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Meromorphic Open-String Vertex Algebras and Riemannian Manifolds*

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Abstract Let M be a Riemannian manifold. For $p \in M$, the tensor algebra $T(\widehat{T_pM}_-)$ of the negative part of the affinization $\widehat{T_pM}$ of the tangent space T_pM of M at p has a natural structure of a meromorphic open-string vertex algebra. These meromorphic open-string vertex algebras form a vector bundle over M with a connection. The author constructs a sheaf $\mathcal{V}$ of meromorphic open-string vertex algebras on the sheaf of parallel sections of this vector bundle. Using covariant derivatives, he constructs a representation on the space of smooth functions of the algebra of parallel tensor fields. These representations are used to construct a sheaf $\mathcal{W}$ of left $\mathcal{V}$-modules generated by the sheaf of smooth functions. In particular, the author obtains a meromorphic open-string vertex algebra V_M as the global sections on M of the sheaf $\mathcal{V}$ and a left V_M-module W_M as the global sections on M of the sheaf $\mathcal{W}$. He shows that the Laplacian on M is in fact a component of a vertex operator for the left V_M-module W_M restricted to the space of smooth functions.

Keywords Meromorphic open-string vertex algebra, Riemannian manifold, Sheaf of left modules
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1 Introduction

Conjectures by physicists on nonlinear sigma models, especially supersymmetric nonlinear sigma models with Calabi-Yau manifolds as targets, are one of the most influential sources of inspiration and motivation for many works in geometry in the past two or three decades. Classically, a nonlinear sigma model is given by the set of all harmonic maps from a two-dimensional Riemannian manifold to a Riemannian manifold (the target). The main challenge for mathematicians is the construction of the corresponding quantum nonlinear sigma model. The difficulties lie in the fact that the target is not flat, the nonlinear sigma model is a quantum field theory with interaction. In physics, a quantum field theory with interaction is studied by using the methods of path integrals, perturbative expansion (more precisely, asymptotic expansion) and renormalization. Unfortunately, it does not seem to be mathematically possible to directly rigorize these physical methods to construct the correlation functions for such a quantum field theory.

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Assuming the existence of nonlinear sigma models, physicists have obtained many surprising mathematical conjectures. Some of these conjectures have been proved by mathematicians using methods developed in mathematics. But there are still many deep conjectures to be understood and proved. Besides proving these conjectures from physics, it is also of great importance to understand mathematically what is going on underlying these deep conjectures. A mathematical construction of nonlinear sigma models would allow us to obtain such a deep conceptual understanding and at the same time to prove these conjectures.

In the present paper, we construct meromorphic open-string vertex algebras and their representations (see [3] for definitions and constructions) from a Riemannian manifold. We hope that these algebras and representations will provide a starting point for a new mathematical approach to the construction of nonlinear sigma models. In the case that the target is a Euclidean space or a torus, the nonlinear sigma model becomes a linear sigma model and can be constructed mathematically using the representations of Heisenberg algebras. In these constructions, a crucial ingredient is the modules for the Heisenberg algebras generated by eigenfunctions of the Laplacian of the target. The role of the eigenfunctions can be conceptually understood as follows: Sigma models describe perturbative string theory. When the strings are degenerate to points in the space, string theory becomes quantum mechanics. In particular, all the states in quantum mechanics should also be states in sigma models. Mathematically, quantum mechanics on a Riemannian manifold M (without additional potential terms describing interactions) is essentially the study of the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi = \Delta\psi,$$

where ψ is a function on $M \times \mathbb{R}$. Using the method of separation of variables, we first study a product solution fT of the equation above where f is a function on M and T is a function on $\mathbb{R}$. Then there exists $\lambda \in \mathbb{C}$ such that f is an eigenfunction of the Laplacian Δ with the eigenvalue λ and $T = C\mathrm{e}^{-\frac{\lambda}{\mathrm{i}\hbar}t}$ for some $C \in \mathbb{C}$. Thus the study of the Schrödinger equation above is reduced to the study of eigenvalues and eigenfunctions of the Laplacian Δ. Eigenfunctions of Δ are states in the quantum mechanics on M whose eigenvalues are the energies when the quantum mechanical particle is in these states.

For a Riemannian manifold M, its tangent spaces are Euclidean spaces. From these tangent spaces, one can construct vertex operator algebras associated with Heisenberg algebras. These vertex operator algebras form a vector bundle of vertex operator algebras over M. By tautology, the space of smooth sections of this bundle is a vertex algebra, a variant of vertex operator algebras satisfying fewer conditions. Geometrically this vertex algebra is not very interesting because as a module for the ring of the smooth functions on M, the information about this vertex algebra can all be obtained from the theory of vector bundles and the vertex operator algebras over the fibers. Algebraically, since this vertex algebra does not satisfy the important grading restriction condition and its weight 0 subspace is not one-dimensional (in fact, it is the infinite-dimensional space of all smooth functions), not many interesting results for this vertex algebra can be expected. To obtain a vertex algebra having better properties, it is natural to consider the subspace of parallel sections of this vector bundle. It was first observed by Tamanoi [8–9] that the space of parallel sections of a vector bundle of vertex operator superalgebras constructed from suitable modules for Clifford algebras has a natural structure of a vertex operator superalgebra. The same observation can be made to see the existence of a natural structure of a vertex operator algebra on the space of parallel sections of the vector bundle of Heisenberg vertex operator algebras mentioned above. However, the only functions on M belonging to this vertex operator algebra are constant functions and, in particular, eigenfunctions on M are not in this vertex operator algebra. In fact, we do not expect that eigenfunctions will in general be in any vertex operator algebra because their eigenvalues in general are not integers.

On the other hand, it is known that the state space of a chiral rational conformal field theory is mathematically the direct sum of irreducible modules for the chiral algebra (the vertex operator algebra of meromorphic fields) of the conformal field theory (see [1–2]). Though the nonlinear sigma model with target M is in general not even a conformal field theory, it would still be natural to look for some modules or generalized modules that contain eigenfunctions on M. To find such modules or generalized modules, one would have to construct a representation of the symmetric algebra on the tangent space at a point $p \in M$ on the space of smooth functions on an open neighborhood of p. When M is not flat, however, such a representation does not exist for obvious reasons: If we choose a coordinate patch near p and use the derivatives with respect to the coordinates to give the representation, the representation images of higher derivatives depend on the coordinate patch and thus are not covariant. If we use the covariant derivatives, then we do not have a representation of the symmetric algebra on the tangent space at a point p; the failure of being a representation is measured exactly by the curvature tensor. This failure indicates that we should consider tensor algebras instead of symmetric algebras.

In [3], the author introduced a notion of meromorphic open-string vertex algebra. A meromorphic open-string vertex algebra is an open-string vertex algebra in the sense of Kong and the author [4] satisfying additional rationality (or meromorphicity) conditions for vertex operators. The vertex operator map for a meromorphic open-string vertex algebra in general does not satisfy the Jacobi identity, commutativity, the commutator formula, skew-symmetry or even the associator formula but still satisfies rationality and associativity. In particular, the operator product expansion holds for vertex operators for a meromorphic open-string vertex algebra. In [3], the author constructed such algebras on the tensor algebra of the negative part of the affinization of a vector space and left modules over these algebras.

In the present paper, using covariant derivatives, parallel tensor fields and the constructions in [3], we construct a sheaf of meromorphic open-string vertex algebras from a Riemannian manifold M and a sheaf of left modules for this sheaf generated by the space of smooth functions on M.

More precisely, for a Riemannian manifold M, let TM be the tangent bundle of M, $T(TM)$ the vector bundle of the tensor algebras on the tangent spaces at points on M and $T(\widehat{TM}_-)$ the vector bundle over M whose fibers are the negative parts of the affinization of the tangent spaces of M. Using the meromorphic open-string vertex algebras constructed in [3], we construct a sheaf $\mathcal{V}$ of meromorphic open-string vertex algebras on the sheaf of spaces of parallel sections of the vector bundle $T(\widehat{TM}_-)$. In particular, the space V_M of the global sections of $\mathcal{V}$ gives a meromorphic open-string vertex algebra canonically associated to M. For an open subset U of M, let $C^\infty(U)$ be the space of smooth functions on M. For each open subset U of M, we construct a representation on the space of smooth functions on U of the algebra of parallel sections of $T(TM)$ on U. Using these representations and the constructions of left modules for meromorphic open-string vertex algebras in [3], we construct a sheaf $\mathcal{W}$ of left modules for $\mathcal{V}$ generated by $C^\infty(U)$. In particular, the space W_M of the global sections of $\mathcal{W}$ gives a left V_M-module canonically associated to M. As an example, we show that the Laplacian on M is in fact a component of a vertex operator for the left V_M-module W_M restricted to the space of smooth functions.

The construction in the present paper can be generalized to give constructions of left modules generated by forms on M for suitable meromorphic open-string vertex algebra associated to a Riemannian manifold M. In the case that M is Kähler or Calabi-Yau, we have stronger results. These will be discussed in future publications.

The author studied differential geometry under the supervision of Professor Hesheng HU as a master student from 1982 to 1984 in Fudan university. The publication of the present paper in this issue is dedicated to the memory of her. This paper was finished in 2012 and was posted to the arXiv on May 14, 2012. The present version is identical to the original version except that some typos are corrected and two paragraphs (including this one) are added. In [6], Qi gave the explicit examples of meromorphic open-string vertex algebras and their modules associated to two-dimensional orientable space forms. In [7], to understand modules for meromorphic open-string vertex algebras generated by eigenfunctions of the Laplacians on space forms, Qi obtained results and formulated a conjecture on covariant derivatives of such eigenfunctions. Research projects based on this paper have also been actively carried out by several people including the author.

Here the author would also like to address one issue on which some mathematicians and the author have different opinions. One opinion is that this paper is based on the parallel sections of vector bundles and thus cannot lead to a construction of the two-dimensional quantum field theory associated to a Riemannian manifold. For example, this opinion states that the conformal field theories associated to tori cannot be constructed based on the approach developed in this paper. People with this opinion obviously did not read the present paper carefully. In the discussion in this introduction above, the author has indicated clearly that, though the meromorphic open-string vertex algebra associated to a Riemannian manifold is obtained using parallel sections, the modules are not. Here the author would like to point out another related misunderstanding about the vertex-operator-algebraic approach to conformal field theory. Some people mistakenly think that a vertex operator algebra determines a conformal field theory completely. This is true only in the case of rational conformal field theories but is wrong in general. One class of counterexamples to this statement is the conformal field theories associated to irrational tori. For all irrational tori of the same dimension, the associated vertex operator algebras are all the same as the vertex operator algebra for the corresponding Euclidean space (the Heisenberg vertex operator algebra in this dimension) since in this case there is no larger vertex operator algebra such as the lattice vertex operator algebras in the rational tori case. What determines a conformal field theory associated to a given irrational torus is the choice of a subcategory of the category of modules for the Heisenberg vertex operator algebra. The main difficulty that the author overcome in this paper is, as discussed above, the construction of modules generated by eigenfunctions of the Laplacian. This construction is not given by parallel sections and uses the geometry of the Riemannian manifold in a crucial way. Note that eigenfunctions of the Laplacian on a Riemannian manifold contain a lot of information about the Riemannian manifold. Though it has been known for a long time that we cannot hear the shape of a Riemannian manifold (that is, the eigenvalues of the Laplacian cannot determine the Riemannian manifold up to isometries), eigenfunctions can indeed determine at least a compact Riemannian manifold since every function in a suitable Sobolev space can be expanded as a (finite or infinite) sum of eigenfunctions. Also for a torus, no matter whether it is rational or irrational, it is easy to use the construction of the present paper to construct the corresponding conformal field theory. This is in fact one of the reasons why the author always believes that the approach developed in this paper is correct.

In this paper, we shall fix a Riemannian manifold M. For basic material on Riemannian geometry, we refer the reader to the book [5]. For meromorphic open-string vertex algebras and left modules, see [3].

The present paper is organized as follows: In Section 2, we recall some basic constructions of vector bundles and sheaves on a Riemannian manifold M. In Section 3, we construct the sheaf $\mathcal{V}$ of meromorphic open-string vertex algebras on M. In particular, we construct the meromorphic open-string vertex algebra V_M of the global sections of $\mathcal{V}$ canonically associated to M. In Section 4, using covariant derivatives, we construct a homomorphism of algebras from the algebra of parallel tensor fields on an open subset of M to the algebra of linear operators on the space of smooth functions on the same open subset. In particular, we obtain a representation on the space of smooth functions of the algebra of parallel tensor fields. We construct in Section 5 the sheaf $\mathcal{W}$ of left modules for $\mathcal{V}$ generated by the sheaf of smooth functions on M. In particular, we construct the left V_M-module W_M of $\mathcal{W}$ canonically associated to M. In particular, we construct the left V_M-module of the global sections of $\mathcal{W}$ canonically associated to M. In Section 6, we show that the Laplacian on M is in fact a component of a vertex operator for the left V_M-module W_M restricted to the space of smooth functions.

2 Vector Bundles and Sheaves from the Tangent Bundle of a Riemannian Manifold M

In this section, we recall some basic constructions of vector bundles and sheaves on a Riemannian manifold.

Let M be a Riemannian manifold and g the metric on M. Consider the tangent bundle TM of M and the trivial bundles $M \times \mathbb{C}[t, t^{-1}]$ and $M \times \mathbb{C}\mathbf{k}$ where t is a formal variable and $\mathbf{k}$ is a basis of a one-dimensional vector space $\mathbb{C}\mathbf{k}$. Let

$$\widehat{TM} = TM \otimes (M \times \mathbb{C}[t, t^{-1}]) \oplus M \times \mathbb{C}\mathbf{k}$$

be the vector bundle whose fiber at $p \in M$ is

$$\widehat{T_pM} = T_pM \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}.$$

Since $\widehat{T_pM}$ for $p \in M$ has a structure of Heisenberg algebra and the transition functions at points of M preserve the gradings of the Heisenberg algebras, $\widehat{TM}$ has a structure of a vector bundle of Heisenberg algebras. For $p \in M$, $\widehat{T_pM}$ has a decomposition

$$\widehat{T_pM} = \widehat{T_pM}_- \oplus \widehat{T_pM}_0 \oplus \widehat{T_pM}_+,$$

where

$$\begin{aligned}
\widehat{T_pM}_- &= T_pM \otimes t^{-1}\mathbb{C}[t^{-1}],\\
\widehat{T_pM}_0 &= T_pM \otimes \mathbb{C}t^0 \oplus \mathbb{C}\mathbf{k}\\
&\simeq T_pM \oplus \mathbb{C}\mathbf{k},\\
\widehat{T_pM}_+ &= TM \otimes t\mathbb{C}[t].
\end{aligned}$$

These triangle decompositions of the Heisenberg algebras give the triangle decomposition

$$\widehat{TM} = \widehat{TM}_- \oplus \widehat{TM}_0 \oplus \widehat{TM}_+,$$

where

$$\begin{aligned}
\widehat{TM}_- &= TM \otimes (M \times t^{-1}\mathbb{C}[t^{-1}]),\\
\widehat{TM}_0 &= TM \otimes (M \times \mathbb{C}t^0) \oplus M \times \mathbb{C}\mathbf{k}\\
&\simeq TM \oplus (M \times \mathbb{C}\mathbf{k}),\\
\widehat{TM}_+ &= TM \otimes (M \times t\mathbb{C}[t]).
\end{aligned}$$

The connection on TM induces connections on $\widehat{TM}$, $\widehat{TM}_-$ and $\widehat{TM}_+$. The product bundle $M \times \mathbb{C}\mathbf{k}$ has a trivial connection.

For $p \in M$, recall the subalgebra $N(\widehat{T_PM})$ of the tensor algebra $T(\widehat{T_pM})$ introduced in [3, Section 3]. In fact, let I be the two-sided ideal of $T(\widehat{T_pM})$ generated by elements of the form

$$\begin{gathered}
(X \otimes t^m) \otimes (Y \otimes t^n) - (Y \otimes t^n) \otimes (X \otimes t^m) - m(a, b)\delta_{m+n,0}\mathbf{k},\\
(X \otimes t^k) \otimes (Y \otimes t^0) - (Y \otimes t^0) \otimes (X \otimes t^k),\\
(X \otimes t^k) \otimes \mathbf{k} - \mathbf{k} \otimes (X \otimes t^k)
\end{gathered}$$

for $X, Y \in T_pM$, $m \in \mathbb{Z}_+$, $n \in -\mathbb{Z}_+$, $k \in \mathbb{Z}$. Then by [3, Proposition 3.1],

$$N(\widehat{T_pM}) = T(\widehat{T_pM})/I$$

is isomorphic to

$$T(\widehat{T_pM}_-) \otimes T(\widehat{T_pM}_+) \otimes T(T_pM) \otimes T(\mathbb{C}\mathbf{k}). \tag{2.1}$$

Let

$$T(\widehat{TM}_{-}),\quad T(\widehat{TM}_{+}),\quad T(TM),\quad T(M\times\mathbb{C}\mathbf{k})$$

be the vector bundles whose fibers at $p\in M$ are the tensor algebras

$$T(\widehat{T_pM}_{-}),\quad T(\widehat{T_pM}_{+}),\quad T(T_pM),\quad T(\mathbb{C}\mathbf{k})$$

on the fibers of

$$\widehat{TM}_{-},\quad \widehat{TM}_{+},\quad TM,\quad M\times\mathbb{C}\mathbf{k},$$

respectively. Since (2.1) is the fiber of the vector bundle

$$T(\widehat{TM}_{-})\otimes T(\widehat{TM}_{+})\otimes T(TM)\otimes T(M\times\mathbb{C}\mathbf{k}) \tag{2.2}$$

at $p\in M$, we also have a vector bundle $N(\widehat{TM})$ whose fiber at $p\in M$ is $N(\widehat{T_pM})$. By definition, $N(\widehat{TM})$ as a vector bundle is isomorphic to the vector bundle (2.2).

For a vector bundle E over M, we shall use $\Gamma_U(E)$ to denote the space of smooth sections of E on an open subset U of M. For a vector bundle E over M with a connection, we shall use $\Pi_U(E)$ to denote the space of parallel sections of E on U. By definition, $\Pi_U(E)\subset\Gamma_U(E)$. When the fibers of E are associative algebras, $\Gamma_U(E)$ has a structure of an associative algebra. If the covariant derivative with respect to every element of $\Gamma_U(TM)$ is a derivation of the associative algebra $\Gamma_U(E)$, then $\Pi_U(E)$ is a subalgebra of $\Gamma_U(E)$.

Taking E to be

$$N(\widehat{TM}),\quad T(\widehat{TM}_{-}),\quad T(\widehat{TM}_{+}),\quad T(TM),\quad T(M\times\mathbb{C}\mathbf{k}), \tag{2.3}$$

we have the associative algebras

$$\begin{gathered}\Gamma_U(N(\widehat{TM})),\quad \Gamma_U(T(\widehat{TM}_{-})),\quad \Gamma_U(T(\widehat{TM}_{+})),\\ \Gamma_U(T(TM)),\quad \Gamma_U(T(M\times\mathbb{C}\mathbf{k})),\end{gathered} \tag{2.4}$$

respectively, of smooth sections. It is clear that

$$\Gamma_U(T(\widehat{TM}_{-})),\quad \Gamma_U(T(\widehat{TM}_{+})),\quad \Gamma_U(T(TM)),\quad \Gamma_U(T(M\times\mathbb{C}\mathbf{k}))$$

can be embedded as subalgebras of $\Gamma_U(U(\widehat{TM}))$. The connections on $\widehat{TM}_{-}$, TM and $\widehat{TM}_{+}$ uniquely determine connections on $T(\widehat{TM}_{-})$, $T(TM)$ and $T(\widehat{TM}_{+})$, respectively, by requiring that for every open subset U of M, the covariant derivatives with respect to every element of $\Gamma_U(TM)$ are derivations of the associative algebras $\Gamma_U(T(\widehat{TM}_{-}))$, $\Gamma_U(T(\widehat{TM}_{+}))$ and $\Gamma_U(T(TM))$, respectively. We also have a canonical flat connection on the trivial bundle

$$T(M\times\mathbb{C}\mathbf{k})\simeq M\times T(\mathbb{C}\mathbf{k}).$$

Since $N(\widehat{TM})$ is isomorphic to (2.2), the connections on $T(\widehat{TM}_{-})$, $T(\widehat{TM}_{+})$, $T(TM)$ and $T(M\times\mathbb{C}\mathbf{k})$ further determine a connection on $N(\widehat{TM})$.

By definition, the covariant derivatives with respect to elements of the space $\Gamma_U(TM)$ of the vector bundles in (2.3) are derivations of the corresponding associative algebras in (2.4). Thus we have the associative algebras

$$\Pi_U(N(\widehat{TM})),\quad \Pi_U(T(\widehat{TM}_{-})),$$

$$\Pi_U(T(\widehat{TM}_+)), \quad \Pi_U(T(TM)), \quad \Pi_U(M \times T(\mathbb{C}\mathbf{k}))$$

of parallel sections.

For a vector bundle E, the spaces $\Gamma_E(U)$ of smooth sections on open subsets U of M and the obvious restriction maps from $\Gamma_E(U)$ to $\Gamma_E(U')$ when $U' \subset U$ give a sheaf Γ_E. Similarly for a vector bundle E with a connection, we also have the sheaf Π_E whose sections on an open subset U is $\Pi_E(U)$. The sheaf Π_E is a subsheaf of Γ_E. Taking E to be the vector bundles in (2.3), we have the sheaves

$$\begin{gathered}\Gamma(N(\widehat{TM})), \quad \Gamma(T(\widehat{TM}_-)), \quad \Gamma(T(\widehat{TM}_+)),\\ \Gamma(T(TM)), \quad \Gamma(T(M \times \mathbb{C}\mathbf{k})),\\ \Pi(N(\widehat{TM})), \quad \Pi(T(\widehat{TM}_-)), \quad \Pi(T(\widehat{TM}_+)),\\ \Pi(T(TM)), \quad \Pi(M \times T(\mathbb{C}\mathbf{k})).\end{gathered}$$

We know that the space of parallel sections of a vector bundle with a connection is canonically isomorphic to the space of fixed points of a fiber under the action of the holonomy group. In particular, we have the following result.

Proposition 2.1 *Let U be an open subset of M. The spaces*

$$\Pi_U(T(\widehat{TM}_-)), \quad \Pi_U(T(\widehat{TM}_+)), \quad \Pi_U(T(TM)), \quad \Pi_U(N(\widehat{TM}))$$

are canonically isomorphic to the spaces of fixed points of

$$T(\widehat{T_pM}_-), \quad T(\widehat{T_pM}_+), \quad T(T_pM), \quad N(\widehat{T_pM}),$$

respectively, for $p \in U$ under the actions of the holonomy groups of the restrictions of the vector bundles

$$T(\widehat{TM}_-), \quad T(\widehat{TM}_+), \quad T(TM), \quad N(\widehat{TM}),$$

respectively, to U.

3 A Sheaf $\mathcal{V}$ of Meromorphic Open-String Vertex Algebras on M

In this section, we construct a sheaf of meromorphic open-string vertex algebras on M. In particular, the global sections of this sheaf give a canonical meromorphic open-string vertex algebra associated to M.

First we have the following result.

Proposition 3.1 *The fibers of the vector bundle $T(\widehat{TM}_-)$ have natural structures of meromorphic open-string vertex algebras and $T(\widehat{TM}_-)$ has a natural structure of vector bundle of meromorphic open-string vertex algebras.*

Proof Since the fibers of $T(\widehat{TM}_-)$ are the tensor algebras on the fibers of $\widehat{TM}_-$, by [3, Theorem 5.1], they have natural structures of meromorphic open-string vertex algebras. It is clear that the transition functions of the vector bundle $T(\widehat{TM}_-)$ at points on M are automorphisms of meromorphic open-string vertex algebras. Thus $T(\widehat{TM}_-)$ has a natural structure of vector bundle of meromorphic open-string vertex algebras.

Then we have the following corollary.

Corollary 3.1 *For an open subset* U *of* M*, the space* $\Gamma_U(T(\widehat{TM}_-))$ *of sections of* $T(\widehat{TM}_-)$ *has a natural structure of meromorphic open-string vertex algebra. The assignment*

$$U \to \Gamma_U(T(\widehat{TM}_-))$$

together with the restrictions of sections form a sheaf of meromorphic open-string vertex algebras.

Proof The $\mathbb{Z}$-gradings on the fibers of $T(\widehat{TM}_-)$ induce a $\mathbb{Z}$-grading on $\Gamma_U(T(\widehat{TM}_-))$. The constant section 1 is the vacuum. The vertex operator map is defined pointwise. It is clear that with the $\mathbb{Z}$-grading, the vacuum and the vertex operator map, $\Gamma_U(T(\widehat{TM}_-))$ is a meromorphic open-string vertex algebra. The second conclusion is also clear.

The construction in Corollary 3.1 is simple. But these meromorphic open-string vertex algebras are not what we are interested in. In fact, the sheaf of meromorphic open-string vertex algebras obtained in Corollary 3.1 contains the sheaf of smooth functions on M and the smooth functions commute with vertex operators. In particular, the vertex operators in this sheaf of meromorphic open-string vertex algebras cannot contain differential operators acting on the space of smooth functions. Since the quantum mechanics on M involves differential operators, the sheaf of meromorphic open-string vertex algebras in Corollary 3.1 is not what we are looking for.

Let $S(\widehat{TM}_-)$ be the vector bundle whose fiber at $p \in M$ is the symmetric algebra $S(\widehat{T_pM}_-)$ of $\widehat{T_pM}_-$. As we mentioned in the introduction, Tamanoi observed in [8–9] that the space $\Pi_U(S(\widehat{TM}_-))$ of parallel sections on U of $S(\widehat{TM}_-)$ is a vertex operator algebra. We now construct our sheaf of meromorphic open-string vertex algebras similarly.

Given a meromorphic open-string vertex algebra $(V, Y_V, \mathbf{1})$ and a group H of automorphisms of V, let V^H be the subspace of V consisting of elements that are fixed by H. Since automorphisms of V preserve $\mathbf{1} \in V$ (see [3]), $\mathbf{1} \in V^H$. Also since for $u, v \in V^H$ and $h \in H$, $hY_V(u,x)v = Y_V(hu,x)hv = Y_V(u,x)v$, the image of $V^H \otimes V^H$ under Y_V is in $V^H[[x,x^{-1}]]$. We shall denote the restriction of Y_V to $V^H \otimes V^H$ by Y_{V^H}. Then Y_{V^H} is a linear map from $V^H \otimes V^H$ to $V^H[[x,x^{-1}]]$. The following result is obvious.

Proposition 3.2 *The triple* $(V^H, Y_{V^H}, \mathbf{1})$ *is a meromorphic open-string vertex subalgebra of* $(V, Y_V, \mathbf{1})$.

For $p \in M$ and a connected open subset U of M containing p, the holonomy group $H_p(U)$ of the restriction of the vector bundle $T(\widehat{TM}_-)$ to U acts on the fiber $T(\widehat{T_pM}_-)$ at p of the vector bundle $T(\widehat{TM}_-)$. By Proposition 3.1, $T(\widehat{T_pM}_-)$ has a structure of meromorphic open-string vertex algebra.

Lemma 3.1 *For a connected open subset* U *of* M*,* $\alpha \in H_p(U)$ *and* $u, v \in T(\widehat{T_pM}_-)$*,*

$$\alpha(Y_{T(\widehat{T_pM}_-)}(u,x)v) = Y_{T(\widehat{T_pM}_-)}(\alpha(u),x)\alpha(v).$$

Proof Recall the notations in [3]. We need only prove the lemma in the case

$$u = X_1(-n_1)\cdots X_k(-n_k)\mathbf{1}$$

for $X_1, \cdots, X_k \in T_pM$ and $n_1, \cdots, n_k \in \mathbb{Z}_+$. Since the connection on $T(\widehat{TM}_-)$ is induced from the connection on TM, the parallel transport in $T(\widehat{TM}_-)$ along a path in M is also induced from the parallel transport in TM along the same path. Let γ be a loop in M based at p. Denote both the parallel transports along γ in TM and in $T(\widehat{TM}_-)$ by α_γ. Then we have

$$\alpha_\gamma(\mathbf{1}) = \mathbf{1}$$

and

$$\alpha_\gamma(X_1(-m_1)\cdots X_k(-m_k)\mathbf{1}) = \alpha_\gamma(X_1)(-m_1)\cdots\alpha_\gamma(X_k)(-m_k)\mathbf{1} \tag{3.1}$$

for $n_1, \cdots, n_k \in \mathbb{Z}$.

By definition,

$$\begin{aligned} &Y_{S(\widehat{T_pM}_-)}(X_1(-n_1)\cdots X_k(-n_k)\mathbf{1}, x)v \\ &= \Big({}^\circ_\circ \frac{1}{(n_1-1)!}\Big(\frac{\mathrm{d}^{n_1-1}}{\mathrm{d}x^{n_1-1}}X_1(x)\Big)\cdots\frac{1}{(n_k-1)!}\Big(\frac{\mathrm{d}^{n_k-1}}{\mathrm{d}x^{n_k-1}}X_k(x)\Big){}^\circ_\circ\Big)v, \end{aligned}$$

where as in [3],

$$X_i(x) = \sum_{n\in\mathbb{Z}} X_i(n)x^{-n-1}$$

for $i = 1, \cdots, k$ and ${}^\circ_\circ \cdot {}^\circ_\circ$ is the normal ordering operation defined in [3]. Thus by Lemma 4.2 in [3] and (3.1), we have

$$\begin{aligned} &\alpha_\gamma(Y_{S(\widehat{T_pM}_-)}(u, x)v) \\ &= \alpha_\gamma(Y_{S(\widehat{T_pM}_-)}(X_1(-n_1)\cdots X_k(-n_k)\mathbf{1}, x)v) \\ &= \alpha_\gamma\Big(\Big({}^\circ_\circ \frac{1}{(n_1-1)!}\Big(\frac{\mathrm{d}^{n_1-1}}{\mathrm{d}x^{n_1-1}}X_1(x)\Big)\cdots\frac{1}{(n_k-1)!}\Big(\frac{\mathrm{d}^{n_k-1}}{\mathrm{d}x^{n_k-1}}X_k(x)\Big){}^\circ_\circ\Big)v\Big) \\ &= {}^\circ_\circ \frac{1}{(n_1-1)!}\Big(\frac{\mathrm{d}^{n_1-1}}{\mathrm{d}x^{n_1-1}}(\alpha_\gamma(X_1))(x)\Big)\cdots\frac{1}{(n_k-1)!}\Big(\frac{\mathrm{d}^{n_k-1}}{\mathrm{d}x^{n_k-1}}(\alpha_\gamma(X_k))(x)\Big){}^\circ_\circ \alpha_\gamma(v) \\ &= Y_{S(\widehat{T_pM}_-)}(\alpha_\gamma(X_1)(-n_1)\cdots\alpha_\gamma(X_k)(-n_k)\mathbf{1}, x)\alpha_\gamma(v) \\ &= Y_{S(\widehat{T_pM}_-)}(\alpha_\gamma(X_1(-n_1)\cdots X_k(-n_k)\mathbf{1}), x)\alpha_\gamma(v) \\ &= Y_{S(\widehat{T_pM}_-)}(\alpha_\gamma(u), x)\alpha_\gamma(v). \end{aligned}$$

From the lemma above, we obtain immediately the following corollary.

Corollary 3.2 *For a connected open subset U of M, the holonomy group $H_p(U)$ is a subgroup of the automorphism group of the meromorphic open-string vertex algebra $T(\widehat{T_pM}_-)$. In particular, $(T(\widehat{T_pM}_-))^{H_p(U)}$ is a meromorphic open-string vertex subalgebra of $T(\widehat{T_pM}_-)$.*

For an open subset U of M, let

$$V_U = \Pi_U(T(\widehat{TM}_-)).$$

Then the assignment $U \to V_U$ and the restrictions of sections give a sheaf $\mathcal{V}$. By Proposition 2.1, V_U is canonically isomorphic to $(T(\widehat{T_pM}_-))^{H_p(U)}$. Thus we have the following theorem.

Theorem 3.1 *For a connected open subset U of M and $p \in U$, the canonical isomorphism from $(T(\widehat{T_pM}\_))^{H_p(U)}$ to V_U gives V_U a natural structure of meromorphic open-string vertex algebra. This structure of meromorphic open-string vertex algebra is independent of the choice of p. For general open subset U of M, V_U as a $\mathbb{Z}$-graded vector space is isomorphic to the underlying $\mathbb{Z}$-graded vector space of the direct product meromorphic open-string vertex algebra $\prod_{\alpha \in \mathcal{A}} V_{U_\alpha}$ (see [3, Definition 2.6]) where U_α for $\alpha \in \mathcal{A}$ are the connected components of U. In particular, V_U also has a natural structure of meromorphic open-string vertex algebra. For an open subset U of M and an open subset $\widetilde{U}$ of U, the restriction map from V_U to $V_{\widetilde{U}}$ is a homomorphism of meromorphic open-string vertex algebras. In particular, the sheaf $\mathcal{V}$ is a sheaf of meromorphic open-string vertex algebras.*

Proof The first and second statements of the theorem are clear.

For general open subset U of M, choose a point p_α in each connected component U_α of U for $\alpha \in \mathcal{A}$ (elements of $\mathcal{A}$ labeling the connected components of U), then $\Pi_{U_\alpha}(T(\widehat{T_{p_\alpha}M}\_))$ is isomorphic to $(T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ as a graded vector space, where $H_{p_\alpha}(U_\alpha)$ is the holonomy group of the connection on the vector bundle $T(\widehat{TM}_-)$ restricted to the connected component U_α. But $\Pi_U(T(\widehat{TM}_-))$ is isomorphic to $\prod_{\alpha \in \mathcal{A}} \Pi_{U_\alpha}(T(\widehat{T_{p_\alpha}M}\_))$ as a graded vector space. Hence V_U is isomorphic to $\prod_{\alpha \in \mathcal{A}} (T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ as a graded vector space. Since $\prod_{\alpha \in \mathcal{A}} (T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ has a structure of the direct product meromorphic open-string vertex algebra of $(T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ for $\alpha \in \mathcal{A}$, V_U has a natural structure of a meromorphic open-string vertex algebra of central charge n.

For an open subset U of M and an open subset $\widetilde{U}$ of U, let U_α for $\alpha \in \mathcal{A}$ be the connected components of U and let $\widetilde{U}_\beta$ for $\beta \in \mathcal{B}$ be the connected components of $\widetilde{U}$. Then for $\beta \in \mathcal{B}$, there exists $\alpha \in \mathcal{A}$ such that $\widetilde{U}_\beta \subset U_\alpha$. For each $\beta \in \mathcal{B}$, we choose a point $\widetilde{p}_\beta \in \widetilde{U}_j$. Then there exists $\alpha \in \mathcal{A}$ such that $\widetilde{p}_\beta \in U_\alpha$. We choose $p_\alpha \in U_\alpha$ from those $\widetilde{p}_\beta$'s such that $\widetilde{p}_\beta \in \widetilde{U}_\beta$. Then $H_{\widetilde{p}_\beta}(\widetilde{U}_\beta)$ can be naturally embedded into $H_{p_\alpha}(U_\alpha)$ when $\widetilde{p}_\beta \in U_\alpha$. Thus the direct product meromorphic open-string vertex algebra $\prod_{\alpha \in \mathcal{A}} (T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ can be embedded into the direct product meromorphic open-string vertex algebra $\prod_{\beta \in \mathcal{B}} (T(\widehat{T_{\widetilde{p}_\beta}M}\_))^{H_{\widetilde{p}_\beta}(\widetilde{U}_\beta)}$. The embedding from $\prod_{\alpha \in \mathcal{A}} (T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ to $\prod_{\beta \in \mathcal{B}} (T(\widehat{T_{\widetilde{p}_\beta}M}\_))^{H_{\widetilde{p}_\beta}(\widetilde{U}_\beta)}$ corresponds to the restriction map from V_U to $V_{\widetilde{U}}$, that is, we have the following commutative diagram:

$$
\begin{array}{ccc}
\displaystyle\prod_{\alpha \in \mathcal{A}} (T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)} & \longrightarrow & V_U \\
\downarrow & & \downarrow \\
\displaystyle\prod_{\beta \in \mathcal{B}} (T(\widehat{T_{\widetilde{p}_\beta}M}\_))^{H_{\widetilde{p}_\beta}(\widetilde{U}_\beta)} & \longrightarrow & V_{\widetilde{U}}.
\end{array}
$$

Since the embedding from $\prod_{\alpha \in \mathcal{A}} (T(\widehat{T_{p_\alpha}M}\_))^{H_{p_\alpha}(U_\alpha)}$ to $\prod_{\beta \in \mathcal{B}} (T(\widehat{T_{\widetilde{p}_\beta}M}\_))^{H_{\widetilde{p}_\beta}(\widetilde{U}_\beta)}$ is a homomorphism of meromorphic open-string vertex algebras, the restriction map from V_U to $V_{\widetilde{U}}$ is also a homomorphism of meromorphic open-string vertex algebras.

Remark 3.1 For an open subset U of M, V_U is always nontrivial. In fact, the metric g can be viewed as an element of the space $\Gamma_U(T^2(T^*M))$ of smooth sections on U of the second symmetric tensor powers of the cotangent bundle T^*M of M. On the other hand, g also gives an isomorphism of vector bundles from T^*M to TM. It induces an isomorphism of vector bundles from $T^2(T^*M)$ to $T^2(TM)$, which in turn induces a linear isomorphism from $\Gamma_U(T^2(T^*M))$ to $\Gamma_U(T^2(TM))$. The image of the element $g \in \Gamma_U(T^2(T^*M))$ under this isomorphism is an element of $\Gamma_U(T^2(TM))$ and is denoted g^{-1}. Since g is parallel, g^{-1} is also parallel. For $k, l \in \mathbb{Z}_+$, the vector bundles $TM \otimes t^{-k}$ and $TM \otimes t^{-l}$ are isomorphic to TM. In particular, the space

$$\Gamma_U((TM \otimes t^{-k}) \otimes (TM \otimes t^{-l}))$$

of sections of the vector bundle

$$(TM \otimes t^{-k}) \otimes (TM \otimes t^{-l})$$

is isomorphic to the space $\Gamma_U(T^2(TM))$. In particular, $g^{-1} \in \Gamma_U(T^2(TM))$ corresponds to an element

$$g^{-1}(-k, -l) \in \Gamma_U((TM \otimes t^{-k}) \otimes (TM \otimes t^{-l}).$$

Since g^{-1} is in fact parallel, that is, $g^{-1} \in \Pi_U(T^2(TM))$, and the connection on $(TM \otimes t^{-k}) \otimes (TM \otimes t^{-l})$ is obtained from the connection on $T^2(TM)$, $g^{-1}(-k, -l)$ is also parallel, that is,

$$g^{-1}(-k, -l) \in \Pi_U((TM \otimes t^{-k}) \otimes (TM \otimes t^{-l})) \subset \Pi_U(T(\widehat{TM}_-)) = V_U$$

for $k, l \in \mathbb{Z}_+$, giving infinitely many nonzero elements of V_U of different weights.

Remark 3.2 It is well known that for $p \in M$, the symmetric algebra $S(\widehat{T_pM}_-)$ has a natural structure of a vertex operator algebra. These symmetric algebras form a vector bundle $S(\widehat{TM}_-)$ of vertex operator algebras with a connection. The same construction as the one for $\mathcal{V}$ above shows that the space $\Pi_U(S(\widehat{TM}_-))$ of parallel sections of $S(\widehat{TM}_-)$ on an open subset U of M form a sheaf of conformal vertex algebras such that when U is connected, $\Pi_U(S(\widehat{TM}_-))$ is a vertex operator algebra. From [3, Remark 5.2], for $p \in M$, we have a homomorphism of meromorphic open-string vertex algebras from $T(\widehat{T_pM}_-)$ to $S(\widehat{T_pM}_-)$. Thus we have a homomorphism of vector bundles from $T(\widehat{TM}_-)$ to $S(\widehat{TM}_-)$ such that the connection on $T(\widehat{TM}_-)$ is mapped to the connection on $S(\widehat{TM}_-)$. In particular, we have a homomorphism of sheaves of meromorphic open-string vertex algebras from the sheaf $\mathcal{V}$ to the sheaf $\Pi_U(S(\widehat{TM}_-))$ of parallel sections of the vector bundle $S(\widehat{TM}_-)$.

4 Covariant Derivatives and Parallel Tensor Fields

Given an open subset U of M, let $C^\infty(U)$ be the space of smooth functions on U. For $m \in \mathbb{N}$, let $T^m(TM)$ be the m-th tensor power of the tangent bundle TM and $\Gamma_U(T^m(TM))$ the space of sections of $T^m(TM)$. Then $\Gamma_U(T(TM))$ is the coproduct of $\Gamma_U(T^m(TM))$. Given $f \in C^\infty(U)$, there is an m-th order covariant derivative $\nabla^m f$ which can be viewed as a $(0, m)$-tensor. Since $\nabla^m f$ is a $(0, m)$ tensor, we can view it as a module map from the $C^\infty(U)$-module $\Gamma_U(T^m(TM))$ to the $C^\infty(U)$-module $C^\infty(U)$. Since $\nabla^m f$ is linear in f, we can view ∇^m as a linear map from $C^{\infty}(U)$ to $\mathrm{Hom}_{C^{\infty}(U)}(\Gamma_U(T^m(TM)), C^{\infty}(U))$. Since such a map corresponds to a linear map from $\Gamma_U(T^m(TM))$ to $L(C^{\infty}(U))$, we have a linear map

$$\psi_U^m : \Gamma_U(T^m(TM)) \to L(C^{\infty}(U))$$

corresponding to ∇^m, where $L(C^{\infty}(U))$ is the space of all linear operators on $C^{\infty}(U)$. By definition, for $\mathcal{X} \in \Gamma_U(T^m(TM))$,

$$(\psi_U^m(\mathcal{X}))f = (\nabla^m f)(\mathcal{X}).$$

The linear maps ψ_m for $m \in \mathbb{N}$ give a single linear map

$$\psi_U : \Gamma_U(T(TM)) \to L(C^{\infty}(U)).$$

As we mentioned in the preceding section, $\Gamma_U(T(TM))$ is an associative algebra. The space $L(C^{\infty}(U))$ is in fact also an associative algebra. But in general, the isomorphism ψ is not an isomorphism of associative algebras. The associative algebra $\Gamma_U(T(TM))$ has a subalgebra $\Pi_U(T(TM))$.

Let

$$\phi_U : \Pi_U(T(TM)) \to L(C^{\infty}(U))$$

be the restriction of ψ_U to $\Pi_U(T(TM))$. Then we have the following theorem.

Theorem 4.1 *For $\mathcal{X} \in \Gamma_U(T(TM))$ and $\mathcal{Y} \in \Pi_U(T(TM))$, we have*

$$\psi_{\widetilde{U}}(\mathcal{X} \otimes \mathcal{Y}) = \psi_U(\mathcal{X})\psi_U(\mathcal{Y}). \tag{4.1}$$

In particular, the linear map ϕ_U is a homomorphism of associative algebras and gives $C^{\infty}(U)$ a $\Pi_U(T(TM))$-module structure.

Proof We need only prove (4.1) for $m, l \in \mathbb{N}$, $\mathcal{X} \in \Gamma_U(T(TM))$ and $\mathcal{Y} \in \Pi_U(T(TM))$. We use induction on m. When $m = 0$, (4.1) certainly holds. Now assume that when $m = k$, (4.1) holds. To prove (4.1) in the case $m = k+1$, we need only prove that for $f \in C^{\infty}(U)$ and $p \in U$,

$$(\psi_{\widetilde{U}}(\mathcal{X} \otimes \mathcal{Y})f)(p) = (\psi_U(\mathcal{X})\psi_U(\mathcal{Y})f)(p). \tag{4.2}$$

For $p \in U$, there exists an open subset $\widetilde{U}$ of U containing p such that the restriction $\mathcal{X}|_{\widetilde{U}}$ of $\mathcal{X}$ to $\widetilde{U}$ is a sum of elements of the form $X \otimes \widetilde{\mathcal{X}}$ for $X \in \Gamma_{\widetilde{U}}(TM)$ and $\widetilde{\mathcal{X}} \in \Pi_{\widetilde{U}}(TM)$. Hence we can prove (4.2) for those $\mathcal{X}$ such that

$$\mathcal{X}|_{\widetilde{U}} = X \otimes \widetilde{\mathcal{X}}$$

for $X \in \Gamma_{\widetilde{U}}(TM)$ and $\widetilde{\mathcal{X}} \in \Pi_{\widetilde{U}}(TM)$. In this case for $f \in C^{\infty}(U)$, by definition,

$$\begin{aligned}
&(\psi_{\widetilde{U}}(\mathcal{X}|_{\widetilde{U}})\psi_{\widetilde{U}}(\mathcal{Y}|_{\widetilde{U}}))f \\
&= (\psi_{\widetilde{U}}(\mathcal{X}|_{\widetilde{U}}))((\psi_{\widetilde{U}}(\mathcal{Y}|_{\widetilde{U}}))f)
\end{aligned}$$

$$
\begin{aligned}
&= (\nabla^{k+1}((\psi_{\widetilde{U}}(\mathcal{Y}|_{\widetilde{U}}))f))(X \otimes \widetilde{\mathcal{X}})\\
&= X((\nabla^{k}((\psi_{\widetilde{U}}(\mathcal{Y}|_{\widetilde{U}}))f))(\tilde{\mathcal{X}})) - (\nabla^{k}((\psi_{\tilde{U}}(\mathcal{Y}|_{\tilde{U}})f))(\nabla_X\tilde{\mathcal{X}})\\
&= X((\psi_{\tilde{U}}(\tilde{\mathcal{X}}))((\psi_{\tilde{U}}(\mathcal{Y}|_{\tilde{U}})f)) - (\psi_{\tilde{U}}(\nabla_X\tilde{\mathcal{X}}))((\psi_{\tilde{U}}(\mathcal{Y}|_{\tilde{U}}))f)\\
&= X((\psi_{\widetilde{U}}(\widetilde{\mathcal{X}})\psi_{\widetilde{U}}(\mathcal{Y}|_{\widetilde{U}}))f) - (\psi_{\widetilde{U}}(\nabla_X\widetilde{\mathcal{X}})\psi_{\widetilde{U}}(\mathcal{Y}|_{\widetilde{U}}))f. \qquad (4.3)
\end{aligned}
$$

By the induction assumption, the right-hand side of (4.3) is equal to

$$X((\psi_{\widetilde{U}}(\widetilde{\mathcal{X}} \otimes \mathcal{Y}|_{\widetilde{U}}))f) - (\psi_{\widetilde{U}}((\nabla_X\widetilde{\mathcal{X}}) \otimes \mathcal{Y}|_{\widetilde{U}}))f. \qquad (4.4)$$

Since $\mathcal{Y}$ is parallel, we have

$$\nabla_X\mathcal{Y}|_{\widetilde{U}} = 0$$

and thus

$$(\nabla_X\widetilde{\mathcal{X}}) \otimes \mathcal{Y}|_{\widetilde{U}} = \nabla_X(\widetilde{\mathcal{X}} \otimes \mathcal{Y}|_{\widetilde{U}}). \qquad (4.5)$$

Using (4.5), (4.4) becomes

$$
\begin{aligned}
&X((\psi_{\widetilde{U}}(\widetilde{\mathcal{X}} \otimes \mathcal{Y}|_{\widetilde{U}}))f) - (\psi_{\widetilde{U}}(\nabla_X(\widetilde{\mathcal{X}} \otimes \mathcal{Y}|_{\widetilde{U}})))f\\
&= X((\nabla^{k+l}f)(\widetilde{\mathcal{X}} \otimes \mathcal{Y}|_{\widetilde{U}})) - (\nabla^{k+l}f)(\nabla_X(\widetilde{\mathcal{X}} \otimes \mathcal{Y})|_{\widetilde{U}})\\
&= (\nabla^{k+1+l}f)(X \otimes \widetilde{\mathcal{X}} \otimes \mathcal{Y}|_{\widetilde{U}})\\
&= (\psi_{\widetilde{U}}(\mathcal{X}|_{\widetilde{U}} \otimes \mathcal{Y}|_{\widetilde{U}}))f. \qquad (4.6)
\end{aligned}
$$

The calculations from (4.3) to (4.6) show that the left-hand side of (4.3) is equal to the right-hand side of (4.6). In particular, the value of the left-hand side of (4.3) at p is equal to the value of the right-hand side of (4.6) at p. But the value of the left-hand side of (4.3) at p is equal to the right-hand side of (4.2) and the value of the right-hand side of (4.6) at p is equal to the left-hand side of (4.2). Thus (4.1) holds. Since p and f are arbitrary, (4.1) in the case $m = k + 1$ is proved.

5 A Sheaf $\mathcal{W}$ of Modules for $\mathcal{V}$ Generated by the Sheaf of Smooth Functions on M

In this section, we construct a sheaf $\mathcal{W}$ modules for the sheaf $\mathcal{V}$ of meromorphic open-string vertex algebras in the preceding section from the sheaf C^∞ of smooth functions on M.

Let U be an open subset of M. For simplicity, we discuss only the case that U is connected. The general case is similar. By Theorem 4.1, $C^\infty(U)$ is a $\Pi_U(T(TM))$-module. For $p \in U$, by Proposition 2.1, $\Pi_U(T(TM))$ is isomorphic to $(T(T_pM))^{H_p(U)}$. We shall identify $\Pi_U(T(TM))$ with $(T(T_pM))^{H_p(U)}$. In particular, $C^\infty(U)$ is a $(T(T_pM))^{H_p(U)}$-module. Since $(T(T_pM))^{H_p(U)}$ is a subalgebra of $T(T_pM)$, we have the induced $T(T_pM)$-module

$$C_p(U) = T(T_pM) \otimes_{(T(T_pM))^{H_p(U)}} C^\infty(U).$$

By Theorems 5.1 (or Proposition 3.1 in Section 3 above) and 6.5 in [3], $T(\widehat{T_pM}\_)$ has a natural structure of meromorphic open-string vertex algebra and $T(\widehat{T_pM}\_)\otimes C(U)$ has a natural structure of left $T(\widehat{T_pM}\_)$-module. By Corollary 3.2, $(T(\widehat{T_pM}\_))^{H_p(U)}$ is a meromorphic open-string vertex subalgebra of $T(\widehat{T_pM}\_)$. In particular, $T(\widehat{T_pM}\_)\otimes C_p(U)$ is also a left $(T(\widehat{T_pM}\_))^{H_p(U)}$-module. Let W_U be the left $(T(\widehat{T_pM}\_))^{H_p(U)}$-submodule of $T(\widehat{T_pM}\_)\otimes C_p(U)$ generated by elements of the form $1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f)$ for $f\in C^\infty(U)$, where $1\otimes_{(T(T_pM))^{H_p(U)}} f$ is the image of $1\otimes f$ under the projection from $T(T_pM)\otimes C^\infty(U)$ to $C_p(U)$. By Theorem 3.1, the meromorphic open-string vertex subalgebra $(T(\widehat{T_pM}\_))^{H_p(U)}$ is canonically isomorphic to $V_U=\Pi_U(T(\widehat{TM}_-))$. We shall identify $(T(\widehat{T_pM}\_))^{H_p(U)}$ and V_U. Thus W_U has a natural structure of left V_U-module.

The construction of W_U here depends on p. But W_U is in fact independent of p. Let q be another point in U. Then the subspace of $T(\widehat{T_pM}\_)\otimes C_p(U)$ consisting of elements of the form $1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f)$ for $f\in C^\infty(U)$ is canonically isomorphic to the subspace of $T(\widehat{T_qM}\_)\otimes C_q(U)$ consisting of elements of the form $1\otimes(1\otimes_{(T(T_qM))^{H_q(U)}} f)$ for $f\in C^\infty(U)$. Also both the meromorphic open-string vertex algebras $(T(\widehat{T_pM}\_))^{H_p(U)}$ and $(T(\widehat{T_qM}\_))^{H_q(U)}$ are canonically isomorphic to the meromorphic open-string vertex algebra $\Pi_U(T(\widehat{TM}_-))=V_U$. Thus the left V_U-module generated by elements of the form $1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f)$ for $f\in C^\infty(U)$ is isomorphic to the left V_U-module generated by elements of the form $1\otimes(1\otimes_{(T(T_qM))^{H_q(U)}} f)$ for $f\in C^\infty(U)$. Since the subspace of $T(\widehat{T_pM}\_)\otimes C_p(U)$ consisting of elements of the form $1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f)$ for $f\in C^\infty(U)$ is canonically isomorphic to $C^\infty(U)$, we can view W_U as a canonical left V_U-module generated by $C^\infty(U)$. We have proved the following result in the case that U is connected; the general case can be obtained using direct products as in the case of open-string vertex algebras in Section 3.

Theorem 5.1 *For an open subset U of M and $p\in U$, W_U has a natural structure of left V_U-module. For different choices of p, we obtain canonically isomorphic left V_U-modules. In particular, we have a canonical left V_U-module W_U generated by $C^\infty(U)$.*

In Theorem 5.1, taking $U=M$, we obtain the following corollary.

Corollary 5.1 *The space $C^\infty(M)$ on M generates a canonical left V_M-module.*

Let V_1 and V_2 be meromorphic open-string vertex algebras, and W_1 and W_2 be left V_1- and V_2-modules, respectively. Let $f: V_1\to V_2$ be a homomorphism of meromorphic open-string vertex algebras. Then W_2 is also a left V_1-module. A homomorphism from W_1 to W_2 associated to f is a homomorphism from W_1 to W_2 as a left V_1-module.

By definition, W_U is the left $(T(\widehat{T_pM}\_))^{H_p(U)}$-submodule of $T(\widehat{T_pM}\_)\otimes C(U)$ generated by elements of the form $1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f)$ for $f\in C^\infty(U)$. For open subsets U and $\widetilde{U}$ such that $\widetilde{U}\subset U$, we have a restriction map from V_U to $V_{\widetilde{U}}$ which corresponding to a restriction map from $(T(\widehat{T_pM}\_))^{H_p(U)}$ to $(T(\widehat{T_pM}\_))^{H_p(\widetilde{U})}$. Using the restriction map from $(T(\widehat{T_pM}\_))^{H_p(U)}$ to $(T(\widehat{T_pM}\_))^{H_p(\widetilde{U})}$ and the restriction map from $C^\infty(U)$ to $C^\infty(\widetilde{U})$, we obtain a restriction map from W_U to $W_{\widetilde{U}}$. By definition, we obtain the following theorem.

Theorem 5.2 *For open subsets U and $\widetilde{U}$ such that $\widetilde{U} \subset U$, the restriction map from W_U to $W_{\widetilde{U}}$ is a homomorphism from W_U to $W_{\widetilde{U}}$ associated with the restriction map from V_U to $V_{\widetilde{U}}$. In particular, $\mathcal{W}$ is a sheaf of $\mathcal{V}$-modules.*

6 The Laplacian on M as a Component of a Vertex Operator

Many conjectures on geometry were obtained from quantum field theory by interpreting some geometric or analytic objects as quantum-field-theoretic objects. In this section, as an example, we show that the Laplacian of M is in fact a component of a vertex operator for W_M acting on $C^\infty(M)$.

Let $\{E_i\}_{i=1}^n$ be an orthonormal frame in an open neighborhood U of a point $p \in M$. Recall the element $g^{-1}(-1,-1) \in V_U$. Then in U,

$$g^{-1}(-1,-1) = \sum_{i=1}^n (E_i \otimes t^{-1}) \otimes (E_i \otimes t^{-1}).$$

We identify $V_U = \Pi_U(T(\widehat{TM}_-))$ with $(T(\widehat{T_pM}_-))^{H_p(U)}$. Under this identification, $g^{-1}(-1,-1)$ is identified with

$$\sum_{i=1}^n (E_i|_p \otimes t^{-1}) \otimes (E_i|_p \otimes t^{-1}) = \sum_{i=1}^n (E_i|_p)(-1)(E_i|_p)(-1)\mathbf{1},$$

where $\mathbf{1}_{T(\widehat{T_pM}_-)}$ is the vacuum of the meromorphic open-string vertex algebra $T(\widehat{T_pM}_-)$ and $(E_i|_p)(-1)$ is the representation image of $E_i|_p \otimes t^{-1}$ on $T(\widehat{T_pM}_-)$.

Recall that W_U by definition is the left $(T(\widehat{T_pM}_-))^{H_p(U)}$-submodule of $T(\widehat{T_pM}_-) \otimes C_p(U)$ generated by elements of the form $1 \otimes (1 \otimes_{(T(T_pM))^{H_p(U)}} f)$ for $f \in C^\infty(U)$. Let $f \in C^\infty(U)$. Consider

$$Y_{W_U}(g^{-1}(-1,-1), x)(1 \otimes (1 \otimes_{(T(T_pM))^{H_p(U)}} f)).$$

Since W_U is a $(T(\widehat{T_pM}_-))^{H_p(U)}$-submodule of $T(\widehat{T_pM}_-) \otimes C_p(U)$ and the $(T(\widehat{T_pM}_-))^{H_p(U)}$-module $T(\widehat{T_pM}_-) \otimes C_p(U)$ is induced from the $T(\widehat{T_pM}_-)$-module structure on $T(\widehat{T_pM}_-) \otimes C_p(U)$, the vertex operator map Y_{W_U} is the restriction of the vertex operator map $Y_{T(\widehat{T_pM}_-)}$ to $(T(\widehat{T_pM}_-))^{H_p(U)} \otimes W_U$. In particular,

$$\begin{aligned} Y_{W_U}(g^{-1}(-1,-1), x) &= Y_{T(\widehat{T_pM}_-)}(g^{-1}(-1,-1), x) \\ &= \sum_{i=1}^n Y_{T(\widehat{T_pM}_-)}((E_i|_p)(-1)(E_i|_p)(-1)\mathbf{1}, x) \\ &= \sum_{i=1}^n {}^{\circ}_{\circ}(E_i|_p)(x)(E_i|_p)(x){}^{\circ}_{\circ}. \end{aligned}$$

Then the coefficient of the x^{-2} term of $Y_{W_U}(g^{-1}(-1,-1), x)$ is

$$\sum_{i=1}^n \sum_{k \in \mathbb{Z}} {}^{\circ}_{\circ}(E_i|_p)(-k)(E_i|_p)(k){}^{\circ}_{\circ}$$

$$= 2\sum_{i=1}^{n}\sum_{k\in\mathbb{Z}_+}(E_i|_p)(-k)(E_i|_p)(k) + \sum_{i=1}^{n}(E_i|_p)(0)(E_i|_p)(0).$$

This coefficient or component of the vertex operator $Y_{W_U}(g^{-1}(-1,-1),x)$ acting on $(1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f))$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{n}(E_i|_p)(0)(E_i|_p)(0)(1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f))\\
&= 1\otimes\Big(\sum_{i=1}^{n}(E_i|_p\otimes E_i|_p)\otimes_{(T(T_pM))^{H_p(M)}} f\Big)\\
&= 1\otimes\Big(1\otimes_{(T(T_pM))^{H_p(U)}}\Big(\phi\Big(\sum_{i=1}^{n}E_i\otimes E_i\Big)f\Big)\Big)\\
&= 1\otimes\Big(1\otimes_{(T(T_pM))^{H_p(U)}}\Big((\nabla^2 f)\Big(\sum_{i=1}^{n}E_i\otimes E_i\Big)\Big)\Big)\\
&= 1\otimes\Big(1\otimes_{(T(T_pM))^{H_p(U)}}\Big(\sum_{i=1}^{n}\nabla^2_{E_i,E_i}f\Big)\Big)\\
&= 1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}}\Delta f).
\end{aligned}$$

Thus we see that this component of the vertex operator $Y_{W_U}(g^{-1}(-1,-1),x)$ acting on $(1\otimes(1\otimes_{(T(T_pM))^{H_p(U)}} f))$ can be interpreted as the Laplacian Δ acting on the smooth function f on U.

Since $\mathcal{V}$ and $\mathcal{W}$ are sheaves, the conclusion above for a small open neighborhood of every $p\in M$ implies that the same conclusion holds for any open subset U of M and $f\in C^\infty(U)$. In particular, the component of the vertex operator $Y_{W_M}(g^{-1}(-1,-1),x)$ acting on $(1\otimes(1\otimes_{(T(T_pM))^{H_p(M)}} f))$ is equal to $1\otimes(1\otimes_{(T(T_pM))^{H_p(M)}}\Delta f)$ and corresponds to the Laplacian Δ acting on the smooth function f defined on M.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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