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The Uniqueness of Minimal Maps into Cartan-Hadamard Manifolds via the Squared Singular Values

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Abstract In this paper, the authors give a uniqueness theorem for the Dirichlet problem of minimal maps into general Riemannian manifolds with non-positive sectional curvature, improving [Lee, YI., Ooi, Y. S. and Tsui, MP., Uniqueness of minimal graph in general codimension, *J. Geom. Anal.*, **29**, 2019, 121–133, Theorem 5.2]. The proof of this theorem is based on the convexity of several functions in terms of squared singular values along the geodesic homotopy of two given minimal maps.

Keywords General codimension, Minimal surface system, Dirichlet problem for minimal maps, Squared singular values
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1 Introduction

Let $f = (f^1, \cdots, f^n)$ be a vector-valued function on a domain $\Omega \subset \mathbb{R}^m$, $\Gamma(f) := \{(x, f(x)) : x \in \Omega\}$ be the graph of f, then Γ_f is a minimal submanifold (i.e., the mean curvature vectors vanish everywhere) of $\mathbb{R}^{m+n}$ if and only if f satisfies the minimal surface system as follows:

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x^i}\Big(\sqrt{g} g^{ij} \frac{\partial f^\alpha}{\partial x^j}\Big) = 0, \quad \forall \alpha = 1, \cdots, n, \tag{1.1}$$

where (g^{ij}) is the inverse of $(g_{ij}) := \big(\delta_{ij} + \sum\limits_{\alpha=1}^{n} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j}\big)$ and $g := \det(g_{ij})$. $\Delta_f := \sqrt{g}$ is called the slope of f. As a generalization, Schoen [9] introduced the concept of minimal map: Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimension m and n, respectively, then a smooth map $f : M \to N$ is a minimal map if and only if its graph is a minimal submanifold in the product manifold $M \times N$. The research on minimal graphs has a long and fertile history and a lot of works focus on the Dirichlet problem: Given a bounded domain Ω in M and a map ϕ from the boundary of Ω (denoted by $\partial\Omega$) into N, what kind of and how many minimal maps exist, so that each one $f : \Omega \to N$ satisfies $f|_{\partial\Omega} = \phi$.

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For the case of minimal graphs of codimension 1, i.e., $N = \mathbb{R}$, the solution to the Dirichlet problem is unique (if exists), and the graph of this solution is area-minimizing, i.e., the volume of this submanifold takes the minimum along all submanifolds with the graph of ϕ as their boundary. These conclusions can be derived from the convexity of the volume functional (see e.g. [1, Chap. 12]). Unfortunately, these beautiful results cannot be generalized to higher codimensional cases, i.e., $n \geq 2$. In their pioneering paper [2], Lawson-Osserman constructed distinct minimal maps f_1, f_2, f_3 from the unit disk $\mathbb{D}$ in $\mathbb{R}^2$ to $\mathbb{R}^2$ sharing the same boundary data, where f_3 corresponds to an unstable minimal graph, i.e., even a little deformation can decrease its area. Afterwards, by making systematic developments on the Lawson-Osserman constructions, the authors showed in [10] that there exist infinitely many vector-valued functions on Euclidean spheres, so that we can find infinitely many solutions to the Dirichlet problem with each of them as the boundary data. Therefore, it is usually referred to as a restricted class of submanifolds when we talk about the uniqueness and stability of minimal graphs of higher codimension.

From the viewpoint of singular values, Lee, Wang, Tsui, Ooi and other mathematicians [3–7] studied this problem, and produced a series of successive works. Let $\mathrm{d}f$ be the tangent map from T_xM into $T_{f(x)}N$, then for any non-negative number λ, λ is a singular value of $\mathrm{d}f$ at x if and only if λ^2 is a critical value of the squared norm function

$$v \mapsto g_N(\mathrm{d}f(v), \mathrm{d}f(v)), \quad \forall v \in T_xM, \quad |v| = 1. \tag{1.2}$$

By the theory of singular value decomposition, the slope of f can be written as

$$\Delta_f = \prod_{i=1}^{m} (1 + \lambda_i^2)^{\frac{1}{2}}, \tag{1.3}$$

where $\lambda_1 \geq \cdots \geq \lambda_m$ are all singular values of $\mathrm{d}f$ at x. As shown in [4],

$$(x_1, \cdots, x_m) \mapsto \prod_{i=1}^{m} (1 + x_i^2)^{\frac{1}{2}} \tag{1.4}$$

is a strictly convex (or convex) function on $\mathcal{M}$ (or $\overline{\mathcal{M}}$), where $\mathcal{M}$ consists of all vectors in $\mathbb{R}^m_{\geq 0} := [0, +\infty)^m$ satisfying

$$x_i x_j < 1, \quad \forall i \neq j, \tag{1.5}$$

$$\prod_{i=1}^{m} (1 - x_i^2) + \sum_{i=1}^{m} (1 - x_1^2) \cdots x_i^2 \cdots (1 - x_m^2) > 0, \tag{1.6}$$

and $\overline{\mathcal{M}}$ is the closure of $\mathcal{M}$. Moreover, in conjunction with the second variation formula of the volume functional in terms of singular values (see [6]), criteria for the stability and uniqueness of minimal graphs can be established as follows.

Theorem 1.1 (see [4]) *Let $f: M \to N$ be a minimal map, where N has non-positive sectional curvature everywhere, then the graph of f is stable (or weakly stable) whenever the singular value vector $(\lambda_1, \cdots, \lambda_m) \in \mathcal{M}$ (or $\overline{\mathcal{M}}$) everywhere on M.*

Theorem 1.2 (see [7]) *Suppose that $f_0, f_1 : \Omega(\subset M) \to \mathbb{R}^n$ are both minimal maps with the same boundary data, and the singular value vectors of both f_0 and f_1 all lie in a fixed symmetric convex subset of $\overline{\mathcal{M}}$, then $f_0 = f_1$.*

To prove Theorem 1.2, it is natural to consider the geodesic homotopy $\{f_t : t \in [0, 1]\}$ of f_0 and f_1, such that for each $x \in M$, $\gamma_x(t) := f_t(x)$ is a geodesic connecting $f_0(x)$ and $f_1(x)$. Along each γ_x, denote by $\lambda(t) := (\lambda_1(t), \cdots, \lambda_m(t))$ the singular value vector function; then it is easy to verify that $\sum\limits_{i=1}^{k} \lambda_i(t)$ is a convex function for each $1 \le k \le m$. Based on this fundamental fact, we can show the singular value vectors of f_t still lie in $\overline{\mathcal{M}}$, and then $f_0 = f_1$ follows from the second variation formula. However, if we replace the target manifold of minimal maps by a general Riemannian manifold N with non-positive sectional curvature, $\sum\limits_{i=1}^{k} \lambda_i(t)$ is no longer convex, so the above scheme is not feasible.

To overcome this obstruction, we consider the squared singular value vector

$$\lambda^2(\mathrm{d}f) := (\lambda_1^2, \cdots, \lambda_m^2) \tag{1.7}$$

in the present paper. Let $\overline{\mathcal{N}}$ be a subset of $\mathbb{R}^m_{\ge 0}$, which consists of all such vectors $a := (a_1, \cdots, a_m)$ satisfying the following two conditions:

$$a_i a_j \le 1, \quad \forall i \ne j, \tag{1.8}$$

$$\prod_{i=1}^{m}(1 - a_i) + \sum_{i=1}^{m}(1 - a_1) \cdots a_i \cdots (1 - a_m) \ge 0. \tag{1.9}$$

Then obviously

$$\lambda^2(\mathrm{d}f) \in \overline{\mathcal{N}} \Leftrightarrow \lambda(\mathrm{d}f) \in \overline{\mathcal{M}}. \tag{1.10}$$

The main goal of this paper is the proof of the following criterion for the uniqueness of minimal maps into Riemannian manifolds with non-positive sectional curvature, which gives some corollaries.

Theorem 1.3 *Let N be a complete Riemannian manifold with non-positive sectional curvature, and $f_0, f_1 : \Omega(\subset M) \to N$ be minimal maps with the same boundary data. If f_0 is homotopic to f_1, and both $\lambda^2(\mathrm{d}f_0)$ and $\lambda^2(\mathrm{d}f_1)$ lie in a symmetric convex set $\mathcal{C} \subset \overline{\mathcal{N}}$, then $f_0 = f_1$.*

Corollary 1.1 *Let N be a complete Riemannian manifold with non-positive sectional curvature. Suppose that $f_0, f_1 : \Omega(\subset M) \to N$ are minimal maps with the same boundary data, which are homotopic to each other, then $f_0 = f_1$ if either of the following occurs:*

- *The singular values of f_0 and f_1 all satisfy $\lambda_i^2+\lambda_j^2\le 2$ $(\forall i\ne j)$ and $\sum_{i=1}^m \lambda_i^2\le 3-\frac{1}{m-1}$.*
- *$m\ge 3$, and the singular values of f_0 and f_1 all satisfy $\lambda_i^2+\lambda_j^2\le 2$ $(\forall i\ne j)$ and $\prod_{i=1}^m (1+\lambda_i^2)^{\frac12}\le\sqrt{3}\big(2-\frac{1}{m-1}\big)^{\frac12}$.*
- *The slopes of f_0 and f_1 are no more than $\sqrt{3}$.*

In particular, the first conclusion of Corollary 1.1 is an improvement of [3, Theorem 5.2], which claims $f_0=f_1$ whenever $\sum_{i=1}^m \lambda_i^2<2$.

This paper will be organized as follows. In Section 2, we prove the existence of geodesic homotopy $\{f_t : t\in[0,1]\}$ of f_0 and f_1 with the aid of the classical Cartan-Hadamard theorem and we calculate the second derivative of the volume function of $\Gamma(f_t)$ in terms of the singular values. Afterwards in Section 3, by applying majorization techniques in convex optimisation as in [3, 7], we establish the following confined property of the squared singular value vector function along the geodesic homotopy: For any interval $[t_1,t_2]\subset[0,1]$, $\lambda^2(t_1),\lambda^2(t_2)\in\overline{\mathcal{N}}$ implies $\lambda^2(t)\in\overline{\mathcal{N}}$. In the process, the convexity of the functions $\sum_{i=1}^k \lambda_i^2(t)$ with $1\le k\le m$ plays a crucial role. Section 4 will be dedicated to the proof of Theorem 1.3 based on preliminary works in the last two sections. Finally in Section 5, the construction of symmetric convex subsets of $\overline{\mathcal{N}}$ enables us to give applications of Theorem 1.3.

2 The Second Variation Formula for the Volume Functional

Let Ω be a bounded domain of an m-dimensional Riemannian manifold (M,g_M), (N,g_N) be an n-dimensional complete Riemanniam manifold with non-positive sectional curvature, i.e., $K_N\le 0$, and f_0,f_1 be both smooth maps from Ω into N. Assume that f_0,f_1 are homotopic to each other, and $f_0|_{\partial\Omega}=f_1|_{\partial\Omega}$. Let $\widetilde{N}$ be the universal covering manifold of N equipped with the pull-back metric $g_{\widetilde{N}}$, $\widetilde{f}_0,\widetilde{f}_1:\Omega\to\widetilde{N}$ be lifts of f_0,f_1, respectively, so that $\widetilde{f}_0(x_0)=\widetilde{f}_1(x_0)$ for a fixed $x_0\in\Omega$, then $K_{\widetilde{N}}\le 0$ and the classical Cartan-Hadamard theorem implies the existence and uniqueness of the geodesic $\widetilde{\gamma}_x(t):[0,1]\to\widetilde{N}$ for each $x\in\Omega$, so that $\widetilde{\gamma}_x(0)=\widetilde{f}_0(x)$, $\widetilde{\gamma}_x(1)=\widetilde{f}_1(x)$. Define

$$\widetilde{f}_t(x):=\widetilde{\gamma}_x(t),\quad f_t:=\pi\circ\widetilde{f}_t$$

with π the universal covering mapping from $\widetilde{N}$ onto N. Such $\{f_t : M\to N \mid t\in[0,1]\}$ is called a geodesic homotopy, which satisfies

- f_t smoothly depends on t;
- for each $y\in\partial\Omega$, $t\in[0,1]\mapsto f_t(y)$ is a constant function;
- for each $x\in\Omega$, $\gamma_x(t):=f_t(x)$ is a geodesic in N connecting $f_0(x)$ and $f_1(x)$.

Given $0\le t\le 1$, then f_t induces an embedding

$$x\in\Omega\mapsto(x,f_t(x))\in M\times N,$$

whose image, denoted by Γ_{f_t}, is the graph of f_t. For each $x \in \Omega$, by the theory of singular value decomposition, there exist orthonormal bases $\{a_i\}_{i=1}^m$, $\{b_j\}_{j=1}^n$ in T_xM and $T_{f_t(x)}N$, respectively, such that

$$\mathrm{d}f_t(a_i) = \begin{cases} \lambda_i(t)b_i, & i = 1, \cdots, r, \\ 0, & i = r+1, \cdots, m, \end{cases} \tag{2.1}$$

where

$$\lambda_1(t) \geq \cdots \geq \lambda_r(t) > \lambda_{r+1}(t) = \cdots = \lambda_m(t) = 0 \tag{2.2}$$

are singular values of $(\mathrm{d}f_t)_x : (T_x, g_M) \to (T_{f_t(x)}, g_N)$, and r is the rank of this tangent map. As in [3],

$$\lambda(t) := (\lambda_1(t), \cdots, \lambda_m(t)) \tag{2.3}$$

is called the singular value vector of $\mathrm{d}f_t$ at x. Let

$$g(t) := g_M + f_t^* g_N$$

be the induced metric on Γ_{f_t}, whose corresponding volume form is

$$\mathrm{d}v_t = \sqrt{\det(g_{ij}(t))}\, \mathrm{d}v_M \tag{2.4}$$

with $\mathrm{d}v_M$ the volume form of M and

$$g_{ij}(t) := g_M(a_i, a_j) + g_N(\mathrm{d}f_t(a_i), \mathrm{d}f_t(a_j)). \tag{2.5}$$

Denote by $V := \frac{\mathrm{d}f_t}{\mathrm{d}t}$ the variation field on Ω, then a straightforward calculation shows

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{ij}(t) = \langle \nabla_{\mathrm{d}f_t(a_i)} V, \mathrm{d}f_t(a_j)\rangle + \langle \mathrm{d}f_t(a_i), \nabla_{\mathrm{d}f_t(a_j)} V\rangle, \tag{2.6}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} g_{ij}(t) = {} & 2\langle \nabla_{\mathrm{d}f_t(a_i)} V, \nabla_{\mathrm{d}f_t(a_j)} V\rangle \\ & + \langle R(V, \mathrm{d}f_t(a_i))V, \mathrm{d}f_t(a_j)\rangle + \langle \mathrm{d}f_t(a_i), R(V, \mathrm{d}f_t(a_j))V\rangle \\ & + \langle \nabla_{\mathrm{d}f_t(a_i)} \nabla_V V, \mathrm{d}f_t(a_j)\rangle + \langle \mathrm{d}f_t(a_j), \nabla_{\mathrm{d}f_t(a_j)} \nabla_V V\rangle. \end{aligned} \tag{2.7}$$

Here $\langle \cdot, \cdot \rangle := g_N(\cdot, \cdot)$, ∇ is the Levi-Civita connection associated to g_N and

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{2.8}$$

is the corresponding curvature operator. Denote by

$$A(t) := \int_\Omega \sqrt{\det(g_{ij}(t))}\, \mathrm{d}v_M \tag{2.9}$$

the volume of Γ_{f_t}, then

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^m \Big(g^{ij}\frac{\mathrm{d}g_{ij}}{\mathrm{d}t}\Big)\mathrm{d}v_t, \tag{2.10}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}A(t) = &-\frac{1}{2}\int_\Omega \sum_{i,j,k,l=1}^m g^{ij}\frac{\mathrm{d}g_{jk}}{\mathrm{d}t}g^{kl}\frac{\mathrm{d}g_{li}}{\mathrm{d}t}\mathrm{d}v_t + \frac{1}{4}\int_\Omega \Big(\sum_{i,j=1}^m g^{ij}\frac{\mathrm{d}g_{ij}}{\mathrm{d}t}\Big)^2\mathrm{d}v_t \\ &+ \frac{1}{2}\int_\Omega \sum_{i,j=1}^m g^{ij}\frac{\mathrm{d}^2g_{ij}}{\mathrm{d}t^2}\mathrm{d}v_t\end{aligned} \tag{2.11}$$

with (g^{ij}) the inverse of (g_{ij}). Denoting

$$p_{i\alpha} := \langle \nabla_{\mathrm{d}f_t(a_i)}V, b_\alpha\rangle \tag{2.12}$$

and replacing $\frac{\mathrm{d}g_{ij}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2g_{ij}}{\mathrm{d}t^2}$ in (2.11) with formulas in (2.6)–(2.7), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}A(t) = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} + \text{(v)}, \tag{2.13}$$

where

$$\text{(i)} = \int_\Omega \Big(\sum_{1\le i\le r}\frac{p_{ii}^2}{(1+\lambda_i^2)^2} + \sum_{1\le i,j\le r, i\ne j}\frac{\lambda_i\lambda_j p_{ii}p_{jj}}{(1+\lambda_i^2)(1+\lambda_j^2)}\Big)\mathrm{d}v_t, \tag{2.14}$$

$$\text{(ii)} = \int_\Omega \sum_{1\le i<j\le r}\frac{p_{ij}^2+p_{ji}^2-2\lambda_i\lambda_j p_{ij}p_{ji}}{(1+\lambda_i^2)(1+\lambda_j^2)}\mathrm{d}v_t, \tag{2.15}$$

$$\text{(iii)} = \int_\Omega \sum_{1\le i\le r, r+1\le\alpha\le n}\frac{p_{i\alpha}^2}{1+\lambda_i^2}\mathrm{d}v_t, \tag{2.16}$$

$$\text{(iv)} = \int_\Omega \sum_{1\le i\le r}\frac{\lambda_i^2}{1+\lambda_i^2}\langle \nabla_{b_i}\nabla_V V, b_i\rangle \mathrm{d}v_t, \tag{2.17}$$

$$\text{(v)} = \int_\Omega \sum_{1\le i\le r}\frac{\lambda_i^2}{1+\lambda_i^2}\langle R(b_i,V)b_i, V\rangle \mathrm{d}v_t. \tag{2.18}$$

As shown in [4], (i) ≥ 0 whenever $\lambda \in \overline{\mathcal{M}}$, (ii) ≥ 0 whenever $0 \le \lambda_i\lambda_j \le 1$ for each $i \ne j$, (iii) is non-negative, (iv) $= 0$ since $\{f_t\}$ is a geodesic homotopy and (v) ≥ 0 since $K_N \le 0$.

3 Confined Properties of Squared Singular Value Vectors

Definition 3.1 (see [7]) *Let $x := (x_1, \cdots, x_m)$, $y := (y_1, \cdots, y_m) \in \mathbb{R}^m$. y is called l-weakly majorized by x, whenever*

$$\sum_{i=1}^k \widetilde{y}_i \le \sum_{j=1}^k \widetilde{x}_j$$

for $k = 1, 2, \cdots, l$, where $\{\widetilde{x}_i\}$ (or $\{\widetilde{y}_i\}$) is a rearrangement of $\{x_i\}$ (or $\{y_i\}$) in descending order, denoted by $y \prec_l x$.

For an arbitrary point $x \in \mathbb{R}^m_{\geq 0}$, let $W(x) := \{y \in \mathbb{R}^m_{\geq 0} : y \prec_m x\}$, $E(x)$ be the set consisting of all these points $(\delta_1 x_{\sigma(1)}, \cdots, \delta_m x_{\sigma(m)})$, where σ is an arbitrary permutation of $\{1, \cdots, m\}$ and $\delta_i = 0$ or 1, and $H(x)$ be the convex hull of $E(x)$, then $W(x) = H(x)$ (see [8, Theorem 6]). The following two lemmas on $W(x)$ shall play a crucial part on the present paper.

Lemma 3.1 *Let D be a domain of $\mathbb{R}^m$, F be a strictly convex, symmetric function on D, such that for each $u, v \in D$, $u_i \leq v_i$ for each $i = 1, \cdots, m$ implies $F(u) \leq F(v)$, and the equality holds if and only if $u = v$. Then for each $x \in D$ satisfying $W(x) \subset D$, we have $F(y) \leq F(x)$ for each $y \in W(x)$, where the equality holds if and only if y is a rearrangement of x.*

Proof Denote

$$E(x) = \{v_1, \cdots, v_p\} \tag{3.1}$$

with

$$v_\alpha = (\delta_1^\alpha x_{\sigma_\alpha(1)}, \cdots, \delta_m^\alpha x_{\sigma_\alpha(m)}), \tag{3.2}$$

then each $y \in W(x) = H(x)$ can be written as

$$y = \lambda_1 v_1 + \cdots + \lambda_p v_p, \tag{3.3}$$

where $\lambda_1, \cdots, \lambda_p$ are all non-negative numbers, satisfying $\sum\limits_{\alpha=1}^{p} \lambda_\alpha = 1$. Let

$$w_\alpha := (x_{\sigma_\alpha(1)}, \cdots, x_{\sigma_\alpha(m)}) \tag{3.4}$$

and

$$z := \lambda_1 w_1 + \cdots + \lambda_p w_p \in H(x). \tag{3.5}$$

Combining (3.1)–(3.5), we have $y_i \leq z_i$ for all $i = 1, \cdots, m$. Along with the strict convexity and symmetry of F, we get

$$F(y) \leq F(z) \leq F(x),$$

where the equality holds if and only if $y = z = w_\alpha$ for some α, i.e., y is a rearrangement of x.

Lemma 3.2 *Let $\mathcal{C}$ be a symmetric convex subset of $\overline{\mathcal{N}}$ defined in* (1.8)–(1.9), *then $W(x) \subset \overline{\mathcal{N}}$ for each $x \in \mathcal{C}$. Moreover, if $y \in W(x) \cap \partial\mathcal{N}$ and* $\max\{y_i\} > 1$, *then $\sum\limits_{i=1}^{m} y_i = \sum\limits_{i=1}^{m} x_i$. Here $\partial\mathcal{N}$ consists of all such vectors $a := (a_1, \cdots, a_m)$ in $\overline{\mathcal{N}}$ satisfying*

$$\max_{1 \leq i < j \leq m} a_i a_j = 1 \tag{3.6}$$

or

$$\prod_{i=1}^{m}(1 - a_i) + \sum_{i=1}^{m}(1 - a_1) \cdots a_i \cdots (1 - a_m) = 0. \tag{3.7}$$

Proof Let y be an arbitrary point in $W(x)$. Since $W(x)$ is preserved under the action of permutations, without loss of generality we can assume

$$x_1 \geq \cdots \geq x_m, \quad y_1 \geq \cdots \geq y_m. \tag{3.8}$$

If $y_1 \leq 1$, then y automatically satisfies (1.8)–(1.9) and hence $y \in \overline{\mathcal{N}}$. Now we assume $y_1 > 1$. In this case, noting that

$$\begin{aligned} &\prod_{i=1}^{m}(1-y_i) + \sum_{i=1}^{m}(1-y_1)\cdots y_i \cdots (1-y_m) \\ =&\prod_{i=1}^{m}(1-y_i)\Big(\sum_{i=1}^{m}\frac{1}{1-y_i} - m + 1\Big), \end{aligned} \tag{3.9}$$

we have

$$\begin{aligned} y \in \overline{\mathcal{N}} \Leftrightarrow y_1 y_2 \leq 1 \quad &\text{and} \quad \frac{1}{1-y_1} + G(y_2, \cdots, y_m) \leq m-1, \\ y \in \partial\mathcal{N} \Leftrightarrow y_1 y_2 = 1 \quad &\text{or} \quad \frac{1}{1-y_1} + G(y_2, \cdots, y_m) = m-1 \end{aligned} \tag{3.10}$$

with

$$G : (y_2, \cdots, y_m) \in [0,1)^{m-1} \mapsto \sum_{i=2}^{m}\frac{1}{1-y_i}. \tag{3.11}$$

Due to the symmetry and convexity of $\mathcal{C}$,

$$\widehat{x} := (x_2, x_1, x_3, \cdots, x_m) \tag{3.12}$$

and

$$\frac{x+\widehat{x}}{2} = \Big(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_m\Big) \tag{3.13}$$

both lie in $\mathcal{C}$, then

$$y_1 y_2 < \Big(\frac{y_1+y_2}{2}\Big)^2 \leq \Big(\frac{x_1+x_2}{2}\Big)^2 \leq 1. \tag{3.14}$$

Denote

$$\begin{aligned} z &= (z_1, z_2, z_3, \cdots, z_m) \\ &:= (y_1, x_2 + (x_1 - y_1), x_3, \cdots, x_m), \end{aligned} \tag{3.15}$$

then z is also a convex combination of x and $\widehat{x}$ and hence $z \in \mathcal{C}$. By $y \prec_m x$, it is easy to verify that $y \prec_m z$ and

$$(y_2, \cdots, y_m) \prec_{m-1} (z_2, \cdots, z_m). \tag{3.16}$$

Applying Lemma 3.1 to the function G gives

$$\frac{1}{1-y_1}+G(y_2,\cdots,y_m)\le\frac{1}{1-z_1}+G(z_2,\cdots,z_m)\le m-1 \tag{3.17}$$

and then $y\in\overline{\mathcal{N}}$. Particularly if $y\in\partial\mathcal{N}$, then (3.10) and (3.14) imply $\frac{1}{1-y_1}+G(y_2,\cdots,y_m)=m-1$. Thus the equality of (3.17) holds, which forces $y=z$ and then

$$\sum_{i=1}^m y_i=\sum_{i=1}^m z_i=\sum_{i=1}^m x_i. \tag{3.18}$$

Now we consider the squared singular value vectors

$$\lambda^2(t):=(\lambda_1^2(t),\cdots,\lambda_m^2(t)) \tag{3.19}$$

along a given geodesic homotopy. Based on Lemma 3.2, we can get a confined property for $\lambda^2(t)$ as follows.

Proposition 3.1 *Let* $[t_1,t_2]\subset[0,1]$ *and*

$$\mu(t):=\frac{t_2-t}{t_2-t_1}\lambda^2(t_1)+\frac{t-t_1}{t_2-t_1}\lambda^2(t_2)$$

be the linear function on this interval satisfying $\mu(t_1)=\lambda^2(t_1)$ *and* $\mu(t_2)=\lambda^2(t_2)$*, then*

$$\sum_{i=1}^l\lambda_i^2(t)\le\sum_{i=1}^l\mu_i(t),\quad\forall 1\le l\le m.$$

Especially if $\sum\limits_{i=1}^l\lambda_i^2(t_0)=\sum\limits_{i=1}^l\mu_i(t_0)$ *for some* $t_0\in(t_1,t_2)$*, we have* $\nabla_{\mathrm{d}f_t(a_i)}V=0$ *for* $i=1,\cdots,l$*,* $t\in[t_1,t_2]$*.*

Moreover, if both $\lambda^2(t_1)$ *and* $\lambda^2(t_2)$ *lie in a symmetric convex subset* $\mathcal{C}$ *in* $\overline{\mathcal{N}}$*, then* $\lambda^2(t)\in\overline{\mathcal{N}}$ *for each* $t\in[t_1,t_2]$*. Especially if* $\lambda^2(t_0)\in\partial\mathcal{N}$ *and* $\lambda_1^2(t_0)>1$ *for some* $t_0\in(t_1,t_2)$*, we have* $\nabla_{\mathrm{d}f_t(a_i)}V=0$ *for* $i=1,\cdots,m$*,* $t\in[t_1,t_2]$*.*

Proof For any fixed $t_0\in[t_1,t_2]$, let $\{a_i\}$ be an orthonormal basis of T_xM, such that

$$\langle\mathrm{d}f_{t_0}(a_i),\mathrm{d}f_{t_0}(a_i)\rangle=\begin{cases}\lambda_i^2(t_0), & 1\le i\le r:=\operatorname{rank}\mathrm{d}f_{t_0},\\ 0, & r+1\le i\le m.\end{cases} \tag{3.20}$$

Now we consider two functions

$$F_k(t):=\sum_{i=1}^k\langle\mathrm{d}f_t(a_i),\mathrm{d}f_t(a_i)\rangle, \tag{3.21}$$

$$S_k(t):=\sum_{i=1}^k\lambda_i^2(t) \tag{3.22}$$

on $[t_1, t_2]$. Due to the properties of singular values, it is easily verified that $S_k(t_0) = F_k(t_0)$ and $F_k(t) \le S_k(t)$. On the other hand,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}F_k(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_{i=1}^{k}\langle \mathrm{d}f_t(a_i), \mathrm{d}f_t(a_i)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{k}2\langle \nabla_{\mathrm{d}f_t(a_i)}V, \mathrm{d}f_t(a_i)\rangle \\ &= 2\sum_{i=1}^{k}\langle \nabla_{\mathrm{d}f_t(a_i)}V, \nabla_{\mathrm{d}f_t(a_i)}V\rangle + 2\sum_{i=1}^{k}\langle \nabla_V\nabla_{\mathrm{d}f_t(a_i)}V, \mathrm{d}f_t(a_i)\rangle \\ &= 2\sum_{i=1}^{k}|\nabla_{\mathrm{d}f_t(a_i)}V|^2 + 2\sum_{i=1}^{k}\langle R(V, \mathrm{d}f_t(a_i))V, \mathrm{d}f_t(a_i)\rangle \\ &\quad + 2\sum_{i=1}^{k}\langle \nabla_{\mathrm{d}f_t(a_i)}\nabla_V V, \mathrm{d}f_t(a_i)\rangle \ge 0 \end{aligned} \tag{3.23}$$

showing $F_k(t)$ is a convex function. Thus

$$\begin{aligned} S_k(t_0) = F_k(t_0) &\le \frac{t_2 - t_0}{t_2 - t_1}F_k(t_1) + \frac{t_0 - t_1}{t_2 - t_1}F_k(t_2) \\ &\le \frac{t_2 - t_0}{t_2 - t_1}S_k(t_1) + \frac{t_0 - t_1}{t_2 - t_1}S_k(t_2) \\ &= \sum_{i=1}^{k}\mu_i(t_0). \end{aligned} \tag{3.24}$$

Moreover, if $S_l(t_0) = \sum\limits_{i=1}^{l}\mu_i(t_0)$, then (3.24) shows $F_l|_{[t_1,t_2]}$ is linear and hence (3.23) implies $\nabla_{\mathrm{d}f_t(a_i)}V = 0$ for $i = 1, \cdots, l$.

Note that (3.24) is equivalent to saying that $\lambda^2(t) \prec_m \mu(t)$ for $t \in [t_1, t_2]$. Once $\lambda^2(t_1)$, $\lambda^2(t_2) \in \mathcal{C}$, the symmetry and convexity of $\mathcal{C} \subset \overline{\mathcal{N}}$ show $\mu(t) \in \mathcal{C}$ and hence we get $\lambda^2(t) \in \overline{\mathcal{N}}$ by Lemma 3.2. Moreover, if $\lambda^2(t_0) \in \partial\mathcal{N}$ and $\lambda_1^2(t_0) > 1$, then

$$S_m(t_0) = \sum_{i=1}^{m}\lambda_i^2(t_0) = \sum_{i=1}^{m}\mu_i(t_0)$$

and hence $\nabla_{\mathrm{d}f_t(a_i)}V = 0$ for $i = 1, \cdots, m$ and $t \in [t_1, t_2]$.

4 Proof of the Main Theorem

Suppose that $f_0, f_1 : \Omega(\subset M) \to N$ are minimal maps, such that f_0, f_1 are homotopic to each other and $f_0|_{\partial\Omega} = f_1|_{\partial\Omega}$. Let $\{f_t : \Omega \to N \mid t \in [0, 1]\}$ be a geodesic homotopy of f_0 and f_1. For each $x \in \Omega$, let $\lambda^2(t) := \lambda^2((\mathrm{d}f_t)_x)$ be the squared singular value vector function. By Proposition 3.1, $\lambda^2(0), \lambda^2(1) \in \mathcal{C}$ ensure $\lambda^2(t) \in \overline{\mathcal{N}}$, i.e., $\lambda(t) \in \overline{\mathcal{M}}$ and hence $\frac{\mathrm{d}^2}{\mathrm{d}t^2}A(t) \ge 0$. In conjunction with $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}A(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=1}A(t) = 0$ (since both f_0 and f_1 are minimal maps), we have $\frac{\mathrm{d}^2}{\mathrm{d}t^2}A(t) = 0$, i.e., (i) = (ii) = (iii) = 0 for each $t \in (0, 1)$ (see (2.14)–(2.16)).

Define

$$\begin{aligned}\Lambda_1 &:= \{t \in (0,1) : \lambda(t) \in \mathcal{M}\},\\ \Lambda_2 &:= \{t \in (0,1) : \lambda_2(t) < 1\},\\ &\cdots \\ \Lambda_m &:= \{t \in (0,1) : \lambda_m(t) < 1\},\\ \Lambda_{m+1} &:= (0,1).\end{aligned} \tag{4.1}$$

When $t \in \Lambda_1$, as shown in [4, Theorem 3.2], (i) = (ii) = (iii) = 0 implies that $p_{i\alpha} = 0$ for any $1 \le i \le m$ and $1 \le \alpha \le n$ and hence

$$\nabla_{\mathrm{d}f_t(a_i)} V = 0, \quad \forall 1 \le i \le m. \tag{4.2}$$

By the continuity, this equality holds for each $t \in \overline{\Lambda}_1$.

If $\Lambda_2 \backslash \overline{\Lambda}_1 = \emptyset$, (4.2) always holds in Λ_2. Otherwise, for each t in this set, we have $\lambda^2(t) \in \partial\mathcal{N}$, then $\lambda_2^2(t) < 1$ forces $\lambda_1^2(t) > 1$, then Proposition 3.1 and the continuity ensure (4.2) holds for all $t \in \overline{\Lambda}_2$.

Next we need to show that the equality (4.2) also holds on each Λ_i by induction on i. Suppose that $\nabla_{\mathrm{d}f_t(a_i)} V = 0$ holds for all $t \in \overline{\Lambda}_k$ with $2 \le k \le m$ and the open set $\Lambda_{k+1} \setminus \overline{\Lambda}_k$ is nonempty. For each $t \in [t_1, t_2] \subset \Lambda_{k+1} \setminus \overline{\Lambda}_k$, it is easy to see that $\lambda_1(t) = \cdots = \lambda_k(t) = 1$ and $1 > \lambda_{k+1}(t) \ge \cdots \ge \lambda_m(t) \ge 0$, which means $\sum\limits_{i=1}^{k} \lambda_i^2(t) = \sum\limits_{i=1}^{k} \mu_i(t)$, and we can conclude that

$$\nabla_{\mathrm{d}f_t(a_i)} V = 0, \quad \forall 1 \le i \le k,\ t \in \Lambda_{k+1} \tag{4.3}$$

by Proposition 3.1. In combination of (2.14)–(2.16), we have $p_{i\alpha} = 0$ for $k+1 \le i \le r$ and $1 \le \alpha \le n$, which means $\nabla_{\mathrm{d}f_t(a_i)} V = 0$ for $k+1 \le i \le m$. Together with (4.3) we know (4.2) also holds for $t \in \Lambda_{k+1}$, finishing the induction step.

Therefore, for each $t \in [0,1]$, V is a parallel vector field on the graph Γ_{f_t}. According to the boundary condition we can derive $V \equiv 0$ and hence $f_0 = f_1$. This completes the proof of Theorem 1.3.

5 Applications

In this section, we give some applications of Theorem 1.3.

Let $\mathcal{C}$ be a symmetric convex subset of $\overline{\mathcal{N}}$, then for each $a \in \mathcal{C}$, we can proceed as in (3.12)–(3.14) to show

$$a_i + a_j \le 2, \quad \forall 1 \le i < j \le m. \tag{5.1}$$

On the other hand, from this condition, it immediately follows that

$$a_i a_j \le \left(\frac{a_i + a_j}{2}\right)^2 \le 1, \tag{5.2}$$

i.e., such a must satisfy (1.8). It is natural to ask, besides (5.1), whichever additional restrictions can make sure a symmetric convex subset of $\mathbb{R}^m_{\geq 0}$ completely lies in $\overline{\mathcal{N}}$. In the following text we shall consider this question.

Corollary 5.1 *Suppose that $f_0, f_1 : \Omega(\subset M) \to N$ are minimal maps with the same boundary data, f_0 is homotopic to f_1 and $K_N \leq 0$. If both $\lambda^2(\mathrm{d}f_0)$ and $\lambda^2(\mathrm{d}f_1)$ lie in*

$$\mathcal{C}_m := \Big\{a \in \mathbb{R}^m_{\geq 0} \Big| \sum_{i=1}^m a_i \leq 3 - \frac{1}{m-1}, a_i + a_j \leq 2, \forall 1 \leq i < j \leq m\Big\},$$

then $f_0 = f_1$.

Proof Obviously $\mathcal{C}_m$ is symmetric and convex. To show $\mathcal{C}_m \subset \overline{\mathcal{N}}$, it remains for us to consider Condition (1.9) when $\max\{a_i\} > 1$. Due to the symmetry of $\mathcal{C}_m$, we can assume $a_1 = \max\{a_i\}$ without loss of generality. As shown in the proof of Lemma 3.2, this condition is equivalent to

$$\frac{1}{1-a_1} + G(a_2, \cdots, a_m) \leq m - 1. \tag{5.3}$$

Here the definition of G is given in (3.11). For each given $t \in (0, 1]$, let

$$\mathcal{D}_{m-1,t} := \Big\{(a_2, \cdots, a_m) \in \mathbb{R}^{m-1}_{\geq 0} \Big| \sum_{i=2}^m a_i \leq 2 - t - \frac{1}{m-1}, \max_i a_i \leq 1 - t\Big\}, \tag{5.4}$$

then

- for $a_1 = 1 + t$, $(a_1, \cdots, a_m) \in \mathcal{C}_m$ if and only if $(a_2, \cdots, a_m) \in \mathcal{D}_{m-1,t}$;
- $\mathcal{D}_{m-1,t}$ is a convex polyhedron in $\mathbb{R}^{m-1}$;
- G is a symmetric, strictly convex function on $\mathcal{D}_{m-1,t}$, which should take its maximum at a vertex of this polyhedron.

Therefore

$$\begin{aligned}
&\frac{1}{1-a_1} + G(a_2, \cdots, a_m) \\
&\leq \sup\Big\{\frac{1}{1-(1+t)} + \max G|_{\mathcal{D}_{m-1,t}} : t \in (0, 1]\Big\} \\
&\leq \sup\Big\{\frac{1}{1-(1+t)} + \frac{1}{1-(1-t)} + \frac{1}{1 - \frac{m-2}{m-1}} : t \in (0, 1]\Big\} \\
&= m - 1.
\end{aligned} \tag{5.5}$$

This completes the proof of $\mathcal{C}_m \subset \overline{\mathcal{N}}$. Finally, $f_0 = f_1$ is a direct corollary of Theorem 1.3.

Corollary 5.2 *Suppose that $f_0, f_1 : \Omega(\subset M) \to N$ are minimal maps with the same boundary data, f_0 is homotopic to f_1 and $K_N \leq 0$. If both $\lambda^2(\mathrm{d}f_0)$ and $\lambda^2(\mathrm{d}f_1)$ lie in*

$$\mathcal{V}_m := \Big\{a \in \mathbb{R}^m_{\geq 0} \Big| \prod_{i=1}^m (1 + a_i)^{\frac{1}{2}} \leq \mu_m, a_i + a_j \leq 2, \forall 1 \leq i < j \leq m\Big\}$$

with

$$\mu_m := \sqrt{3} \cdot \Big(2 - \frac{1}{m-1}\Big)^{\frac{1}{2}},$$

then $f_0 = f_1$.

Proof Based on Corollary 5.1, it suffices for us to show $\mathcal{V}_m \subset \mathcal{C}_m$; equivalently,

$$\prod_{i=1}^{m}(1+a_i)^{\frac{1}{2}} \le \mu_m \Rightarrow \sum_{i=1}^{m} a_i \le 3 - \frac{1}{m-1} \tag{5.6}$$

always holds for each $a := (a_1, \cdots, a_m)$ satisfying $a_1 \ge \cdots \ge a_m \ge 0$ and $a_1 + a_2 \le 2$. We shall prove (5.6) by using reduction to absurdity. Assume $\sum\limits_{i=1}^{m} a_i > 3 - \frac{1}{m-1}$, then

$$\begin{aligned}
\prod_{i=1}^{m}(1+a_i) &\ge 1 + \sum_{i=1}^{m} a_i + (a_1+a_2)\Big(\sum_{i=3}^{m} a_i\Big) \\
&> 1 + \Big(3 - \frac{1}{m-1}\Big) + 2\Big(1 - \frac{1}{m-1}\Big) \\
&= 3\Big(2 - \frac{1}{m-1}\Big),
\end{aligned} \tag{5.7}$$

causing a contradiction. This completes the proof of the present corollary.

For any vector a satisfying $\prod\limits_{i=1}^{m}(1+a_i)^{\frac{1}{2}} \le \sqrt{3}$, we have

$$a_i + a_j \le \sum_{i=1}^{m} a_i \le \prod_{i=1}^{m}(1+a_i) - 1 \le 2. \tag{5.8}$$

In conjunction with Corollaries 5.1–5.2, we can establish a uniqueness result for minimal maps via the slope functions.

Corollary 5.3 *Suppose that* $f_0, f_1 : \Omega(\subset M) \to N$ *are minimal maps with the same boundary data,* f_0 *is homotopic to* f_1 *and* $K_N \le 0$. *If their singular values satisfy* $\prod\limits_{i=1}^{m}(1+\lambda_i^2)^{\frac{1}{2}} \le \sqrt{3}$, *then* $f_0 = f_1$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Giusti, E. and Williams, G. H., Minimal Surfaces and Functions of Bounded Variation, Vol. 80, Birkhäuser, Boston, 1984.

[2] Lawson, H. B. and Osserman, R., Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system, *Acta Math.*, **139**, 1977, 1–17.

[3] Lee, YI., Ooi, Y. S. and Tsui, MP., Uniqueness of minimal graph in general codimension, *J. Geom. Anal.*, **29**, 2019, 121–133.

[4] Lee, YI. and Tsui, MP., Stability of the minimal surface system and convexity of area functional, *Trans. Am. Math. Soc.*, **366**(7), 2014, 3357–3371.

[5] Lee, YI. and Wang, M. T., A stability criterion for nonparametric minimal submanifolds, *Manuscr. Math.*, **112**(2), 2003, 161–169.

[6] Lee, YI. and Wang, M. T., A note on the stability and uniqueness for solutions to the minimal surface system, *Math. Res. Lett.*, **15**(1), 2008, 197–206.

[7] Li, M. H., Yang, L. and Zhu, T. Y., A note on the uniqueness of minimal maps into $\mathbb{R}^n$ via singular values, 2023, arXiv: 2311.11075v1.

[8] Mirsky, L., On a convex set of matrices, *Arch. Math.*, **10**, 1959, 88–92.

[9] Schoen, R., The role of harmonic mappings in rigidity and deformation problems, Complex geometry (Osaka, 1990), *Lecture Notes in Pure Appl. Math.*, **143**, 1993, 179–200.

[10] Xu, X. W., Yang, L. and Zhang, Y. S., Dirichlet boundary values on Euclidean balls with infinitely many solutions for the minimal surface system, *J. Math. Pures. Appl.*, **129**, 2019, 266–300.