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Symmetric Reduction of a Regular Controlled Lagrangian System with a Momentum Map*

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In memory of my advisor—Professor Hesheng HU

Abstract In this paper, the author first defines a regular controlled Lagrangian (RCL for short) system on a symplectic fiber bundle, establishing a good expression of the dynamical vector field of an RCL system. This dynamical vector field synthesizes the Euler-Lagrange vector field and its changes under the actions of the external force and the control. Moreover, the author describes the RCL-equivalence, the RpCL-equivalence, and the RoCL-equivalence, proving regular point and regular orbit reduction theorems for the RCL system and the regular Lagrangian system with symmetry and a momentum map. Finally, as an application the author considers the regular point reducible RCL systems on a generalization of Lie group.

Keywords Regular controlled Lagrangian system, Legendre transformation, RCL-equivalence, Momentum map, Regular point reduction, Regular orbit reduction

2020 MR Subject Classification 70H33, 53D20, 70Q05

1 Introduction

Following the developments of science and technology, researchers paid close attention to the study of Hamiltonian systems with controls. In [12], Marsden et al. set up a kind of regular reduction theory for a regular controlled Hamiltonian (RCH for short) system defined on a symplectic fiber bundle with symmetry and a momentum map, from the viewpoint of the completeness of the Marsden-Weinstein reduction, and by using the careful analysis of the geometrical and the topological structures of phase space and the reduced phase spaces of the system. The reduction is an extension of symmetric reduction theory for a Hamiltonian system with a momentum map under the regular controlled Hamiltonian equivalence conditions. These researches reveal internal relationships of geometrical structures of phase spaces, the dynamical vector fields and the controls of the RCH system and its reduced systems.

In this paper, we first define an RCL system on a symplectic fiber bundle, by using Legendre transformation and Euler-Lagrange vector field, and following the ideas in [12]. The RCL system is a regular Lagrangian system with the external force and the control. In general, the RCL

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system under the action of the external force and the control is not a regular Lagrangian system, however, it is a dynamical system closely related to a regular Lagrangian system, and it can be explored and studied by extending the methods for the external force and the control in the study of the regular Lagrangian systems. In consequence, we can set up the regular reduction theory for an RCL system with symmetry and a momentum map, by analyzing carefully the geometrical and the topological structures of the phase space and the reduced phase spaces of the corresponding regular Lagrangian system.

A brief outline of this paper is as follows. In Section 2, we review some relevant definitions and basic facts about the regular Lagrangian system and its regular point and regular orbit reductions; we also analyse the geometrical structures of phase space and the reduced phase spaces of a regular Lagrangian system, which will be used in subsequent sections. An RCL system is defined by using a (Lagrangian) symplectic form on a symplectic fiber bundle and on the tangent bundle of a configuration manifold, respectively, and a good expression of the dynamical vector field for the RCL system is given, and the RCL-equivalence is introduced in Section 3. From Section 4 we begin to discuss the RCL systems with symmetries and the momentum maps by combining with regular reduction theory of a regular Lagrangian system. The regular point and regular orbit reducible RCL systems are considered, respectively, in Section 4 and Section 5, and we prove the regular point and regular orbit reduction theorems for the RCL systems, which explain the relationships between the RpCL-equivalence, the RoCL-equivalence for the reducible RCL systems with symmetries and the RCL-equivalence for the associated reduced RCL systems. We also study the equivalence relationships of the regular reducible Lagrangian systems, the R_p-reduced Lagrangian systems and the R_o-reduced Lagrangian systems. As the applications of the theoretical results, in Section 6, we give a regular point reducible RCL system on the generalization $G \times V$ of a Lie group G, where V is a vector space. The R_p-reduced system is an RCL system on the generalization $\mathcal{O}_\mu \times V \times V$ of a co-adjoint orbit $\mathcal{O}_\mu$ of G. These research works develop the theory of symmetric reduction for the RCL systems with symmetries and the momentum maps, and make us have a much deeper understanding and recognition for the structures of the regular controlled mechanical systems.

2 Legendre Transformation, Regular Lagrangian System and Its Reduction

In the following, we first give some relevant definitions and basic facts about Legendre transformation and the regular Lagrangian system including its regular reductions. We also analyse the geometrical structures of the phase space and the reduced phase spaces for a regular Lagrangian system with symmetry, which will be used in subsequent sections. We shall follow the notations and conventions introduced in [1–2, 7–8, 11, 16–17]. For convenience, we assume that all manifolds in this paper are real, smooth and finite dimensional. In particular, in the following we always assume that Q is a smooth manifold with coordinates q^i, and TQ its tangent bundle with coordinates $(q^i, \dot{q}^i)$, and T^*Q its cotangent bundle with coordinates (q^i, p_i), which is the canonical cotangent bundle coordinates of T^*Q and $\theta = p_i \mathbf{d}q^i$ and $\omega = -\mathbf{d}\theta = \mathbf{d}q^i \wedge \mathbf{d}p_i$ are the canonical one-form and the canonical symplectic form on T^*Q, respectively, where the summation on repeated indices is understood.

Definition 2.1 *Assume that Q is an n-dimensional smooth manifold and the function L :*

$TQ \to \mathbb{R}$. Then the map $\mathcal{F}L : TQ \to T^*Q$ defined by

$$\mathcal{F}L(v)w := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_q(v+tw), \quad \forall v, w \in T_qQ, \tag{2.1}$$

is a fiber-preserving smooth map, which is called the fiber derivative of L, where L_q denotes the restriction of L to the fiber over $q \in Q$. If $\mathcal{F}L : TQ \to T^*Q$ is a local diffeomorphism, then $L : TQ \to \mathbb{R}$ is called a regular Lagrangian; and if $\mathcal{F}L : TQ \to T^*Q$ is a diffeomorphism, then L is called hyperregular.

In the finite dimensional case, the local expression of the map $\mathcal{F}L : TQ \to T^*Q$ is given by

$$\mathcal{F}L(q^i, \dot{q}^i) = \Big(q^i, \frac{\partial L}{\partial \dot{q}^i}\Big) = (q^i, p_i). \tag{2.2}$$

The change of data from $(q^i, \dot{q}^i)$ on TQ to (q^i, p_i) on T^*Q, which is given by the map $\mathcal{F}L : TQ \to T^*Q$, is called a Legendre transformation. From Marsden and Ratiu [11], we know that the Lagrangian L is regular, if the matrix $(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j})$ is invertible. In the following by using the Legendre transformation, we can give a definition of a regular Lagrangian system as follows.

Definition 2.2 (Regular Lagrangian system) *Assume that Q is a smooth manifold, and θ and ω are the canonical one-form and the canonical symplectic form on the cotangent bundle T^*Q, and the function $L : TQ \to \mathbb{R}$ is hyperregular. Denote $\theta^L := (\mathcal{F}L)^*\theta$ and $\omega^L := (\mathcal{F}L)^*\omega$, where the bundle map $(\mathcal{F}L)^* : T^*T^*Q \to T^*TQ$. Then θ^L and ω^L are called the Lagrangian one-form and the Lagrangian symplectic form on the tangent bundle TQ, respectively. Define an action $A : TQ \to \mathbb{R}$ given by $A(v) := \mathcal{F}L(v)v$, $\forall v \in T_qQ$ and an energy $E_L : TQ \to \mathbb{R}$ given by $E_L := A - L$. If there exists a vector field ξ_L on TQ, such that the Euler-Lagrange equation $\mathbf{i}_{\xi_L}\omega^L = \mathbf{d}E_L$ holds, then ξ_L is called an Euler-Lagrange vector field of L, and the triple (TQ, ω^L, L) is called a regular Lagrangian system.*

In the finite dimensional case, the local expression of θ^L and ω^L are given by

$$\theta^L = \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i, \quad \omega^L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\mathrm{d}q^i \wedge \mathrm{d}\dot{q}^j,$$

where the summation on repeated indices is understood. Moreover, we know that the energy E_L is conserved along the flow of the Euler-Lagrange vector field ξ_L, if ξ_L satisfies a second order equation, that is, $T\tau_Q \circ \xi_L = id_{TQ}$, where the map $T\tau_Q : TTQ \to TQ$, is the tangent map of the projection $\tau_Q : TQ \to Q$. Moreover, in a local coordinates of TQ, an integral curve $(q(t), \dot{q}(t))$ of ξ_L satisfies the following Euler-Lagrange equations:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \dot{q}^i, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, 2, \cdots, n.$$

If L is regular, then ξ_L always satisfies the second order equation.

Furthermore, by using the Legendre transformation, the following proposition gives a description of the equivalence between the regular Lagrangian system (TQ, ω^L, L) and the Hamiltonian system (T^*Q, ω_0, H) under the hyperregular case of L (see [11]).

Proposition 2.1 *Assume that $L : TQ \to \mathbb{R}$ is a hyperregular Lagrangian on TQ. Define a function $H := E_L \cdot (\mathcal{F}L)^{-1} : T^*Q \to \mathbb{R}$. Then H is a hyperregular Hamiltonian on T^*Q, and* *the Hamiltonian vector field* $X_H \in TT^*Q$ *and the Euler-Lagrange vector field* $\xi_L \in TTQ$ *are* $\mathcal{F}L$*-related, i.e.,* $T(\mathcal{F}L)\cdot\xi_L = X_H\cdot\mathcal{F}L$*, where* $T(\mathcal{F}L): TTQ \to TT^*Q$ *is the tangent map of* $\mathcal{F}L : TQ \to T^*Q$*, and the integral curves of* ξ_L *are mapped by* $\mathcal{F}L$ *onto integral curves of* X_H*.*

It is well-known that Hamiltonian reduction theory is one of the most active subjects in the study of modern analytical mechanics and applied mathematics, in which a lot of deep and beautiful results have been obtained; for these results, we refer to the studies given in [1–3, 5, 7–11, 13–16], among which the Marsden-Weinstein reduction for the Hamiltonian systems with symmetry and momentum maps is the most important and foundational. Now, for a regular Lagrangian system with symmetry and momentum map, we can also give its regular point reduction as follows.

Let Q be a smooth manifold and TQ its tangent bundle with the induced Lagrangian symplectic form ω^L. Assume that $\Phi : G\times Q \to Q$ is a smooth left action of a Lie group G on Q, which is free and proper, then the tangent lifted left action $\Phi^T : G\times TQ \to TQ$ is also free and proper. Moreover, assume that the action is symplectic with respect to ω^L and admits an Ad^*-equivariant momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$. For a regular value of $\mathbf{J}_L$, $\mu\in\mathfrak{g}^*$, denote by $G_\mu = \{g\in G \mid \mathrm{Ad}_g^*\mu = \mu\}$ the isotropy subgroup of the co-adjoint G-action at the point $\mu\in\mathfrak{g}^*$. Since $G_\mu(\subset G)$ acts freely and properly on Q and on TQ, $Q_\mu = Q/G_\mu$ is a smooth manifold, and the canonical projection $\rho_\mu : Q\to Q_\mu$ is a surjective submersion. It follows that G_μ also acts freely and properly on $\mathbf{J}_L^{-1}(\mu)$, so that the space $(TQ)_\mu = \mathbf{J}_L^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form ω_μ^L uniquely characterized by the relation

$$\tau_\mu^*\cdot\omega_\mu^L = j_\mu^*\cdot\omega^L. \tag{2.3}$$

The map $j_\mu : \mathbf{J}_L^{-1}(\mu)\to TQ$ is the inclusion and $\tau_\mu : \mathbf{J}_L^{-1}(\mu)\to (TQ)_\mu$ is the projection. The pair $((TQ)_\mu, \omega_\mu^L)$ is called the regular point reduced space of (TQ, ω^L) at μ.

Let $L : TQ \to \mathbb{R}$ be a G-invariant hyperregular Lagrangian, the flow F_t of the Euler-Lagrange vector field ξ_L leaves the connected components of $\mathbf{J}_L^{-1}(\mu)$ invariant and commutes with the G-action, so it induces a flow f_t^μ on $(TQ)_\mu$, defined by $f_t^\mu\cdot\tau_\mu = \tau_\mu\cdot F_t\cdot j_\mu$, and the vector field ξ_{l_μ} generated by the flow f_t^μ on $((TQ)_\mu, \omega_\mu^L)$ is the reduced Euler-Lagrange vector field with the associated regular point reduced Lagrangian function $l_\mu : (TQ)_\mu\to\mathbb{R}$ defined by $l_\mu\cdot\tau_\mu = L\cdot j_\mu$, and the reduced Euler-Lagrange equation $\mathbf{i}_{\xi_{l_\mu}}\omega_\mu^L = \mathbf{d}E_{l_\mu}$ holds, where the reduced energy $E_{l_\mu} : (TQ)_\mu\to\mathbb{R}$ is given by $E_{l_\mu} := A_\mu - l_\mu$, and the reduced action $A_\mu : (TQ)_\mu\to\mathbb{R}$ is given by $A_\mu\cdot\tau_\mu = A\cdot j_\mu$, and the Euler-Lagrange vector fields ξ_L and ξ_{l_μ} are τ_μ-related. Thus, we can introduce a kind of regular point reducible Lagrangian systems as follows.

Definition 2.3 (Regular point reducible Lagrangian system) *A* 4*-tuple* (TQ, G, ω^L, L)*, where the hyperregular Lagrangian* $L : TQ\to\mathbb{R}$ *is* G*-invariant, is called a regular point reducible Lagrangian system, if there exists a point* $\mu\in\mathfrak{g}^*$*, which is a regular value of the momentum map* $\mathbf{J}_L$*, such that the regular point reduced system, that is, the* 3*-tuple* $((TQ)_\mu, \omega_\mu^L, l_\mu)$*, where* $(TQ)_\mu = \mathbf{J}_L^{-1}(\mu)/G_\mu$*,* $\tau_\mu^*\cdot\omega_\mu^L = j_\mu^*\cdot\omega^L$*,* $l_\mu\cdot\tau_\mu = L\cdot j_\mu$*, is a regular Lagrangian system, which is simply written as an* R_p*-reduced Lagrangian system. Where* $((TQ)_\mu, \omega_\mu^L)$ *is the* R_p*-reduced space, the function* $l_\mu : (TQ)_\mu\to\mathbb{R}$ *is called the* R_p*-reduced Lagrangian.*

We know that the orbit reduction of a Hamiltonian system is an alternative approach to symplectic reduction given by Kazhdan, Kostant and Sternberg [4] and Marle [6], which is different from the Marsden-Weinstein reduction. Now, for a regular Lagrangian system with symmetry and momentum map, we can also give its regular orbit reduction as follows, which is different from the above regular point reduction.

Assume that $\Phi : G \times Q \to Q$ is a smooth left action of a Lie group G on Q, if this action is free and proper, then the tangent lifted left action $\Phi^T : G \times TQ \to TQ$ is also free and proper. Moreover, assume that the action is symplectic with respect to ω^L and admits an Ad^*-equivariant momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$. For a regular value of the momentum map $\mathbf{J}_L$, $\mu \in \mathfrak{g}^*$, $\mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^*$ is the G-orbit of the co-adjoint G-action through the point μ. Since G acts freely, properly and symplectically on TQ with respect to ω^L, the quotient space $(TQ)_{\mathcal{O}_\mu} = \mathbf{J}_L^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the reduced symplectic form $\omega^L_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$j^*_{\mathcal{O}_\mu} \cdot \omega^L = \tau^*_{\mathcal{O}_\mu} \cdot \omega^L_{\mathcal{O}_\mu} + (\mathbf{J}_L)^*_{\mathcal{O}_\mu} \cdot \omega^{L+}_{\mathcal{O}_\mu}, \tag{2.4}$$

where $(\mathbf{J}_L)_{\mathcal{O}_\mu}$ is the restriction of the momentum map $\mathbf{J}_L$ to $\mathbf{J}_L^{-1}(\mathcal{O}_\mu)$, that is, $(\mathbf{J}_L)_{\mathcal{O}_\mu} = \mathbf{J}_L \cdot j_{\mathcal{O}_\mu}$. Here $\omega^{L+}_{\mathcal{O}_\mu}$ and $\omega^+_{\mathcal{O}_\mu}$ are the $+$-symplectic structures on the orbit $\mathcal{O}_\mu$ given by

$$\omega^{L+}_{\mathcal{O}_\mu}(\nu)(\xi,\ \eta) = \omega^+_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in \mathcal{O}_\mu,\ \xi, \eta \in \mathfrak{g},\ \xi_{\mathfrak{g}^*}, \eta_{\mathfrak{g}^*} \in \mathfrak{g}^*. \tag{2.5}$$

The maps $j_{\mathcal{O}_\mu} : \mathbf{J}_L^{-1}(\mathcal{O}_\mu) \to TQ$ and $\tau_{\mathcal{O}_\mu} : \mathbf{J}_L^{-1}(\mathcal{O}_\mu) \to (TQ)_{\mathcal{O}_\mu}$ are natural injection and projection, respectively. The pair $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ is called the regular orbit reduced space of (TQ, ω^L) at the point μ.

Let $L : TQ \to \mathbb{R}$ be a G-invariant hyperregular Lagrangian; the flow F_t of the Euler-Lagrange vector field ξ_L leaves the connected components of $\mathbf{J}_L^{-1}(\mathcal{O}_\mu)$ invariant and commutes with the G-action. It thus induces a flow $f_t^{\mathcal{O}_\mu}$ on $(TQ)_{\mathcal{O}_\mu}$, defined by $f_t^{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu} \cdot F_t \cdot j_{\mathcal{O}_\mu}$. The vector field $\xi_{l_{\mathcal{O}_\mu}}$ generated by the flow $f_t^{\mathcal{O}_\mu}$ on $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ is the reduced Euler-Lagrange vector field, associated with the regular orbit reduced Lagrangian function $l_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R}$ defined by $l_{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = L \cdot j_{\mathcal{O}_\mu}$. The reduced Euler-Lagrange equation $\mathbf{i}_{\xi_{l_{\mathcal{O}_\mu}}} \omega^L_{\mathcal{O}_\mu} = \mathbf{d}E_{l_{\mathcal{O}_\mu}}$ holds, where the reduced energy $E_{l_{\mathcal{O}_\mu}} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R}$ is given by $E_{l_{\mathcal{O}_\mu}} := A_{\mathcal{O}_\mu} - l_{\mathcal{O}_\mu}$, and the reduced action $A_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R}$ is given by $A_{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = A \cdot j_{\mathcal{O}_\mu}$. The Euler-Lagrange vector fields ξ_L and $\xi_{l_{\mathcal{O}_\mu}}$ are $\tau_{\mathcal{O}_\mu}$-related. Thus, we can introduce a kind of the regular orbit reducible Lagrangian systems as follows.

Definition 2.4 (Regular orbit reducible Lagrangian system) *A* 4*-tuple* (TQ, G, ω^L, L), *where the hyperregular Lagrangian* $L : TQ \to \mathbb{R}$ *is* G*-invariant, is called a regular orbit reducible Lagrangian system, if there exists an orbit* $\mathcal{O}_\mu$, $\mu \in \mathfrak{g}^*$, *where* μ *is a regular value of the momentum map* $\mathbf{J}_L$, *such that the regular orbit reduced system, that is, the* 3*-tuple* $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu})$, *where* $(TQ)_{\mathcal{O}_\mu} = \mathbf{J}_L^{-1}(\mathcal{O}_\mu)/G$, $\tau^*_{\mathcal{O}_\mu} \cdot \omega^L_{\mathcal{O}_\mu} = j^*_{\mathcal{O}_\mu} \cdot \omega^L - (\mathbf{J}_L)^*_{\mathcal{O}_\mu} \cdot \omega^{L+}_{\mathcal{O}_\mu}$, $l_{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = L \cdot j_{\mathcal{O}_\mu}$, *is a regular Lagrangian system, which is simply written as an* R_o*-reduced Lagrangian system. Where* $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ *is the* R_o*-reduced space, the function* $l_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R}$ *is called the* R_o*-reduced Lagrangian.*

In the following we shall give a precise analysis for the geometrical structures of the regular point reduced space $((TQ)_\mu, \omega^L_\mu)$ and the regular orbit reduced space $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$. Assume that the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular; then the Legendre transformation $\mathcal{F}L$:

$TQ \to T^*Q$ is a diffeomorphism. If the cotangent lift G-action $\Phi^{T^*} : G \times T^*Q \to T^*Q$ is free, proper and symplectic with respect to the canonical symplectic form ω on T^*Q, and has an Ad^*-equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by $\langle \mathbf{J}(\alpha_q), \xi \rangle = \alpha_q(\xi_Q(q))$, where $\alpha_q \in T^*_q Q$ and $\xi \in \mathfrak{g}$, $\xi_Q(q)$ is the value of the infinitesimal generator ξ_Q of the G-action at $q \in Q$; $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the duality pairing on dual $\mathfrak{g}^*$ and $\mathfrak{g}$. Then we have the following theorem.

Theorem 2.1 *Assume that the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, and that the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is (Φ^T, Φ^{T^*})-equivariant; then the following two assertions hold:*

(i) *The momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ given by $\mathbf{J}_L = \mathbf{J} \cdot \mathcal{F}L$, is Ad^*-equivariant;*

(ii) *If $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $\mathbf{J}$, then μ is also a regular value of the momentum map $\mathbf{J}_L$.*

Proof We first prove that the momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ is Ad^*-equivariant. Since the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is a diffeomorphism. Because the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is Ad^*-equivariant, $\mathrm{Ad}^* \cdot \mathbf{J} = \mathbf{J} \cdot \Phi^{T^*}$. Note that the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is $(\Phi^T,\ \Phi^{T^*})$-equivariant, so we have that $\Phi^{T^*} \cdot \mathcal{F}L = \mathcal{F}L \cdot \Phi^T$. From the following commutative Diagram-1,

$$\begin{array}{ccccc}
TQ & \xrightarrow{\Phi^T} & TQ & \xrightarrow{\mathbf{J}_L} & \mathfrak{g}^* \\
\mathcal{F}L \big\downarrow & & \mathcal{F}L \big\downarrow & & \mathrm{Ad}^* \big\downarrow \\
T^*Q & \xrightarrow{\Phi^{T^*}} & T^*Q & \xrightarrow{\mathbf{J}} & \mathfrak{g}^*
\end{array}$$

Diagram-1

we can obtain that

$$\mathrm{Ad}^* \cdot \mathbf{J}_L = \mathrm{Ad}^* \cdot \mathbf{J} \cdot \mathcal{F}L = \mathbf{J} \cdot \Phi^{T^*} \cdot \mathcal{F}L = \mathbf{J} \cdot \mathcal{F}L \cdot \Phi^T = \mathbf{J}_L \cdot \Phi^T.$$

Thus, the momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ is Ad^*-equivariant.

Next, we prove (ii). If $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $\mathbf{J}$, then there exists an $\alpha \in T^*Q$ such that $\mathbf{J}(\alpha) = \mu$. Since the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is a diffeomorphism, we have that $v = \mathcal{F}L^{-1}(\alpha) \in TQ$ satisfies

$$\mathbf{J}_L(v) = \mathbf{J} \cdot \mathcal{F}L(\mathcal{F}L^{-1}(\alpha)) = \mathbf{J}(\alpha) = \mu.$$

Thus, $\mu \in \mathfrak{g}^*$ is also a regular value of the momentum map $\mathbf{J}_L$.

For a given $\mu \in \mathfrak{g}^*$, a regular value of the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, denote by G_μ the isotropy subgroup of the co-adjoint G-action at the point μ; then the Marsden-Weinstein reduced space $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form ω_μ uniquely characterized by the relation

$$\pi^*_\mu \cdot \omega_\mu = i^*_\mu \cdot \omega. \tag{2.6}$$

The map $i_\mu : \mathbf{J}^{-1}(\mu) \to T^*Q$ is the inclusion and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. From [8], we know that the classification of symplectic reduced spaces of a cotangent bundle is given as follows. (1) If $\mu = 0$, the symplectic reduced space of cotangent bundle T^*Q at $\mu = 0$ is given by $((T^*Q)_\mu, \omega_\mu) = (T^*(Q/G), \widehat{\omega})$, where $\widehat{\omega}$ is the canonical symplectic form of cotangent bundle $T^*(Q/G)$. Thus, the symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ at $\mu = 0$ is a symplectic vector bundle. (2) If $\mu \neq 0$, and G is Abelian, then $G_\mu = G$; in this case, the regular point symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to symplectic vector bundle $(T^*(Q/G), \widehat{\omega} - B_\mu)$, where B_μ is a magnetic term. (3) If $\mu \neq 0$, G is not Abelian and $G_\mu \neq G$; in this case, the regular point symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle over $T^*(Q/G_\mu)$ with fiber being the co-adjoint orbit $\mathcal{O}_\mu$, see the cotangent bundle reduction theorem—bundle version, and also see [10]. Comparing the regular point reduced spaces $((TQ)_\mu, \omega^L_\mu)$ and $((T^*Q)_\mu, \omega_\mu)$ at the point μ, we have the following theorem.

Theorem 2.2 *Assume that the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, and that the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is (Φ^T, Φ^{T*})-equivariant; then the regular point reduced space $((TQ)_\mu, \omega^L_\mu)$ of (TQ, ω^L) at μ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ of (T^*Q, ω_0) at μ, and hence is also symplectically diffeomorphic to a symplectic fiber bundle.*

Proof Since the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is a diffeomorphism. Because $\mathcal{F}L$ is (Φ^T, Φ^{T*})-equivariant, that is, $\Phi^{T*} \cdot \mathcal{F}L = \mathcal{F}L \cdot \Phi^T$, we can define a map $(\mathcal{F}L)_\mu : (TQ)_\mu \to (T^*Q)_\mu$ given by $(\mathcal{F}L)_\mu \cdot \tau_\mu = \pi_\mu \cdot \mathcal{F}L$, and $i_\mu \cdot \mathcal{F}L = \mathcal{F}L \cdot j_\mu$; see the following commutative Diagram-2, which is well-defined and a diffeomorphism.

$$\begin{array}{ccc} \mathbf{J}_L^{-1}(\mu) \subset TQ & \xrightarrow{\mathcal{F}L} & \mathbf{J}^{-1}(\mu) \subset T^*Q \\ {\scriptstyle \tau_\mu}\downarrow & & \downarrow{\scriptstyle \pi_\mu} \\ (TQ)_\mu & \xrightarrow{(\mathcal{F}L)_\mu} & (T^*Q)_\mu \end{array}$$

Diagram-2

We shall prove that $(\mathcal{F}L)_\mu$ is symplectic, that is, $(\mathcal{F}L)^*_\mu \cdot \omega_\mu = \omega^L_\mu$. In fact, from (2.6) and (2.3), we have that

$$\begin{aligned} \tau^*_\mu \cdot (\mathcal{F}L)^*_\mu \cdot \omega_\mu &= ((\mathcal{F}L)_\mu \cdot \tau_\mu)^* \cdot \omega_\mu = (\pi_\mu \cdot \mathcal{F}L)^* \cdot \omega_\mu = (\mathcal{F}L)^* \cdot \pi^*_\mu \cdot \omega_\mu \\ &= (\mathcal{F}L)^* \cdot i^*_\mu \cdot \omega = (i_\mu \cdot \mathcal{F}L)^* \cdot \omega = (\mathcal{F}L \cdot j_\mu)^* \cdot \omega \\ &= j^*_\mu \cdot (\mathcal{F}L)^* \cdot \omega = j^*_\mu \cdot \omega^L = \tau^*_\mu \cdot \omega^L_\mu. \end{aligned}$$

Notice that τ_μ is surjective, and hence $(\mathcal{F}L)^*_\mu \cdot \omega_\mu = \omega^L_\mu$. Thus, the regular point reduced space $((TQ)_\mu, \omega^L_\mu)$ of (TQ, ω^L) at μ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ of (T^*Q, ω) at μ. From [8], we know that the space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle, and hence $((TQ)_\mu, \omega^L_\mu)$ is also symplectically diffeomorphic to a symplectic fiber bundle.

For a given $\mu \in \mathfrak{g}^*$, a regular value of the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, the regular orbit reduced space $(T^*Q)_{\mathcal{O}_\mu} = \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$i^*_{\mathcal{O}_\mu} \cdot \omega = \pi^*_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu} + \mathbf{J}^*_{\mathcal{O}_\mu} \cdot \omega^+_{\mathcal{O}_\mu}, \tag{2.7}$$

where $\mathbf{J}_{\mathcal{O}_\mu}$ is the restriction of the momentum map $\mathbf{J}$ to $\mathbf{J}^{-1}(\mathcal{O}_\mu)$, that is, $\mathbf{J}_{\mathcal{O}_\mu} = \mathbf{J} \cdot i_{\mathcal{O}_\mu}$, and $\omega^+_{\mathcal{O}_\mu}$ is the +-symplectic structure on the orbit $\mathcal{O}_\mu$ given by

$$\omega^+_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \langle \nu, [\xi, \eta] \rangle, \ \ \forall \nu \in \mathcal{O}_\mu, \ \xi, \eta \in \mathfrak{g}, \ \xi_{\mathfrak{g}^*}, \eta_{\mathfrak{g}^*} \in \mathfrak{g}^*. \tag{2.8}$$

The maps $i_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to T^*Q$ and $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to (T^*Q)_{\mathcal{O}_\mu}$ are natural injections and projection, respectively. In the general case, we may think that the structure of the symplectic orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is more complex than that of the symplectic point reduced space $((T^*Q)_\mu, \omega_\mu)$, but from [16] and the regular reduction diagram, we know that the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$, and hence is also symplectically diffeomorphic to a symplectic fiber bundle. Comparing the regular orbit reduced spaces $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ and $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ at the orbit $\mathcal{O}_\mu$, we have the following theorem.

Theorem 2.3 *Assume that the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, and that the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is (Φ^T, Φ^{T*})-equivariant; then the regular orbit reduced space $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ of (TQ, ω^L) at the orbit $\mathcal{O}_\mu$ is symplectically diffeomorphic to the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ of (T^*Q, ω) at the orbit $\mathcal{O}_\mu$, and hence is also symplectically diffeomorphic to a symplectic fiber bundle.*

Proof Since the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is a diffeomorphism. Because $\mathcal{F}L$ is (Φ^T, Φ^{T*})-equivariant, that is, $\Phi^{T*} \cdot \mathcal{F}L = \mathcal{F}L \cdot \Phi^T$, we can define a map $(\mathcal{F}L)_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to (T^*Q)_{\mathcal{O}_\mu}$ given by $(\mathcal{F}L)_{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu} \cdot \mathcal{F}L$, and $i_{\mathcal{O}_\mu} \cdot \mathcal{F}L = \mathcal{F}L \cdot j_{\mathcal{O}_\mu}$; see the following commutative Diagram-3. This map is well-defined and a diffeomorphism.

$$\begin{array}{ccc}
\mathbf{J}_L^{-1}(\mathcal{O}_\mu) \subset TQ & \xrightarrow{\mathcal{F}L} & \mathbf{J}^{-1}(\mathcal{O}_\mu) \subset T^*Q \\
\tau_{\mathcal{O}_\mu} \Big\downarrow & & \Big\downarrow \pi_{\mathcal{O}_\mu} \\
(TQ)_{\mathcal{O}_\mu} & \xrightarrow{(\mathcal{F}L)_{\mathcal{O}_\mu}} & (T^*Q)_{\mathcal{O}_\mu}
\end{array}$$

Diagram-3

We shall prove that $(\mathcal{F}L)_{\mathcal{O}_\mu}$ is symplectic, that is, $(\mathcal{F}L)^*_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu} = \omega^L_{\mathcal{O}_\mu}$. In fact, from (2.7) and (2.4)–(2.5), we have that

$$\begin{aligned}
\tau^*_{\mathcal{O}_\mu} \cdot (\mathcal{F}L)^*_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu} &= ((\mathcal{F}L)_{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu})^* \cdot \omega_{\mathcal{O}_\mu} = (\pi_{\mathcal{O}_\mu} \cdot \mathcal{F}L)^* \cdot \omega_{\mathcal{O}_\mu} \\
&= (\mathcal{F}L)^* \cdot \pi^*_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu} = (\mathcal{F}L)^* \cdot (i^*_{\mathcal{O}_\mu} \cdot \omega - \mathbf{J}^*_{\mathcal{O}_\mu} \cdot \omega^+_{\mathcal{O}_\mu}) \\
&= (\mathcal{F}L)^* \cdot i^*_{\mathcal{O}_\mu} \cdot \omega - (\mathcal{F}L)^* \cdot (\mathbf{J}^*_{\mathcal{O}_\mu} \cdot \omega^+_{\mathcal{O}_\mu}) \\
&= (i_{\mathcal{O}_\mu} \cdot \mathcal{F}L)^* \cdot \omega - (\mathbf{J}_{\mathcal{O}_\mu} \cdot \mathcal{F}L)^* \cdot \omega^{L+}_{\mathcal{O}_\mu} \\
&= (\mathcal{F}L \cdot j_{\mathcal{O}_\mu})^* \cdot \omega - (\mathbf{J} \cdot i_{\mathcal{O}_\mu} \cdot \mathcal{F}L)^* \cdot \omega^{L+}_{\mathcal{O}_\mu} \\
&= j^*_{\mathcal{O}_\mu} \cdot (\mathcal{F}L)^* \cdot \omega - (\mathbf{J} \cdot \mathcal{F}L \cdot j_{\mathcal{O}_\mu})^* \cdot \omega^{L+}_{\mathcal{O}_\mu} \\
&= j^*_{\mathcal{O}_\mu} \cdot \omega^L - (\mathbf{J}_L \cdot j_{\mathcal{O}_\mu})^* \cdot \omega^{L+}_{\mathcal{O}_\mu} \\
&= j^*_{\mathcal{O}_\mu} \cdot \omega^L - (\mathbf{J}_L)^*_{\mathcal{O}_\mu} \cdot \omega^{L+}_{\mathcal{O}_\mu} = \tau^*_{\mathcal{O}_\mu} \cdot \omega^L_{\mathcal{O}_\mu}.
\end{aligned}$$

Notice that $\tau_{\mathcal{O}_\mu}$ is surjective, and hence $(\mathcal{F}L)^*_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu} = \omega^L_{\mathcal{O}_\mu}$. Thus, the regular orbit reduced space $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ of (TQ, ω^L) at the orbit $\mathcal{O}_\mu$ is symplectically diffeomorphic to the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ of (T^*Q, ω) at the orbit $\mathcal{O}_\mu$. From [16] and the regular reduction diagram, we know that the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ at the orbit $\mathcal{O}_\mu$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ of (T^*Q, ω) at μ, and hence $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ at μ, and is also symplectically diffeomorphic to a symplectic fiber bundle.

Thus, from the above discussion, we know that the regular point or regular orbit reduced space for a regular Lagrangian system defined on a tangent bundle may not be a tangent bundle. Considering the completeness of the symmetric reduction, if we may define an RCL system on a symplectic fiber bundle, then it is possible to describe uniformly the RCL systems on TQ and their regular reduced RCL systems on the associated reduced spaces.

3 Regular Controlled Lagrangian System and Its Dynamics

In order to give a proper definition of CL system, by following the ideas in [12], we first define a CL system on TQ by using the Lagrangian symplectic form, and such a system is called a regular controlled Lagrangian (RCL for short) system; then we regard a regular Lagrangian system on TQ as a special case of an RCL system without external force and control. Thus, the set of the regular Lagrangian systems on TQ is a subset of the set of RCL systems on TQ. On the other hand, since the regular reduced system of a regular Lagrangian system with symmetry defined on the tangent bundle TQ may not be a regular Lagrangian system on a tangent bundle, we cannot define an RCL system on the tangent bundle TQ directly. However, from Theorems 2.2–2.3, we know that the regular point reduced space $((TQ)_\mu, \omega^L_\mu)$ of (TQ, ω^L) at μ is symplectically diffeomorphic to a symplectic fiber bundle over $T(Q/G_\mu)$ with fiber being the co-adjoint orbit $\mathcal{O}_\mu$, and the regular orbit reduced space $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ of (TQ, ω^L) at the orbit $\mathcal{O}_\mu$ is also symplectically diffeomorphic to a symplectic fiber bundle. In consequence, if we can define an RCL system on a symplectic fiber bundle, then it is possible to describe uniformly the RCL systems on TQ and their regular reduced RCL systems on the associated reduced spaces; we can also study regular reduction of the RCL systems with symmetries and momentum maps, as an extension of the regular reduction theory of the regular Lagrangian systems under regular controlled Lagrangian equivalence conditions, and set up the regular reduction theory of the RCL systems on a tangent bundle by using momentum map, the associated reduced Lagrangian symplectic form, and the viewpoint of the completeness of regular reduction.

In this section, we first define an RCL system on a symplectic fiber bundle. Then, by using the Legendre transformation and the Lagrangian symplectic form on the tangent bundle of a configuration manifold, we obtain the RCL system on a tangent bundle as a special case and give a good expression for the dynamical vector field of the RCL system—one that allows us to discuss RCL-equivalence. Consequencely, we can study RCL systems with symmetries by combining them with the symmetric reduction of the regular Lagrangian systems with symmetries. For convenience, we assume that all controls appearing in this paper are the admissible controls.

Let $(\mathbb{E}, M, \pi)$ be a fiber bundle. For each point $x \in M$, assume that the fiber $\mathbb{E}_x = \pi^{-1}(x)$ is a smooth submanifold of $\mathbb{E}$ equipped with a symplectic form $\omega_{\mathbb{E}}(x)$; that is, $(\mathbb{E}, \omega_{\mathbb{E}})$ is a symplectic fiber bundle. Suppose a function $L : \mathbb{E} \to \mathbb{R}$ is a hyperregular Lagrangian, and there is an action function $A:\mathbb{E}\to\mathbb{R}$ and an Euler-Lagrange vector field ξ_L that satisfy the equation $\mathbf{i}_{\xi_L}\omega_{\mathbb{E}}=\mathbf{d}E_L$, where $E_L:\mathbb{E}\to\mathbb{R}$ is an energy function given by $E_L:=A-L$. Then $(\mathbb{E},\omega_{\mathbb{E}},L)$ is a regular Lagrangian system. Moreover, if considering the external force and control, we can define a kind of RCL system on the symplectic fiber bundle $\mathbb{E}$ as follows.

Definition 3.1 (RCL system) *An RCL system on* $\mathbb{E}$ *is a* 5*-tuple* $(\mathbb{E},\omega_{\mathbb{E}},L,F^L,\mathcal{C}^L)$*, where* $(\mathbb{E},\omega_{\mathbb{E}},L)$ *is a regular Lagrangian system, the function* $L:\mathbb{E}\to\mathbb{R}$ *is called the* (*hyperregular*) *Lagrangian, a fiber-preserving map* $F^L:\mathbb{E}\to\mathbb{E}$ *is called the* (*external*) *force map, and a fiber submanifold* $\mathcal{C}^L$ *of* $\mathbb{E}$ *is called the control subset.*

Sometimes, $\mathcal{C}^L$ is also denoted as the set of fiber-preserving maps from $\mathbb{E}$ to $\mathcal{C}^L$. When a feedback control law $u^L:\mathbb{E}\to\mathcal{C}^L$ is chosen, the 5-tuple $(\mathbb{E},\omega_{\mathbb{E}},L,F^L,u^L)$ is a closed-loop dynamical system. In particular, if Q is a smooth manifold, with TQ its tangent bundle and T^*Q its cotangent bundle with a canonical symplectic form ω, assume that $L:TQ\to\mathbb{R}$ is a hyperregular Lagrangian on TQ and the Legendre transformation $\mathcal{F}L:TQ\to T^*Q$ is a diffeomorphism, then (TQ,ω^L) is a symplectic vector bundle, where $\omega^L=\mathcal{F}L^*(\omega)$. If we take $\mathbb{E}=TQ$, from the above definition we can obtain an RCL system on the tangent bundle TQ, that is, the 5-tuple $(TQ,\omega^L,L,F^L,\mathcal{C}^L)$.

In order to describe the dynamics of the RCL system $(\mathbb{E},\omega_{\mathbb{E}},L,F^L,\mathcal{C}^L)$ with a control law $u^L:\mathbb{E}\to\mathcal{C}^L$, we need to give a good expression for the dynamical vector field of the RCL system. We shall use the notation of vertical lift maps of a vector along a fiber introduced in [12]. In fact, for a smooth manifold M, its tangent bundle TM is a vector bundle, and for the fiber bundle $\pi:\mathbb{E}\to M$, we consider the tangent mapping $T\pi:T\mathbb{E}\to TM$ and its kernel $\ker(T\pi)=\{\rho\in T\mathbb{E}\mid T\pi(\rho)=0\}$, which is a vector subbundle of $T\mathbb{E}$. We denote $V\mathbb{E}:=\ker(T\pi)$, which is called the vertical bundle of $\mathbb{E}$. Assume that there is a metric on $\mathbb{E}$, we take a Levi-Civita connection $\mathcal{A}$ on $T\mathbb{E}$, and denote by $H\mathbb{E}:=\ker(\mathcal{A})$, which is called the horizontal bundle of $\mathbb{E}$, such that $T\mathbb{E}=H\mathbb{E}\oplus V\mathbb{E}$. For any $x\in M$, $a_x,b_x\in\mathbb{E}_x$, any tangent vector $\rho(b_x)\in T_{b_x}\mathbb{E}$ can be split into horizontal and vertical parts, that is, $\rho(b_x)=\rho^h(b_x)\oplus\rho^v(b_x)$, where $\rho^h(b_x)\in H_{b_x}\mathbb{E}$ and $\rho^v(b_x)\in V_{b_x}\mathbb{E}$. Let γ be a geodesic in $\mathbb{E}_x$ connecting a_x and b_x. We denote by $\rho^v_\gamma(a_x)$ a tangent vector at a_x, which is the parallel displacement of the vertical vector $\rho^v(b_x)$ along the geodesic γ from b_x to a_x. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, we have $T\pi(\rho^v_\gamma(a_x))=0$, and hence $\rho^v_\gamma(a_x)\in V_{a_x}\mathbb{E}$. Now, for $a_x,b_x\in\mathbb{E}_x$ and tangent vector $\rho(b_x)\in T_{b_x}\mathbb{E}$, we can define the vertical lift map of a vector along a fiber as follows

$$\mathrm{vlift}:T\mathbb{E}_x\times\mathbb{E}_x\to T\mathbb{E}_x;\ \ \mathrm{vlift}(\rho(b_x),a_x)=\rho^v_\gamma(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of γ. If $F^L:\mathbb{E}\to\mathbb{E}$ is a fiber-preserving map, for any $x\in M$, we have $F^L_x:\mathbb{E}_x\to\mathbb{E}_x$ and $TF^L_x:T\mathbb{E}_x\to T\mathbb{E}_x$; thus, for any $a_x\in\mathbb{E}_x$ and $\rho\in T\mathbb{E}_x$, the vertical lift of ρ under the action of F^L along a fiber is defined by

$$(\mathrm{vlift}(F^L_x)\rho)(a_x)=\mathrm{vlift}((TF^L_x\rho)(F^L_x(a_x)),a_x)=(TF^L_x\rho)^v_\gamma(a_x),$$

where γ is a geodesic in $\mathbb{E}_x$ connecting $F^L_x(a_x)$ and a_x.

In particular, when $\pi:\mathbb{E}\to M$ is a vector bundle, for any $x\in M$, the fiber $\mathbb{E}_x=\pi^{-1}(x)$ is a vector space. In this case, we can choose the geodesic γ to be a straight line, and the vertical vector is invariant under parallel displacement along straight line, that is, $\rho^v_\gamma(a_x) = \rho^v(b_x)$. Moreover, when $\mathbb{E} = TQ$, by using the local trivialization of TTQ, we have that $TTQ \cong TQ \times TQ$ (locally). Since $\tau_Q : TQ \to Q$, and $T\tau_Q : TTQ \to TQ$, in this case, for any $v_x,\ w_x \in T_xQ,\ x \in Q$, we know that $(0, w_x) \in V_{w_x}T_xQ$, and hence we can get that

$$\mathrm{vlift}((0, w_x)(w_x), v_x) = (0, w_x)(v_x) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(v_x + sw_x),$$

which coincides with the definition of the vertical lift map along a fiber in [11].

For a given RCL system $(TQ, \omega^L, L, F^L, \mathcal{C}^L)$, the dynamical vector field of the associated regular Lagrangian system (TQ, ω^L, L) is the Euler-Lagrange vector field ξ_L, such that $\mathbf{i}_{\xi_L}\omega^L = \mathbf{d}E_L$. When we consider the external force $F^L : TQ \to TQ$, by using the above notation of vertical lift map of a vector along a fiber, the change of ξ_L under the action of F^L is that

$$\mathrm{vlift}(F^L)\xi_L(v_x) = \mathrm{vlift}((TF^L\xi_L)(F^L(v_x)), v_x) = (TF^L\xi_L)^v_\gamma(v_x),$$

where $v_x \in T_xQ,\ x \in Q$ and the geodesic γ is a straight line in T_xQ connecting $F^L_x(v_x)$ and v_x. Similary, when a feedback control law $u^L : TQ \to \mathcal{C}^L$ is chosen, the change of ξ_L under the action of u^L is that

$$\mathrm{vlift}(u^L)\xi_L(v_x) = \mathrm{vlift}((Tu^L\xi_L)(u^L(v_x)), v_x) = (Tu^L\xi_L)^v_\gamma(v_x).$$

Consequently, we can give an expression for the dynamical vector field of the RCL system as follows.

Theorem 3.1 *The dynamical vector field of an RCL system $(TQ, \omega^L, L, F^L, \mathcal{C}^L)$ with a control law u^L is the synthesis of the Euler-Lagrange vector field ξ_L and its changes under the actions of the external force F^L and control u^L, that is,*

$$\xi_{(TQ,\omega^L,L,F^L,u^L)}(v_x) = \xi_L(v_x) + \mathrm{vlift}(F^L)\xi_L(v_x) + \mathrm{vlift}(u^L)\xi_L(v_x)$$

for any $v_x \in T_xQ,\ x \in Q$. For convenience, it is simply written as

$$\xi_{(TQ,\omega^L,L,F^L,u^L)} = \xi_L + \mathrm{vlift}(F^L) + \mathrm{vlift}(u^L). \tag{3.1}$$

Where $\mathrm{vlift}(F^L) = \mathrm{vlift}(F^L)\xi_L$, and $\mathrm{vlift}(u^L) = \mathrm{vlift}(u^L)\xi_L$ are the changes of ξ_L under the actions of F^L and u^L. We also denote that $\mathrm{vlift}(\mathcal{C}^L) = \cup\{\mathrm{vlift}(u^L)\xi_L |\ u^L \in \mathcal{C}^L\}$. It is worth noting that, in order to facilitate deduction and calculation, we always use the simple expression of the dynamical vector field $\xi_{(TQ,\omega^L,L,F^L,u^L)}$. Moreover, we also use the simple expressions for the R_p-reduced vector field $\xi_{((TQ)_\mu,\omega^L_\mu,l_\mu,f^L_\mu,u^L_\mu)}$ and the R_o-reduced vector field $\xi_{((TQ)_{\mathcal{O}_\mu},\omega^L_{\mathcal{O}_\mu},l_{\mathcal{O}_\mu},f^L_{\mathcal{O}_\mu},u^L_{\mathcal{O}_\mu})}$ in Sections 4–5.

From the expression (3.1) of the dynamical vector field of the RCL system, we know that under the actions of the external force F^L and control u^L, in general, the dynamical vector field may not be an Euler-Lagrange vector field, and hence the RCL system may not yet be a regular Lagrangian system. However, it is a dynamical system closed with respect to a regular Lagrangian system, and it can be explored and studied by extending the methods for handling external force and control in the study of the regular Lagrangian system. In particular, it is worth noting that the energy E_L is conserved along the flow of the Euler-Lagrange vector field ξ_L, if ξ_L satisfies the second order equation $T\tau_Q \circ \xi_L = id_{TQ}$. Since $T\tau_Q \cdot \mathrm{vlift}(F^L) = T\tau_Q \cdot \mathrm{vlift}(u^L) = 0$, from the expression (3.1) we have that $T\tau_Q \circ \xi_{(TQ,\omega^L,L,F^L,u^L)} = id_{TQ}$, that is, the dynamical vector field of the RCL system always satisfies the second-order equation.

On the other hand, for two given regular Lagrangian systems (TQ_i, ω_i^L, L_i), $i = 1, 2$, we say they are equivalent, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that their Euler-Lagrange vector fields ξ_{L_i}, $i = 1, 2$ satisfy the condition $\xi_{L_2} \cdot T\varphi = T(T\varphi) \cdot \xi_{L_1}$. Here the map $T\varphi : TQ_1 \to TQ_2$ is the tangent map of φ, and the map $T(T\varphi) : TTQ_1 \to TTQ_2$ is the tangent map of $T\varphi$. It is easy to see that the condition $\xi_{L_2} \cdot T\varphi = T(T\varphi) \cdot \xi_{L_1}$ is equivalent to the fact that the map $T\varphi : TQ_1 \to TQ_2$ is symplectic with respect to their Lagrangian symplectic forms ω_i^L on TQ_i, $i = 1, 2$.

For two given RCL systems $(TQ_i, \omega_i^L, L_i, F_i^L, \mathcal{C}_i^L)$, $i = 1, 2$, we also want to define their equivalence, that is, to find a diffeomorphism $\varphi : Q_1 \to Q_2$ such that $\xi_{(TQ_2,\omega_2^L,L_2,F_2^L,\mathcal{C}_2^L)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,\omega_1^L,L_1,F_1^L,\mathcal{C}_1^L)}$. However, it is worth noting that, when an RCL system is given, the force map $F^L : TQ \to TQ$ is determined, but the feedback control law $u^L : TQ \to \mathcal{C}^L$ can be chosen. In order to explicitly emphasize the impact of external force and control in study of the RCL systems, by using the above expression (3.1) for the dynamical vector field of the RCL system, we can describe how the feedback control law modifies the structure of the RCL system, and thus induce the regular controlled Lagrangian matching conditions and RCL-equivalence are induced as follows.

Definition 3.2 (RCL-equivalence) *Suppose that we have two RCL systems $(TQ_i, \omega_i^L, L_i, F_i^L, \mathcal{C}_i^L)$, $i = 1, 2$, we say they are RCL-equivalent, or simply, $(TQ_1, \omega_1^L, L_1, F_1^L, \mathcal{C}_1^L) \overset{RCL}{\sim} (TQ_2, \omega_2^L, L_2, F_2^L, \mathcal{C}_2^L)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the following regular controlled Lagrangian matching conditions hold:*

RCL-1 *The control subsets $\mathcal{C}_i^L$, $i = 1, 2$ satisfy the condition $\mathcal{C}_2^L = T\varphi(\mathcal{C}_1^L)$, where the map $T\varphi : TQ_1 \to TQ_2$ is the tangent map of φ.*

RCL-2 *For each control law $u_1^L : TQ_1 \to \mathcal{C}_1^L$, there exists a control law $u_2^L : TQ_2 \to \mathcal{C}_2^L$, such that the two closed-loop dynamical systems have the same dynamical vector fields, that is, $\xi_{(TQ_2,\omega_2^L,L_2,F_2^L,u_2^L)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,\omega_1^L,L_1,F_1^L,u_1^L)}$, where the map $T(T\varphi) : TTQ_1 \to TTQ_2$ is the tangent map of $T\varphi$.*

From the expression (3.1) of the dynamical vector field of the RCL system and the condition $\xi_{(TQ_2,\omega_2^L,L_2,F_2^L,u_2^L)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,\omega_1^L,L_1,F_1^L,u_1^L)}$, we have that

$$(\xi_{L_2} + \mathrm{vlift}(F_2^L)\xi_{L_2} + \mathrm{vlift}(u_2^L)\xi_{L_2}) \cdot T\varphi = T(T\varphi) \cdot [\xi_{L_1} + \mathrm{vlift}(F_1^L)\xi_{L_1} + \mathrm{vlift}(u_1^L)\xi_{L_1}].$$

By using the notation of the vertical lift map of a vector along a fiber, for $v_x \in T_xQ_1$, $x \in Q_1$, we have that

$$\begin{aligned}
&T(T\varphi) \cdot \mathrm{vlift}(F_1^L)\xi_{L_1}(v_x) \\
&= T(T\varphi) \cdot \mathrm{vlift}((TF_1^L \cdot \xi_{L_1})(F_1^L(v_x)), v_x) \\
&= \mathrm{vlift}((T(T\varphi) \cdot TF_1^L \cdot T(T\varphi^{-1}) \cdot \xi_{L_1})(T\varphi \cdot F_1^L \cdot T\varphi^{-1} \cdot (T\varphi \cdot v_x)), T\varphi \cdot v_x) \\
&= \mathrm{vlift}((T((T\varphi) \cdot F_1^L \cdot T\varphi^{-1}) \cdot \xi_{L_1})(T\varphi \cdot F_1^L \cdot T\varphi^{-1}(T\varphi \cdot v_x)), T\varphi \cdot v_x) \\
&= \mathrm{vlift}(T\varphi \cdot F_1^L \cdot T\varphi^{-1}) \cdot \xi_{L_1}(T\varphi \cdot v_x),
\end{aligned}$$

where the map $T\varphi^{-1}: TQ_2 \to TQ_1$ is the inverse of the tangent map $T\varphi$. Similarly, we have that $T(T\varphi)\cdot \text{vlift}(u_1^L)\xi_{L_1} = \text{vlift}(T\varphi \cdot u_1^L \cdot T\varphi^{-1})\cdot \xi_{L_1}\cdot T\varphi$. Thus, the explicit relation between the two control laws $u_i^L : TQ_i \to \mathcal{C}_i^L,\ i=1,2$ in RCL-2 is given by

$$\begin{aligned}&(\text{vlift}(u_2^L) - \text{vlift}(T\varphi\cdot u_1^L\cdot T\varphi^{-1}))\cdot T\varphi \\ =& -\xi_{L_2}\cdot T\varphi + T(T\varphi)(\xi_{L_1}) + (-\text{vlift}(F_2^L) + \text{vlift}(T\varphi\cdot F_1^L\cdot T\varphi^{-1}))\cdot T\varphi. \end{aligned} \tag{3.2}$$

From the above relation we know that, when two RCL systems $(TQ_i, \omega_i^L, L_i, F_i^L, \mathcal{C}_i^L)$, $i = 1,2$, are RCL-equivalent with respect to $T\varphi$, the corresponding regular Lagrangian systems (TQ_i, ω_i^L, L_i), $i=1,2$, may not be equivalent with respect to $T\varphi$. If the two corresponding regular Lagrangian systems are also equivalent with respect to $T\varphi$, then the control laws $u_i^L : TQ_i \to \mathcal{C}_i^L,\ i=1,2$ and the external forces $F_i^L : TQ_i \to TQ_i,\ i=1,2$ in RCL-2 must satisfy the following condition

$$\text{vlift}(u_2^L) - \text{vlift}(T\varphi\cdot u_1^L\cdot T\varphi^{-1}) = -\text{vlift}(F_2^L) + \text{vlift}(T\varphi\cdot F_1^L\cdot T\varphi^{-1}). \tag{3.3}$$

In the following we shall introduce regular point and regular orbit reducible RCL systems with symmetries, and show various relationships of their regular reducible RCL-equivalences.

4 Regular Point Reduction of the RCL System

We know that, when the external force and control of an RCL system $(TQ, \omega^L, L, F^L, \mathcal{C}^L)$ are both zero, that is, $F^L = 0$, and $\mathcal{C}^L = \emptyset$, then the RCL system is just a regular Lagrangian system (TQ, ω^L, L). Thus, we can regard a regular Lagrangian system on TQ as a special case of the RCL system without external force and control. Consequently, the set of regular Lagrangian systems with symmetries on TQ is a subset of the set of RCL systems with symmetries on TQ. If we first consider the regular point reduction of a regular Lagrangian system with symmetry, then we may study the regular point reduction of an RCL system with symmetry, as an extension of the regular point reduction of a regular Lagrangian system under the regular controlled Lagrangian equivalence conditions. In order to do this, in this section we consider the RCL system with symmetry and momentum map, and first give the regular point reducible RCL system and the RpCL-equivalence, and then prove the regular point reduction theorems for the RCL system and regular Lagrangian system.

We know that, if an RCL system with symmetry and momentum map is regular point reducible, then the associated regular Lagrangian system must be regular point reducible. Thus, from Definition 2.3 and Theorem 2.2, if the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is $(\Phi^T,\ \Phi^{T*})$-equivariant, then we can introduce a type of regular point reducible RCL systems as follows.

Definition 4.1 (Regular point reducible RCL system) *A 6-tuple* $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$, *where the hyperregular Lagrangian* $L : TQ \to \mathbb{R}$, *the fiber-preserving map* $F^L : TQ \to TQ$ *and the fiber submanifold* $\mathcal{C}^L$ *of* TQ *are all* G*-invariant, is called a regular point reducible RCL system, if the Legendre transformation* $\mathcal{F}L : TQ \to T^*Q$ *is* $(\Phi^T,\ \Phi^{T*})$*-equivariant, and there exists a point* $\mu \in \mathfrak{g}^*$*, which is a regular value of the momentum map* $\mathbf{J}_L$*, such that the regular point reduced system, that is, the 5-tuple* $((TQ)_\mu, \omega_\mu^L, l_\mu, f_\mu^L, \mathcal{C}_\mu^L)$*, where* $(TQ)_\mu = \mathbf{J}_L^{-1}(\mu)/G_\mu$*,* $\tau_\mu^*\omega_\mu^L = j_\mu^*\omega^L$*,* $l_\mu\cdot\tau_\mu = L\cdot j_\mu$*,* $F^L(\mathbf{J}_L^{-1}(\mu)) \subset \mathbf{J}_L^{-1}(\mu)$*,* $f_\mu^L\cdot\tau_\mu = \tau_\mu\cdot F^L\cdot j_\mu$*,* $\mathcal{C}^L\cap\mathbf{J}_L^{-1}(\mu) \neq \emptyset$*,*

$\mathcal{C}^L_\mu = \tau_\mu(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu))$, *is an RCL system, which is simply written as the* R_p*-reduced RCL system. Here,* $((TQ)_\mu, \omega^L_\mu)$ *is the* R_p*-reduced space, the function* $l_\mu : (TQ)_\mu \to \mathbb{R}$ *is called the* R_p*-reduced Lagrangian, the fiber-preserving map* $f^L_\mu : (TQ)_\mu \to (TQ)_\mu$ *is called the* R_p*-reduced (external) force map, and* $\mathcal{C}^L_\mu$ *is a fiber submanifold of* $(TQ)_\mu$ *that is called the* R_p*-reduced control subset.*

It is worth noting that for the regular point reducible RCL system $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$, the G-invariant external force map $F^L : TQ \to TQ$ has to satisfy the conditions $F^L(\mathbf{J}_L^{-1}(\mu)) \subset \mathbf{J}_L^{-1}(\mu)$, and $f^L_\mu \cdot \tau_\mu = \tau_\mu \cdot F^L \cdot j_\mu$, so that we can define the R_p-reduced external force map $f^L_\mu : (TQ)_\mu \to (TQ)_\mu$. The condition $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu) \neq \emptyset$ in the above definition ensures that the G-invariant control subset $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu)$ can be reduced and that the R_p-reduced control subset is $\mathcal{C}^L_\mu = \tau_\mu(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu))$.

Assume that the dynamical vector field $\xi_{(TQ,G,\omega^L,L,F^L,u^L)}$ of a given regular point reducible RCL system $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$ with a control law $u^L \in \mathcal{C}^L$ can be expressed by

$$\xi_{(TQ,G,\omega^L,L,F^L,u^L)} = \xi_L + \mathrm{vlift}(F^L) + \mathrm{vlift}(u^L). \tag{4.1}$$

Then, for the regular point reducible RCL system we can also introduce the regular point reducible controlled Lagrangian equivalence (RpCL-equivalence) as follows.

Definition 4.2 (RpCL-equivalence) *Suppose that we have two regular point reducible RCL systems* $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i = 1, 2$, *we say they are RpCL-equivalent, or simply,* $(TQ_1, G_1, \omega^L_1, L_1, F^L_1, \mathcal{C}^L_1) \overset{RpCL}{\sim} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, \mathcal{C}^L_2)$, *if there exists a diffeomorphism* $\varphi : Q_1 \to Q_2$ *such that the following regular point reducible controlled Lagrangian matching conditions hold:*

RpCL-1 *For* $\mu_i \in \mathfrak{g}^*_i$, *the regular reducible points of the RCL systems* $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i = 1, 2$, *the map* $(T\varphi)_\mu = j^{-1}_{\mu_2} \cdot T\varphi \cdot j_{\mu_1} : (\mathbf{J}_L)^{-1}_1(\mu_1) \to (\mathbf{J}_L)^{-1}_2(\mu_2)$ *is* $(G_{1\mu_1}, G_{2\mu_2})$*-equivariant and* $\mathcal{C}^L_2 \cap (\mathbf{J}_L)^{-1}_2(\mu_2) = (T\varphi)_\mu(\mathcal{C}^L_1 \cap (\mathbf{J}_L)^{-1}_1(\mu_1))$, *where* $\mu = (\mu_1, \mu_2)$, *and we denote by* $j^{-1}_{\mu_2}(S)$ *the pre-image of a subset* $S \subset TQ_2$ *under the map* $j_{\mu_2} : (\mathbf{J}_L)^{-1}_2(\mu_2) \to TQ_2$.

RpCL-2 *For each control law* $u^L_1 : TQ_1 \to \mathcal{C}^L_1$, *there exists a control law* $u^L_2 : TQ_2 \to \mathcal{C}^L_2$, *such that the two closed-loop dynamical systems have the same dynamical vector fields, that is,* $\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}$.

It is worth noting that for the regular point reducible RCL system, the induced equivalent map $(T\varphi)_\mu$ also preserves the equivariance of the G-action at the regular point. If a feedback control law $u^L : TQ \to \mathcal{C}^L$ is chosen, and $u^L \in \mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu)$, and $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu) \neq \emptyset$, then the R_p-reduced control law is $u^L_\mu : (TQ)_\mu \to \mathcal{C}^L_\mu = \tau_\mu(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu))$, and $u^L_\mu \cdot \tau_\mu = \tau_\mu \cdot u^L \cdot j_\mu$. The R_p-reduced RCL system $((TQ)_\mu, \omega^L_\mu, l_\mu, f^L_\mu, u^L_\mu)$ is a closed-loop regular dynamical system with the R_p-reduced control law u^L_μ. Assume that its dynamical vector field $\xi_{((TQ)_\mu,\omega^L_\mu,l_\mu,f^L_\mu,u^L_\mu)}$ can be expressed by

$$\xi_{((TQ)_\mu,\omega^L_\mu,l_\mu,f^L_\mu,u^L_\mu)} = \xi_{l_\mu} + \mathrm{vlift}(f^L_\mu) + \mathrm{vlift}(u^L_\mu), \tag{4.2}$$

where ξ_{l_μ} is the R_p-reduced Euler-Lagrange vector field, and $\mathrm{vlift}(f^L_\mu) = \mathrm{vlift}(f^L_\mu)\xi_{l_\mu}$, $\mathrm{vlift}(u^L_\mu) = \mathrm{vlift}(u^L_\mu)\xi_{l_\mu}$ are the changes of ξ_{l_μ} under the actions of the R_p-reduced external force f^L_μ and the R_p-reduced control law u^L_μ, and the dynamical vector fields of the RCL system and the

R_p-reduced RCL system satisfy the condition

$$\xi_{((TQ)_\mu,\omega^L_\mu,l_\mu,f^L_\mu,u^L_\mu)}\cdot\tau_\mu = T\tau_\mu\cdot\xi_{(TQ,G,\omega^L,L,F^L,u^L)}\cdot j_\mu \tag{4.3}$$

(see [12, 20]). Then we can obtain the following regular point reduction theorem for the RCL system, which explains the relationship between the RpCL-equivalence of the regular point reducible RCL system with symmetry and the RCL-equivalence of the associated R_p-reduced RCL system.

Theorem 4.1 *Two regular point reducible RCL systems* $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i=1,2$, *are RpCL-equivalent if and only if the associated R_p-reduced RCL systems* $((TQ_i)_{\mu_i}, \omega^L_{i\mu_i}, l_{i\mu_i}, f^L_{i\mu_i}, \mathcal{C}^L_{i\mu_i}), i=1,2,$ *are RCL-equivalent.*

Proof If $(TQ_1, G_1, \omega^L_1, L_1, F^L_1, \mathcal{C}^L_1) \overset{RpCL}{\sim} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, \mathcal{C}^L_2)$, then there exists a diffeomorphism $\varphi: Q_1\to Q_2$ such that for $\mu_i\in\mathfrak{g}^*_i, i=1,2$, $(T\varphi)_\mu = j^{-1}_{\mu_2}\cdot T\varphi\cdot j_{\mu_1}: (\mathbf{J}_L)^{-1}_1(\mu_1)\to (\mathbf{J}_L)^{-1}_2(\mu_2)$ is $(G_{1\mu_1}, G_{2\mu_2})$-equivariant, and $\mathcal{C}^L_2\cap(\mathbf{J}_L)^{-1}_2(\mu_2) = (T\varphi)_\mu(\mathcal{C}^L_1\cap(\mathbf{J}_L)^{-1}_1(\mu_1))$ and RpCL-2 holds. From the following commutative Diagram-4:

$$\begin{array}{ccccc}
TQ_1 & \xleftarrow{j_{\mu_1}} & (\mathbf{J}_L)^{-1}_1(\mu_1) & \xrightarrow{\tau_{\mu_1}} & (TQ_1)_{\mu_1}\\
{\scriptstyle T\varphi}\downarrow & & {\scriptstyle (T\varphi)_\mu}\downarrow & & {\scriptstyle (T\varphi)_{\mu/G}}\downarrow\\
TQ_2 & \xleftarrow{j_{\mu_2}} & (\mathbf{J}_L)^{-1}_2(\mu_2) & \xrightarrow{\tau_{\mu_2}} & (TQ_2)_{\mu_2}
\end{array}$$

Diagram-4

we can define a map $(T\varphi)_{\mu/G}: (TQ_1)_{\mu_1}\to(TQ_2)_{\mu_2}$ such that $(T\varphi)_{\mu/G}\cdot\tau_{\mu_1} = \tau_{\mu_2}\cdot(T\varphi)_\mu$. Since $(T\varphi)_\mu: (\mathbf{J}_L)^{-1}_1(\mu_1)\to(\mathbf{J}_L)^{-1}_2(\mu_2)$ is $(G_{1\mu_1}, G_{2\mu_2})$-equivariant, $(T\varphi)_{\mu/G}$ is well-defined. We shall show that $\mathcal{C}^L_{2\mu_2} = (T\varphi)_{\mu/G}(\mathcal{C}^L_{1\mu_1})$. In fact, since $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i=1,2$, are regular point reducible RCL systems, we have that $\mathcal{C}^L_i\cap(\mathbf{J}_L)^{-1}_i(\mu_i)\neq\emptyset$ and $\mathcal{C}^L_{i\mu_i} = \tau_{\mu_i}(\mathcal{C}^L_i\cap(\mathbf{J}_L)^{-1}_i(\mu_i))$, $i=1,2$. From $\mathcal{C}^L_2\cap(\mathbf{J}_L)^{-1}_2(\mu_2) = (T\varphi)_\mu(\mathcal{C}^L_1\cap(\mathbf{J}_L)^{-1}_1(\mu_1))$, we have that

$$\begin{aligned}
\mathcal{C}^L_{2\mu_2} &= \tau_{\mu_2}(\mathcal{C}^L_2\cap(\mathbf{J}_L)^{-1}_2(\mu_2)) = \tau_{\mu_2}\cdot(T\varphi)_\mu(\mathcal{C}^L_1\cap(\mathbf{J}_L)^{-1}_1(\mu_1))\\
&= (T\varphi)_{\mu/G}\cdot\tau_{\mu_1}(\mathcal{C}^L_1\cap(\mathbf{J}_L)^{-1}_1(\mu_1)) = (T\varphi)_{\mu/G}(\mathcal{C}^L_{1\mu_1}).
\end{aligned}$$

Thus, the condition RCL-1 holds. On the other hand, for the R_p-reduced control law $u^L_{1\mu_1}: (TQ_1)_{\mu_1}\to\mathcal{C}^L_{1\mu_1}$, we have the control law $u^L_1: TQ_1\to\mathcal{C}^L_1$ such that $u^L_{1\mu_1}\cdot\tau_{\mu_1} = \tau_{\mu_1}\cdot u^L_1\cdot j_{\mu_1}$. From the condition RpCL-2 we know that there exists the control law $u^L_2: TQ_2\to\mathcal{C}^L_2$, such that $\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)}\cdot T\varphi = T(T\varphi)\cdot\xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}$. However, for the control law $u^L_2: TQ_2\to\mathcal{C}^L_2$, we have the R_p-reduced control law $u^L_{2\mu_2}: (TQ_2)_{\mu_2}\to\mathcal{C}^L_{2\mu_2}$ such that $u^L_{2\mu_2}\cdot\tau_{\mu_2} = \tau_{\mu_2}\cdot u^L_2\cdot j_{\mu_2}$. Note that for $i=1,2$, from (4.3), we have that

$$\xi_{((TQ_i)_{\mu_i},\omega^L_{i\mu_i},l_{i\mu_i},f^L_{i\mu_i},u^L_{i\mu_i})}\cdot\tau_{\mu_i} = T\tau_{\mu_i}\cdot\xi_{(TQ_i,G_i,\omega^L_i,L_i,F^L_i,u^L_i)}\cdot j_{\mu_i}, \tag{4.4}$$

and from the commutative Diagram-4, $(T\varphi)_{\mu/G}\cdot\tau_{\mu_1} = \tau_{\mu_2}\cdot(T\varphi)_\mu$ and $j_{\mu_2}\cdot(T\varphi)_\mu = (T\varphi)\cdot j_{\mu_1}$ we have that

$$\begin{aligned}
&\xi_{((TQ_2)_{\mu_2},\omega^L_{2\mu_2},l_{2\mu_2},f^L_{2\mu_2},u^L_{2\mu_2})}\cdot(T\varphi)_{\mu/G}\cdot\tau_{\mu_1} = \xi_{((TQ_2)_{\mu_2},\omega^L_{2\mu_2},l_{2\mu_2},f^L_{2\mu_2},u^L_{2\mu_2})}\cdot\tau_{\mu_2}\cdot(T\varphi)_\mu\\
&= T\tau_{\mu_2}\cdot\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)}\cdot j_{\mu_2}\cdot(T\varphi)_\mu = T\tau_{\mu_2}\cdot\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)}\cdot(T\varphi)\cdot j_{\mu_1}
\end{aligned}$$

$$= T\tau_{\mu_2} \cdot T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1} = T(\tau_{\mu_2} \cdot (T\varphi)_\mu) \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1}$$
$$= T((T\varphi)_{\mu/G} \cdot \tau_{\mu_1}) \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1} = T((T\varphi)_{\mu/G}) \cdot T\tau_{\mu_1} \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1}$$
$$= T((T\varphi)_{\mu/G}) \cdot \xi_{((TQ_1)_{\mu_1},\omega_{1\mu_1}^L,l_{1\mu_1},f_{1\mu_1}^L,u_{1\mu_1}^L)} \cdot \tau_{\mu_1}.$$

Since $\tau_{\mu_1} : (\mathbf{J}_L)_1^{-1}(\mu_1) \to (TQ_1)_{\mu_1}$ is surjective, we have that

$$\xi_{((TQ_2)_{\mu_2},\omega_{2\mu_2}^L,l_{2\mu_2},f_{2\mu_2}^L,u_{2\mu_2}^L)} \cdot (T\varphi)_{\mu/G} = T((T\varphi)_{\mu/G}) \cdot \xi_{((TQ_1)_{\mu_1},\omega_{1\mu_1}^L,l_{1\mu_1},f_{1\mu_1}^L,u_{1\mu_1}^L)}, \tag{4.5}$$

that is, the condition RCL-2 holds. So, the R_p-reduced RCL systems $((TQ_i)_{\mu_i}, \omega_{i\mu_i}^L, l_{i\mu_i}, f_{i\mu_i}^L, \mathcal{C}_{i\mu_i}^L)$, $i = 1, 2$, are RCL-equivalent.

Conversely, assume that the R_p-reduced RCL systems $((TQ_i)_{\mu_i}, \omega_{i\mu_i}^L, l_{i\mu_i}, f_{i\mu_i}^L, \mathcal{C}_{i\mu_i}^L)$, $i = 1, 2$, are RCL-equivalent; then there exists a diffeomorphism $(T\varphi)_{\mu/G} : (TQ_1)_{\mu_1} \to (TQ_2)_{\mu_2}$, such that $\mathcal{C}_{2\mu_2}^L = (T\varphi)_{\mu/G}(\mathcal{C}_{1\mu_1}^L)$, $\mu_i \in \mathfrak{g}_i^*$, $i = 1, 2$ and for the R_p-reduced control law $u_{1\mu_1}^L : (TQ_1)_{\mu_1} \to \mathcal{C}_{1\mu_1}^L$, there exists an R_p-reduced control law $u_{2\mu_2}^L : (TQ_2)_{\mu_2} \to \mathcal{C}_{2\mu_2}^L$, such that (4.5) holds. Then from the commutative Diagram-4, we can define a map $(T\varphi)_\mu : (\mathbf{J}_L)_1^{-1}(\mu_1) \to (\mathbf{J}_L)_2^{-1}(\mu_2)$ such that $\tau_{\mu_2} \cdot (T\varphi)_\mu = (T\varphi)_{\mu/G} \cdot \tau_{\mu_1}$, and the map $T\varphi : TQ_1 \to TQ_2$ such that $T\varphi \cdot j_{\mu_1} = j_{\mu_2} \cdot (T\varphi)_\mu$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$ whose tangent lift is just $T\varphi : TQ_1 \to TQ_2$. Moreover, for the above definition of $(T\varphi)_\mu$, we know that $(T\varphi)_\mu$ is $(G_{1\mu_1}, G_{2\mu_2})$-equivariant. In fact, for any $z_i \in (\mathbf{J}_L)_i^{-1}(\mu_i)$, $g_i \in G_{i\mu_i}$, $i = 1, 2$ such that $z_2 = (T\varphi)_\mu(z_1)$ and $[z_2] = (T\varphi)_{\mu/G}[z_1]$, we have that

$$\begin{aligned}(T\varphi)_\mu(\Phi_{1g_1}(z_1)) &= \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2} \cdot (T\varphi)_\mu(\Phi_{1g_1}(z_1)) = \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2} \cdot (T\varphi)_\mu(g_1 z_1)\\ &= \tau_{\mu_2}^{-1} \cdot (T\varphi)_{\mu/G} \cdot \tau_{\mu_1}(g_1 z_1) = \tau_{\mu_2}^{-1} \cdot (T\varphi)_{\mu/G}[z_1] = \tau_{\mu_2}^{-1} \cdot [z_2]\\ &= \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2}(g_2 z_2) = \Phi_{2g_2}(z_2) = \Phi_{2g_2} \cdot (T\varphi)_\mu(z_1).\end{aligned}$$

Here we denote by $\tau_{\mu_1}^{-1}(S)$ the pre-image of a subset $S \subset (TQ_1)_{\mu_1}$ under the map $\tau_{\mu_1} : (\mathbf{J}_L)_1^{-1}(\mu_1) \to (TQ_1)_{\mu_1}$, and for any $z_1 \in (\mathbf{J}_L)_1^{-1}(\mu_1)$, $\tau_{\mu_1}^{-1} \cdot \tau_{\mu_1}(z_1) = z_1$. So, we obtain that $(T\varphi)_\mu \cdot \Phi_{1g_1} = \Phi_{2g_2} \cdot (T\varphi)_\mu$. Moreover, we have that

$$\begin{aligned}\mathcal{C}_2^L \cap (\mathbf{J}_L)_2^{-1}(\mu_2) &= \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2}(\mathcal{C}_2^L \cap (\mathbf{J}_L)_2^{-1}(\mu_2)) = \tau_{\mu_2}^{-1} \cdot \mathcal{C}_{2\mu_2}^L = \tau_{\mu_2}^{-1} \cdot (T\varphi)_{\mu/G}(\mathcal{C}_{1\mu_1}^L)\\ &= \tau_{\mu_2}^{-1} \cdot (T\varphi)_{\mu/G} \cdot \tau_{\mu_1}(\mathcal{C}_1^L \cap (\mathbf{J}_L)_1^{-1}(\mu_1)) = \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2} \cdot (T\varphi)_\mu(\mathcal{C}_1^L \cap (\mathbf{J}_L)_1^{-1}(\mu_1))\\ &= (T\varphi)_\mu(\mathcal{C}_1^L \cap (\mathbf{J}_L)_1^{-1}(\mu_1)).\end{aligned}$$

Thus, the condition RpCL-1 holds. In the following we shall prove that the condition RpCL-2 holds. For the above R_p-reduced control laws $u_{i\mu_i}^L : (TQ_i)_{\mu_i} \to \mathcal{C}_{i\mu_i}^L$, $i = 1, 2$, there exist control laws $u_i^L : TQ_i \to \mathcal{C}_i^L$, such that $u_{i\mu_i}^L \cdot \tau_{\mu_i} = \tau_{\mu_i} \cdot u_i^L \cdot j_{\mu_i}$, $i = 1, 2$. We shall prove that

$$\xi_{(TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)}.$$

In fact, from (4.4) we have that

$$\begin{aligned}&T((T\varphi)_{\mu/G}) \cdot \xi_{((TQ_1)_{\mu_1},\omega_{1\mu_1}^L,l_{1\mu_1},f_{1\mu_1}^L,u_{1\mu_1}^L)} \cdot \tau_{\mu_1}\\ &= T((T\varphi)_{\mu/G}) \cdot T\tau_{\mu_1} \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1}\\ &= T((T\varphi)_{\mu/G} \cdot \tau_{\mu_1}) \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1}\\ &= T(\tau_{\mu_2} \cdot (T\varphi)_\mu) \cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)} \cdot j_{\mu_1}\end{aligned}$$

$$= T\tau_{\mu_2}\cdot T(T\varphi)\cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)}\cdot j_{\mu_1}.$$

On the other hand,

$$\xi_{((TQ_2)_{\mu_2},\omega^L_{2\mu_2},l_{2\mu_2},f^L_{2\mu_2},u^L_{2\mu_2})}\cdot (T\varphi)_{\mu/G}\cdot \tau_{\mu_1}=\xi_{((TQ_2)_{\mu_2},\omega^L_{2\mu_2},l_{2\mu_2},f^L_{2\mu_2},u^L_{2\mu_2})}\cdot \tau_{\mu_2}\cdot (T\varphi)_{\mu}$$
$$= T\tau_{\mu_2}\cdot \xi_{(TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)}\cdot j_{\mu_2}\cdot (T\varphi)_\mu = T\tau_{\mu_2}\cdot \xi_{(TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)}\cdot T\varphi\cdot j_{\mu_1}.$$

From (4.5) we have that

$$T\tau_{\mu_2}\cdot \xi_{(TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)}\cdot T\varphi\cdot j_{\mu_1} = T\tau_{\mu_2}\cdot T(T\varphi)\cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)}\cdot j_{\mu_1}.$$

Note that the map $j_{\mu_1}:(\mathbf{J}_L)_1^{-1}(\mu_1)\to TQ_1$ is injective, and $T\tau_{\mu_2}:T(\mathbf{J}_L)_2^{-1}(\mu_2)\to T(TQ_2)_{\mu_2}$ is surjective, hence we have that

$$\xi_{(TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)}\cdot T\varphi = T(T\varphi)\cdot \xi_{(TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)}.$$

It follows that the theorem holds.

It is worth noting that, when the external force and control of a regular point reducible RCL system $(TQ,G,\omega^L,L,F^L,\mathcal{C}^L)$ are both zero, that is, $F^L=0$ and $\mathcal{C}^L=\emptyset$, then the RCL system is just a regular point reducible Lagrangian system (TQ,G,ω^L,L). Then the following theorem explains the relationship between the equivalence of the regular point reducible Lagrangian systems with symmetries and the equivalence of the associated R_p-reduced Lagrangian systems.

Theorem 4.2 *Two regular point reducible Lagrangian systems* $(TQ_i,G_i,\omega_i^L,L_i)$, $i=1,2$, *are equivalent if and only if the associated* R_p*-reduced Lagrangian systems* $((TQ_i)_{\mu_i},\omega^L_{i\mu_i},l_{i\mu_i})$, $i=1,2$, *are equivalent.*

Proof If two regular point reducible Lagrangian systems $(TQ_i,G_i,\omega_i^L,L_i)$, $i=1,2$, are equivalent, then there exists a diffeomorphism $\varphi:Q_1\to Q_2$ such that $T\varphi:TQ_1\to TQ_2$ is symplectic with respect to their Lagrangian symplectic forms ω_i^L, $i=1,2$, that is, $\omega_1^L=(T\varphi)^*\cdot\omega_2^L$, and for $\mu_i\in\mathfrak{g}_i^*$, $i=1,2$, $(T\varphi)_\mu=j_{\mu_2}^{-1}\cdot T\varphi\cdot j_{\mu_1}:(\mathbf{J}_L)_1^{-1}(\mu_1)\to(\mathbf{J}_L)_2^{-1}(\mu_2)$ is $(G_{1\mu_1},G_{2\mu_2})$-equivariant. From the above commutative Diagram-4, we can define a map $(T\varphi)_{\mu/G}:(TQ_1)_{\mu_1}\to(TQ_2)_{\mu_2}$ such that $(T\varphi)_{\mu/G}\cdot\tau_{\mu_1}=\tau_{\mu_2}\cdot(T\varphi)_\mu$. Since $(T\varphi)_\mu:(\mathbf{J}_L)_1^{-1}(\mu_1)\to(\mathbf{J}_L)_2^{-1}(\mu_2)$ is $(G_{1\mu_1},G_{2\mu_2})$-equivariant, $(T\varphi)_{\mu/G}$ is well-defined. In order to prove that the associated R_p-reduced Lagrangian systems $((TQ_i)_{\mu_i},\omega^L_{i\mu_i},l_{i\mu_i})$, $i=1,2$, are equivalent, in the following we shall show that $(T\varphi)_{\mu/G}$ is symplectic with respect to their R_p-reduced Lagrangian symplectic forms $\omega^L_{i\mu_i}$, $i=1,2$, that is, $(T\varphi)^*_{\mu/G}\omega^L_{2\mu_2}=\omega^L_{1\mu_1}$. In fact, since $T\varphi:TQ_1\to TQ_2$ is symplectic with respect to their Lagrangian symplectic forms, the map $(T\varphi)^*:\Omega^2(TQ_2)\to\Omega^2(TQ_1)$ satisfies $(T\varphi)^*\omega_2^L=\omega_1^L$. From (2.3) we know that, $j^*_{\mu_i}\omega_i^L=\tau^*_{\mu_i}\omega^L_{i\mu_i}$, $i=1,2$, from the following commutative Diagram-5,

$$\begin{array}{ccccc}
\Omega^2(TQ_2) & \xrightarrow{j^*_{\mu_2}} & \Omega^2((\mathbf{J}_L)_2^{-1}(\mu_2)) & \xleftarrow{\tau^*_{\mu_2}} & \Omega^2((TQ_2)_{\mu_2})\\
{\scriptstyle (T\varphi)^*}\downarrow & & {\scriptstyle (T\varphi)^*_\mu}\downarrow & & {\scriptstyle (T\varphi)^*_{\mu/G}}\downarrow\\
\Omega^2(TQ_1) & \xrightarrow{j^*_{\mu_1}} & \Omega^2((\mathbf{J}_L)_1^{-1}(\mu_1)) & \xleftarrow{\tau^*_{\mu_1}} & \Omega^2((TQ_1)_{\mu_1})
\end{array}$$

Diagram-5

we have that

$$\begin{aligned}\tau_{\mu_1}^* \cdot (T\varphi)_{\mu/G}^* \omega_{2\mu_2}^L &= ((T\varphi)_{\mu/G} \cdot \tau_{\mu_1})^* \omega_{2\mu_2}^L = (\tau_{\mu_2} \cdot (T\varphi)_\mu)^* \omega_{2\mu_2}^L \\ &= (j_{\mu_2}^{-1} \cdot T\varphi \cdot j_{\mu_1})^* \cdot \tau_{\mu_2}^* \omega_{2\mu_2}^L \\ &= j_{\mu_1}^* \cdot (T\varphi)^* \cdot (j_{\mu_2}^{-1})^* \cdot j_{\mu_2}^* \omega_2^L \\ &= j_{\mu_1}^* \cdot (T\varphi)^* \omega_2^L = j_{\mu_1}^* \omega_1^L = \tau_{\mu_1}^* \omega_{1\mu_1}^L.\end{aligned}$$

Note that τ_{μ_1} is surjective, thus $(T\varphi)_{\mu/G}^* \omega_{2\mu_2}^L = \omega_{1\mu_1}^L$.

Conversely, assume that the R_p-reduced Lagrangian systems $((TQ_i)_{\mu_i}, \omega_{i\mu_i}^L, l_{i\mu_i})$, $i = 1, 2$, are equivalent, then there exists a diffeomorphism $(T\varphi)_{\mu/G} : (TQ_1)_{\mu_1} \to (TQ_2)_{\mu_2}$ which is symplectic with respect to their R_p-reduced Lagrangian symplectic forms $\omega_{i\mu_i}^L$, $i = 1, 2$. From the above commutative Diagram-4, we can define a map $(T\varphi)_\mu : (\mathbf{J}_L)_1^{-1}(\mu_1) \to (\mathbf{J}_L)_2^{-1}(\mu_2)$ such that $\tau_{\mu_2} \cdot (T\varphi)_\mu = (T\varphi)_{\mu/G} \cdot \tau_{\mu_1}$ and the map $T\varphi : TQ_1 \to TQ_2$ such that $T\varphi \cdot j_{\mu_1} = j_{\mu_2} \cdot (T\varphi)_\mu$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$ whose tangent map is just $T\varphi : TQ_1 \to TQ_2$. From the definition of $(T\varphi)_\mu$, we know that $(T\varphi)_\mu$ is $(G_{1\mu_1}, G_{2\mu_2})$-equivariant. In the following we shall show that $T\varphi$ is symplectic with respect to the Lagrangian symplectic forms ω_i^L, $i = 1, 2$, that is, $\omega_1^L = (T\varphi)^* \cdot \omega_2^L$. Since $(T\varphi)_{\mu/G} : (TQ_1)_{\mu_1} \to (TQ_2)_{\mu_2}$ is symplectic with respect to their R_p-reduced Lagrangian symplectic forms, the map $((T\varphi)_{\mu/G})^* : \Omega^2((TQ_2)_{\mu_2}) \to \Omega^2((TQ_1)_{\mu_1})$ satisfies $((T\varphi)_{\mu/G})^* \cdot \omega_{2\mu_2}^L = \omega_{1\mu_1}^L$. From (2.3) we know that $j_{\mu_i}^* \cdot \omega_i^L = \tau_{\mu_i}^* \cdot \omega_{i\mu_i}^L$, $i = 1, 2$, from the commutative Diagram-5, we have that

$$\begin{aligned}j_{\mu_1}^* \cdot \omega_1^L &= \tau_{\mu_1}^* \cdot \omega_{1\mu_1}^L = \tau_{\mu_1}^* \cdot (T\varphi)_{\mu/G}^* \cdot \omega_{2\mu_2}^L = ((T\varphi)_{\mu/G} \cdot \tau_{\mu_1})^* \cdot \omega_{2\mu_2}^L \\ &= (\tau_{\mu_2} \cdot (T\varphi)_\mu)^* \cdot \omega_{2\mu_2}^L = (j_{\mu_2}^{-1} \cdot T\varphi \cdot j_{\mu_1})^* \cdot \tau_{\mu_2}^* \cdot \omega_{2\mu_2}^L \\ &= j_{\mu_1}^* \cdot (T\varphi)^* \cdot (j_{\mu_2}^{-1})^* \cdot j_{\mu_2}^* \cdot \omega_2^L = j_{\mu_1}^* \cdot (T\varphi)^* \omega_2^L.\end{aligned}$$

Note that j_{μ_1} is injective, and hence $\omega_1^L = (T\varphi)^* \omega_2^L$. Thus, the regular point reducible Lagrangian systems $(TQ_i, G_i, \omega_i^L, L_i)$, $i = 1, 2$, are equivalent.

Thus, the regular point reduction Theorem 4.1 for the RCL systems can be regarded as an extension of the regular point reduction Theorem 4.2 for the regular Lagrangian systems under the regular controlled Lagrangian equivalence conditions.

5 Regular Orbit Reduction of the RCL System

Since the set of regular Lagrangian systems with symmetries on TQ is a subset of the set of RCL systems with symmetries on TQ. If we first consider the regular orbit reduction of a regular Lagrangian system with symmetry, then we may study the regular orbit reduction of an RCL system with symmetry, as an extension of the regular orbit reduction of a regular Lagrangian system under the regular controlled Lagrangian equivalence conditions. In order to do this, in this section we consider the RCL system with symmetry and momentum map, and first give the regular orbit reducible RCL system and the RoCL-equivalence, and then prove the regular orbit reduction theorems for the RCL system and regular Lagrangian system.

Note that, if an RCL system with symmetry and momentum map is regular orbit reducible, then the associated regular Lagrangian system must be regular orbit reducible. Thus, from Definition 2.4 and Theorem 2.3, if the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is $(\Phi^T,\ \Phi^{T*})$-equivariant, then we can introduce a type of regular orbit reducible RCL systems as follows.

Definition 5.1 (Regular orbit reducible RCL system) *A 6-tuple* $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$, *where the hyperregular Lagrangian* $L : TQ \to \mathbb{R}$, *and the fiber-preserving map* $F^L : TQ \to TQ$ *and the fiber submanifold* $\mathcal{C}^L$ *of* TQ *are all* G*-invariant, is called a regular orbit reducible RCL system if the Legendre transformation* $\mathcal{F}L : TQ \to T^*Q$ *is* $(\Phi^T,\ \Phi^{T*})$*-equivariant, and there exists an orbit* $\mathcal{O}_\mu,\ \mu \in \mathfrak{g}^*$, *where* μ *is a regular value of the momentum map* $\mathbf{J}_L$, *such that the regular orbit reduced system, that is, the 5-tuple* $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu}, f^L_{\mathcal{O}_\mu}, \mathcal{C}^L_{\mathcal{O}_\mu})$, *where* $(TQ)_{\mathcal{O}_\mu} = \mathbf{J}_L^{-1}(\mathcal{O}_\mu)/G$, $\tau^*_{\mathcal{O}_\mu}\omega^L_{\mathcal{O}_\mu} = j^*_{\mathcal{O}_\mu}\omega^L - (\mathbf{J}_L)^*_{\mathcal{O}_\mu}\omega^{L+}_{\mathcal{O}_\mu}$, $l_{\mathcal{O}_\mu}\cdot\tau_{\mathcal{O}_\mu} = L\cdot j_{\mathcal{O}_\mu}$, $F^L(\mathbf{J}_L^{-1}(\mathcal{O}_\mu)) \subset \mathbf{J}_L^{-1}(\mathcal{O}_\mu)$, $f^L_{\mathcal{O}_\mu}\cdot\tau_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu}\cdot F^L\cdot j_{\mathcal{O}_\mu}$, *and* $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu) \neq \emptyset$, $\mathcal{C}^L_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu}(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu))$, *is an RCL system, which is simply written as* R_o*-reduced RCL system. Here,* $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})$ *is the* R_o*-reduced space, the function* $l_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R}$ *is called the* R_o*-reduced Lagrangian, the fiber-preserving map* $f^L_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to (TQ)_{\mathcal{O}_\mu}$ *is called the* R_o*-reduced* (*external*) *force map, and* $\mathcal{C}^L_{\mathcal{O}_\mu}$ *is a fiber submanifold of* $(TQ)_{\mathcal{O}_\mu}$ *that is called the* R_o*-reduced control subset.*

It is worth noting that for the regular orbit reducible RCL system $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$, the G-invariant external force map $F^L : TQ \to TQ$ has to satisfy the conditions $F^L(\mathbf{J}_L^{-1}(\mathcal{O}_\mu)) \subset \mathbf{J}_L^{-1}(\mathcal{O}_\mu)$, and $f^L_{\mathcal{O}_\mu}\cdot\tau_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu}\cdot F^L\cdot j_{\mathcal{O}_\mu}$ so that we can define the R_o-reduced external force map $f^L_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to (TQ)_{\mathcal{O}_\mu}$. The condition $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu) \neq \emptyset$ in the above definition ensures that the G-invariant control subset $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu)$ can be reduced and that the R_o-reduced control subset is $\mathcal{C}^L_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu}(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu))$.

Assume that the dynamical vector field $\xi_{(TQ,G,\omega^L,L,F^L,u^L)}$ of a given regular orbit reducible RCL system $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$ with a control law $u^L \in \mathcal{C}^L$ can be expressed by

$$\xi_{(TQ,G,\omega^L,L,F^L,u^L)} = \xi_L + \mathrm{vlift}(F^L) + \mathrm{vlift}(u^L). \tag{5.1}$$

Then, for the regular orbit reducible RCL system we can also introduce the regular orbit reducible controlled Lagrangian equivalence (RoCL-equivalence) as follows.

Definition 5.2 (RoCL-equivalence) *Suppose that we have two regular orbit reducible RCL systems* $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i = 1, 2$, *we say they are RoCL-equivalent, or simply,* $(TQ_1, G_1, \omega^L_1, L_1, F^L_1, \mathcal{C}^L_1) \overset{RoCL}{\sim} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, \mathcal{C}^L_2)$, *if there exists a diffeomorphism* $\varphi : Q_1 \to Q_2$ *such that the following regular orbit reducible controlled Lagrangian matching conditions hold*:

RoCL-1 *For* $\mathcal{O}_{\mu_i},\ \mu_i \in \mathfrak{g}^*_i$, *the regular reducible orbits of RCL systems* $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i = 1, 2$, *the map* $(T\varphi)_{\mathcal{O}_\mu} = j^{-1}_{\mathcal{O}_{\mu_2}}\cdot T\varphi\cdot j_{\mathcal{O}_{\mu_1}} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})$ *is* (G_1, G_2)*-equivariant, and* $\mathcal{C}^L_2 \cap (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2}) = (T\varphi)_{\mathcal{O}_\mu}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}))$, *where* $\mu = (\mu_1, \mu_2)$, *and denote by* $j^{-1}_{\mathcal{O}_{\mu_2}}(S)$ *the pre-image of a subset* $S \subset TQ_2$ *under the map* $j_{\mathcal{O}_{\mu_2}} : (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2}) \to TQ_2$.

RoCL-2 *For each control law* $u^L_1 : TQ_1 \to \mathcal{C}^L_1$, *there exists the control law* $u^L_2 : TQ_2 \to \mathcal{C}^L_2$, *such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is,* $\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)}\cdot T\varphi = T(T\varphi)\cdot\xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}$.

It is worth noting that for the regular orbit reducible RCL system, the induced equivalent map $(T\varphi)_{\mathcal{O}_\mu}$ also preserves the equivariance of the G-action on their regular orbits. If a feedback control law is $u^L : TQ \to \mathcal{C}^L$ is chosen, and $u^L \in \mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu)$, and $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu) \neq \emptyset$, then the R_o-reduced control law $u^L_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathcal{C}^L_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu}(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mathcal{O}_\mu))$, and $u^L_{\mathcal{O}_\mu}\cdot\tau_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu}\cdot u^L\cdot j_{\mathcal{O}_\mu}$. The R_o-reduced RCL system $((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu}, f^L_{\mathcal{O}_\mu}, u^L_{\mathcal{O}_\mu})$ is a closed-loop regular dynamical system with the R_o-reduced control law $u^L_{\mathcal{O}_\mu}$. Assume that its dynamical vector field $\xi_{((TQ)_{\mathcal{O}_\mu},\omega^L_{\mathcal{O}_\mu},l_{\mathcal{O}_\mu},f^L_{\mathcal{O}_\mu},u^L_{\mathcal{O}_\mu})}$ can be expressed by

$$\xi_{((TQ)_{\mathcal{O}_\mu},\omega^L_{\mathcal{O}_\mu},l_{\mathcal{O}_\mu},f^L_{\mathcal{O}_\mu},u^L_{\mathcal{O}_\mu})} = \xi_{l_{\mathcal{O}_\mu}} + \text{vlift}(f^L_{\mathcal{O}_\mu}) + \text{vlift}(u^L_{\mathcal{O}_\mu}), \tag{5.2}$$

where $\xi_{l_{\mathcal{O}_\mu}}$ is the R_o-reduced Euler-Lagrange vector field, and $\text{vlift}(f^L_{\mathcal{O}_\mu}) = \text{vlift}(f^L_{\mathcal{O}_\mu})\xi_{l_{\mathcal{O}_\mu}}$, $\text{vlift}(u^L_{\mathcal{O}_\mu}) = \text{vlift}(u^L_{\mathcal{O}_\mu})\xi_{l_{\mathcal{O}_\mu}}$ are the changes of $\xi_{l_{\mathcal{O}_\mu}}$ under the actions of the R_o-reduced external force $f^L_{\mathcal{O}_\mu}$ and the R_o-reduced control law $u^L_{\mathcal{O}_\mu}$, and the dynamical vector fields of the RCL system and the R_o-reduced RCL system satisfy the condition

$$\xi_{((TQ)_{\mathcal{O}_\mu},\omega^L_{\mathcal{O}_\mu},l_{\mathcal{O}_\mu},f^L_{\mathcal{O}_\mu},u^L_{\mathcal{O}_\mu})} \cdot \tau_{\mathcal{O}_\mu} = T\tau_{\mathcal{O}_\mu} \cdot \xi_{(TQ,G,\omega^L,L,F^L,u^L)} \cdot j_{\mathcal{O}_\mu} \tag{5.3}$$

(see [12, 20]). Then we can obtain the following regular orbit reduction theorem for the RCL system, which explains the relationship between the RoCL-equivalence of the regular orbit reducible RCL system with symmetry and the RCL-equivalence of the associated R_o-reduced RCL system.

Theorem 5.1 *If two regular orbit reducible RCL systems $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i = 1, 2$, are RoCL-equivalent if and only if the associated R_o-reduced RCL systems $((TQ_i)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}}, f^L_{i\mathcal{O}_{\mu_i}}, \mathcal{C}^L_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, are RCL-equivalent.*

Proof If $(TQ_1, G_1, \omega^L_1, L_1, F^L_1, \mathcal{C}^L_1) \overset{RoCL}{\sim} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, \mathcal{C}^L_2)$, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that for $\mathcal{O}_{\mu_i}$, $\mu_i \in \mathfrak{g}^*_i$, the regular reducible orbits, the map $(T\varphi)_{\mathcal{O}_\mu} = j^{-1}_{\mathcal{O}_{\mu_2}} \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}} : (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)^{-1}_2(\mathcal{O}_{\mu_2})$ is (G_1, G_2)-equivariant, and $\mathcal{C}^L_2 \cap (\mathbf{J}_L)^{-1}_2(\mathcal{O}_{\mu_2}) = (T\varphi)_{\mathcal{O}_\mu}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1}))$, and RoCL-2 holds. From the following commutative Diagram-6:

$$\begin{array}{ccccc}
TQ_1 & \xleftarrow{j_{\mathcal{O}_{\mu_1}}} & (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1}) & \xrightarrow{\tau_{\mathcal{O}_{\mu_1}}} & (TQ_1)_{\mathcal{O}_{\mu_1}} \\
{\scriptstyle T\varphi}\downarrow & & {\scriptstyle (T\varphi)_{\mathcal{O}_\mu}}\downarrow & & {\scriptstyle (T\varphi)_{\mathcal{O}_\mu/G}}\downarrow \\
TQ_2 & \xleftarrow{j_{\mathcal{O}_{\mu_2}}} & (\mathbf{J}_L)^{-1}_2(\mathcal{O}_{\mu_2}) & \xrightarrow{\tau_{\mathcal{O}_{\mu_2}}} & (TQ_2)_{\mathcal{O}_{\mu_2}}
\end{array}$$

Diagram-6

we can define a map $(T\varphi)_{\mathcal{O}_\mu/G} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to (TQ_2)_{\mathcal{O}_{\mu_2}}$ such that $(T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}} = \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}$. Because $(T\varphi)_{\mathcal{O}_\mu} : (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)^{-1}_2(\mathcal{O}_{\mu_2})$ is (G_1, G_2)-equivariant, $(T\varphi)_{\mathcal{O}_\mu/G}$ is well-defined. We shall show that $\mathcal{C}^L_{2\mathcal{O}_{\mu_2}} = (T\varphi)_{\mathcal{O}_\mu/G}(\mathcal{C}^L_{1\mathcal{O}_{\mu_1}})$. In fact, since $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, \mathcal{C}^L_i)$, $i = 1, 2$, are regular orbit reducible RCL systems, we have that $\mathcal{C}^L_i \cap (\mathbf{J}_L)^{-1}_i(\mathcal{O}_{\mu_i}) \neq \emptyset$ and $\mathcal{C}^L_{i\mathcal{O}_{\mu_i}} = \tau_{\mathcal{O}_{\mu_i}}(\mathcal{C}^L_i \cap (\mathbf{J}_L)^{-1}_i(\mathcal{O}_{\mu_i}))$, $i = 1, 2$. From $\mathcal{C}^L_2 \cap (\mathbf{J}_L)^{-1}_2(\mathcal{O}_{\mu_2}) = (T\varphi)_{\mathcal{O}_\mu}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1}))$, we have that

$$\begin{aligned}
\mathcal{C}^L_{2\mathcal{O}_{\mu_2}} &= \tau_{\mathcal{O}_{\mu_2}}(\mathcal{C}^L_2 \cap (\mathbf{J}_L)^{-1}_2(\mathcal{O}_{\mu_2})) = \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1})) \\
&= (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)^{-1}_1(\mathcal{O}_{\mu_1})) = (T\varphi)_{\mathcal{O}_\mu/G}(\mathcal{C}^L_{1\mathcal{O}_{\mu_1}}).
\end{aligned}$$

Thus, the condition RCL-1 holds. On the other hand, for the R_o-reduced control law $u^L_{1\mathcal{O}_{\mu_1}} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to \mathcal{C}^L_{1\mathcal{O}_{\mu_1}}$, we have the control law $u^L_1 : TQ_1 \to \mathcal{C}^L_1$ such that $u^L_{1\mathcal{O}_{\mu_1}} \cdot \tau_{\mathcal{O}_{\mu_1}} = \tau_{\mathcal{O}_{\mu_1}} \cdot u^L_1 \cdot j_{\mathcal{O}_{\mu_1}}$. From the condition RoCL-2 we know that there exists the control law $u^L_2 : TQ_2 \to \mathcal{C}^L_2$, such that $\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}$. However, for the control law $u_2^L : TQ_2 \to \mathcal{C}_2^L$, we have the R_o-reduced control law $u^L_{2\mathcal{O}_{\mu_2}} : (TQ_2)_{\mathcal{O}_{\mu_2}} \to \mathcal{C}^L_{2\mathcal{O}_{\mu_2}}$ such that $u^L_{2\mathcal{O}_{\mu_2}} \cdot \tau_{\mathcal{O}_{\mu_2}} = \tau_{\mathcal{O}_{\mu_2}} \cdot u_2^L \cdot j_{\mathcal{O}_{\mu_2}}$. Note that for $i = 1, 2$, from (5.3), we have that

$$\xi_{((TQ_i)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}}, f^L_{i\mathcal{O}_{\mu_i}}, u^L_{i\mathcal{O}_{\mu_i}})} \cdot \tau_{\mathcal{O}_{\mu_i}} = T\tau_{\mathcal{O}_{\mu_i}} \cdot \xi_{(TQ_i, G_i, \omega_i^L, L_i, F_i^L, u_i^L)} \cdot j_{\mathcal{O}_{\mu_i}}, \tag{5.4}$$

and from the commutative Diagram-6, $(T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}} = \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}$ and $j_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} = (T\varphi) \cdot j_{\mathcal{O}_{\mu_1}}$, we have that

$$\begin{aligned}
&\xi_{((TQ_2)_{\mathcal{O}_{\mu_2}}, \omega^L_{2\mathcal{O}_{\mu_2}}, l_{2\mathcal{O}_{\mu_2}}, f^L_{2\mathcal{O}_{\mu_2}}, u^L_{2\mathcal{O}_{\mu_2}})} \cdot (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}} \\
&= \xi_{((TQ_2)_{\mathcal{O}_{\mu_2}}, \omega^L_{2\mathcal{O}_{\mu_2}}, l_{2\mathcal{O}_{\mu_2}}, f^L_{2\mathcal{O}_{\mu_2}}, u^L_{2\mathcal{O}_{\mu_2}})} \cdot \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} \\
&= T\tau_{\mathcal{O}_{\mu_2}} \cdot \xi_{(TQ_2, G_2, \omega_2^L, L_2, F_2^L, u_2^L)} \cdot j_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} \\
&= T\tau_{\mathcal{O}_{\mu_2}} \cdot \xi_{(TQ_2, G_2, \omega_2^L, L_2, F_2^L, u_2^L)} \cdot (T\varphi) \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T\tau_{\mathcal{O}_{\mu_2}} \cdot T(T\varphi) \cdot \xi_{(TQ_1, G_1, \omega_1^L, L_1, F_1^L, u_1^L)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T(\tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}) \cdot \xi_{(TQ_1, G_1, \omega_1^L, L_1, F_1^L, u_1^L)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T((T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}) \cdot \xi_{(TQ_1, G_1, \omega_1^L, L_1, F_1^L, u_1^L)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T((T\varphi)_{\mathcal{O}_\mu/G}) \cdot T\tau_{\mathcal{O}_{\mu_1}} \cdot \xi_{(TQ_1, G_1, \omega_1^L, L_1, F_1^L, u_1^L)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T((T\varphi)_{\mathcal{O}_\mu/G}) \cdot \xi_{((TQ_1)_{\mathcal{O}_{\mu_1}}, \omega^L_{1\mathcal{O}_{\mu_1}}, l_{1\mathcal{O}_{\mu_1}}, f^L_{1\mathcal{O}_{\mu_1}}, u^L_{1\mathcal{O}_{\mu_1}})} \cdot \tau_{\mathcal{O}_{\mu_1}}.
\end{aligned}$$

Since $\tau_{\mathcal{O}_{\mu_1}} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (TQ_1)_{\mathcal{O}_{\mu_1}}$ is surjective, we have that

$$\begin{aligned}
&\xi_{((TQ_2)_{\mathcal{O}_{\mu_2}}, \omega^L_{2\mathcal{O}_{\mu_2}}, l_{2\mathcal{O}_{\mu_2}}, f^L_{2\mathcal{O}_{\mu_2}}, u^L_{2\mathcal{O}_{\mu_2}})} \cdot (T\varphi)_{\mathcal{O}_\mu/G} \\
&= T((T\varphi)_{\mathcal{O}_\mu/G}) \cdot \xi_{((TQ_1)_{\mathcal{O}_{\mu_1}}, \omega^L_{1\mathcal{O}_{\mu_1}}, l_{1\mathcal{O}_{\mu_1}}, f^L_{1\mathcal{O}_{\mu_1}}, u^L_{1\mathcal{O}_{\mu_1}})},
\end{aligned} \tag{5.5}$$

that is, the condition RCL-2 holds. So, the R_o-reduced RCL systems $((TQ_i)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}}, f^L_{i\mathcal{O}_{\mu_i}}, \mathcal{C}^L_{i\mathcal{O}_{\mu_i}}),\ i = 1, 2$, are RCL-equivalent.

Conversely, assume that the R_o-reduced RCL systems $((TQ_i)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}}, f^L_{i\mathcal{O}_{\mu_i}}, \mathcal{C}^L_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, are RCL-equivalent; then there exists a diffeomorphism $(T\varphi)_{\mathcal{O}_\mu/G} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to (TQ_2)_{\mathcal{O}_{\mu_2}}$, such that $\mathcal{C}^L_{2\mathcal{O}_{\mu_2}} = (T\varphi)_{\mathcal{O}_\mu/G}(\mathcal{C}^L_{1\mathcal{O}_{\mu_1}})$, $\forall \mathcal{O}_{\mu_i}$, $\mu_i \in \mathfrak{g}_i^*$, $i = 1, 2$ and for the R_o-reduced control law $u^L_{1\mathcal{O}_{\mu_1}} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to \mathcal{C}^L_{1\mathcal{O}_{\mu_1}}$, there exists the R_o-reduced control law $u^L_{2\mathcal{O}_{\mu_2}} : (TQ_2)_{\mathcal{O}_{\mu_2}} \to \mathcal{C}^L_{2\mathcal{O}_{\mu_2}}$, such that (5.5) holds. Then from commutative Diagram-6, we can define a map $(T\varphi)_{\mathcal{O}_\mu} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})$ such that $\tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} = (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}$, and the map $T\varphi : TQ_1 \to TQ_2$ such that $T\varphi \cdot j_{\mathcal{O}_{\mu_1}} = j_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$, whose tangent lift is just $T\varphi : TQ_1 \to TQ_2$. Moreover, for the above definition of $(T\varphi)_{\mathcal{O}_\mu}$, we know that $(T\varphi)_{\mathcal{O}_\mu}$ is (G_1, G_2)-equivariant. In fact, for any $z_i \in (\mathbf{J}_L)_i^{-1}(\mathcal{O}_{\mu_i})$, $g_i \in G_i$, $i = 1, 2$ such that $z_2 = (T\varphi)_{\mathcal{O}_\mu}(z_1)$, $[z_2] = (T\varphi)_{\mathcal{O}_\mu/G}[z_1]$, we have that

$$\begin{aligned}
(T\varphi)_{\mathcal{O}_\mu}(\Phi_{1g_1}(z_1)) &= \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}(\Phi_{1g_1}(z_1)) = \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}(g_1 z_1) \\
&= \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}(g_1 z_1) = \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu/G}[z_1] = \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot [z_2] \\
&= \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot \tau_{\mathcal{O}_{\mu_2}}(g_2 z_2) = \Phi_{2g_2}(z_2) = \Phi_{2g_2} \cdot (T\varphi)_{\mathcal{O}_\mu}(z_1).
\end{aligned}$$

Here we denote by $\tau^{-1}_{\mathcal{O}_{\mu_1}}(S)$ the pre-image of a subset $S \subset (TQ_1)_{\mathcal{O}_{\mu_1}}$ under the map $\tau_{\mathcal{O}_{\mu_1}} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (TQ_1)_{\mathcal{O}_{\mu_1}}$, and for any $z_1 \in (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1})$, $\tau^{-1}_{\mathcal{O}_{\mu_1}} \cdot \tau_{\mathcal{O}_{\mu_1}}(z_1) = z_1$. So, we obtain that $(T\varphi)_{\mathcal{O}_\mu} \cdot \Phi_{1g_1} = \Phi_{2g_2} \cdot (T\varphi)_{\mathcal{O}_\mu}$. Moreover, we have that

$$\mathcal{C}_2^L \cap (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2}) = \tau^{-1}_{\mathcal{O}_{\mu_2}} \cdot \tau_{\mathcal{O}_{\mu_2}}(\mathcal{C}_2^L \cap (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2}))$$

$$
\begin{aligned}
&= \tau_{\mathcal{O}_{\mu_2}}^{-1} \cdot \mathcal{C}^L_{2\mathcal{O}_{\mu_2}} = \tau_{\mathcal{O}_{\mu_2}}^{-1} \cdot (T\varphi)_{\mathcal{O}_\mu/G}(\mathcal{C}^L_{1\mathcal{O}_{\mu_1}}) \\
&= \tau_{\mathcal{O}_{\mu_2}}^{-1} \cdot (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1})) \\
&= \tau_{\mathcal{O}_{\mu_2}}^{-1} \cdot \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1})) \\
&= (T\varphi)_{\mathcal{O}_\mu}(\mathcal{C}^L_1 \cap (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1})).
\end{aligned}
$$

Thus, the condition RoCL-1 holds. In the following we shall prove that the condition RoCL-2 holds. For the above R_o-reduced control laws $u^L_{i\mathcal{O}_{\mu_i}} : (TQ_i)_{\mathcal{O}_{\mu_i}} \to \mathcal{C}^L_{i\mathcal{O}_{\mu_i}}$, $i = 1, 2$, there exist control laws $u^L_i : TQ_i \to \mathcal{C}^L_i$, such that $u^L_{i\mathcal{O}_{\mu_i}} \cdot \tau_{\mathcal{O}_{\mu_i}} = \tau_{\mathcal{O}_{\mu_i}} \cdot u^L_i \cdot j_{\mathcal{O}_{\mu_i}}$, $i = 1, 2$. we shall prove that

$$
\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}.
$$

In fact, from (5.4) we have that

$$
\begin{aligned}
&T((T\varphi)_{\mathcal{O}_\mu/G}) \cdot \xi_{((TQ_1)_{\mathcal{O}_{\mu_1}},\omega^L_{1\mathcal{O}_{\mu_1}},l_{1\mathcal{O}_{\mu_1}},f^L_{1\mathcal{O}_{\mu_1}},u^L_{1\mathcal{O}_{\mu_1}})} \cdot \tau_{\mathcal{O}_{\mu_1}} \\
&= T((T\varphi)_{\mathcal{O}_\mu/G}) \cdot T\tau_{\mathcal{O}_{\mu_1}} \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T((T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T(\tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)} \cdot j_{\mathcal{O}_{\mu_1}} \\
&= T\tau_{\mathcal{O}_{\mu_2}} \cdot T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)} \cdot j_{\mathcal{O}_{\mu_1}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\xi_{((TQ_2)_{\mathcal{O}_{\mu_2}},\omega^L_{2\mathcal{O}_{\mu_2}},l_{2\mathcal{O}_{\mu_2}},f^L_{2\mathcal{O}_{\mu_2}},u^L_{2\mathcal{O}_{\mu_2}})} \cdot (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}} \\
&= \xi_{((TQ_2)_{\mathcal{O}_{\mu_2}},\omega^L_{2\mathcal{O}_{\mu_2}},l_{2\mathcal{O}_{\mu_2}},f^L_{2\mathcal{O}_{\mu_2}},u^L_{2\mathcal{O}_{\mu_2}})} \cdot \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} \\
&= T\tau_{\mathcal{O}_{\mu_2}} \cdot \xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot j_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} \\
&= T\tau_{\mathcal{O}_{\mu_2}} \cdot \xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}}.
\end{aligned}
$$

From (5.5) we have that

$$
T\tau_{\mathcal{O}_{\mu_2}} \cdot \xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}} = T\tau_{\mathcal{O}_{\mu_2}} \cdot T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)} \cdot j_{\mathcal{O}_{\mu_1}}.
$$

Note that the map $j_{\mathcal{O}_{\mu_1}} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to TQ_1$ is injective, and $T\tau_{\mathcal{O}_{\mu_2}} : T(\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2}) \to T(TQ_2)_{\mathcal{O}_{\mu_2}}$ is surjective, hence we have that

$$
\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}.
$$

It follows that the theorem holds.

It is worth noting that, when the external force and control of a regular orbit reducible RCL system $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$ are both zero, that is, $F^L = 0$ and $\mathcal{C}^L = \emptyset$, then the RCL system is just a regular orbit reducible Lagrangian system (TQ, G, ω^L, L). Then the following theorem explains the relationship between the equivalence of the regular orbit reducible Lagrangian systems with symmetries and the equivalence of the associated R_o-reduced Lagrangian systems. It is worthy of noting that for the regular orbit reducible Lagrangian system, the induced equivalent map $(T\varphi)_{\mathcal{O}_\mu}$ not only keeps the equivariance of G-action on their regular orbits, but also keeps the restriction of the (+)-symplectic structure on the regular orbit to $\mathbf{J}_L^{-1}(\mathcal{O}_\mu)$.

Theorem 5.2 *If two regular orbit reducible Lagrangian systems* $(TQ_i, G_i, \omega_i^L, L_i)$, $i = 1, 2$, *are equivalent, then their associated* R_o*-reduced Lagrangian systems* $((TQ)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, *must be equivalent. Conversely, if the* R_o*-reduced Lagrangian systems* $((TQ)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, *are equivalent, and the induced map* $(T\varphi)_{\mathcal{O}_\mu} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})$, *such that* $(\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} = (T\varphi)^*_{\mathcal{O}_\mu} \cdot (\mathbf{J}_L)^*_{2\mathcal{O}_{\mu_2}} \cdot \omega^{L+}_{2\mathcal{O}_{\mu_2}}$, *then the regular orbit reducible Lagrangian systems* $(TQ_i, G_i, \omega_i^L, L_i)$, $i = 1, 2$, *are equivalent.*

Proof If two regular orbit reducible Lagrangian systems $(TQ_i, G_i, \omega_i^L, L_i)$, $i = 1, 2$, are equivalent, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that $T\varphi : TQ_1 \to TQ_2$ is symplectic with respect to their Lagrangian symplectic forms ω_i^L, $i = 1, 2$, and for $\mathcal{O}_{\mu_i}$, $\mu_i \in \mathfrak{g}_i^*$, $i = 1, 2$, $(T\varphi)_{\mathcal{O}_\mu} = j^{-1}_{\mathcal{O}_{\mu_2}} \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})$ is (G_1, G_2)-equivariant. From the above commutative Diagram-6, we can define a map $(T\varphi)_{\mathcal{O}_\mu/G} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to (TQ_2)_{\mathcal{O}_{\mu_2}}$ such that $(T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}} = \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu}$. Since $(T\varphi)_{\mathcal{O}_\mu} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})$ is (G_1, G_2)-equivariant, $(T\varphi)_{\mathcal{O}_\mu/G}$ is well-defined. In order to prove that the associated R_o-reduced Lagrangian systems $((TQ)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, are equivalent, in the following we shall prove that $(T\varphi)_{\mathcal{O}_\mu/G}$ is symplectic with respect to their R_o-reduced Lagrangian symplectic forms $\omega^L_{i\mathcal{O}_i}$, $i = 1, 2$, that is, $(T\varphi)^*_{\mathcal{O}_\mu/G} \cdot \omega^L_{2\mathcal{O}_{\mu_2}} = \omega^L_{1\mathcal{O}_{\mu_1}}$. In fact, since $T\varphi : TQ_1 \to TQ_2$ is symplectic with respect to their Lagrangian symplectic forms, the map $(T\varphi)^* : \Omega^2(TQ_2) \to \Omega^2(TQ_1)$ satisfies $(T\varphi)^* \cdot \omega_2^L = \omega_1^L$. From (2.4), we have that $j^*_{\mathcal{O}_{\mu_i}} \cdot \omega_i^L = \tau^*_{\mathcal{O}_{\mu_i}} \cdot \omega^L_{i\mathcal{O}_{\mu_i}} + (\mathbf{J}_L)^*_{i\mathcal{O}_{\mu_i}} \cdot \omega^{L+}_{i\mathcal{O}_{\mu_i}}$, $i = 1, 2$, and $(\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} = ((T\varphi)_{\mathcal{O}_\mu})^* \cdot (\mathbf{J}_L)^*_{2\mathcal{O}_{\mu_2}} \cdot \omega^{L+}_{2\mathcal{O}_{\mu_2}}$. From the following commutative Diagram-7:

$$\begin{array}{ccccc}
\Omega^2(TQ_2) & \xrightarrow{j^*_{\mathcal{O}_{\mu_2}}} & \Omega^2((\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})) & \xleftarrow{\tau^*_{\mathcal{O}_{\mu_2}}} & \Omega^2((TQ_2)_{\mathcal{O}_{\mu_2}}) \\
{\scriptstyle (T\varphi)^*}\downarrow & & {\scriptstyle (T\varphi)^*_{\mathcal{O}_\mu}}\downarrow & & {\scriptstyle (T\varphi)^*_{\mathcal{O}_\mu/G}}\downarrow \\
\Omega^2(TQ_1) & \xrightarrow{j^*_{\mathcal{O}_{\mu_1}}} & \Omega^2((\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1})) & \xleftarrow{\tau^*_{\mathcal{O}_{\mu_1}}} & \Omega^2((TQ_1)_{\mathcal{O}_{\mu_1}})
\end{array}$$

Diagram-7

we have that

$$\begin{aligned}
\tau^*_{\mathcal{O}_{\mu_1}} \cdot (T\varphi)^*_{\mathcal{O}_\mu/G} \omega^L_{2\mathcal{O}_{\mu_2}} &= ((T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} \\
&= (\tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} \\
&= ((T\varphi)_{\mathcal{O}_\mu})^* \cdot \tau^*_{\mathcal{O}_{\mu_2}} \cdot \omega^L_{2\mathcal{O}_{\mu_2}} \\
&= (j^{-1}_{\mathcal{O}_{\mu_2}} \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}})^* \cdot j^*_{\mathcal{O}_{\mu_2}} \cdot \omega_2^L - (T\varphi)^*_{\mathcal{O}_\mu} \cdot (\mathbf{J}_L)^*_{2\mathcal{O}_{\mu_2}} \cdot \omega^{L+}_{2\mathcal{O}_{\mu_2}} \\
&= j^*_{\mathcal{O}_{\mu_1}} \cdot (T\varphi)^* \cdot \omega_2^L - (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= j^*_{\mathcal{O}_{\mu_1}} \cdot \omega_1^L - (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= \tau^*_{\mathcal{O}_{\mu_1}} \cdot \omega^L_{1\mathcal{O}_{\mu_1}}.
\end{aligned}$$

Since $\tau_{\mathcal{O}_{\mu_1}}$ is surjective, we have that $((T\varphi)_{\mathcal{O}_\mu/G})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} = \omega^L_{1\mathcal{O}_{\mu_1}}$.

Conversely, assume that the R_o-reduced Lagrangian systems $((TQ_i)_{\mathcal{O}_{\mu_i}}, \omega^L_{i\mathcal{O}_{\mu_i}}, l_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, are equivalent; then there exists a diffeomorphism $(T\varphi)_{\mathcal{O}_\mu/G} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to (TQ_2)_{\mathcal{O}_{\mu_2}}$, which is symplectic with respect to the R_o-reduced Lagrangian symplectic forms $\omega^L_{i\mathcal{O}_i}$, $i = 1, 2$, that is, $(T\varphi)^*_{\mathcal{O}_\mu/G} \cdot \omega^L_{2\mathcal{O}_{\mu_2}} = \omega^L_{1\mathcal{O}_{\mu_1}}$. Thus, from the above commutative Diagram-6, we can define a map $(T\varphi)_{\mathcal{O}_\mu} : (\mathbf{J}_L)_1^{-1}(\mathcal{O}_{\mu_1}) \to (\mathbf{J}_L)_2^{-1}(\mathcal{O}_{\mu_2})$, such that $\tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} = (T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}}$, and map $T\varphi : TQ_1 \to TQ_2$, such that $j_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu} = T\varphi \cdot j_{\mathcal{O}_{\mu_1}}$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$, whose tangent map is just $T\varphi : TQ_1 \to TQ_2$. From definition of $(T\varphi)_{\mathcal{O}_\mu}$, we know that $(T\varphi)_{\mathcal{O}_\mu}$ is (G_1, G_2)-equivariant.

Now we shall show that $T\varphi$ is symplectic with respect to the Lagrangian symplectic forms $\omega_i^L,\ i = 1, 2$, that is, $\omega_1^L = (T\varphi)^* \cdot \omega_2^L$. In fact, since $(T\varphi)_{\mathcal{O}_\mu/G} : (TQ_1)_{\mathcal{O}_{\mu_1}} \to (TQ_2)_{\mathcal{O}_{\mu_2}}$ is symplectic with respect to their R_o-reduced Lagrangian symplectic forms, the map $((T\varphi)_{\mathcal{O}_\mu/G})^* : \Omega^2((TQ_2)_{\mathcal{O}_{\mu_2}}) \to \Omega^2((TQ_1)_{\mathcal{O}_{\mu_1}})$ satisfies $((T\varphi)_{\mathcal{O}_\mu/G})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} = \omega^L_{1\mathcal{O}_{\mu_1}}$. From (2.4) we have that $j^*_{\mathcal{O}_{\mu_i}} \cdot \omega_i^L = \tau^*_{\mathcal{O}_{\mu_i}} \cdot \omega^L_{i\mathcal{O}_{\mu_i}} + (\mathbf{J}_L)^*_{i\mathcal{O}_{\mu_i}} \cdot \omega^{L+}_{i\mathcal{O}_{\mu_i}}$, $i = 1, 2$. From the commutative Diagram-7, we have that

$$
\begin{aligned}
j^*_{\mathcal{O}_{\mu_1}} \cdot \omega_1^L &= \tau^*_{\mathcal{O}_{\mu_1}} \cdot \omega^L_{1\mathcal{O}_{\mu_1}} + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= \tau^*_{1\mathcal{O}_{\mu_1}} \cdot ((T\varphi)_{\mathcal{O}_\mu/G})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= ((T\varphi)_{\mathcal{O}_\mu/G} \cdot \tau_{\mathcal{O}_{\mu_1}})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= (\tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)_{\mathcal{O}_\mu})^* \cdot \omega^L_{2\mathcal{O}_{\mu_2}} + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= (j^{-1}_{\mathcal{O}_{\mu_2}} \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}})^* \cdot \tau^*_{\mathcal{O}_{\mu_2}} \cdot \omega^L_{2\mathcal{O}_{\mu_2}} + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= j^*_{\mathcal{O}_{\mu_1}} \cdot (T\varphi)^* \cdot (j^{-1}_{\mathcal{O}_{\mu_2}})^* \cdot [j^*_{\mathcal{O}_{\mu_2}} \cdot \omega_2^L - (\mathbf{J}_L)^*_{2\mathcal{O}_{\mu_2}} \cdot \omega^{L+}_{2\mathcal{O}_{\mu_2}}] + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} \\
&= j^*_{\mathcal{O}_{\mu_1}} \cdot (T\varphi)^* \cdot \omega_2^L - ((T\varphi)_{\mathcal{O}_\mu})^* \cdot (\mathbf{J}_L)^*_{2\mathcal{O}_{\mu_2}} \cdot \omega^{L+}_{2\mathcal{O}_{\mu_2}} + (\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}}.
\end{aligned}
$$

Note that $j_{\mathcal{O}_{\mu_1}}$ is injective, and by our hypothesis,

$$(\mathbf{J}_L)^*_{1\mathcal{O}_{\mu_1}} \cdot \omega^{L+}_{1\mathcal{O}_{\mu_1}} = ((T\varphi)_{\mathcal{O}_\mu})^* \cdot (\mathbf{J}_L)^*_{2\mathcal{O}_{\mu_2}} \cdot \omega^{L+}_{2\mathcal{O}_{\mu_2}},$$

we have that $\omega_1^L = (T\varphi)^* \omega_2^L$. Thus, the regular orbit reducible Lagrangian systems $(TQ_i, G_i, \omega_i^L, L_i),\ i = 1, 2$, are equivalent.

Thus, the regular orbit reduction Theorem 5.1 for the RCL systems can be regarded as an extension of the regular orbit reduction Theorem 5.2 for the regular Lagrangian systems under regular controlled Lagrangian equivalence conditions.

6 RCL System on a Generalization of Lie Group

As an application of regular point reduction of the RCL system with symmetry and a momentum map, in this section, we study the regular point reducible RCL system on the generalization of a Lie group, and give its R_p-reduced RCL system, which is an RCL system on the generalization of a coadjoint orbit of the Lie group.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and TG its tangent bundle and T^*G its cotangent bundle with the canonical symplectic form ω. Assume that $L : TG \to \mathbb{R}$ is a hyperregular Lagrangian, and that the Legendre transformation $\mathcal{F}L : TG \to T^*G$ is a diffeomorphism. By using the Legendre transformation we can induce a Lagrangian symplectic form ω^L on the tangent bundle TG, that is, $\omega^L := (\mathcal{F}L)^*\omega$. Define an action $A : TG \to \mathbb{R}$ given by $A(v) := \mathcal{F}L(v)v,\ \forall v \in T_gG, g \in G$ and an energy $E_L : TG \to \mathbb{R}$ given by $E_L := A - L$. If there exists an Euler-Lagrange vector field ξ_L on TG, such that the Euler-Lagrange equation $\mathbf{i}_{\xi_L}\omega^L = \mathbf{d}E_L$ holds, then the triple (TG, ω^L, L) is a regular Lagrangian system. An RCL system on G is a 5-tuple $(TG, \omega^L, L, F^L, \mathcal{C}^L)$, where (TG, ω^L, L) is a regular Lagrangian system, and the fiber-preserving map $F^L : TG \to TG$ is an (external) force map, and the fiber submanifold $\mathcal{C}^L$ of TG is a control subset. In the following we shall give an R_p-reduced RCL system on a coadjoint orbit of the Lie group G.

We know that the left and right translations on the Lie group G induce the left and right actions of G on itself. If $I_g : G \to G$; $I_g(h) = ghg^{-1} = L_g \cdot R_{g^{-1}}(h)$, for $g, h \in G$, is the inner automorphism on G, then the adjoint representation of a Lie group G is defined by $\mathrm{Ad}_g = T_eI_g = T_{g^{-1}}L_g \cdot T_eR_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$, and the coadjoint representation is given by $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$; $\langle \mathrm{Ad}^*_{g^{-1}}(\mu), \xi\rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}(\xi)\rangle$, where $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, and $\mu \in \mathfrak{g}^*$, and $\xi \in \mathfrak{g}$ and $\langle , \rangle$ denotes the pairing on $\mathfrak{g}^*$ and $\mathfrak{g}$. We now identify locally TG and $G \times \mathfrak{g}$, as well as T^*G and $G \times \mathfrak{g}^*$, by using the left translation. In fact, the map $\lambda : TG \to G \times \mathfrak{g},\ \lambda(v_g) := (g, (T_eL_g) \cdot v_g)$, for any $v_g \in T_gG$, which defines a vector bundle isomorphism usually referred to as the local left trivialization of TG. If the left G-action $L_g : G \to G$ is free and proper, then the tangent lift of the action to its tangent bundle TG, given by $\Phi^T : G \times TG \to TG,\ g \cdot (h, w) := (gh, w)$, for any $g, h \in G,\ w \in \mathfrak{g}$, and the cotangent lift of the action to its cotangent bundle T^*G, given by $\Phi^{T*} : G \times T^*G \to T^*G,\ g \cdot (h, \nu) := (gh, \nu)$, for any $g, h \in G,\ \nu \in \mathfrak{g}^*$, are also the free and proper actions, and the orbit spaces $(TG)/G$ and $(T^*G)/G$ are both smooth manifolds and $\tau_{/G} : TG \to (TG)/G$ and $\pi_{/G} : T^*G \to (T^*G)/G$ are both smooth submersions. We note that $(TG)/G$ is diffeomorphic to $(G \times \mathfrak{g})/G$ and $(T^*G)/G$ is diffeomorphic to $(G \times \mathfrak{g}^*)/G$, since G acts trivially on $\mathfrak{g}$ and $\mathfrak{g}^*$, it follows that $(G \times \mathfrak{g})/G \cong \mathfrak{g}$ and $(G \times \mathfrak{g}^*)/G \cong \mathfrak{g}^*$. And hence $(TG)/G$ is diffeomorphic to $\mathfrak{g}$ and $(T^*G)/G$ is diffeomorphic to $\mathfrak{g}^*$.

Assume that the tangent lifted left action $\Phi^T : G \times TG \to TG$ is symplectic with respect to Lagrangian symplectic form ω^L, and that the action admits an Ad^*-equivariant momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}*$. For a regular value of $\mathbf{J}_L$, $\mu \in \mathfrak{g}^*$, denote $G_\mu = \{g \in G \mid \mathrm{Ad}^*_g \mu = \mu\}$ the isotropy subgroup of the coadjoint G-action at the point $\mu \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\mu = \mathbf{J}_L^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form ω^L_μ uniquely characterized by the relation

$$\tau^*_\mu \cdot \omega^L_\mu = j^*_\mu \cdot \omega^L. \tag{6.1}$$

The map $j_\mu : \mathbf{J}_L^{-1}(\mu) \to TG$ is the inclusion and $\tau_\mu : \mathbf{J}_L^{-1}(\mu) \to \mathcal{O}_\mu$ is the projection. The pair $((\mathcal{O}_\mu, \omega^L_\mu)$ is called the regular point reduced space of (TG, ω^L) at μ.

On the other hand, from [1], we know that $\mathfrak{g}^*$ is a Poisson manifold with respect to the $(\pm)$-Lie-Poisson bracket $\{\cdot, \cdot\}_\pm$ defined by

$$\{f, g\}_\pm(\mu) := \pm\Big\langle \mu, \Big[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\Big]\Big\rangle, \quad \forall f, g \in C^\infty(\mathfrak{g}^*),\ \ \mu \in \mathfrak{g}^*, \tag{6.2}$$

where the element $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$ is defined by the equality $\big\langle v, \frac{\delta f}{\delta \mu}\big\rangle := Df(\mu) \cdot v$ for any $v \in \mathfrak{g}^*$. Thus, for the coadjoint orbit $\mathcal{O}_\mu,\ \mu \in \mathfrak{g}^*$, the orbit symplectic structure can be defined by

$$\omega^\pm_{\mathcal{O}_\mu}(\nu)(\mathrm{ad}^*_\xi(\nu), \mathrm{ad}^*_\eta(\nu)) = \pm\langle \nu, [\xi, \eta]\rangle, \quad \forall\, \xi, \eta \in \mathfrak{g},\ \nu \in \mathcal{O}_\mu \subset \mathfrak{g}^*, \tag{6.3}$$

which coincide with the restriction of the Lie-Poisson brackets on $\mathfrak{g}^*$ to the coadjoint orbit $\mathcal{O}_\mu$. Consequently, the coadjoint orbit $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu}),\ \mu \in \mathfrak{g}^*$, is symplectically diffeomorphic to an R_p-reduced space $(\mathcal{O}_\mu, \omega^L_\mu)$ of (TG, ω^L) at μ.

In the following we consider the Lagrangian $L(g, \xi) : TG \cong G \times \mathfrak{g} \to \mathbb{R}$, which is usual the kinetic minus the potential energy of the system, where $(g, \xi) \in G \times \mathfrak{g}$, and $\xi \in \mathfrak{g}$, regarded as the velocity of system. We can introduce the conjugate momentum $p_i = \frac{\partial L}{\partial \xi^i}$, $i = 1, \cdots, n$, $n = \dim G$, and define the Legendre transformation $\mathcal{F}L : TG \cong G \times \mathfrak{g} \to T^*G \cong G \times \mathfrak{g}^*$, $(g^i, \xi^i) \to (g^i, p_i)$. If $\mathcal{F}L : TG \to T^*G$ is a diffeomorphism, then the Lagrangian $L : TG \to \mathbb{R}$ is hyperregular. Assume that the hyperregular Lagrangian $L : TG \to \mathbb{R}$ is G-invariant, and that the Legendre transformation $\mathcal{F}L : TG \to T^*G$ is $(\Phi^T,\ \Phi^{T*})$-equivariant, and that the fiber-preserving map $F^L : TG \to TG$ and the fiber submanifold $\mathcal{C}^L$ of TG are both left tangent lifted G-action Φ^T invariant. If the Euler-Lagrange vector field ξ_L satisfies the Euler-Lagrange equation $\mathbf{i}_{\xi_L}\omega^L = \mathbf{d}E_L$, where the energy $E_L : TG \to \mathbb{R}$ given by $E_L := A - L$, and the action $A : TG \to \mathbb{R}$ given by $A(v) := \mathcal{F}L(v)v,\ \forall v \in T_gG,\ g \in G$, and the flow F_t of the Euler-Lagrange vector field ξ_L leaves the connected components of $\mathbf{J}_L^{-1}(\mu)$ invariant and commutes with the G-action, then it induces a flow f_t^μ on $\mathcal{O}_\mu$, defined by $f_t^\mu \cdot \tau_\mu = \tau_\mu \cdot F_t \cdot j_\mu$, and the vector field ξ_{l_μ} generated by the flow f_t^μ on $(\mathcal{O}_\mu, \omega_\mu^L)$ is the R_p-reduced Euler-Lagrange vector field with the associated R_p-reduced Lagrangian function $l_\mu : \mathcal{O}_\mu \to \mathbb{R}$ defined by $l_\mu \cdot \tau_\mu = L \cdot j_\mu$, and the R_p-reduced Euler-Lagrange equation $\mathbf{i}_{\xi_{l_\mu}}\omega_{L\mu} = \mathbf{d}E_{l_\mu}$ holds, where the R_p-reduced energy $E_{l_\mu} : \mathcal{O}_\mu \to \mathbb{R}$ given by $E_{l_\mu} := A_\mu - l_\mu$, and the R_p-reduced action $A_\mu : \mathcal{O}_\mu \to \mathbb{R}$ given by $A_\mu \cdot \tau_\mu = A \cdot j_\mu$, and the Euler-Lagrange vector fields ξ_L and ξ_{l_μ} are τ_μ-related. Thus, we obtain the R_p-reduced Lagrangian system $(\mathcal{O}_\mu, \omega_\mu^L, l_\mu)$ as follows.

Theorem 6.1 *Assume that the Lagrangian $L : TG \to \mathbb{R}$ is hyperregular, and that the Legendre transformation $\mathcal{F}L : TG \to T^*G$ is (Φ^T, Φ^{T*})-equivariant. Then the* 6*-tuple $(TG, G, \omega^L, L, F^L, \mathcal{C}^L)$ is a regular point reducible RCL system on Lie group G, where the Lagrangian $L : TG \to \mathbb{R}$, the fiber-preserving map $F^L : TG \to TG$ and the fiber submanifold $\mathcal{C}^L$ of TG are all left tangent lifted G-action Φ^T invariant. For a point $\mu \in \mathfrak{g}^*$, the regular value of the momentum map $\mathbf{J}_L : TG \to \mathfrak{g}^*$, the R_p-reduced system, that is, the* 5*-tuple $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}^-, l_\mu, f_\mu^L, \mathcal{C}_\mu^L)$, is an RCL system, where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the coadjoint orbit, $\omega_{\mathcal{O}_\mu}^-$ is orbit symplectic form, $l_\mu \cdot \tau_\mu = L \cdot j_\mu$, $F^L(\mathbf{J}_L^{-1}(\mu)) \subset \mathbf{J}_L^{-1}(\mu)$, $f_\mu^L \cdot \tau_\mu = \tau_\mu \cdot F^L \cdot j_\mu$, $\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu) \neq \emptyset$, $\mathcal{C}_\mu^L = \tau_\mu(\mathcal{C}^L \cap \mathbf{J}_L^{-1}(\mu))$. Moreover, two regular point reducible RCL systems $(TG_i, G_i, \omega_i^L, L_i, F_i^L, \mathcal{C}_i^L)$, $i = 1, 2$, are RpCL-equivalent if and only if the associated R_p-reduced RCL systems $(\mathcal{O}_{i\mu_i}, \omega_{\mathcal{O}_{i\mu_i}}^-, l_{i\mu_i}, f_{i\mu_i}^L, \mathcal{C}_{i\mu_i}^L)$, $i = 1, 2$, are RCL-equivalent.*

Moreover, we can study the regular point reduction of an RCL system with symmetry and a momentum map on the generalization of a Lie group TQ, where the configuration space $Q = G \times V$, G is a Lie group and V is a k-dimensional vector space. Define the left G-action $\Phi : G \times Q \to Q$, $\Phi(g, (h, \theta)) := (gh, \theta)$, for any $g, h \in G$, $\theta \in V$, that is, the G-action on Q is the left translation on the first factor G, and G acts trivially on the second factor V. Since $TQ \cong TG \times TV$, and $TV \cong V \times V$, by using the left trivialization of TG, that is, $TG \cong G \times \mathfrak{g}$, we have that $TQ = G \times \mathfrak{g} \times V \times V$. If the left G-action $\Phi : G \times Q \to Q$ is free and proper, then the tangent lift of the action to its tangent bundle TQ, given by $\Phi^T : G \times TQ \to TQ$, $\Phi^T(g, (h, \eta, \theta, \kappa)) := (gh, \eta, \theta, \kappa)$, for any $g, h \in G$, $\eta \in \mathfrak{g}$, $\theta, \kappa \in V$, is also a free and proper action, the orbit space $(TQ)/G$ is a smooth manifold and $\tau : TQ \to (TQ)/G$ is a smooth submersion. Since G acts trivially on $\mathfrak{g}$ and on $V \times V$, it follows that $(TQ)/G$ is diffeomorphic to $\mathfrak{g} \times V \times V$.

For $\mu \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$ has the orbit symplectic forms $\omega_{\mathcal{O}_\mu}^\pm$. Let ω_V be the canonical symplectic form on $T^*V \cong V \times V^*$ given by

$$\omega_V((\theta_1, \lambda_1), (\theta_2, \lambda_2)) = \langle \lambda_2, \theta_1 \rangle - \langle \lambda_1, \theta_2 \rangle,$$

where $(\theta_i,\lambda_i)\in V\times V^*,\ i=1,2,\ \langle\cdot,\cdot\rangle$ is the natural pairing on V^* and V. Since V and V^* is isomorphic, there is a map $\sigma: V\to V^*$ such that $\lambda_i=\sigma(\delta_i)$, for $\delta_i\in V, i=1,2$, we can induce that a symplectic form ω_V^L on $TV\cong V\times V$ is given by

$$\omega_V^L((\theta_1,\delta_1),(\theta_2,\delta_2))=\omega_V((\theta_1,\lambda_1),(\theta_2,\lambda_2))=\langle\lambda_2,\theta_1\rangle-\langle\lambda_1,\theta_2\rangle.$$

Thus, we can induce a symplectic form $\widetilde{\omega}^{\pm L}_{\mathcal{O}_\mu\times V\times V}=\tau^*_{\mathcal{O}_\mu}\omega^{\pm}_{\mathcal{O}_\mu}+\tau_V^*\omega_V^L$ on the smooth manifold $\mathcal{O}_\mu\times V\times V$, where the maps $\tau_{\mathcal{O}_\mu}:\mathcal{O}_\mu\times V\times V\to\mathcal{O}_\mu$ and $\tau_V:\mathcal{O}_\mu\times V\times V\to V\times V$ are canonical projections.

On the other hand, the cotangent bundle T^*Q has a canonical symplectic form ω_Q, and the tangent bundle TQ has a Lagrangian symplectic form $\omega_Q^L=(\mathcal{F}L)^*\omega_Q$, from $TQ\cong TG\times TV$ we have that $\omega_Q^L=\tau_1^*\omega_0^L+\tau_2^*\omega_V^L$ on TQ, where ω_0^L is the Lagrangian symplectic form on TG and the maps $\tau_1: Q=G\times V\to G$ and $\tau_2: Q=G\times V\to V$ are canonical projections. Assume that the tangent lift of the left G-action $\Phi^T: G\times TQ\to TQ$ is symplectic with respect to ω_Q^L, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q^L: TQ\to\mathfrak{g}^*$ such that $\mathbf{J}_Q^L\cdot\tau_1^*=\mathbf{J}_G^L$, where $\mathbf{J}_G^L: TG\to\mathfrak{g}^*$ is a momentum map of left G-action on TG and we assume that it exists, if $\mu\in\mathfrak{g}^*$ is a regular value of $\mathbf{J}_G^L$, then $\mu\in\mathfrak{g}^*$ is also a regular value of $\mathbf{J}_Q^L$ and $(\mathbf{J}_Q^L)^{-1}(\mu)\cong(\mathbf{J}_G^L)^{-1}(\mu)\times V\times V$. Denote $G_\mu=\{g\in G\mid \mathrm{Ad}_g^*\mu=\mu\}$ the isotropy subgroup of coadjoint G-action at the point $\mu\in\mathfrak{g}^*$. It follows that G_μ acts also freely and properly on $(\mathbf{J}_Q^L)^{-1}(\mu)$, the regular point reduced space $(TQ)_\mu=(\mathbf{J}_Q^L)^{-1}(\mu)/G_\mu\cong(TG)_\mu\times V\times V$ of (TQ,ω_Q^L) at μ, is a symplectic manifold with the reduced Lagrangian symplectic form ω_μ^L uniquely characterized by the relation $\tau_\mu^*\omega_\mu^L=j_\mu^*\omega_Q^L=j_\mu^*\tau_1^*\omega_0^L+j_\mu^*\tau_2^*\omega_V^L$, where the map $j_\mu:(\mathbf{J}_Q^L)^{-1}(\mu)\to TQ$ is the inclusion and $\tau_\mu:(\mathbf{J}_Q^L)^{-1}(\mu)\to(TQ)_\mu$ is the projection. Since $((TG)_\mu,\omega_\mu^L)$ is symplectically diffeomorphic to $(\mathcal{O}_\mu,\omega^-_{\mathcal{O}_\mu})$, we have that $((TQ)_\mu,\omega_\mu^L)$ is symplectically diffeomorphic to $(\mathcal{O}_\mu\times V\times V,\widetilde{\omega}^{-L}_{\mathcal{O}_\mu\times V\times V})$.

Now we identify TG and $G\times\mathfrak{g}$, by using the left translation, and $TV\cong V\times V$, then $TQ\cong G\times\mathfrak{g}\times V\times V$. Consequently, we consider the Lagrangian $L(g,\xi,\theta,\dot\theta): TQ\cong G\times\mathfrak{g}\times V\times V\to\mathbb{R}$, which is usually the total kinetic minus potential energy of the system, where $(g,\xi)\in G\times\mathfrak{g}$, and $\theta\in V$, ξ^i and $\dot\theta^j=\frac{\mathrm{d}\theta^j}{\mathrm{d}t}$ $(i=1,\cdots,n,\ j=1,\cdots,k,\ n=\dim G,\ k=\dim V)$, regarded as the velocities of the system. We can introduce the conjugate momentum $p_i=\frac{\partial L}{\partial\xi^i},\ l_j=\frac{\partial L}{\partial\dot\theta^j}$, $i=1,\cdots,n,\ j=1,\cdots,k$, and define the Legendre transformation $FL: TQ\cong G\times\mathfrak{g}\times V\times V\to T^*Q\cong G\times\mathfrak{g}^*\times V\times V^*$, $(g^i,\xi^i,\theta^j,\dot\theta^j)\to(g^i,p_i,\theta^j,l_j)$. If $\mathcal{F}L: TQ\to T^*Q$ is a diffeomorphism, then the Lagrangian $L: TQ\to\mathbb{R}$ is hyperregular. We can define the Lagrangian symplectic form on TQ given by $\omega_Q^L:=(\mathcal{F}L)^*\omega_Q$, and the momentum map $\mathbf{J}_Q^L: TQ\to\mathfrak{g}^*$ given by $\mathbf{J}_Q^L=\mathbf{J}_Q\cdot\mathcal{F}L$. If the Legendre transformation $\mathcal{F}L: TQ\to T^*Q$ is $(\Phi^T,\ \Phi^{T*})$-equivariant, from Theorem 2.1 we know that $\mathbf{J}_Q^L$ is Ad^*-equivariant, and if $\mu\in\mathfrak{g}^*$ is a regular value of the momentum map $\mathbf{J}_Q$, then μ is also a regular value of the momentum map $\mathbf{J}_Q^L$. Moreover, we consider the regular point reduced space $((TQ)_\mu=(\mathbf{J}_Q^L)^{-1}(\mu)/G_\mu,\omega_\mu^L)$ of (TQ,ω^L) at μ, from Theorems 2.2 and 6.1, we know that $((TQ)_\mu,\omega_\mu^L)$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu,\omega_\mu)$ of (T^*Q,ω_Q) at μ, and hence symplectically diffeomorphic to the orbit space $(\mathcal{O}_\mu\times V\times V,\widetilde{\omega}^{-L}_{\mathcal{O}_\mu\times V\times V}),\ \mu\in\mathfrak{g}^*$.

Assume that the hyperregular Lagrangian $L: TQ\to\mathbb{R}$ is G-invariant, and that the Euler-Lagrange vector field ξ_L satisfies the Euler-Lagrange equation $\mathbf{i}_{\xi_L}\omega^L=\mathbf{d}E_L$, where the energy $E_L: TQ\to\mathbb{R}$ is given by $E_L:=A-L$, and the action $A: TQ\to\mathbb{R}$ is given by $A(v):=\mathcal{F}L(v)v,\ \forall v\in T_qQ,\ q\in Q$. If the flow F_t of the Euler-Lagrange vector field ξ_L leaves the connected components of $(\mathbf{J}_Q^L)^{-1}(\mu)$ invariant and commutes with the G-action, then it induces a flow f_t^{μ} on $(TQ)_{\mu}$, defined by $f_t^{\mu} \cdot \tau_{\mu} = \tau_{\mu} \cdot F_t \cdot j_{\mu}$, and the vector field $\xi_{l_{\mu}}$ generated by the flow f_t^{μ} on $((TQ)_{\mu}, \omega_{\mu}^L)$ is the R_p-reduced Euler-Lagrange vector field with the associated R_p-reduced Lagrangian function $l_{\mu} : (TQ)_{\mu} \to \mathbb{R}$ defined by $l_{\mu} \cdot \tau_{\mu} = L \cdot j_{\mu}$, and the R_p-reduced Euler-Lagrange equation $\mathbf{i}_{\xi_{l_{\mu}}} \omega_{L\mu} = \mathbf{d}E_{l_{\mu}}$ holds, where the R_p-reduced energy $E_{l_{\mu}} : (TQ)_{\mu} \to \mathbb{R}$ given by $E_{l_{\mu}} := A_{\mu} - l_{\mu}$, and the R_p-reduced action $A_{\mu} : (TQ)_{\mu} \to \mathbb{R}$, given by $A_{\mu} \cdot \tau_{\mu} = A \cdot j_{\mu}$, and the Euler-Lagrange vector fields ξ_L and $\xi_{l_{\mu}}$ are τ_{μ}-related. Thus, we obtain the R_p-reduced Lagrangian system $((TQ)_{\mu}, \omega_{\mu}^L, l_{\mu})$. Moreover, if the hyperregular Lagrangian $L : TQ \to \mathbb{R}$, the fiber-preserving map $F^L : TQ \to TQ$ and the fiber submanifold $\mathcal{C}^L$ of TQ are all left tangent lifted G-action Φ^T invariant, then we have the following theorem.

Theorem 6.2 *Assume that the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular, and that the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is (Φ^T, Φ^{T*})-equivariant. Then the 6-tuple $(TQ, G, \omega^L, L, F^L, \mathcal{C}^L)$ is a regular point reducible RCL system, where $Q = G \times V$, and G is a Lie group and V is a k-dimensional vector space, and the Lagrangian $L : TQ \to \mathbb{R}$, the fiber-preserving map $F^L : TQ \to TQ$ and the fiber submanifold $\mathcal{C}^L$ of TQ are all left tangent lifted G-action Φ^T invariant. For a point $\mu \in \mathfrak{g}^*$, the regular value of the momentum map $\mathbf{J}_Q^L : TQ \to \mathfrak{g}^*$, the R_p-reduced system, that is, the 5-tuple $(\mathcal{O}_{\mu} \times V \times V, \widetilde{\omega}^{-L}_{\mathcal{O}_{\mu} \times V \times V}, l_{\mu}, f_{\mu}^L, \mathcal{C}_{\mu}^L)$, is an RCL system, where $\mathcal{O}_{\mu} \subset \mathfrak{g}^*$ is the coadjoint orbit, $\widetilde{\omega}^{-L}_{\mathcal{O}_{\mu} \times V \times V}$ is orbit symplectic form on $\mathcal{O}_{\mu} \times V \times V$, $l_{\mu} \cdot \tau_{\mu} = L \cdot j_{\mu}$, $F^L((\mathbf{J}_Q^L)^{-1}(\mu)) \subset (\mathbf{J}_Q^L)^{-1}(\mu)$, $f_{\mu}^L \cdot \tau_{\mu} = \tau_{\mu} \cdot F^L \cdot j_{\mu}$, $\mathcal{C}^L \cap (\mathbf{J}_Q^L)^{-1}(\mu) \neq \emptyset$, $\mathcal{C}_{\mu}^L = \tau_{\mu}(\mathcal{C}^L \cap (\mathbf{J}_Q^L)^{-1}(\mu))$. Moreover, two regular point reducible RCL systems $(TQ_i, G_i, \omega_i^L, L_i, F_i^L, \mathcal{C}_i^L)$, $i = 1, 2$, are RpCL-equivalent if and only if the associated R_p-reduced RCL systems $(\mathcal{O}_{i\mu_i} \times V_i \times V_i, \widetilde{\omega}^{-L}_{\mathcal{O}_{i\mu_i} \times V_i \times V_i}, l_{i\mu_i}, f_{i\mu_i}^L, \mathcal{C}_{i\mu_i}^L), i = 1, 2$, are RCL-equivalent.*

7 Conclusions

The theory of controlled mechanical system is a very important subject, and its research gathers together some separate areas of research; such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system to aid both analysis and design. Following the theoretical development of geometric mechanics, a lot of important problems about this subject were being explored and studied (see [12, 18–27]). In particular, it is worth noting that the research idea and work in [12] are very important, the authors set up the regular reduction theory for the standing regular controlled Hamiltonian systems defined on a symplectic fiber bundle, from the viewpoint of completeness of Marsden-Weinstein reduction.

In this paper, following the ideas in [12], we define an RCL system which is a standing regular Lagrangian system in a symplectic fiber bundle, by using the vertical lift map of the external force and the control, and describing the dynamical vector fields of the RCL system as the synthesis of Euler-Lagrange vector field and its changes under the actions of the external force and the control. Moreover, we can describe the RCL-equivalence, the RpCL-equivalence, and the RoCL-equivalence, and prove the regular point and regular orbit reduction theorems for the RCL system and the regular Lagrangian system with symmetry and a momentum map. Thus, we set up the regular reduction theory for the RCL system defined on a symplectic fiber bundle, by carefully analyzing the geometrical and topological structures of the phase space and the reduced phase space of the corresponding regular Lagrangian system. The reduction extends the symmetric reduction theory for a regular Lagrangian system under regular controlled

Lagrangian equivalence conditions.

We know that the different geometric structures determine the different controlled mechanical systems. It is a natural idea to develop a variety of reduction theory and applications for RCH systems and RCL systems, in particular, in celestial mechanics, hydrodynamics and plasma physics. In addition, it is also an important topic for us to explore and reveal the deeply internal relationships between the geometrical structures of phase spaces and the dynamical vector fields of the controlled mechanical systems. However, it is an important task for us to correct and develop well the research work of Professor Jerrold E. Marsden, such that we never feel sorry for his great cause.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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