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Construction of Initial Data Sets for Einstein-Scalar and Einstein-Maxwell Equations by Conformally Covariant Split System*

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Abstract By using the implicit function, the authors prove the existence of solutions of the conformally covariant split system on compact three-dimensional Riemannian manifolds. They give rise to certain initial data for the Einstein-scalar system and the Einstein-Maxwell system.

Keywords Constraint equations, Conformally covariant split, Initial data, Einstein-scalar, Einstein-Maxwell
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1 Introduction

The celebrated work of Choquet-Bruhat shows that one can always find a spacetime solution of the Einstein equations containing an embedded hypersurface whose metric and extrinsic curvature agree with the given set of smooth initial data (see [4]). These data cannot be freely specified. In fact, they must satisfy certain geometric constraints. Seeking to construct solutions of the Einstein constraint equations is notoriously difficult. Our focus here is on the up-to-date most useful approach: The conformal method. This method was developed by Lichnerowicz [15] and Choquet-Bruhat and York [7]. The idea is to turn the constraint equations to a determined elliptic partial differential equation system via a conformal transformation.

There are at least two different ways to do the construction. The sets of free data are the same. Precisely, we have a three-dimensional Riemannian manifold (M^3, g), a symmetric trace- and divergence-free (TT) tensor of type $(0, 2)$ σ_{ij}, a smooth function τ, and certain initial values of matter fields on the manifold M.

We would like to seek a positive function ϕ and a one-form W satisfying a coupled set of elliptic partial differential equations. The physical metric $\widetilde{g}$ is given by $\widetilde{g}_{ij} = \phi^4 g_{ij}$ while the extrinsic curvature (or the second fundamental form) K is different between these two procedures.

In one of the procedures, which is called the "semi-decoupling split" (historically "Method

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A") (see [5]), the equations for (ϕ, W) take the form

$$\Delta\phi - \frac{1}{8}R_g\phi + \frac{1}{8}|\sigma + L_gW|_g^2\phi^{-7} - \frac{1}{12}\tau^2\phi^5 = 0, \tag{1.1a}$$

$$\nabla_i(L_gW)^i_j - \frac{2}{3}\phi^6\nabla_j\tau = 0, \tag{1.1b}$$

where the Laplacian Δ and the scalar curvature R_g are of the g-compatible covariant derivative ∇_i respectively, and L_g is the conformal Killing operator,

$$(L_gW)_{ij} = \nabla_iW_j + \nabla_jW_i - \frac{2}{3}(\text{div}_gW)g_{ij}.$$

Then $\big(\widetilde{g} = \phi^4g, K = \frac{\tau}{3}\widetilde{g}+\phi^{-2}(\sigma+L_gW)\big)$ satisfies the vacuum constraint equation and therefore it becomes an initial data set of the vacuum Einstein field equations.

There is another way, which we call "conformally covariant split" (historically "Method B") (see [19]), and the equations for (ϕ, W) are

$$\Delta\phi - \frac{1}{8}R_g\phi + \frac{1}{8}|\sigma|_g^2\phi^{-7} + \frac{1}{4}\langle\sigma, L_gW\rangle_g\phi^{-1} - \Big(\frac{1}{12}\tau^2 - \frac{1}{8}|L_gW|_g^2\Big)\phi^5 = 0, \tag{1.2a}$$

$$\nabla_i(L_gW)^i_j - \frac{2}{3}\nabla_j\tau + 6(L_gW)^i_j\nabla_i\log\phi = 0. \tag{1.2b}$$

Then $(M, \widetilde{g} = \phi^4g, K = \frac{\tau}{3}\widetilde{g} + \phi^{-2}\sigma + \phi^4L_gW)$ becomes an initial data set for the vacuum spacetime.

Much attention has been received in the mathematical community for the semi-decoupling split system (standard conformal method) (see [2–3]). If τ is a constant, the system (1.1) splits in a natural way so that we are only left with the well-studied Lichnerowicz equation (see [12]). Although some results are obtained for the case of non-constant τ (see [1, 9, 13–14, 17]), it remains open in general. Unfortunately, there is no such nice decoupling property for the conformally covariant split system (1.2). However, (1.2) and some of its invariants (see [10]) possess the conformal covariance property (see [16, Eqn (1.8)]). Many fewer mathematical results are known for this conformally covariant split system, albeit numerical relativists recently started to apply it for certain studies (see [8]).

In this paper, we prove the existence of solutions of the conformally covariant split system on compact three-dimensional Riemannian manifolds. They give rise to certain initial data for the Einstein-scalar system and the Einstein-Maxwell system. The seed solutions are from the vacuum initial data (see [16]) and by using the implicit function theorem, we obtain the constructed initial data sets for the coupled matter fields via perturbations.

We will use geometric units with $c = G = 1$ and the spacetime signature convention is assumed to be $(-, +, +, +)$. The standard notation $W^{k,p}$ denotes the Sobolev space of functions or tensor fields defined on the Riemannian manifold M. And $W^{k,p}_+$ denotes the subset consisting of all positive $W^{k,p}$ functions.

This paper is organized as follows. We give a brief introduction to the constraint equations in Section 2. Initial data sets for the Einstein-scalar equations and the Einstein-Maxwell equations are constructed in Section 3 and Section 4, respectively.

2 The Constraint Equations

Let us start by foliating the spacetime $(\mathcal{M}^{3,1}, \mathbf{h})$ as a set of spacelike three-dimensional hypersurfaces

$$\mathbf{h} = -(\mathrm{d}t)^2 + \widetilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

These slices are labelled by a parameter t and denoted by $M_t = \{t = \text{const.}\}$. And (x^1, x^2, x^3) are the spatial coordinates. Denote by $\widetilde{g}_{ij} = \widetilde{g}_{ij}(0)$ and $K_{ij} = \frac{1}{2}\partial_t|_{t=0}\widetilde{g}_{ij}(t)$ the induced metric and the extrinsic curvature of $M = M_0$, respectively.

Further assume that the following Einstein field equations and the matter field equation are satisfied

$$\mathbf{Ric_h} - \frac{\mathbf{Scal_h}}{2}\mathbf{h} = 8\pi\mathbf{T}, \tag{2.1a}$$

$$\mathcal{C}(\mathcal{F}, \mathbf{h}) = 0. \tag{2.1b}$$

Here $\mathbf{Ric_h}$ and $\mathbf{Scal_h}$ are the Ricci tensor and the scalar curvature of the spacetime metric $\mathbf{h}$, $\mathcal{F}$ denotes certain physical quantities of the matter field, and $\mathbf{T}$ is the associated energy-momentum tensor. The Einstein field equations (2.1a) are derived by varying the action of the field theory with respect to the metric $\mathbf{h}$ while the matter field equation (2.1b) follows from variation with respect to the matter field data $\mathcal{F}$.

Due to the Gauss equation and the Codazzi equation, these data are not freely specified. They must satisfy the Einstein constraint equations:

$$R_{\widetilde{g}} - |K|^2_{\widetilde{g}} + (\mathrm{tr}_{\widetilde{g}}K)^2 = 8\pi T_{00}, \tag{2.2a}$$

$$\widetilde{\nabla}^i K_{ij} - \widetilde{\nabla}_j(\mathrm{tr}_{\widetilde{g}}K) = 8\pi T_{0j}, \tag{2.2b}$$

where $T_{\mu\nu} = \mathbf{T}(\partial_\mu, \partial_\nu)|_{(0,x)}$. Here $\widetilde{\nabla}$ and $R_{\widetilde{g}}$ denote the covariant derivative and the scalar curvature of the metric $\widetilde{g}$, respectively.

An Einstein-scalar field theory is introduced by the choice of an action taking the form

$$\mathcal{S} = \int_{\mathcal{M}} \Big(\frac{1}{16\pi}\mathbf{Scal_h} - \frac{1}{2}|\mathrm{d}\mathbf{\Psi}|^2_{\mathbf{h}} - V(\mathbf{\Psi})\Big)\mathrm{d}\mathbf{h}.$$

The potential $V(\cdot)$ is a given smooth function of a real variable. Varying this action with respect to the scalar field $\mathbf{\Psi}$, it yields the wave equation

$$\Box_{\mathbf{h}}\mathbf{\Psi} = V'(\mathbf{\Psi}).$$

However, coupling to a scalar field does not lead to any new constraints. The constraint equations (2.2) of the Einstein-scalar system are

$$R_{\widetilde{g}} - |K|^2_{\widetilde{g}} + (\mathrm{tr}_{\widetilde{g}}K)^2 = \widetilde{\pi}^2 + |\mathrm{d}\Psi|^2_{\widetilde{g}} + 2V(\Psi), \tag{2.3a}$$

$$\widetilde{\nabla}^i K_{ij} - \widetilde{\nabla}_j(\mathrm{tr}_{\widetilde{g}}K) = -\widetilde{\pi}\widetilde{\nabla}_j\Psi, \tag{2.3b}$$

where $\Psi = \mathbf{\Psi}(0,x)$ and $\widetilde{\pi} = \partial_0 \mathbf{\Psi}(0,x)$ denote the initial value and the initial speed of the scalar field $\mathbf{\Psi}$ on M.

When the gravitational field is coupled with a source-free electromagnetic field $\mathbf{F}$, the action takes the form

$$\mathcal{S} = \int_{\mathcal{M}} \Big(\frac{1}{16\pi} \mathbf{Scal}_{\mathbf{h}} - \frac{1}{16\pi} \mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu} \Big) \mathrm{d}\mathbf{h}.$$

Varying this action with respect to the electromagnetic field leads to the homogeneous Maxwell equations

$$\mathrm{d}\mathbf{F} = 0, \tag{2.4a}$$

$$\mathrm{d}{*}\mathbf{F} = 0, \tag{2.4b}$$

where $*$ is the Hodge-star operator of the spacetime metric $\mathbf{h}$. Note that the energy-momentum tensor of the electromagnetic field $\mathbf{F}$ is

$$\mathbf{T}_{\mu\nu} = \frac{1}{4\pi} \Big(\mathbf{F}_{\mu\lambda} \mathbf{F}_{\nu}{}^{\lambda} - \frac{1}{4} \mathbf{h}_{\mu\nu} \mathbf{F}^{\lambda\rho} \mathbf{F}_{\lambda\rho} \Big).$$

Set

$$\mathbf{D}_{\mu\lambda} = ({*}\mathbf{F})_{\mu\lambda} = \frac{1}{2} {}^{(4)}\varepsilon_{\mu\lambda\alpha\beta} \mathbf{h}^{\alpha\theta} \mathbf{h}^{\beta\delta} \mathbf{F}_{\theta\delta},$$

where ${}^{(4)}\varepsilon$ is the volume element of the spacetime metric $\mathbf{h}$. Then the energy-momentum tensor can be rewritten as

$$\mathbf{T}_{\mu\nu} = \frac{1}{8\pi} (\mathbf{F}_{\mu\lambda} \mathbf{F}_{\nu}{}^{\lambda} + \mathbf{D}_{\mu\lambda} \mathbf{D}_{\nu}{}^{\lambda}).$$

Denote by

$$\widetilde{f}_{ij} = \mathbf{F}_{ij}(0,x), \quad \widetilde{d}_{ij} = \mathbf{D}_{ij}(0,x).$$

The geometric constraint equations now become

$$R_{\widetilde{g}} - |K|^2_{\widetilde{g}} + (\mathrm{tr}_{\widetilde{g}} K)^2 = |\widetilde{f}|^2_{\widetilde{g}} + |\widetilde{d}|^2_{\widetilde{g}}, \tag{2.5a}$$

$$\widetilde{\nabla}^i K_{ij} - \widetilde{\nabla}_j (\mathrm{tr}_{\widetilde{g}} K) = (\sqrt{\widetilde{g}})^{-1} \epsilon^{kil} \widetilde{d}_{il} \widetilde{f}_{jk}, \tag{2.5b}$$

where ϵ^{kil} is the standard Levi-Civita symbol.

The Maxwell equations (2.4) lead to the following additional physical constraints:

$$\widetilde{f}_{[12,3]} = 0, \tag{2.6a}$$

$$\widetilde{d}_{[12,3]} = 0. \tag{2.6b}$$

The advantage of formulating the electronic field in terms of the Faraday form $\mathbf{F}_{\mu\nu}$ instead of using the 3-vectors $\vec{E}$ (electric field) and $\vec{B}$ (magnetic field) is that, in this case, there appear only ordinary derivatives in the Maxwell constraints (2.6) which allows us to avoid studying the conformal invariance of these equations.

3 Initial Data of the Einstein-Scalar System

We consider a scalar field $\mathbf{\Psi}$ coupled with gravity. For the massive Klein-Gordon field theory, the potential function $V(\mathbf{\Psi}) = \frac{1}{2}m^2\mathbf{\Psi}^2$. There are different forms of the coefficient in the potential for the massive Klein-Gordon field. For instance, $V(\mathbf{\Psi}) = \frac{1}{2}M\mathbf{\Psi}^2$ in [18] and $V(\mathbf{\Psi}) = \frac{1}{2}m^2\mathbf{\Psi}^2$ in [6]. In this paper, we follow the convention in [6] where the parameter m indeed denotes the mass.

In this section, by using the implicit theorem, we prove the existence of solutions of the extended conformally split system. These solutions give rise to the physical initial data set for the massive Klein-Gordon field.

Suppose that we give a symmetric TT tensor σ, three smooth functions τ, Ψ, and π on a three-dimensional Riemannian manifold (M, g). We try to find a positive function ϕ and a one-form W:

$$\begin{aligned} &\Delta\phi - \frac{1}{8}(R_g - |\mathrm{d}\Psi|_g^2)\phi + \frac{1}{8}(|\sigma|_g^2 + \pi^2)\phi^{-7} + \frac{1}{4}\langle\sigma, L_g W\rangle_g \phi^{-1} \\ &- \Big(\frac{1}{12}\tau^2 - \frac{1}{4}V(\Psi) - \frac{1}{8}|L_g W|_g^2\Big)\phi^5 = 0, \end{aligned} \tag{3.1a}$$

$$\nabla_i (L_g W)^i_j - \frac{2}{3}\nabla_j \tau + 6(L_g W)^i_j \nabla_i \log\phi + \phi^{-6}\pi\nabla_j\Psi = 0. \tag{3.1b}$$

Here $\Delta = \nabla_i\nabla^i$ and R_g are the Laplacian and the scalar curvature computed of the metric g, respectively, and L_g is the conformal Killing operator.

Let $\widetilde{g} = \phi^4 g$, $K = \frac{\tau}{3}\phi^4 g + \phi^{-2}\sigma + \phi^4 L_g W$ and $\widetilde{\pi} = \phi^{-6}\pi$.

Proposition 3.1 *For* (ϕ, W) *solving system* (3.1), *then* $(M^3, \widetilde{g}, K, \Psi, \widetilde{\pi})$ *becomes an initial data set satisfying the constraint equations* (2.3).

Proof Straightforward calculations lead to

$$\begin{aligned} &R_{\widetilde{g}} - |K|_{\widetilde{g}}^2 + (\mathrm{tr}_{\widetilde{g}}K)^2 \\ =\ &\phi^{-5}(R_g\phi - 8\Delta\phi) - \Big(\frac{1}{3}\tau^2 + \phi^{-12}|\sigma|_g^2 + |L_g W|_g^2 + 2\phi^{-6}\langle\sigma, L_g W\rangle_g\Big) + \tau^2 \\ =\ &\pi^2\phi^{-12} + |\mathrm{d}\Psi|_g^2 + 2V(\Psi). \end{aligned}$$

Then the Hamiltonian constraint holds. For the momentum constraint, one checks

$$\begin{aligned} &\widetilde{\nabla}^i K_{ij} - \widetilde{\nabla}_j \mathrm{tr}_{\widetilde{g}}K \\ =\ &\nabla^i (L_g W)_{ij} + 6\phi^{-1}(\nabla^i\phi)(L_g W)_{ij} - \frac{2\nabla_j\tau}{3} \\ =\ &-\pi\phi^{-6}\nabla_j\Psi \\ =\ &-\widetilde{\pi}\nabla_j\Psi. \end{aligned}$$

Our first theorem in this section shows that the initial data set for the Einstein-scalar system can be produced from a maximal slice in vacuum. Precisely, we have the following theorem.

Theorem 3.1 *Suppose that we already have vacuum initial data* (M, g, K) *with* $\mathrm{tr}_g K = 0$. *Suppose* $K \neq 0$ *for some region. Assume further that* (M, g) *has no conformal Killing vector* *fields and the potential is of the Klein-Gordon form* $V(\Psi)=\frac{1}{2}m^2\Psi^2$. *Given any* $(\tau,\Psi,\pi)\in W^{1,p}\times W^{1,p}\times L^p$, *there is a positive constant* $\eta>0$ *such that for any* $\mu\in(0,\eta)$, *there exists at least one solution* $(\widehat{\phi},\widehat{W})\in W^{2,p}_+\times W^{2,p}$ *of system* (3.1) *for the data* $(\widehat{\sigma}=\mu^{12}K,\widehat{\tau}=\mu^{-1}\tau,\widehat{\Psi}=\mu^{13}\Psi,\widehat{\pi}=\mu^{13}\pi)$.

Proof The proof is based on the implicit function theorem and the ideas are borrowed from [5, 11, 16].

Since (M,g,K) constitutes vacuum maximal initial data, system (3.1) admits a particular solution $(\overline{\phi}\equiv 1,\overline{W}\equiv 0)$ for $\overline{\tau}=0$, $\overline{\sigma}=K$, $\overline{\Psi}=0$ and $\overline{\pi}=0$.

Let us consider the following μ-deformed system corresponding to (3.1):

$$\mathcal{G}\colon \mathbb{R}\times W^{2,p}_+\times W^{2,p}\to L^p\times L^p,$$

$$\begin{pmatrix}\mu\\ \phi\\ W\end{pmatrix}\mapsto\begin{pmatrix}\Delta\phi-\frac{1}{8}R\phi+\frac{1}{8}|K|^2\phi^{-7}+\frac{1}{4}\mu^4\langle K,L_gW\rangle\phi^{-1}-\Big(\mu^{10}\frac{1}{12}\tau^2-\mu^8\frac{1}{8}|L_gW|^2\Big)\phi^5\\ +\mu^{26}\frac{1}{8}|\mathrm{d}\Psi|^2\phi+\mu^2\frac{\pi^2}{8}\phi^{-7}+\mu^{38}\frac{1}{8}m^2\Psi^2\phi^5\\ \nabla_i(L_gW)^i_j-\frac{2}{3}\mu\nabla_j\tau+6(L_gW)^i_j\nabla_i\log\phi+\mu^{10}\phi^{-6}\pi\nabla_j\Psi\end{pmatrix}.$$

It is easy to see that $\mathcal{G}$ is a C^1-mapping. The condition that (M,g,K) constitute vacuum initial data with $\mathrm{tr}_gK=0$ implies that $\mathcal{G}(0,1,0)=(0,0)$. We now prove that the partial derivative of $\mathcal{G}$ with respect to the variables (ϕ,W) is an isomorphism at $(0,\overline{\phi}\equiv 1,\overline{W}\equiv 0)$. The differential at the point $(0,\overline{\phi}\equiv 1,\overline{W}\equiv 0)$ is given by

$$D\mathcal{G}|_{(0,1,0)}\begin{pmatrix}\delta\phi\\ \delta W\end{pmatrix}=\begin{pmatrix}\Delta-\frac{1}{8}R-\frac{7}{8}|K|^2, & 0\\ 0, & \Delta_L\end{pmatrix}\begin{pmatrix}\delta\phi\\ \delta W\end{pmatrix},$$

where $\Delta_LW=\mathrm{div}_g(L_gW)$. Since $(0,\overline{\phi}\equiv 1,\overline{W}\equiv 0)$ solves system (3.1), one has

$$\Delta-\frac{1}{8}R-\frac{7}{8}|K|^2=\Delta-|K|^2.$$

The invertibility of the derivative $D\mathcal{G}|_{(0,1,0)}$ follows from the fact that the diagonal terms are invertible.

By the implicit function theorem, for a sufficiently small parameter μ, there exists (ϕ_μ,W_μ) such that $\mathcal{G}(\mu,\phi_\mu,W_\mu)=0$.

Define $\widehat{\phi}_\mu=\mu^3\phi_\mu$ and $\widehat{W}_\mu=\mu^{-2}W_\mu$. Direct calculations show that $(\widehat{\phi}_\mu,\widehat{W}_\mu)$ solves system (3.1) for the rescaled data $(\widehat{\sigma}=\mu^{12}K,\widehat{\tau}=\mu^{-1}\tau,\widehat{\Psi}=\mu^{13}\Psi,\widehat{\pi}=\mu^{13}\pi)$.

Remark 3.1 The existence interval of the parameter μ depends on the given data (τ,Ψ,π). If μ can be chosen as 1, it means that we can construct a solution of the conformal constraint equation (3.1) with arbitrarily given (Ψ,π). However, this is not always true. For small μ, the above mechanism leads to the construction of solutions to the rescaled small initial data $(\mu^{13}\Psi,\mu^{13}\pi)$.

Assume that we already have a constant mean curvature (CMC for short) initial data set $(M^3,\overline{g},\overline{K},\overline{\Psi},\overline{\pi}\equiv 0)$ with $\overline{\tau}=\mathrm{tr}_{\overline{g}}\overline{K}=\text{const}$. This seed data can be obtained by taking constant τ and $\pi \equiv 0$ in Theorem 3.1 for instance. Notice that in this case the traceless part of $\overline{K}$, $\overline{\sigma}_{ij} = \overline{K}_{ij} - \frac{\mathrm{tr}_{\overline{g}}\overline{K}}{3}\overline{g}_{ij}$ is divergence free and this $\overline{\sigma}_{ij}$ could be used as given data in the conformal constraint equations (3.1). System (3.1) admits a special solution ($\overline{\phi} \equiv 1, \overline{W} \equiv 0$) in this particular situation. This obvious solution can be understood as transforming the seed data $(M^3, \overline{g}, \overline{K}, \overline{\Psi}, \overline{\pi})$ into itself. In subsequent part of this section, we use the implicit function theorem to deduce existence of new solutions of (3.1) in the neighbourhood of $(\overline{\tau}, \overline{\Psi}, \overline{\pi} \equiv 0)$. Our second theorem is stated as follows.

Theorem 3.2 *Suppose that we already have a CMC initial data set* $(M^3, \overline{g}, \overline{K}, \overline{\Psi}, \overline{\pi} \equiv 0)$ *with* $\overline{\tau} = \mathrm{tr}_{\overline{g}}\overline{K} = \mathrm{const}$. *Assume that* $-|\overline{K}|^2_{\overline{g}} + V(\overline{\Psi}) \le 0$ *on* M *and* $-|\overline{K}|^2_{\overline{g}} + V(\overline{\Psi}) < 0$ *in some region of* M. *Assume further that* $(M, \overline{g})$ *has no conformal Killing vector fields. Then there is a small neighborhood of* $(\overline{\tau}, \overline{\Psi}, 0)$ *in* $W^{1,p} \times W^{1,p} \times L^p$ *such that for any* (τ, Ψ, π) *in this neighborhood there exists* $(\phi, W) \in W^{2,p}_+ \times W^{2,p}$ *solving the system* (3.1) *for the data* $(\overline{\sigma}_{ij} = \overline{K}_{ij} - \frac{\overline{\tau}}{3}g_{ij}, \tau, \Psi, \pi)$.

Proof First, let us define the operator

$$\mathcal{F}\colon W^{1,p} \times W^{1,p} \times L^p \times W^{2,p}_+ \times W^{2,p} \to L^p \times L^p,$$

$$\begin{pmatrix} \tau \\ \Psi \\ \pi \\ \phi \\ W \end{pmatrix} \mapsto \begin{pmatrix} \Delta\phi - \frac{1}{8}(R - |\mathrm{d}\Psi|^2)\phi + \frac{1}{8}(|\overline{\sigma}|^2 + \pi^2)\phi^{-7} + \frac{1}{4}\langle\overline{\sigma}, LW\rangle\phi^{-1} \\ -\Big(\frac{1}{12}\tau^2 - \frac{1}{4}V(\Psi) - \frac{1}{8}|LW|^2\Big)\phi^5 \\ \nabla_i(LW)^i_j - \frac{2}{3}\nabla_j\tau + 6(LW)^i_j\nabla_i\log\phi + \phi^{-6}\pi\partial_j\Psi \end{pmatrix}.$$

It is easy to see that $\mathcal{F}$ is a C^1-mapping and $\mathcal{F}(\overline{\tau}, \overline{\Psi}, 0, \overline{\phi} \equiv 1, \overline{W} \equiv 0) = (0, 0)$. We prove that the partial derivative of $\mathcal{F}$ with respect to the variables (ϕ, W) is an isomorphism at $(\overline{\tau}, \overline{\Psi}, 0, \overline{\phi} \equiv 1, \overline{W} \equiv 0)$. The differential at the point $(\overline{\tau}, \overline{\Psi}, 0, \overline{\phi} \equiv 1, \overline{W} \equiv 0)$ is given by

$$D\mathcal{F}|_{(\overline{\tau},\overline{\Psi},0,1,0)}\begin{pmatrix}\delta\phi \\ \delta W\end{pmatrix}$$
$$= \begin{pmatrix} \Delta - \frac{1}{8}(R - |\mathrm{d}\overline{\Psi}|^2) - \frac{7}{8}|\overline{\sigma}|^2 - \frac{5}{12}\overline{\tau}^2 + \frac{5}{4}V(\overline{\Psi}), & \frac{1}{4}\langle\overline{\sigma}, L(\cdot)\rangle \\ 0, & \Delta_L \end{pmatrix}\begin{pmatrix}\delta\phi \\ \delta W\end{pmatrix},$$

and it is triangular, meaning that the second row of the above 2×2 block matrix does not depend on $\delta\phi$. Thus, the invertibility of $D\mathcal{F}|_{(\overline{\tau},\overline{\Psi},0,1,0)}$ follows from the fact that the diagonal terms are invertible. More specifically, we have the following claims.

Claim 1

$$\mathcal{H}\colon W^{2,p} \to L^p,$$
$$\delta\phi \mapsto \Big(\Delta - \frac{1}{8}(R - |\mathrm{d}\overline{\Psi}|^2) - \frac{7}{8}|\overline{\sigma}|^2 - \frac{5}{12}\overline{\tau}^2 + \frac{5}{4}V(\overline{\Psi})\Big)\delta\phi$$

is invertible.

Claim 2

$$\begin{aligned}\Delta_L\colon W^{2,p} &\to L^p,\\ \delta W &\mapsto \Delta_L \delta W\end{aligned}$$

is also invertible.

The proof of Claim 2 is a consequence of the assumption that $(M,\overline{g})$ is closed and has no conformal Killing vector fields. The proof of Claim 1 is as follows. Note that $\mathcal{H}$ is a Fredholm operator of index 0. It suffices to show that $\mathcal{H}$ is injective. Since $(\overline{\phi}\equiv 1, \overline{W}\equiv 0)$ solves the system (3.1), one has

$$-\frac{1}{8}R+\frac{1}{8}|\overline{\sigma}|^2+\frac{1}{8}|\mathrm{d}\overline{\Psi}|^2-\frac{1}{12}\overline{\tau}^2+\frac{1}{4}V(\overline{\Psi})=0.$$

Hence,

$$\begin{aligned}&\Delta-\frac{1}{8}\big(R-|\mathrm{d}\overline{\Psi}|^2\big)-\frac{7}{8}|\overline{\sigma}|^2-\frac{5}{12}\overline{\tau}^2+\frac{5}{4}V(\overline{\Psi})\\ &=\Delta-|\overline{\sigma}|^2-\frac{1}{3}\overline{\tau}^2+V(\overline{\Psi})\\ &=\Delta-|\overline{K}|^2+V(\overline{\Psi}).\end{aligned}$$

Clearly, it is a negatively definite operator. This completes the proof.

The initial data sets constructed by using the above two theorems have small π. This drawback can be compensated by the following scaling symmetry of the conformal constraint system (3.1).

Proposition 3.2 *Suppose that system* (3.1) *has a solution* (ϕ, W) *for the data* (g,σ,τ,Ψ,π) *with the Klein-Gordon potential* $V(\Psi)=\frac{1}{2}m^2\Psi^2$. *Set* $\widehat{\phi}=\lambda^{-\frac{1}{4}}\phi$, $\widehat{W}=\lambda^{\frac{1}{2}}W$ *for some positive number* $\lambda\in\mathbb{R}^+$. *Then* $(\widehat{\phi},\widehat{W})$ *satisfies system* (3.1) *for the data* $(\widehat{g}=g,\widehat{\sigma}_{ij}=\lambda^{-1}\sigma_{ij},\widehat{\tau}=\lambda^{\frac{1}{2}}\tau,\widehat{\Psi}=\Psi,\widehat{\pi}=\lambda^{-1}\pi)$ *and* $\widehat{m}=\lambda^{\frac{1}{2}}m$.

Suppose that we have already constructed a physical hypersurface $\big(M^3,\phi^4 g,\phi^{-2}\sigma+\phi^4 L_g W+\frac{\tau}{3}\phi^4 g\big)$ with mass m Klein-Gordon field initial data $(\Psi,\phi^{-6}\pi)$. Then by Proposition 3.2, the physical hypersurface $\big(M^3,\widehat{\phi}^4 g,\widehat{\phi}^{-2}\widehat{\sigma}+\phi^4 L_g\widehat{W}+\frac{\widehat{\tau}}{3}\widehat{\phi}^4 g\big)$ with mass $\widehat{m}$ Klein-Gordon field initial data $(\widehat{\Psi},\widehat{\phi}^{-6}\widehat{\pi})$ satisfies the constraint equations (2.3). We can have large $\widehat{\pi}$ if we choose λ sufficiently small.

4 Initial Data of the Einstein-Maxwell System

In this section, we are going to construct certain initial data set for the source-free electromagnetic field coupled with gravity. Different from the Einstein-scalar system, the Maxwell equations (2.4) give additional physical constraints (2.6). These data can be obtained from those of the flat space. Thus, we should merely concentrate on the geometric constraints (2.5).

Recall that we are given a symmetric TT tensor σ, a smooth functions τ, and two skew-symmetric tensor fields of type $(0,2)$ f_{ij} and d_{ij} on a Riemannian manifold (M,g).

Proposition 4.1 *Let f_{ij} and d_{ij} be two $W^{1,p}$ skew-symmetric tensor fields of type $(0,2)$ on (M,g) satisfying $f_{[12,3]}=d_{[12,3]}=0$. If there is $(\phi,W)\in W^{2,p}_{+}\times W^{2,p}$ solving the following conformally covariant split system*

$$\Delta\phi-\frac{1}{8}R_g\phi+\frac{1}{8}|\sigma|_g^2\phi^{-7}+\frac{1}{4}\langle\sigma,L_gW\rangle_g\phi^{-1}-\Big(\frac{1}{12}\tau^2-\frac{1}{8}|L_gW|_g^2\Big)\phi^5$$
$$+\frac{1}{8}(|f|_g^2+|d|_g^2)\phi^{-3}=0, \tag{4.1a}$$
$$\nabla_i(L_gW)^i_j-\frac{2}{3}\nabla_j\tau+6(L_gW)^i_j\nabla_i\log\phi-\phi^{-6}g^{-\frac{1}{2}}\epsilon^{kil}f_{il}d_{jk}=0, \tag{4.1b}$$

then $\big(M,\widetilde{g}=\phi^4g,K=\frac{\tau}{3}\phi^4g+\phi^{-2}\sigma+\phi^4L_gW,\widetilde{f}=f,\widetilde{d}=d\big)$ satisfies the constraint equations (2.5).

Proof Let us firstly check the Hamiltonian constraint as follows:

$$\begin{aligned}
&R_{\widetilde{g}}-|K|_{\widetilde{g}}^2+(\mathrm{tr}_{\widetilde{g}}K)^2\\
&=\phi^{-5}(R_g\phi-8\Delta\phi)-\Big(\frac{1}{3}\tau^2+\phi^{-12}|\sigma|_g^2+|L_gW|_g^2+2\phi^{-6}\langle\sigma,L_gW\rangle_g\Big)+\tau^2\\
&=-8\phi^{-5}\Big(\Delta\phi-\frac{1}{8}R_g\phi+\frac{1}{8}|\sigma|_g^2\phi^{-7}+\frac{1}{4}\langle\sigma,L_gW\rangle_g\phi^{-1}-\Big(\frac{1}{12}\tau^2-\frac{1}{8}|L_gW|_g^2\Big)\phi^5\Big)\\
&=\phi^{-8}(|f|_g^2+|d|_g^2)\\
&=\phi^{-8}g^{ik}g^{jl}(d_{kl}d_{ij}+f_{kl}f_{ij})\\
&=\widetilde{g}^{ik}\widetilde{g}^{jl}(\widetilde{d}_{kl}\widetilde{d}_{ij}+\widetilde{f}_{kl}\widetilde{f}_{ij})\\
&=|\widetilde{f}|_{\widetilde{g}}^2+|\widetilde{d}|_{\widetilde{g}}^2.
\end{aligned}$$

For the momentum constraint equations, one has

$$\begin{aligned}
&\widetilde{\nabla}^iK_{ij}-\widetilde{\nabla}_j\mathrm{tr}_{\widetilde{g}}K\\
&=\widetilde{\nabla}^i\Big(\frac{\tau}{3}\phi^4g_{ij}+\phi^{-2}\sigma_{ij}+\phi^4(L_gW)_{ij}\Big)-\widetilde{\nabla}_j\tau\\
&=\widetilde{\nabla}^i(\phi^{-2}(\sigma_{ij}+\phi^6(L_gW)_{ij}))-\widetilde{\nabla}_j\tau\\
&=\phi^{-6}\nabla^i(\phi^6(L_gW)_{ij})-\frac{2\nabla_j\tau}{3}\\
&=\nabla^i(L_gW)_{ij}+6\phi^{-1}(\nabla^i\phi)(L_gW)_{ij}-\frac{2\nabla_j\tau}{3}\\
&=\phi^{-6}g^{-\frac{1}{2}}\epsilon^{kil}f_{il}d_{jk}\\
&=(\widetilde{g})^{-\frac{1}{2}}\epsilon^{kil}\widetilde{f}_{il}\widetilde{d}_{jk}.
\end{aligned}$$

Note that there appear only ordinary derivatives of f and d in the electromagnetic constraints, the form of the equations are invariant under conformal transformation. Hence,

$$\begin{aligned}
\widetilde{f}_{[12,3]}&=f_{[12,3]}=0,\\
\widetilde{d}_{[12,3]}&=d_{[12,3]}=0.
\end{aligned}$$

Therefore, $(M,\widetilde{g},K,\widetilde{f},\widetilde{d})$ solves (2.5).

Theorem 4.1 *Suppose that we already have vacuum initial data* (M,g,K) *with* $\mathrm{tr}_g K = 0$. *Suppose* $K \neq 0$ *for some region. Assume further that* (M,g) *has no conformal Killing vector fields. For any given* $(\tau, f, d) \in W^{1,p} \times W^{1,p} \times W^{1,p}$ *such that* $f_{[12,3]} = d_{[12,3]} = 0$, *there is a positive constant* $\eta > 0$ *such that for any* $\mu \in (0,\eta)$, *there exists at least one solution* $(\widehat{\phi}, \widehat{W}) \in W^{2,p}_+ \times W^{2,p}$ *of system* (4.1) *for the data* $(\widehat{\sigma} = \mu^{12}K, \widehat{\tau} = \mu^{-1}\tau, \widehat{f} = \mu^9 f, \widehat{d} = \mu^9 d)$.

Proof The proof is similar to that of Theorem 3.1.

Since (M,g,K) constitutes vacuum maximal initial data, system (3.1) admits a particular solution $(\overline{\phi} \equiv 1, \overline{W} \equiv 0)$ for $\overline{\tau} = 0$, $\overline{\sigma} = K$, $f = 0$ and $d = 0$.

Let us consider the following μ-deformed system corresponding to (4.1):

$$\mathcal{G}\colon \mathbb{R} \times W^{2,p}_+ \times W^{2,p} \to L^p \times L^p,$$

$$\begin{pmatrix}\mu\\ \phi\\ W\end{pmatrix} \mapsto \begin{pmatrix} \Delta\phi - \frac{1}{8}R\phi + \frac{1}{8}|K|^2\phi^{-7} + \frac{1}{4}\mu^4\langle K, L_gW\rangle\phi^{-1} - \Big(\mu^{10}\frac{1}{12}\tau^2 - \mu^8\frac{1}{8}|L_gW|^2\Big)\phi^5 \\ +\frac{1}{8}\mu^6(|f|_g^2 + |d|_g^2)\phi^{-3} \\ \nabla_i(L_gW)^i_j - \frac{2}{3}\mu\nabla_j\tau + 6(L_gW)^i_j\nabla_i\log\phi - \mu^2\phi^{-6}g^{-\frac{1}{2}}\epsilon^{kil}d_{il}f_{jk} \end{pmatrix}.$$

It is easy to see that $\mathcal{G}$ is a C^1-mapping. The condition that (M,g,K) constitute vacuum initial data with $\mathrm{tr}_g K = 0$ implies that $\mathcal{G}(0,1,0) = (0,0)$. We now prove that the partial derivative of $\mathcal{G}$ with respect to the variables (ϕ, W) is an isomorphism at $(0, \overline{\phi} \equiv 1, \overline{W} \equiv 0)$. The differential at the point $(0, \overline{\phi} \equiv 1, \overline{W} \equiv 0)$ is given by

$$D\mathcal{G}|_{(0,1,0)}\begin{pmatrix}\delta\phi\\ \delta W\end{pmatrix} = \begin{pmatrix}\Delta - \frac{1}{8}R - \frac{7}{8}|K|^2, & 0\\ 0, & \Delta_L\end{pmatrix}\begin{pmatrix}\delta\phi\\ \delta W\end{pmatrix},$$

where $\Delta_L W = \mathrm{div}_g(LW)$. Since $(0, \overline{\phi} \equiv 1, \overline{W} \equiv 0)$ solves system (4.1), one has

$$\Delta - \frac{1}{8}R - \frac{7}{8}|K|^2 = \Delta - |K|^2.$$

The invertibility of the derivative $D\mathcal{G}|_{(0,1,0)}$ follows from the fact that the diagonal terms are invertible.

By the implicit function theorem, for a sufficiently small parameter μ, there exists (ϕ_μ, W_μ) such that $\mathcal{G}(\mu, \phi_\mu, W_\mu) = 0$.

Define $\widehat{\phi}_\mu = \mu^3\phi_\mu$ and $\widehat{W}_\mu = \mu^{-2}W_\mu$. Direct calculations show that $(\widehat{\phi}_\mu, \widehat{W}_\mu)$ solves system (4.1) for the rescaled data $(\widehat{\sigma} = \mu^{12}K, \widehat{\tau} = \mu^{-1}\tau, \widehat{f} = \mu^9 f, \widehat{d} = \mu^9 d)$. Since μ is a constant, it is clear that $\widehat{f}$ and $\widehat{d}$ also satisfy the Maxwell constraints (2.6).

Remark 4.1 The existence interval of the parameter μ depends on the given data (τ, f, d). If μ can be chosen as 1, it means that we can construct a solution of the conformal constraint equation (4.1) with arbitrarily given (f, d). However, this is not always true. For small μ, the above mechanism leads to the construction of solutions to the rescaled small initial data $(\widehat{f} = \mu^9 f, \widehat{d} = \mu^9 d)$ for the electromagnetic field.

Assume that we already have a CMC initial data set $(M^3, \overline{g}, \overline{K}, \overline{f}, \overline{d} \equiv 0)$ with $\overline{\tau} = \mathrm{tr}_{\overline{g}}\overline{K} =$ const. This seed data can be obtained by taking constant τ and $\pi \equiv 0$ in Theorem 4.1. Notice that in this case the traceless part of $\overline{K}$, $\overline{\sigma}_{ij} = \overline{K}_{ij} - \frac{\mathrm{tr}_{\overline{g}}\overline{K}}{3}\overline{g}_{ij}$ is divergence free and $\overline{d} = 0$, this $\overline{\sigma}_{ij}$ could be used as given data in the conformal constraint equations (4.1). System (4.1) admits a special solution $(\overline{\phi} \equiv 1, \overline{W} \equiv 0)$ in this particular situation. This obvious solution can be understood as transforming the seed data $(M^3, \overline{g}, \overline{K}, \overline{f}, \overline{d} \equiv 0)$ into itself. In subsequent part of this section, we use the implicit function theorem to deduce existence of new solutions of (4.1) in the neighbourhood of $(\overline{\tau}, \overline{f}, \overline{d} \equiv 0)$. Here is our second theorem in this section.

Theorem 4.2 *Suppose that we already have a CMC initial data set* $(M^3, \overline{g}, \overline{K}, \overline{f}, \overline{d} \equiv 0)$ *with* $\overline{\tau} = \mathrm{tr}_{\overline{g}}\overline{K} =$ *const. Assume that* $-|\overline{K}|^2_{\overline{g}} - \frac{1}{2}|\overline{f}|^2_{\overline{g}} \leq 0$ *on* M *and* $-|\overline{K}|^2_{\overline{g}} - \frac{1}{2}|\overline{f}|^2_{\overline{g}} < 0$ *in some region of* M*. Assume further that* $(M, \overline{g})$ *has no conformal Killing vector fields. Then there is a small neighbourhood of* $(\overline{\tau}, \overline{f}, \overline{d} \equiv 0)$ *in* $W^{1,p} \times W^{1,p} \times L^p$ *such that for any* (τ, f, d) *in this neighbourhood there exists* $(\phi, W) \in W^{2,p}_+ \times W^{2,p}$ *solving the system* (4.1) *for the data* $(\overline{\sigma}_{ij} = \overline{K}_{ij} - \frac{\overline{\tau}}{3}g_{ij}, \tau, f, d)$.

The proof is similar to that of Theorem 3.2. One defines

$$\mathcal{F} \colon W^{1,p} \times W^{1,p} \times W^{1,p} \times W^{2,p}_+ \times W^{2,p} \to L^p \times L^p,$$

$$\begin{pmatrix} \tau \\ f \\ d \\ \phi \\ W \end{pmatrix} \mapsto \begin{pmatrix} \Delta\phi - \frac{1}{8}R_{\overline{g}}\phi + \frac{1}{8}|\overline{\sigma}|^2_{\overline{g}}\phi^{-7} + \frac{1}{4}\langle\overline{\sigma}, L_{\overline{g}}W\rangle_{\overline{g}}\phi^{-1} - \left(\frac{1}{12}\tau^2 - \frac{1}{8}|L_{\overline{g}}W|^2_{\overline{g}}\right)\phi^5 \\ +\frac{1}{8}\phi^{-3}(|f|^2_{\overline{g}} + |d|^2_{\overline{g}}) \\ \nabla_i(L_{\overline{g}}W)^i_j - \frac{2}{3}\nabla_j\tau + 6(L_{\overline{g}}W)^i_j\nabla_i\log\phi - \phi^{-6}\overline{g}^{-\frac{1}{2}}\epsilon^{kil}d_{il}f_{jk} \end{pmatrix}.$$

It remains to prove that the differential at $(\overline{\tau}, \overline{f}, \overline{d} \equiv 0, \overline{\phi} \equiv 1, \overline{W} \equiv 0)$ is an isomorphism. The initial data sets constructed from the above two theorems have small d. Again, this drawback can be compensated by the following scaling symmetry of the conformal constraint system (4.1).

Proposition 4.2 *Suppose that system* (4.1) *has a solution* (ϕ, W) *for the data* (g, σ, τ, f, d) *with the Einstein–Maxwell system. Set* $\widehat{\phi} = \lambda^{-\frac{1}{4}}\phi$, $\widehat{W} = \lambda^{\frac{1}{2}}W$ *for some positive number* $\lambda \in \mathbb{R}^+$*. Then* $(\widehat{\phi}, \widehat{W})$ *satisfies system* (4.1) *for the data* $(\widehat{g} = g, \widehat{\sigma}_{ij} = \lambda^{-1}\sigma_{ij}, \widehat{\tau} = \lambda^{\frac{1}{2}}\tau, \widehat{f} = \lambda^{\frac{1}{2}}f, \widehat{d} = \lambda^{\frac{1}{2}}d)$.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Allen, P. T., Clausen, A. and Isenberg, J., Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics, *Class. Quantum Grav.*, **25**(7), 2008, 075009.

[2] Bartnik, R. and Isenberg, J., The Constraint Equations, The Einstein Equations and the Large Scale Behavior of Gravitational Fields, Chruściel, P. T. and Friedrich, H. (eds.), Birkhäuser, Basel, 2004, 1–38.

[3] Carlotto, A., The general relativistic constraint equations, *Living Rev. Relativity*, **24**, 2021 Paper No. 2.

[4] Choquet-Bruhat, Y., Théorème d'existence pour certains systèmes d'quations aux dérivées partielles non linéaires, *Acta Math.*, **88**, 1952, 141–225.

[5] Choquet-Bruhat, Y., Einstein constraints on compact n-dimensional manifolds, *Class. Quantum Grav.*, **21**(3), 2004, S127–S151.

[6] Choquet-Bruhat, Y., Isenberg, J., Pollack, D. and Geroch, R., The constraint equations for the Einstein-scalar field system on compact manifolds, *Class. Quantum Grav.*, **24**(4), 2007, 809–828.

[7] Choquet-Bruhat, Y. and York, J. W., The Cauchy Problem, General Relativity and Gravitation, Vol. 1, Held, A. (ed.), Plenum, New York, 1980, 99–172.

[8] Cook, G. B., Initial data for numerical relativity, *Living Rev. Relativity*, **3**, 2000 Paper No. 5.

[9] Dahl, M., Gicquaud, R. and Humbert, E., A limit equation associated to the solvability of the vacuum Einstein constraint equations by using the conformal method, *Duke Math. J.*, **161**(14), 2012, 2669–2697.

[10] Delay, E., Conformally covariant parametrizations for relativistic initial data, *Class. Quantum Grav.*, **34**(1), 2017, 01LT01.

[11] Gicquaud, R. and Ngô, Q. A., A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor, *Class. Quantum Grav.*, **31**(19), 2014, 195014.

[12] Isenberg, J., Constant mean curvature solutions of the Einstein constraint equations on closed manifolds, *Class. Quantum Grav.*, **12**(9), 1995, 2249–2274.

[13] Isenberg, J. and Moncrief, V., Some Results on Nonconstant Mean Curvature Solutions of the Einstein Constraint Equations, Physics on manifolds (Paris, 1992), Math. Phys. Stud., Vol. 15, Flato, M. Kerner, R. and Lichnerowicz, A. (eds.), Kluwer Acad. Publ., Dordrecht, 1994, 295–302.

[14] Isenberg, J. and Ó Murchadha, N., Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations, *Class. Quantum Grav.*, **21**(3), 2004, S233–S241.

[15] Lichnerowicz, A., L'intégration des équations de la gravitation relativiste et le problème des n-corps, *J. Math. Pures Appl.*, **23**(9), 1944, 37–63.

[16] Mach, P., Wang, Y. and Xie, N., Construction of vacuum initial data by the conformally covariant split system, *Class. Quantum Grav.*, **36**(14), 2019, 145001.

[17] Maxwell, D., A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature, *Math. Res. Lett.*, **16**(4), 2009, 627–645.

[18] Premoselli, B., The Einstein-scalar field constraint system in the positive case, *Commun. Math. Phys.*, **326**(2), 2014, 543–557.

[19] York, J. W., Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial value problem of general relativity, *J. Math. Phys.*, **14**(4), 1973, 456–464.