Chin. Ann. Math. Ser. B
47(1), 2026, 115–126
DOI: 10.1007/s11401-026-0010-2

Chinese Annals of Mathematics, Series B

Lower Bound Estimates of the First Steklov Eigenvalue for Compact Manifolds*

Yiwei LIU[1] Yi-Hu YANG[1]

In memory of Professor Hesheng HU

Abstract Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary $\partial\Omega = \Sigma$. Assume that the Ricci curvature of Ω is nonnegative and the principal curvatures of Σ are bounded from below by a positive constant c. In this paper, by constructing a new weight function, the authors obtain a lower bound of the first nonzero Steklov eigenvalue under the assumption that $\mathrm{Sec}_\Omega \geq -k$, where k is a positive constant. The authors also extend this result to the Steklov-type eigenvalue problem of the weighted Laplacian on a metric measure space.

Keywords Lower bound, Steklov eigenvalue, Weight function
2020 MR Subject Classification 53C21, 53C24

1 Introduction

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional $(n \geq 1)$ smooth compact connected Riemannian manifold with smooth boundary $\partial\Omega = \Sigma$. The Steklov eigenvalue problem is as follows (see [11, 20]):

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where Δ is the Laplace-Beltrami operator of Ω and ν is the outward unit normal vector of the boundary Σ. It is well known that the spectrum of the eigenvalue problem (1.1) is nonnegative, discrete and unbounded:

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to +\infty.$$

For more information about the Steklov eigenvalue problem, interested readers can also refer to [6].

In this paper, we concern the lower bound of the first nonzero Steklov eigenvalue σ_1 of Ω.

In [16], Payne used maximum principle to prove that for a bounded domain $\Omega \subset \mathbb{R}^2$, if the geodesic curvature k_g of the boundary curve Σ satisfies $k_g \geq c > 0$, then the first nonzero

Manuscript received March 3, 2025.
[1]School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China.
E-mail: lyw201611012@sjtu.edu.cn yangyihu@sjtu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 12071283).

Steklov eigenvalue σ_1 of Ω satisfies $\sigma_1 \geq c$, with equality holding if and only if Ω is isometric to a round disk of radius $\frac{1}{c}$. Later, Escobar [3] generalized Payne's result to 2-dimensional compact manifolds with nonnegative Gaussian curvature and strictly convex boundary by a similar method. In higher dimensions, by using Reilly's formula (see [19]), he also provided a non-sharp estimate $\sigma_1 > \frac{c}{2}$ for compact manifolds (Ω^{n+1}, g) which satisfies $\mathrm{Ric}_\Omega \geq 0$ and $h \geq c g_\Sigma > 0$, where h is the second fundamental form of Σ with respect to ν in [3].

Based on the above results, Escobar made the following conjecture (see [4]).

Escobar's conjecture Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary Σ. Assume that the Ricci curvature Ric_Ω is nonnegative and that the second fundamental form $h \geq c g_\Sigma > 0$, where c is a positive constant. Then the first nonzero Steklov eigenvalue σ_1 satisfies

$$\sigma_1 \geq c.$$

Moreover, the equality holds if and only if Ω is isometric to a Euclidean ball of radius $\frac{1}{c}$.

When Ω is a ball equipped with rotationally invariant metric, this conjecture has been proved by Montaño [14] (see also [24]). We also notice that Montaño [15] confirmed this conjecture for Euclidean ellipsoids. Later, Xia and Xiong [22] showed that Escobar's conjecture is true for manifolds with nonnegative sectional curvature. It should be pointed out that they used the weighted Reilly-type formula (see [18]) and the Pohozaev-type identity (see [17, 23]) and constructed a special weight function. Recently, Duncan and Kumar [2] provided a new lower bound of σ_1 which can be seen as an improvement on the result of Escobar (i.e., $\sigma_1 > \frac{c}{2}$).

By constructing a new weight function V (see Section 3) and using the weighted Reilly-type formula and the Pohozaev-type identity, we first prove the following theorem.

Theorem 1.1 *Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary Σ. Assume that the Ricci curvature Ric_Ω is nonnegative, the sectional curvature $\mathrm{Sec}_\Omega \geq -k$ and the second fundamental form $h \geq c g_\Sigma > 0$ for constants $k > 0$ and $c > \sqrt{k}$. Then the first nonzero Steklov eigenvalue σ_1 satisfies*

$$\sigma_1 > c - \frac{k}{c}.$$

Remark 1.1 When $c \geq \sqrt{2k}$, we conclude that $\sigma_1 > c - \frac{k}{c} \geq \frac{c}{2}$, so that our estimate can also be considered as an improvement on the estimate ($\sigma_1 > \frac{c}{2}$) of Escobar [3]. On the other hand, if we assume that k is arbitrarily small, our estimate also implies Escobar's conjecture under the condition of nonnegative sectional curvature by Xia and Xiong [22].

Theorem 1.1 can also be extended to metric measure spaces. Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary Σ and ϕ be a smooth function on Ω. Then $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$ is called a metric measure space, where $\mathrm{d}A$

is the canonical volume element of Ω, and the weighted Laplacian $\mathbb{L}_\phi$ is defined by

$$\mathbb{L}_\phi = \Delta - \langle \nabla\phi, \cdot\rangle.$$

We refer interested readers to [21] for more information about metric measure spaces.

The Steklov-type eigenvalue problem of the weighted Laplacian on a metric measure space $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$ is the following (see also [1]):

$$\begin{cases} \mathbb{L}_\phi u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \tau u & \text{on } \Sigma. \end{cases} \tag{1.2}$$

Similarly, the spectrum of the eigenvalue problem (1.2) is also nonnegative, discrete and unbounded:

$$0 = \tau_0 < \tau_1 \le \tau_2 \le \cdots \to +\infty.$$

Readers can refer to [1, 25] for the lower bound estimates of the first nonzero Steklov-type eigenvalue τ_1.

By using the same weight function as in Theorem 1.1, the weighted Reilly-type formula and the Pohozaev-type identity for the weighted Laplacian (see [1]), we have the following theorem, which can be seen as a generalization of Theorem 1.1 on metric measure spaces.

Theorem 1.2 *Let $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$ be an $(n+1)$-dimensional smooth compact connected metric measure space with smooth boundary Σ, such that the Ricci curvature Ric_Ω is nonnegative, the sectional curvature $\mathrm{Sec}_\Omega \ge -k$ and the second fundamental form $h \ge c g_\Sigma > 0$ for constants $k > 0$ and $c > \sqrt{k}$. Assume that ϕ is convex; then the first nonzero Steklov-type eigenvalue τ_1 satisfies*

$$\tau_1 > c - \frac{k}{c}.$$

The paper is organized as follows. Section 2 gives some basic definitions and some known results which are needed later. Section 3 concentrates on the construction of the weight function. In Section 4, we present the proofs of Theorems 1.1–1.2.

2 Preliminaries

This section mainly introduces some basic definitions and some known results which are needed in the later proofs.

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary $\partial\Omega = \Sigma$ and g_Σ be the induced metric on Σ; denote by $\langle\cdot,\cdot\rangle$ the inner product on Ω as well as Σ. Denote by ∇^Ω, ∇, Δ and ∇^2 the connection, the gradient, the Laplacian and the Hessian on Ω, respectively, while by ∇_Σ and Δ_Σ the gradient and the Laplacian on Σ, respectively. Let ν be the unit outward normal of Σ. We denote by h and

H the second fundamental form and the mean curvature of Σ with respect to ν, respectively, where

$$h(X,Y) = -\langle \nabla^{\Omega}_X Y, \nu\rangle$$

and

$$H = \mathrm{tr}_g h.$$

The principal curvatures of Σ are defined to be the eigenvalues of h. Let R_Ω be the curvature tensor of Ω, i.e., for tangent vectors X, Y, Z, W,

$$R_\Omega(X,Y,Z,W) = \langle \nabla^{\Omega}_X \nabla^{\Omega}_Y Z - \nabla^{\Omega}_Y \nabla^{\Omega}_X Z - \nabla^{\Omega}_{[X,Y]} Z, W\rangle,$$

and Ric_Ω be the Ricci curvature tensor of Ω. Let $\mathrm{d}A$ and $\mathrm{d}a$ be the canonical volume element of Ω and Σ, respectively.

Let $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$ be an $(n+1)$-dimensional smooth compact connected metric measure space with smooth boundary $\partial\Omega = \Sigma$, where ϕ is a smooth function on Ω. Denote by $\mathbb{L}_\phi$ the weighted Laplacian on Ω, while by $\mathbb{L}^{\Sigma}_\phi$ the weighted Laplacian on Σ. Let $\mathrm{Ric}^{\phi}_\Omega$ be the Bakry-Émery-Ricci tensor of Ω, where

$$\mathrm{Ric}^{\phi}_\Omega = \mathrm{Ric}_\Omega + \nabla^2\phi.$$

We denote by H_ϕ the weighted mean curvature of Σ with respect to ν, where

$$H_\phi = H - \frac{\partial\phi}{\partial\nu}.$$

Now, we introduce some known results which will be used in our proofs. The first result is the following weighted Reilly-type formula for the weighted Laplacian (see [1]). Readers can also refer to [18] for more information about the weighted Reilly-type formula.

Proposition 2.1 *Let f and V be two smooth functions on $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$. Then we have*

$$\begin{aligned}
&\int_\Omega V((\mathbb{L}_\phi f)^2 - |\nabla^2 f|^2)\mathrm{e}^{-\phi}\mathrm{d}A\\
=&\int_\Sigma V\Big(2(\mathbb{L}^{\Sigma}_\phi f)\frac{\partial f}{\partial\nu} + H_\phi\Big(\frac{\partial f}{\partial\nu}\Big)^2 + h(\nabla_\Sigma f, \nabla_\Sigma f)\Big)\mathrm{e}^{-\phi}\mathrm{d}a\\
&+\int_\Sigma \frac{\partial V}{\partial\nu}|\nabla_\Sigma f|^2\mathrm{e}^{-\phi}\mathrm{d}a\\
&+\int_\Omega ((\nabla^2 V - \mathbb{L}_\phi V g + V\mathrm{Ric}^{\phi}_\Omega)(\nabla f, \nabla f))\mathrm{e}^{-\phi}\mathrm{d}A.
\end{aligned}\tag{2.1}$$

Remark 2.1 We point out that when $V \equiv 1$, the formula (2.1) is the Reilly-type formula for the weighted Laplacian (see [13]); when $\phi = \mathrm{const.}$, the formula (2.1) is the weighted Reilly-type formula in [18]; when $V \equiv 1$ and $\phi = \mathrm{const.}$, the formula (2.1) is just the classical Reilly's formula (see [19]).

Next, we introduce the following Pohozaev-type identity for the weighted Laplacian (see [1]).

Proposition 2.2 *Let X be a smooth vector field on $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$. Let f be a smooth function such that $\mathbb{L}_\phi f = 0$ on $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$. Then we have*

$$\begin{aligned}&\int_\Omega \Big(\langle \nabla^\Omega_{\nabla f} X, \nabla f\rangle - \frac{1}{2}|\nabla f|^2 \mathrm{div}_\phi(X)\Big)\mathrm{e}^{-\phi}\mathrm{d}A\\ &= \int_\Sigma \Big(\frac{\partial f}{\partial \nu}\langle X, \nabla f\rangle - \frac{1}{2}|\nabla f|^2\langle X, \nu\rangle\Big)\mathrm{e}^{-\phi}\mathrm{da},\end{aligned} \tag{2.2}$$

where $\mathrm{div}_\phi = \mathrm{div} - \langle \nabla\phi, \cdot\rangle$ *denotes the weighted divergence operator on* Ω*.*

Remark 2.2 We point out that when $\phi =$const., the formula (2.2) is the Pohozaev-type identity in [23].

3 The Construction of the Weight Function and Its Smooth Approximation

In this section, we concentrate on the construction of the weight function V. We first introduce the distance function to the boundary Σ.

Let $\rho = \rho(x) = d(x, \Sigma)$ be the distance function to the boundary Σ. Obviously, ρ is smooth on $\Omega\backslash\mathrm{Cut}(\Sigma)$, where $\mathrm{Cut}(\Sigma)$ is the set of the cut points of Σ. For this function, we have the following proposition (see [5, 12]).

Proposition 3.1 *Let (Ω^{n+1}, g) be as in Theorem* 1.1*. Then we have*

$$\max_\Omega \rho < \frac{1}{\sqrt{k}}\coth^{-1}\Big(\frac{c}{\sqrt{k}}\Big). \tag{3.1}$$

Now, we define the weight function V as

$$V = V(\rho) = \frac{c}{k}(\cosh)^2(\sqrt{k}\theta)\ln\Big(\frac{\cosh(\sqrt{k}\theta)}{\cosh(\sqrt{k}(\rho-\theta))}\Big), \tag{3.2}$$

where

$$\theta = \frac{1}{\sqrt{k}}\coth^{-1}\Big(\frac{c}{\sqrt{k}}\Big).$$

By Proposition 3.1, we conclude that $V \geq 0$ on Ω. A direct calculation shows

$$V'(\rho) = -\frac{c}{\sqrt{k}}(\cosh)^2(\sqrt{k}\theta)\frac{\sinh(\sqrt{k}(\rho-\theta))}{\cosh(\sqrt{k}(\rho-\theta))},$$

$$V|_\Sigma = 0$$

and

$$\frac{\partial V}{\partial \nu}\Big|_\Sigma = -\frac{c}{\sqrt{k}}\sinh(\sqrt{k}\theta)\cosh(\sqrt{k}\theta) = -\frac{c^2}{c^2-k}.$$

For this weight function V, we also have the following proposition.

Proposition 3.2 *Let* (Ω^{n+1}, g) *be as in Theorem* 1.1 *and* V *be the weight function discussed above. Then* $\forall x \in \Omega$ *and* $X \in T_x\Omega$ *with* $|X| = 1$*, we have*

$$C(-V(\rho))(x; X) \geq c, \tag{3.3}$$

where

$$C(-V(\rho))(x; X) = \liminf_{r\to 0} \frac{-V(\rho(\exp_x(rX))) - V(\rho(\exp_x(-rX))) + 2V(\rho(x))}{r^2}.$$

In particular, if $x \in \Omega\backslash\mathrm{Cut}(\Sigma)$*, we also have*

$$\nabla^2(-V)|_x(X, X) \geq c.$$

Proof $\forall x \in \Omega$, let $\gamma : [0, l] \to \Omega$ be the minimizing geodesic with arc length parameter such that $t = \rho(\gamma(t))$ for $t \in [0, l]$, where $\gamma(0) \in \Sigma$ and $\gamma(l) = x$.

By Proposition 3.1, we conclude that $-V$ is nonincreasing as a function of ρ. Then by [10, Theorem 2.31], for any $X \in T_x\Omega$ with $|X| = 1$, we have

$$C(-V(\rho))(x; X) \geq -V''(l)\langle\gamma'(l), X\rangle^2 - V'(l)\frac{f'(l)}{f(l)}(1 - \langle\gamma'(l), X\rangle^2),$$

where f is the solution of the following equation

$$\begin{cases} f'' - kf = 0, \\ f(0) = 1, \\ f'(0) = -c. \end{cases}$$

We know that

$$f = \cosh(\sqrt{k}t) - \frac{c}{\sqrt{k}}\sinh(\sqrt{k}t).$$

Then a direct calculation shows that

$$\begin{aligned} C(-V(\rho))(x; X) &\geq c\frac{(\cosh)^2(\sqrt{k}\theta)}{(\cosh)^2(\sqrt{k}(l-\theta))}\langle\gamma'(l), X\rangle^2 \\ &\quad + c(\cosh)^2(\sqrt{k}\theta)(1 - \langle\gamma'(l), X\rangle^2) \\ &> c. \end{aligned}$$

Since ρ is smooth on $\Omega\backslash\mathrm{Cut}(\Sigma)$, we know that if $x \in \Omega\backslash\mathrm{Cut}(\Sigma)$, $C(-V(\rho))(x; X) = \nabla^2(-V)|_x(X, X)$. Therefore,

$$\nabla^2(-V)|_x(X, X) \geq c.$$

We can use Proposition 3.2 to consider the convexity of $-V$. Let us first recall the definition of a ξ-convex function (see [9]).

Definition 3.1 *Let* M *be a Riemannian manifold,* $f : M \to \mathbb{R}$ *be a continuous function on* M *and* ξ *be a real number. We call* f *a* ξ*-convex function at a point* $P \in M$ *if there exists a positive constant* δ *such that the function* $\psi(x) = f(x) - \frac{\xi+\delta}{2}d^2(P, x)$ *is convex in a neighborhood of* P*.*

By a similar proof to that of [22, Lemma 3.4], we conclude the following result.

Proposition 3.3 *Let O be a neighborhood of* $\mathrm{Cut}(\Sigma)$ *such that* $\overline{O} \subset \Omega$. *Then* $\forall \eta > 0$, *the function* $-V$ *discussed above is* $(c-\eta)$*-convex on* O.

Since V is not smooth on Ω, we need to consider the smooth Greene-Wu-type approximation of V (i.e., the Riemannian convolution introduced by Greene-Wu [7–9]). In fact, we have the following result.

Proposition 3.4 *Let O be a neighborhood of* $\mathrm{Cut}(\Sigma)$ *such that* $\overline{O} \subset \Omega$. *Then* $\forall \epsilon > 0$, *there exists a smooth nonnegative function* V_ϵ *on* Ω *such that* $V_\epsilon = V$ *on* $\Omega \backslash O$ *and*

$$\nabla^2(-V_\epsilon) \geq (c-\epsilon) g. \tag{3.4}$$

In particular, we also have

$$\lim_{\epsilon \to 0} \|V_\epsilon - V\|_{C^0(\Omega)} = 0.$$

The proof of Proposition 3.4 is also similar to that of [22, Proposition 3.3], so we omit it.

4 Proofs of Theorems 1.1–1.2

Now, let us concentrate on the proofs of Theorems 1.1–1.2. We have already completed the construction of the weight function. The following proposition will be a crucial step in the proof of Theorem 1.1.

Proposition 4.1 *Let* (Ω^{n+1}, g) *be as in Theorem* 1.1 *and f be a harmonic function on* Ω. *Then we have*

$$\int_\Sigma \Big(\frac{\partial f}{\partial \nu}\Big)^2 \mathrm{d}a \geq \Big(c - \frac{k}{c}\Big) \int_\Omega |\nabla f|^2 \mathrm{d}A. \tag{4.1}$$

Proof By the construction of V_ϵ, we know that

$$V_\epsilon|_\Sigma = 0$$

and

$$\frac{\partial V_\epsilon}{\partial \nu} = -\frac{c^2}{c^2-k}.$$

Then by the weighted Reilly-type formula in Proposition 2.1 (let $\phi \equiv 0$), we have

$$\begin{aligned} -\int_\Omega V_\epsilon |\nabla^2 f|^2 \mathrm{d}A = & -\frac{c^2}{c^2-k} \int_\Sigma |\nabla_\Sigma f|^2 \mathrm{d}a \\ & + \int_\Omega (\nabla^2 V_\epsilon - \Delta V_\epsilon g + V_\epsilon \mathrm{Ric}_\Omega)(\nabla f, \nabla f) \mathrm{d}A. \end{aligned} \tag{4.2}$$

In addition, by the Pohozaev-type identity in Proposition 2.2 (let $\phi \equiv 0$ and $X = \nabla V_\epsilon$), we have

$$\int_\Omega \Big(\langle \nabla^\Omega_{\nabla f} \nabla V_\epsilon, \nabla f\rangle - \frac{1}{2} |\nabla f|^2 \mathrm{div}(\nabla V_\epsilon)\Big) \mathrm{d}A$$

$$= \int_{\Sigma} \Big(\frac{\partial f}{\partial \nu} \langle \nabla V_\epsilon, \nabla f \rangle - \frac{1}{2} |\nabla f|^2 \langle \nabla V_\epsilon, \nu \rangle \Big) \mathrm{d}a, \tag{4.3}$$

that is,

$$\int_{\Omega} (2\nabla^2 V_\epsilon - \Delta V_\epsilon g)(\nabla f, \nabla f) \mathrm{d}A = \frac{c^2}{c^2 - k} \int_{\Sigma} \Big(|\nabla_\Sigma f|^2 - \Big(\frac{\partial f}{\partial \nu} \Big)^2 \Big) \mathrm{d}a. \tag{4.4}$$

We then have

$$\begin{aligned} &\frac{c^2}{c^2 - k} \int_{\Sigma} \Big(\frac{\partial f}{\partial \nu} \Big)^2 \mathrm{d}a \\ = &\int_{\Omega} (-\nabla^2 V_\epsilon (\nabla f, \nabla f) + V_\epsilon |\nabla^2 f|^2 + V_\epsilon \mathrm{Ric}_\Omega (\nabla f, \nabla f)) \mathrm{d}A. \end{aligned} \tag{4.5}$$

By Proposition 3.4 and the curvature assumptions in Theorem 1.1, we have

$$\frac{c^2}{c^2 - k} \int_{\Sigma} \Big(\frac{\partial f}{\partial \nu} \Big)^2 \mathrm{d}a \geq (c - \epsilon) \int_{\Omega} |\nabla f|^2 \mathrm{d}A.$$

Then by letting $\epsilon \to 0$, we conclude that

$$\int_{\Sigma} \Big(\frac{\partial f}{\partial \nu} \Big)^2 \mathrm{d}a \geq \Big(c - \frac{k}{c} \Big) \int_{\Omega} |\nabla f|^2 \mathrm{d}A.$$

Proof of Theorem 1.1 Let f be an eigenfunction corresponding to the first nonzero Steklov eigenvalue σ_1. We then have

$$\int_{\Sigma} \Big(\frac{\partial f}{\partial \nu} \Big)^2 \mathrm{d}a = \sigma_1^2 \int_{\Sigma} f^2 \mathrm{d}a$$

and

$$\int_{\Omega} |\nabla f|^2 \mathrm{d}A = \sigma_1 \int_{\Sigma} f^2 \mathrm{d}a.$$

Then by (4.1), we conclude that

$$\sigma_1 \geq c - \frac{k}{c}.$$

Now we assume that $\sigma_1 = c - \frac{k}{c}$. By (4.5), we have

$$\begin{aligned} c \int_{\Omega} |\nabla f|^2 \mathrm{d}A &= \frac{c^2}{c^2 - k} \int_{\Sigma} \Big(\frac{\partial f}{\partial \nu} \Big)^2 \mathrm{d}a \\ &= \int_{\Omega} (-\nabla^2 V_\epsilon (\nabla f, \nabla f) + V_\epsilon |\nabla^2 f|^2 + V_\epsilon \mathrm{Ric}_\Omega (\nabla f, \nabla f)) \mathrm{d}A \\ &\geq \int_{\Omega} ((c - \epsilon) |\nabla f|^2 + V_\epsilon |\nabla^2 f|^2 + V_\epsilon \mathrm{Ric}_\Omega (\nabla f, \nabla f)) \mathrm{d}A. \end{aligned} \tag{4.6}$$

Then by Proposition 3.4 and letting $\epsilon \to 0$, we have

$$\int_{\Omega} (V |\nabla^2 f|^2 + V \mathrm{Ric}_\Omega (\nabla f, \nabla f)) \mathrm{d}A = 0.$$

We then have the following Obata equation:

$$\begin{cases} \nabla^2 f = 0 & \text{in } \Omega, \\ \dfrac{\partial f}{\partial \nu} = \Big(c - \dfrac{k}{c} \Big) f & \text{on } \Sigma. \end{cases}$$

Then by [22, Proposition 4.3], Ω is isometric to a Euclidean ball of radius $\frac{c}{c^2-k}$. Thus the second fundamental form h satisfies

$$h = \Big(c - \frac{k}{c}\Big) g_\Sigma,$$

which is a contradiction (since $k > 0$ and $h \geq c g_\Sigma$). We then conclude that

$$\sigma_1 > c - \frac{k}{c}.$$

By the proof of Theorem 1.1, we also have the following corollary.

Corollary 4.1 *Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary Σ such that the Ricci curvature Ric_Ω is nonnegative, the sectional curvature $\mathrm{Sec}_\Omega \geq -k$ and the second fundamental form $h \geq c g_\Sigma > 0$ for constants $k \geq 0$ and $c > \sqrt{k}$. Assume that the first nonzero Steklov eigenvalue σ_1 satisfies*

$$\sigma_1 = c - \frac{k}{c}.$$

Then $k = 0$ and Ω is isometric to a Euclidean ball of radius $\frac{1}{c}$.

Now we consider the proof of Theorem 1.2. Note that Proposition 3.4 still holds under the assumptions of Theorem 1.2. Thus, by a similar method, we have the following proposition.

Proposition 4.2 *Let $(\Omega^{n+1}, g, \mathrm{e}^{-\phi}\mathrm{d}A)$ be as in Theorem 1.2 and f be a smooth function on Ω such that $\mathbb{L}_\phi f = 0$. Then we have*

$$\int_\Sigma \Big(\frac{\partial f}{\partial \nu}\Big)^2 \mathrm{e}^{-\phi}\mathrm{d}a \geq \Big(c - \frac{k}{c}\Big) \int_\Omega |\nabla f|^2 \mathrm{e}^{-\phi}\mathrm{d}A. \tag{4.7}$$

Proof By the weighted Reilly-type formula in Proposition 2.1 applied to V_ϵ, we have

$$\begin{aligned} -\int_\Omega V_\epsilon |\nabla^2 f|^2 \mathrm{e}^{-\phi}\mathrm{d}A = & -\frac{c^2}{c^2-k}\int_\Sigma |\nabla_\Sigma f|^2 \mathrm{e}^{-\phi}\mathrm{d}a \\ & + \int_\Omega ((\nabla^2 V_\epsilon - \mathbb{L}_\phi V_\epsilon g + V_\epsilon \mathrm{Ric}_\Omega^\phi)(\nabla f, \nabla f))\mathrm{e}^{-\phi}\mathrm{d}A. \end{aligned} \tag{4.8}$$

In addition, by the Pohozaev-type identity in Proposition 2.2 (let $X = \nabla V_\epsilon$), we have

$$\begin{aligned} & \int_\Omega \Big(\langle \nabla^\Omega_{\nabla f} \nabla V_\epsilon, \nabla f\rangle - \frac{1}{2}|\nabla f|^2 \mathrm{div}_\phi(\nabla V_\epsilon)\Big)\mathrm{e}^{-\phi}\mathrm{d}A \\ = & \int_\Sigma \Big(\frac{\partial f}{\partial \nu}\langle \nabla V_\epsilon, \nabla f\rangle - \frac{1}{2}|\nabla f|^2 \langle \nabla V_\epsilon, \nu\rangle\Big)\mathrm{e}^{-\phi}\mathrm{d}a, \end{aligned} \tag{4.9}$$

that is,

$$\begin{aligned} & \int_\Omega ((2\nabla^2 V_\epsilon - \mathbb{L}_\phi V_\epsilon g)(\nabla f, \nabla f))\mathrm{e}^{-\phi}\mathrm{d}A \\ = & \frac{c^2}{c^2-k}\int_\Sigma \Big(|\nabla_\Sigma f|^2 - \Big(\frac{\partial f}{\partial \nu}\Big)^2\Big)\mathrm{e}^{-\phi}\mathrm{d}a. \end{aligned} \tag{4.10}$$

We then have

$$\begin{aligned}&\frac{c^2}{c^2-k}\int_\Sigma\Big(\frac{\partial f}{\partial\nu}\Big)^2\mathrm{e}^{-\phi}\mathrm{d}a\\&=\int_\Omega(-\nabla^2V_\epsilon(\nabla f,\nabla f)+V_\epsilon|\nabla^2f|^2+V_\epsilon\mathrm{Ric}_\Omega^\phi(\nabla f,\nabla f))\mathrm{e}^{-\phi}\mathrm{d}A.\end{aligned}\tag{4.11}$$

By Proposition 3.4 and the assumptions in Theorem 1.2, we have

$$\frac{c^2}{c^2-k}\int_\Sigma\Big(\frac{\partial f}{\partial\nu}\Big)^2\mathrm{e}^{-\phi}\mathrm{d}a\ge(c-\epsilon)\int_\Omega|\nabla f|^2\mathrm{e}^{-\phi}\mathrm{d}A.$$

Then by letting $\epsilon\to0$, we conclude that

$$\int_\Sigma\Big(\frac{\partial f}{\partial\nu}\Big)^2\mathrm{e}^{-\phi}\mathrm{d}a\ge\Big(c-\frac{k}{c}\Big)\int_\Omega|\nabla f|^2\mathrm{e}^{-\phi}\mathrm{d}A.$$

By choosing f as an eigenfunction corresponding to τ_1, we conclude that

$$\tau_1\ge c-\frac{k}{c}.$$

Therefore, we just need to prove that the situation where the equality holds will not happen.

Proof of Theorem 1.2 Assume that $\tau_1=c-\frac{k}{c}$. By a similar calculation in Proposition 4.1, we conclude that

$$\int_\Omega(V|\nabla^2f|^2+V\mathrm{Ric}_\Omega^\phi(\nabla f,\nabla f))\mathrm{e}^{-\phi}\mathrm{d}A=0.$$

We then have the following Obata equation:

$$\begin{cases}\nabla^2f=0&\text{in }\Omega,\\ \dfrac{\partial f}{\partial\nu}=\Big(c-\dfrac{k}{c}\Big)f&\text{on }\Sigma.\end{cases}$$

Then by a similar argument in the proof of Theorem 1.1, we conclude that

$$\tau_1>c-\frac{k}{c}.$$

Similarly, we also have the following corollary.

Corollary 4.2 *Let $(\Omega^{n+1},g,\mathrm{e}^{-\phi}\mathrm{d}A)$ be an $(n+1)$-dimensional smooth compact connected metric measure space with smooth boundary Σ, such that the Ricci curvature Ric_Ω is nonnegative, the sectional curvature $\mathrm{Sec}_\Omega\ge-k$ and the second fundamental form $h\ge cg_\Sigma>0$ for constants $k\ge0$ and $c>\sqrt{k}$. Assume that ϕ is convex and that the first nonzero Steklov-type eigenvalue τ_1 satisfies*

$$\tau_1=c-\frac{k}{c}.$$

Then $k=0$ and Ω is isometric to a Euclidean ball of radius $\frac{1}{c}$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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