$\widetilde{\mathcal{M}}(z)=HG(z)H^{-1}\mathcal{M}(z)G^{-1}(z), \tag{4.9}$$

where

$$G(z)=\begin{pmatrix} z-\overline{z}_1 & 0\\ 0 & z-z_1\end{pmatrix}.$$

This requires satisfying the conditions

$$\operatorname*{Res}_{z=z_1}\widetilde{\mathcal{M}}(z)=0,\quad \operatorname*{Res}_{z=\overline{z}_1}\widetilde{\mathcal{M}}(z)=0. \tag{4.10}$$

Considering

$$H^{-1}\widetilde{\mathcal{M}}=\begin{pmatrix} (H^{-1}\mathcal{M})_{11} & (H^{-1}\mathcal{M})_{12}\dfrac{z-\overline{z}_1}{z-z_1}\\ (H^{-1}\mathcal{M})_{21}\dfrac{z-z_1}{z-\overline{z}_1} & (H^{-1}\mathcal{M})_{22}\end{pmatrix}, \tag{4.11}$$

the residues of the second row element of the (4.11) are

$$\operatorname*{Res}_{z=z_1}(H^{-1}\mathcal{M})_{21}\frac{z-z_1}{z-\overline{z}_1}=\Big\langle He_2,\operatorname*{Res}_{z=z_1}\frac{z-z_1}{z-\overline{z}_1}\mathcal{M}e_1\Big\rangle=0,$$

$$\operatorname*{Res}_{z=z_1}(H^{-1}\mathcal{M})_{22}=\langle He_2,\operatorname*{Res}_{z=z_1}\mathcal{M}e_2\rangle=0.$$

To calculate the residues in the first row, we proceed as follows

$$\operatorname*{Res}_{z=z_1}(H^{-1}\mathcal{M})_{11}=\langle He_1,\operatorname*{Res}_{z=z_1}\mathcal{M}e_1\rangle$$

$$
\begin{aligned}
&\Rightarrow He_2 = d_1 \operatorname*{Res}_{z=z_1} \mathcal{M} e_1,\\
&\quad \operatorname*{Res}_{z=z_1} (H^{-1}\mathcal{M})_{12} \frac{z-\overline{z}_1}{z-z_1} = \left\langle He_1, \operatorname*{Res}_{z=z_1} \frac{z-\overline{z}_1}{z-z_1} \mathcal{M} e_2 \right\rangle\\
&= \langle He_1, (z_1-\overline{z}_1) \operatorname*{Res}_{z=z_1} \mathcal{M}(z) e_2 \rangle = 0\\
&\Rightarrow He_2 = d_2 (z_1-\overline{z}_1) \mathrm{Res} \mathcal{M}(z) e_2.
\end{aligned}
$$

For any functions d_1, d_2, we can use them for the purpose of equivalence as

$$
d_1 = z_1 - \overline{z}_1, \quad d_2 = c_1 \mathrm{e}^{\frac{\mathrm{i}}{2}(z_1 - z_1^{-1})x - \frac{\mathrm{i}}{2}(z_1 + z_1^{-1})t}, \tag{4.12}
$$

so we have

$$
h_2 = He_2 = \operatorname*{Res}_{z=z_1} \mathcal{M} e_1 - (z_1 - \overline{z}_1) \operatorname*{Res}_{z=z_1} \mathcal{M}(z) e_2. \tag{4.13}
$$

Similarly, for the case $z = \overline{z}_1$, we have

$$
h_1 = He_1 = \operatorname*{Res}_{z=\overline{z}_1} \mathcal{M} e_2 - (z_1 - \overline{z}_1) \operatorname*{Res}_{z=\overline{z}_1} \mathcal{M}(z) e_1, \tag{4.14}
$$

rewrite (4.13)–(4.14) in matrix form

$$
H = (He_1, He_2) = \operatorname*{Res}_{z=\overline{z}_1} \mathcal{M}(z) \begin{pmatrix} \overline{z}_z - z_1 & 1 \\ 1 & z_1 - \overline{z}_1 \end{pmatrix}. \tag{4.15}
$$

Finally, we compute the corresponding potential $\widetilde{u}$, we obtain

$$
\begin{aligned}
\widetilde{\mathcal{M}}(z) &= H(I - G_1 z) H^{-1} (I + \mathcal{M}_1 z + \mathcal{O}(z^2))(I - G_1 z)^{-1}\\
&= I + (\mathcal{M}_1 - HG_1H^{-1} + G_1) z + \mathcal{O}(z^2),
\end{aligned}
$$

where $G_1 = \begin{pmatrix} \overline{z}_1 & 0 \\ o & z_1 \end{pmatrix}$, and hence

$$
\overline{\widetilde{u}}(t,x) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y} (\mathcal{M}_1 - HG_1H^{-1} + G_1)_{12}. \tag{4.16}
$$

By symmetry of (2.7) and (2.12), thus H can expressed as

$$
H = (h_1, h_2) = \begin{pmatrix} \overline{\eta}_2 & \eta_1 \\ -z\overline{\eta}_1 & \eta_2 \end{pmatrix}, \tag{4.17}
$$

taking (4.17) into (4.16), we get

$$
\widetilde{u}(t,x) = u(t,x) + 2\mathrm{i} \frac{\operatorname{Im} z_1 \overline{\eta}_1 \eta_2}{|\eta_2|^2 + z|\eta_1|^2} \mathrm{e}^{\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y}. \tag{4.18}
$$

Considering the potential $\widetilde{v}$, we are seeking to eliminate a simple pole at $\overline{z}_1$ in the first column and another simple pole at z_1 in the second column in the RH problem 2.2. To fulfill this objective, assuming that $\widehat{H} = (\widehat{h}_1, \widehat{h}_2)$ is an unknown invertible matrix solution of the spectral problem (2.23). Similarly, by applying a similar method to calculate the potential $\widetilde{u}$. Define a transformation

$$
\widetilde{\mathcal{N}}(z) \equiv \widehat{H}\widehat{G}(z)\widehat{H}^{-1}(z)\mathcal{N}(z)\widehat{G}^{-1}(z), \tag{4.19}
$$

where

$$\widehat{G}(z)=\begin{pmatrix} z-z_1 & 0\\ 0 & z-\overline{z}_1\end{pmatrix},$$

this requires satisfying the conditions

$$\operatorname*{Res}_{z=z_1}\widetilde{\mathcal{N}}(z)=0,\quad \operatorname*{Res}_{z=\overline{z}_1}\widetilde{\mathcal{N}}(z)=0. \tag{4.20}$$

Using the similar method as (4.21)–(4.22), we acquire

$$\widehat{h}_2=\widehat{H}e_2=\operatorname*{Res}_{z=z_1}\mathcal{N}e_1-(z_1-\overline{z}_1)\operatorname*{Res}_{z=z_1}\mathcal{N}(z)e_2, \tag{4.21}$$

similarly, for the case $z=\overline{z}_1$, we have

$$\widehat{h}_1=\widehat{H}e_1=\operatorname*{Res}_{z=\overline{z}_1}\mathcal{N}e_2-(z_1-\overline{z}_1)\operatorname*{Res}_{z=\overline{z}_1}\mathcal{N}(z)e_1, \tag{4.22}$$

thus $\widehat{H}$ can expressed as

$$\widehat{H}=(\widehat{h}_1,\widehat{h}_2)=\begin{pmatrix}\overline{\gamma}_2 & \gamma_1\\ -z\overline{\gamma}_1 & \gamma_2\end{pmatrix}. \tag{4.23}$$

We calculate

$$\begin{aligned}\widetilde{\mathcal{N}}&=\widehat{H}\Big(I-\frac{\widehat{G}_1}{z}\Big)\widehat{H}^{-1}\Big(I+\frac{\widetilde{\mathcal{N}}_1}{z}\Big)\Big(I-\frac{\widehat{G}_1}{z}\Big)^{-1}+\mathcal{O}(z^{-2})\\ &=I+\frac{\widetilde{\mathcal{N}}_1-\widehat{H}\widehat{G}_1H^{-1}+\widehat{G}_1}{z}+\mathcal{O}(z^{-2}),\end{aligned}$$

where $\widehat{G}_1=\begin{pmatrix} z_1 & 0\\ o & \overline{z}_1\end{pmatrix}$ and hence

$$\overline{\widetilde{v}}(t,x)=\mathrm{e}^{-\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y}[\widetilde{\mathcal{N}}_1-\widehat{H}\widehat{G}_1H^{-1}+\widehat{G}_1]_{12}. \tag{4.24}$$

Taking (4.23) into (4.24), we acquire

$$\widetilde{v}(t,x)=v(t,x)-2\mathrm{i}\frac{\operatorname{Im} z_1\overline{\gamma}_1\gamma_2}{|\gamma_2|^2+z|\gamma_1|^2}\mathrm{e}^{\frac{\mathrm{i}}{2}\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y}. \tag{4.25}$$

4.2 The solution procedure

Lemma 4.1 *Fix $\rho_0>0$ such that $\|r(z)\|_{H^{1,1}(\mathbb{R})\cap L^{2,-2}(\mathbb{R})}<\rho_0$, then there exists a $C(r)>0$ and a $T>0$. For $t\geq T$, we have*

$$|M_{11}(z_1)-\delta(z_1)|+|M_{22}(z_1)-\delta^{-1}(z_1)|\leq C(r)\rho_0t^{-\frac{1}{2}}, \tag{4.26}$$

$$\begin{aligned}&\Big|M_{12}(z_1)+\frac{\delta^{-1}(z_1)}{\sqrt{2t}}\Big(\frac{\beta_{12}}{z_1-1}+\frac{\overline{\beta}_{12}}{z_1+1}\Big)\Big|\\ &+\Big|M_{21}(z_1)-\frac{\delta(z_1)}{\sqrt{2t}}\Big(\frac{\beta_{21}}{z_1-1}+\frac{\overline{\beta}_{21}}{z_1+1}\Big)\Big|\leq C(r)\rho_0t^{-\frac{3}{4}}.\end{aligned} \tag{4.27}$$

Proof We have the following representation:

$$M(z_1)=M^{(3)}(z_1)M^{rhp}(z_1)\mathcal{R}^{(2)}(z_1)^{-1}\delta(z_1)^{\sigma_3}. \tag{4.28}$$

We bound $\delta(z)$ using (3.9). According to Proposition 3.3, this leads us to establish that $M^{(3)}(z_1)=I+\mathcal{O}(t^{-\frac{3}{4}})$, and $M^{rhp}(z_1)$ is as defined by (3.23). Following this, (3.24) allows us to deduce that $|\beta_{12}(z_1)|+|\beta_{21}(z_1)|\le\rho_0$, which in turn confirms Lemma 4.1.

Given Proposition 4.1, we prove Theorem 1.1 as follows:

$$\|u(t,x)-u_\lambda(t,x)\|_{L^\infty}+\|v(t,x)-v_\lambda(t,x)\|_{L^\infty}\le J_1+J_2,$$

where

$$\begin{aligned}J_1&=\|\widetilde{u}(t,x)\|_{L^\infty}+\|\widetilde{v}(t,x)\|_{L^\infty},\\ J_2&=\|\mathbf{B}_v\|_{L^\infty}+\|v_\lambda(t,x)\|_{L^\infty}+\|\mathbf{B}_u\|_{L^\infty}+\|u_\lambda(t,x)\|_{L^\infty}.\end{aligned}$$

We have applied Theorem 3.1 to bound $J_1\le Ct^{-\frac{1}{2}}$. Moving on to J_2, we find that

$$\begin{aligned}\mathbf{B}_u&=\frac{4\mathrm{Im}\,z_1\mathrm{e}^{\mathrm{i}\Theta(z_1)t}\mathrm{e}^{-\mathrm{i}\overline{\Theta(z_1)}t}\overline{\mathcal{M}}_{21}\mathcal{M}_{22}}{b_u^2\exp\Big\{-\dfrac{\mathrm{i}}{2}\displaystyle\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y\Big\}},\\ \mathbf{B}_v&=\frac{4\mathrm{Im}\,z_1\mathrm{e}^{-\mathrm{i}\Theta(z_1)t}\mathrm{e}^{\mathrm{i}\overline{\Theta(z_1)}t}\overline{\mathcal{N}}_{11}\mathcal{N}_{12}}{b_v^2\exp\Big\{-\dfrac{\mathrm{i}}{2}\displaystyle\int_x^{+\infty}(|u|^2+|v|^2)\mathrm{d}y\Big\}},\end{aligned}$$

where

$$\begin{aligned}b_u^2&=z_1|\mathrm{e}^{\mathrm{i}\Theta(z_1)t}\mathcal{M}_{21}|^2+|\mathrm{e}^{\mathrm{i}\Theta(z_1)t}\mathcal{M}_{22}|^2,\\ b_v^2&=z_1|\mathrm{e}^{-\mathrm{i}\Theta(z_1)t}\mathcal{N}_{11}|+|\mathrm{e}^{-\mathrm{i}\Theta(z_1)t}\mathcal{N}_{12}|.\end{aligned}$$

Let $\rho=\|\widetilde{r}(z)\|_{L^\infty(\mathbb{R})}$ and ε be given in (1.4), then Lemma 4.1 implies for $t>T$, we have

$$\begin{aligned}|\mathcal{M}_{21}^+|&\le2(1+\rho^2)^{-1}|\beta_{21}|t^{-\frac{1}{2}}+C\varepsilon t^{-\frac{3}{4}}\le t^{-\frac{1}{2}}\varepsilon K|\mathcal{M}_{11}^+|,\\ |\mathcal{N}_{12}^+|&\le2(1+\rho^2)^{-1}|\beta_{12}|t^{-\frac{1}{2}}+C\varepsilon t^{-\frac{3}{4}}\le t^{-\frac{1}{2}}\varepsilon K|\mathcal{N}_{22}^+|\end{aligned}$$

for a fixed and sufficiently large constant K. We conclude that

$$\begin{aligned}|\mathbf{B}_u|&\le t^{-\frac{1}{2}}\varepsilon K\frac{|\mathcal{M}_{11}^+|}{|\mathcal{M}_{22}^+|}\le t^{-\frac{1}{2}}\varepsilon K(1+\mathcal{O}(\varepsilon t^{-\frac{1}{2}}))\le C\varepsilon t^{-\frac{1}{2}},\\ |\mathbf{B}_v|&\le t^{-\frac{1}{2}}\varepsilon K\frac{|\mathcal{N}_{22}^+|}{|\mathcal{N}_{11}^+|}\le t^{-\frac{1}{2}}\varepsilon K(1+\mathcal{O}(\varepsilon t^{-\frac{1}{2}}))\le C\varepsilon t^{-\frac{1}{2}}\end{aligned}$$

for which $\mathcal{M}(\mathcal{N})_{ii}=\delta^{-(-1)^i}(z_1)+\mathcal{O}(\varepsilon t^{-\frac{1}{2}})$ and $|\delta^{\pm1}(z_1)|\le\langle\rho\rangle$. At last, as $t\to\infty$, we observe that

$$(u_\lambda(t,x),v_\lambda(t,x))\sim\mathcal{O}\Big(\mathrm{sech}\Big(f(x-vt)\pm\mathrm{i}\frac{\alpha}{2}\Big)\Big)\sim\mathcal{O}(\mathrm{e}^{-(|f(x-vt)|^2+|\frac{\alpha}{2}|^2)^{\frac{1}{2}}}),$$

then owing to

$$\mathrm{e}^{-(|f(x-vt)|^2+|\frac{\alpha}{2}|^2)^{\frac{1}{2}}} \leq t^{-\frac{1}{2}}.$$

The result is confirmed.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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