Chin. Ann. Math. Ser. B
47(1), 2026, 145–156
DOI: 10.1007/s11401-026-0011-1

Chinese Annals of Mathematics, Series B

A Note on Vanishing Theorems on Non-pseudoconvex Complex Manifolds*

Jun YAO[1] Qingchun JI[1]

Dedicated to the memory of Professor Hesheng HU

Abstract In this paper, the authors introduce a Morse-theoretic condition under which the Levi form is allowed to have negative eigenvalues outside critical locus, and show that the existence of an exhaustion function satisfying such a condition leads to vanishing theorems.

Keywords Levi form, Vanishing theorems, Dolbeault cohomology, Non-pseudoconvex
2020 MR Subject Classification 32F10, 32F32, 32W05

1 Introduction

Various notions of q-convexity have been extensively studied in complex analysis and complex geometry since Rothstein [15] first introduced the concept in 1955. Andreotti-Grauert [1] established a celebrated finiteness theorem for any coherent analytic sheaf over q-convex spaces, which generalized a result of Grauert [5]. In [8], Ho defined a notion of q-subharmonicity for a function in terms of the trace of its complex Hessian restricted to a q-dimensional space, in order to study the $\overline{\partial}$-problem in a more general setting than pseudoconvex domains. Zampieri [17–18] also considered the q-pseudoconvexity to explore the regularity at the boundary for solutions of the $\overline{\partial}$-problem. For a comprehensive overview of the q-convexity conditions, we refer the reader to [6, 14] and the references therein.

Given a complex manifold M of dimension n, we recall that the Hermitian form of a real valued function $\varphi \in C^2(M)$ at a point $z \in M$ is defined as

$$\varphi_{i\overline{j}}(z) := \sqrt{-1}\partial\overline{\partial}\varphi(L_i, \overline{L}_j), \tag{1.1}$$

where $\{L_j\}_{j=1}^n \subseteq T_z^{1,0}M$ is a basis. We are interested in the vanishing theorems for Dolbeault cohomology groups $H^{p,q}(M)$ in bidegree (p, q), relying on some refined q-convexity conditions due to Hörmander [9] and Andreotti-Grauret [1] as follows.

Definition 1.1 *Let $\varphi \in C^2(M)$ be a real valued function and $z_0 \in M$.*

Manuscript received February 22, 2025.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: jyao_fd@fudan.edu.cn qingchunji@fudan.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 12431005, 12571086).

(1) *We shall say that* φ *satisfies the condition* a_q^+ (*or* a_q^-) *at* z_0, *if* $\mathrm{grad}\varphi(z_0) \neq 0$ *and the form* (1.1) *restricted to the space* $T_{z_0}^{1,0}M \cap \mathrm{Ker}(\mathrm{d}\varphi)$ *has at least* $n-q$ *positive eigenvalues* (*respectively at least* $q+1$ *negative eigenvalues*)*. Furthermore,* φ *is said to fulfill the condition* a_q (*which Folland-Kohn* [4] *also refers to as the condition* $Z(q)$) *at* z_0*, if* φ *satisfies either the condition* a_q^+ *or the condition* a_q^- *at* z_0*.*

(2) *The condition* G_q^+(*or* G_q^-) *for* φ *at* z_0 *means that the form* (1.1) *has at least* $n-q+1$ *positive eigenvalues* (*respectively at least* $q+1$ *negative eigenvalues*)*. Additionally,* φ *is said to fulfill the condition* G_q *at* z_0*, if* φ *satisfies either the condition* G_q^+ *or the condition* G_q^- *at* z_0*.*

It is clear that functions satisfying the condition G_1^+ are precisely the smooth strictly plurisubharmonic functions, and a manifold with a smooth boundary is strongly pseudoconvex indicates that the defining function satisfies the condition a_1^+ on the boundary. The condition a_q^- and the condition G_q^- enable the Hermitian form (1.1) to exhibit pure negativity, and these conditions are closely related to the concept of q-concavity. Moreover, for $1 \leq q \leq n-1$,

$$\begin{cases} \text{condition } a_q^+ \text{ (or } G_q^+) \Rightarrow \text{condition } a_{q+1}^+ \text{ (respectively } G_{q+1}^+), \\ \text{condition } G_q^- \Rightarrow \text{condition } G_{q-1}^- \end{cases} \tag{1.2}$$

and

$$\text{condition } a_q^- \Rightarrow \text{condition } a_{q-1}^- \quad \text{for } 1 \leq q \leq n-2.$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the quadratic form (1.1) and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of the same form restricted to the space $T_{z_0}^{1,0}M \cap \mathrm{Ker}(\mathrm{d}\varphi)$. According to the minimum-maximum principle for the eigenvalues, we have

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \lambda_q \leq \mu_q \leq \lambda_{q+1} \leq \mu_{q+1} \leq \cdots \leq \mu_{n-1} \leq \lambda_n,$$

which implies that if $\mathrm{grad}\,\varphi \neq 0$ at z_0, then

$$\begin{cases} \text{condition } G_q^+ \Rightarrow \text{condition } a_q^+ \quad \text{for } 1 \leq q \leq n, \\ \text{condition } a_q^- \Rightarrow \text{condition } G_q^- \quad \text{for } 0 \leq q \leq n-2. \end{cases} \tag{1.3}$$

In [9], Hörmander proved that the condition a_q is equivalent to the basic estimate, which is used to establish the regularity of the $\overline{\partial}$-problem on the strongly pseudoconvex manifolds with smooth boundary by Morrey [13] and Kohn [11–12]. There are many notable developments on this topic; see [7, 16–18] and the references therein. In contrast to this direction, this paper is devoted to establishing several existence results for the $\overline{\partial}$-problem using the condition a_q. Denote the set of critical points of φ by

$$\mathcal{C}_\varphi := \{z \in M \mid \mathrm{d}\varphi(z) = 0\}. \tag{1.4}$$

Inspired by (1.3), we formulate our main result as follows.

Theorem 1.1 *Let* $1 \leq q \leq n$*. Assume that* M *has an exhaustion function* $\varphi \in C^\infty(M)$ *satisfying the condition* G_q^+ *on* $\mathcal{C}_\varphi$ *and the condition* a_q *outside* $\mathcal{C}_\varphi$*. Then* $H^{p,q}(M) = 0$*.*

Based on the relations (1.2), Theorem 1.1 allows us to give a condition that ensures the vanishing of not only the qth cohomology, but also the kth cohomology for $k \geq q$.

Corollary 1.1 *Let $1 \leq q \leq n$. Suppose that M admits an exhaustion function $\varphi \in C^\infty(M)$ satisfying the condition G_q^+ on $\mathcal{C}_\varphi$ and the condition a_q^+ outside $\mathcal{C}_\varphi$. It follows that $H^{p,k}(M) = 0$ for $k \geq q$.*

In particular, on the further assumption that φ satisfies the condition G_q^+ outside $\mathcal{C}_\varphi$, by means of the Dolbeault theorem, Corollary 1.1 recovers Andreotti-Grauert Theorem (see [1]) for the sheaf of germs of holomorphic p-forms on q-complete manifolds (a q-complete manifold M means that there is a smooth exhaustion function with condition G_q^+ on M).

Corollary 1.2 *Let $1 \leq q \leq n$. Assume that M has an exhaustion function $\varphi \in C^\infty(M)$ satisfying the condition G_q^+ on M. Then $H^{p,k}(M) = 0$ for $k \geq q$.*

In comparison to Theorem 1.1, if we assume that $\mathcal{C}_\varphi$ is compact, Hörmander provided an alternative eigenvalue condition to guarantee the vanishing of the cohomology in [9]. Here is a variant of his theorem.

Theorem 1.2 *Let $1 \leq q \leq n$, and $\varphi \in C^\infty(M)$ be an exhaustion function. We assume that $\mathcal{C}_\varphi \subseteq M_{t_1} \subset\subset M_{t_0}$, where $M_t := \{z \in M \mid \varphi(x) < t\}$ for $t \in \mathbb{R}$ and that φ satisfies*

(1) *the condition G_q on $\overline{M}_{t_1}$ and the condition a_q outside M_{t_0},*

(2) *the condition G_q^+ on $M_{t_0} \setminus \overline{M}_{t_1}$,*

then $H^{p,q}(M) = 0$.

We will present a direct proof of Theorem 1.2 by L^2-estimates, which differs from Hörmander's original approach.

2 Preliminaries

To prove Theorems 1.1–1.2, we first require some preliminary results. The following refined proposition originally due to Andreotti-Vesentine in [2] (see also [3]), plays a crucial role in our argument.

Proposition 2.1 *Let $\varphi \in C^\infty(M)$ fulfill the condition G_q^+ (or G_q^-) on a closed subset E of M, then for any number $\varepsilon > 0$, there exists a Hermitian metric ω on M such that the eigenvalues of the quadratic form (1.1) (respectively the form (1.1) for $-\varphi$) with respect to ω satisfy $\lambda_1 \geq -\varepsilon$ and $\lambda_q = \cdots = \lambda_n = 1$ (respectively $\lambda_n \leq \varepsilon$ and $\lambda_{q+1} = \cdots = \lambda_1 = -1$) on E.*

Proof The proof for the case where φ satisfies the condition G_q^+ is parallel to that of the case where φ satisfies the condition G_q^-, thus we will focus exclusively on the latter case. Our arguments essentially follow those in Lemma 3.1 of Chapter IX in [3].

Let ω_0 be a fixed Hermitian metric, and $-\lambda_n^0 \leq \cdots \leq -\lambda_1^0$ be the eigenvalues of A^0 with respect to the metric ω_0, where A^0 is the Hermitian endomorphism associated to $\sqrt{-1}\partial\overline{\partial}\varphi$ with respect to ω_0. We can select a function $0 < \eta \in C^\infty(M)$ such that for $z \in E$,

$$\eta(z) \leq -\lambda_{q+1}^0(z). \tag{2.1}$$

Indeed, since $-\lambda_{q+1}^0 > 0$ on E, there is an open neighborhood U of E with $-\lambda_{q+1}^0 > 0$ on U. In view of the Whitney approximation theorem, one can find a function $0 < \eta_1 \in C^\infty(U)$ satisfying (2.1) on U. It is clear that $\{U, M \setminus E\}$ forms an open covering of M because E is closed. Let $\{\psi, 1-\psi\}$ be a smooth partition of unity subordinate to the covering $\{U, M \setminus E\}$. Then

$$\eta := \psi\eta_1 + 1 - \psi$$

fulfills the desired property. We then choose a positive function $\theta \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} \theta(t) &\geq \frac{|t|}{\varepsilon} && \text{for } t \leq 0, \\ \theta(t) &\geq t && \text{for } t \geq 0, \\ \theta(t) &= t && \text{for } t \geq 1. \end{aligned}$$

Let ω be the Hermitian metric defined by the following Hermitian endomorphism:

$$A(z) := \eta(z)\theta[(\eta(z))^{-1}A^0(z)].$$

Thus, the eigenvalues of $A(z)$ are $\alpha_{n-j+1}(z) = \eta(z)\theta\big(\frac{-\lambda_j^0(z)}{\eta(z)}\big) > 0$ and we have

$$\begin{aligned} \alpha_{n-j+1}(z) &\geq \frac{|\lambda_j^0(z)|}{\varepsilon}, && \text{if } -\lambda_j^0(z) \leq 0, \\ \alpha_{n-j+1}(z) &\geq -\lambda_j^0(z), && \text{if } -\lambda_j^0(z) \geq 0, \\ \alpha_{n-j+1}(z) &= -\lambda_j^0(z), && \text{if } j \leq q+1 \text{ and } z \in E. \end{aligned}$$

By construction, the eigenvalues of the quadratic form (1.1) for $-\varphi$ with respect to ω are $-\lambda_j(z) = -\frac{\lambda_j^0(z)}{\alpha_{n-j+1}(z)}$, and they have the required properties.

Remark 2.1 According to Proposition 2.1, a smooth function φ satisfies the condition G_q^+ on a manifold means that φ is q-subharmonic with respect to some metric ω in the sense of Ho [8] (i.e., the sum of any q eigenvalues of the Hermitian form for φ is positive with respect to ω).

The weighted L^2-space $L^2_{p,q}(M,\phi)$ for a real valued function $\phi \in C(M)$ is given by

$$L^2_{p,q}(M,\phi) = \Big\{ f \in L^2_{p,q}(M, \mathrm{loc}) \ \Big| \ \int_M |f(x)| \mathrm{e}^{-\phi} < +\infty \Big\}.$$

The notation $(\cdot,\cdot)_\phi$ stands for the inner product on $L^2_{p,q}(M,\phi)$. We will apply the following lemma to

$$L^2_{p,q-1}(M,\phi) \xrightarrow{T} L^2_{p,q}(M,\phi) \xrightarrow{S} L^2_{p,q+1}(M,\phi),$$

where T, S are the closed densely defined extensions of $\overline{\partial}$ in the weak sense, and ϕ will be determined in the sequel.

Lemma 2.1 (see [9]) *Let $T: H_1 \to H_2$ and $S: H_2 \to H_3$ be closed, densely defined linear operators such that* $\mathrm{Im}(T) \subseteq \mathrm{Ker}(S)$. *If there exists a constant $C > 0$ such that*

$$\|g\|^2_{H_2} \leq C^2(\|T^*g\|^2_{H_1} + \|Sg\|^2_{H_3}), \quad g \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S). \tag{2.2}$$

Then for any $f \in \mathrm{Ker}(S)$, one can find $u \in H_1$ such that $Tu = f$ and $\|u\|_{H_1} \leq C\|f\|_{H_2}$.

3 Proofs of Theorems 1.1–1.2

First, we recall basic facts from [9] about L^2-estimates on a Hermitian manifold M.

Proposition 3.1 *Let $U \subseteq M$ be a coordinate patch, and let $\{\omega^1, \cdots, \omega^n\}$ be a local orthonormal frame for forms of type $(1,0)$ in U and $\varphi \in C^2(M, \mathbb{R})$. Then for any positive number $\varepsilon < 1$, any $f \in \mathcal{D}_{p,q}(U)$ with support in a fixed compact subset of U, we have*

$$(1+\varepsilon)(\|T^*f\|_\varphi^2 + \|Sf\|_\varphi^2) \geq (1-\varepsilon) \sum_{I,J} \sum_{j=1}^n \int_U \Big| \frac{\partial f_{I,J}}{\partial \overline{\omega}_j} \Big|^2 \mathrm{e}^{-\varphi} + \sum_{I,K} \sum_{j,k=1}^n \int_U \varphi_{j\overline{k}} f_{I,jK} \overline{f_{I,kK}} \mathrm{e}^{-\varphi} + \int_U C_\varepsilon(z) |f|^2 \mathrm{e}^{-\varphi}, \tag{3.1}$$

where summations are extended to strictly increasing multi-indices, $\mathcal{D}_{p,q}(U)$ denotes the space of smooth (p,q)-forms with compact support in U and $C_\varepsilon(z) \in C^0(U)$ is independent of f.

The notation $\frac{\partial}{\partial \overline{\omega}_j}$ in the above proposition is consistent with that used by Hörmander in [9], and we will adopt the same notation in what follows. In order to make another integration by parts for the gradient term in (3.1), we shall use the following proposition.

Proposition 3.2 *Let $U \subseteq M$ be a coordinate patch and $\varphi \in C^2(M, \mathbb{R})$. Then for any function $w \in C_c^2(U)$ vanishing outside a fixed compact subset of U, we have*

$$\int_U \Big| \frac{\partial w}{\partial \overline{\omega}_j} \Big|^2 \mathrm{e}^{-\varphi} \geq - \int_U \varphi_{j\overline{j}} |w|^2 \mathrm{e}^{-\varphi} + C \|w\|_\varphi |||w|||_\varphi,$$

where C is a constant independent of w and $|||w|||_\varphi^2 := \sum\limits_{j=1}^n \|\frac{\partial w}{\partial \overline{\omega}_j}\|_\varphi^2 + \|w\|_\varphi^2$.

To obtain the final a priori estimate (2.2), we need some lemmas.

Lemma 3.1 *Let $0 \leq q \leq n$, and suppose that M has an exhaustion function $\varphi \in C^\infty(M)$ satisfying the condition a_q outside a subset $E' \subseteq M$. Then for any open neighborhood Ω' of E' in M and any function $C(z) \in C^0(M)$, there exists a convex increasing function $\chi(t) \in C^\infty(\mathbb{R})$ such that, on $\Omega := M \setminus \overline{\Omega'}$, for every $f \in C_{p,q}^\infty(\Omega) \cap \mathcal{D}_{p,q}(M)$,*

$$\|T^*f\|_{\chi(\varphi)}^2 + \|Sf\|_{\chi(\varphi)}^2 \geq \int_\Omega (C(z)+1) |f|^2 \mathrm{e}^{-\chi(\varphi)}. \tag{3.2}$$

Proof Given $z_0 \in \Omega$, there is a coordinate patch U around z_0 such that $\operatorname{grad} \varphi \neq 0$ in U. We choose a local orthonormal frame $\{\omega^1, \cdots, \omega^n\}$ in U so that $\omega^n = \frac{\partial \varphi}{|\partial \varphi|}$, which yields $\varphi_j := \frac{\partial \varphi}{\partial \omega_j} = 0$ for $j < n$ and $\varphi_n = |\partial \varphi|$. By a unitary transformation of ω^j for $j < n$, we can attain that

$$\sum_{j,k=1}^{n-1} \varphi_{i\overline{j}}(z_0) t_j \overline{t}_k = \sum_{j=1}^{n-1} \mu_j(z_0) |t_j|^2,$$

where $\mu_j(z_0) < 0$ for $j \leq \sigma$ and $\mu_j(z_0) \geq 0$ for $j > \sigma$. According to Propositions 3.1–3.2, we know that for any $g \in \mathcal{D}_{p,q}(U)$ with support in a fixed compact subset of U and $0 < \varepsilon < \frac{1}{2}$,

$$(1+\varepsilon)(\|T^*g\|_\varphi^2 + \|Sg\|_\varphi^2)$$

$$\geq (1-\varepsilon)\sum_{I,J}\sum_{j=1}^{n}\int_U \Big|\frac{\partial g_{I,J}}{\partial\overline{\omega}_j}\Big|^2 \mathrm{e}^{-\varphi} + \sum_{I,K}\sum_{j,k=1}^{n}\int_U \varphi_{j\overline{k}} g_{I,jK}\overline{g_{I,kK}}\mathrm{e}^{-\varphi} + \int_U C_U^{\varepsilon}(z)|g|^2\mathrm{e}^{-\varphi}$$
$$\geq \int_U Q_{\varphi}^{\varepsilon}(z;g,g)\mathrm{e}^{-\varphi} + \int_U C_U^{\varepsilon}(z)|g|^2\mathrm{e}^{-\varphi}, \tag{3.3}$$

where $C_U^{\varepsilon}(z)$ denotes various continuous functions on U which are independent of g, and

$$Q_{\varphi}^{\varepsilon}(z;g,g) := \sum_{I,J}\sum_{j=1}^{\sigma} -(1-2\varepsilon)\varphi_{j\overline{j}}|g_{I,J}|^2 + \sum_{I,K}\sum_{j,k=1}^{n}\varphi_{j\overline{k}} g_{I,jK}\overline{g_{I,kK}}.$$

Replace the function φ in (3.3) by $\chi(\varphi)$, where $\chi \in C^{\infty}(\mathbb{R})$ is a convex increasing function to be determined in the sequel. This gives

$$(1+\varepsilon)(\|T^*g\|^2_{\chi(\varphi)} + \|Sg\|^2_{\chi(\varphi)}) \geq \int_U \Big(\chi'(\varphi)Q_{\varphi}^{\varepsilon}(z;g,g) + \chi''(\varphi)\sum_{I,K}|\varphi_n g_{I,nK}|^2\Big)\mathrm{e}^{-\chi(\varphi)}$$
$$+ \int_U C_U^{\varepsilon}(z)|g|^2\mathrm{e}^{-\chi(\varphi)}. \tag{3.4}$$

We may write

$$g = g^1 + g^2 := {\sum_{I,J}}' g^1_{I,J}\omega^I \wedge \overline{\omega}^J + {\sum_{I,J}}'' g^2_{I,J}\omega^I \wedge \overline{\omega}^J,$$

where the notations $\sum'$ and $\sum''$ mean that the summations only extend over strictly increasing multi-indices J with $n \notin J$ and J with $n \in J$, respectively. Note that $g^1 = 0$ when $q = n$. Subsequently, for any positive number ε', $Q_{\varphi}^{\varepsilon}(z;g,g)$ can be bounded from below by the sum of the following two terms:

$$Q_{\varphi}^{\varepsilon,\varepsilon'}(z;g^1,g^1) := {\sum_{I,J}}'\sum_{j=1}^{\sigma} -(1-2\varepsilon)\varphi_{j\overline{j}}|g^1_{I,J}|^2 + {\sum_{I,K}}'\sum_{j,k=1}^{n-1}\varphi_{j\overline{k}} g^1_{I,jK}\overline{g^1_{I,kK}} - \varepsilon'{\sum_{I,J}}'|g^1_{I,J}|^2$$

and

$$Q_{\varphi}^{\varepsilon,\varepsilon'}(z;g^2,g^2) := {\sum_{I,J}}''\sum_{j=1}^{\sigma} -(1-2\varepsilon)\varphi_{j\overline{j}}|g^2_{I,J}|^2 + {\sum_{I,J}}''\varphi_{n\overline{n}}|g^2_{I,J}|^2 + {\sum_{I,K}}''\sum_{j,k=1}^{n-1}\varphi_{j\overline{k}} g^2_{I,jK}\overline{g^2_{I,kK}}$$
$$- C_U^{\varepsilon'}{\sum_{I,J}}''\sum_{j=1}^{n-1}|\varphi_{j\overline{n}} g^2_{I,J}|^2,$$

where $C_U^{\varepsilon'}$ is a positive constant. If $q \leq n-1$, by the hypothesis that φ satisfies condition a_q at z_0,

$$\sum_{j=1}^{n-1}\mu_j^-(z_0) + \sum_{j=1}^{q}\mu_j(z_0) > 0,$$

where $\mu^-(z_0) := \max\{-\mu(z_0), 0\}$. It follows that

$$Q_{\varphi}^{0,0}(z_0;g^1,g^1) = {\sum_{I,J}}'\Big(\sum_{j=1}^{n-1}\mu_j^-(z_0) + \sum_{j\in J}\mu_j(z_0)\Big)|g^1_{I,J}|^2$$

is positive definite. Thus there is a neighborhood $V \subseteq U$ of z_0 and a positive function $\mu_V^{\varepsilon,\varepsilon'}(z) \in C^0(V)$ such that for fixed but sufficiently small ε and ε',

$$Q_\varphi^{\varepsilon,\varepsilon'}(z; g^1, g^1) \geq \mu_V^{\varepsilon,\varepsilon'}(z)|g^1|^2.$$

From the estimate (3.4) for $g \in \mathcal{D}_{p,q}(V)$ with support in a fixed compact subset of V we read off,

$$\begin{aligned}
&(1+\varepsilon)(\|T^*g\|^2_{\chi(\varphi)} + \|Sg\|^2_{\chi(\varphi)}) \\
&\geq \int_V (\chi'(\varphi)(\mu_V^{\varepsilon,\varepsilon'}(z)|g^1|^2 + Q_\varphi^{\varepsilon,\varepsilon'}(z; g^2, g^2)) + \chi''(\varphi)|\varphi_n g^2|^2)\mathrm{e}^{-\chi(\varphi)} \\
&\quad + \int_V C_V^\varepsilon(z)|g|^2 \mathrm{e}^{-\chi(\varphi)},
\end{aligned} \tag{3.5}$$

where $C_V^\varepsilon(z) \in C^0(V)$ is independent of g and χ.

Let $\{V_\nu\}_{\nu \geq 1}$ be coordinate patches in Ω where (3.5) is applicable, and they form a locally finite covering of Ω. We select a partition of unity $\{\psi_\nu\}_{\nu\geq 1}$ subordinate to the covering $\{V_\nu\}_{\nu\geq 1}$ such that $\psi_\nu \in C_c^\infty(V_\nu)$ and $\sum\limits_{\nu\geq 1} \psi_\nu^2 = 1$ in Ω (shrinking V_ν if necessary). Applying (3.5) to $\psi_\nu f$ and adding over ν, we obtain

$$\begin{aligned}
&(1+\varepsilon)(\|T^*f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)}) \\
&\geq \int_\Omega (\chi'(\varphi)\mu^{\varepsilon,\varepsilon'}(z)|f^1|^2 + (\chi'(\varphi)R_\varphi^{\varepsilon,\varepsilon'}(z)|f^2|^2 + \chi''(\varphi)|\varphi_n f^2|^2))\mathrm{e}^{-\chi(\varphi)} \\
&\quad + \int_\Omega C_\Omega^\varepsilon(z)|f|^2 \mathrm{e}^{-\chi(\varphi)},
\end{aligned} \tag{3.6}$$

where

$$\mu^{\varepsilon,\varepsilon'}(z) := \sum_{\nu\geq 1} \psi_\nu^2 \mu_{V_\nu}^{\varepsilon,\varepsilon'}(z)$$

and

$$R_\varphi^{\varepsilon,\varepsilon'}(z) := \sum_{\nu\geq 1}\Big(\sum_{j=1}^{\sigma} -(1-2\varepsilon)\varphi_{j\bar{j}}^{(\nu)} + \varphi_{n\bar{n}}^{(\nu)} - C_{V_\nu}^{\varepsilon'}\sum_{j=1}^{n-1}|\varphi_{j\bar{n}}^{(\nu)}|^2 - \sqrt{\sum_{j,k=1}^{n-1}|\varphi_{j\bar{k}}^{(\nu)}|^2}\,\Big)\psi_\nu^2,$$

in which $\varphi_{i\bar{j}}^{(\nu)}$'s are the functions defined by (1.1) over V_ν. This implies that if we can choose χ increasing so rapidly that for $z \in \Omega$,

$$\begin{cases}\chi'(\varphi)\mu^{\varepsilon,\varepsilon'}(z) \geq (1+\varepsilon)(|C_\Omega^\varepsilon(z)| + C(z) + 1), \\ \chi'(\varphi)R_\varphi^{\varepsilon,\varepsilon'}(z) + \chi''(\varphi)|\varphi_n(z)|^2 \geq (1+\varepsilon)(|C_\Omega^\varepsilon(z)| + C(z) + 1),\end{cases} \tag{3.7}$$

then the desired estimate follows from (3.6). Therefore it only remains to prove (3.7). Indeed, since φ is an exhaustion function, $M_t := \{\varphi < t\} \subset\subset M$ for any $t \in \mathbb{R}$. The fact $M_t \cap \Omega \subset\subset M \setminus \mathcal{C}_\varphi$ (recall that $\mathcal{C}_\varphi$ is defined by (1.4)) allows us to define the following functions on $t \geq t_0$, where t_0 is the largest number such that $M_t \cap \Omega = \emptyset$,

$$\mu(t) := \sup_{M_{t+1}\cap\Omega} \frac{(1+\varepsilon)(|C_\Omega^\varepsilon(z)| + C(z) + 1)}{\mu^{\varepsilon,\varepsilon'}}, \quad R(t) := \sup_{M_{t+1}\cap\Omega} \frac{1 - R_\varphi^{\varepsilon,\varepsilon'}(z)}{|\varphi_n|^2},$$

$$C(t) := \sup_{M_{t+1}} (1+\varepsilon)(|C_\Omega^\varepsilon(z)| + C(z) + 1).$$

Hence, (3.7) is valid if we can choose $\chi \in C^\infty(\mathbb{R})$ such that on $t \geq t_0$,

$$\begin{cases} \chi'(t) \geq \mu(t), \\ \chi''(t)/\chi'(t) \geq R(t), \\ \chi'(t) \geq C(t), \end{cases} \tag{3.8}$$

which is obviously possible.

Our next goal is to derive the estimate as (3.2) near the critical points in $\mathcal{C}_\varphi$ (see (1.4)). For later use, we state a slightly more general version.

Lemma 3.2 *Let $1 \leq q \leq n$, and suppose that M admits an exhaustion function $\varphi \in C^\infty(M)$ satisfying the condition G_q^+ on a closed subset $E \subseteq M$. Then there exists an open neighborhood Ω of E in M and a complete Hermitian metric such that, for any function $C(z) \in C^0(M)$, one can construct a convex increasing function $\chi(t) \in C^\infty(\mathbb{R})$ with the following property: For any $f \in C_{p,k}^\infty(\Omega) \cap \mathcal{D}_{p,k}(M)$ $(k \geq q)$,*

$$\|T^* f\|_{\chi(\varphi)}^2 + \|Sf\|_{\chi(\varphi)}^2 \geq \int_\Omega (C(z)+1)|f|^2 \mathrm{e}^{-\chi(\varphi)}. \tag{3.9}$$

Proof Let $\widetilde{\omega}$ denote the metric given by Proposition 2.1, and $\{\widetilde{\lambda}_j\}_{j=1}^n$ denote the eigenvalues of the Hermitian form (1.1) with respect to $\widetilde{\omega}$. We choose a real valued function $\rho \in C^\infty(M)$ increasing so rapidly at infinity that the Hermitian metric $\omega := \mathrm{e}^\rho \widetilde{\omega}$ is complete on M; then the eigenvalues λ_j of (1.1) with respect to ω are $\mathrm{e}^{-\rho}\widetilde{\lambda}_j$. We shall fix the metric in the proof. For any $z_0 \in E$ and $\varepsilon < \frac{1}{q}$,

$$\lambda_1 + \cdots + \lambda_k = \mathrm{e}^{-\rho}(\widetilde{\lambda}_1 + \cdots + \widetilde{\lambda}_{q-1} + \widetilde{\lambda}_q + \cdots + \widetilde{\lambda}_k) \geq \mathrm{e}^{-\rho}(1-(q-1)\varepsilon) > 0. \tag{3.10}$$

We can select a coordinate patch U around z_0 such that (3.10) holds on U. By means of Proposition 3.1, we know that for any $g \in \mathcal{D}_{p,k}(U)$ with support in a fixed compact subset of U,

$$\begin{aligned} \frac{3}{2}(\|T^* g\|_\varphi^2 + \|Sg\|_\varphi^2) &\geq \frac{1}{2}\sum_{I,J}\sum_{j=1}^n \int_U \Big|\frac{\partial g_{I,J}}{\partial \overline{\omega}_j}\Big|^2 \mathrm{e}^{-\varphi} + \sum_{I,K}\sum_{j,k=1}^n \int_U \varphi_{j\overline{k}} g_{I,jK}\overline{g_{I,kK}}\mathrm{e}^{-\varphi} \\ &\quad + \int_U C_U(z)|g|^2 \mathrm{e}^{-\varphi} \\ &\geq \int_U (\lambda_1 + \cdots + \lambda_k)|g|^2 \mathrm{e}^{-\varphi} + \int_U C_U(z)|g|^2\mathrm{e}^{-\varphi}, \end{aligned} \tag{3.11}$$

where $C_U(z) \in C^0(U)$ is independent of g.

Let $\{U_\nu\}_{\nu \geq 1}$ be coordinate patches set in M such that (3.11) holds on each patch, and they form a locally finite covering of E. Set $\Omega := \bigcup\limits_\nu U_\nu$; then we choose a partition of unity $\{\psi_\mu\}_{\mu\geq 1}$ subordinate to a refinement $\{V_\mu\}_{\mu \geq 1}$ of the covering $\{U_\nu\}_{\nu\geq 1}$, where $\psi_\mu \in C_c^\infty(V_\mu)$ with $\sum\limits_{\mu\geq 1} \psi_\mu^2 = 1$ in Ω. Applying (3.11) to $\psi_\mu f$ and $\chi(\varphi)$ gives

$$\frac{3}{2}(\|T^* f\|_{\chi(\varphi)}^2 + \|Sf\|_{\chi(\varphi)}^2) \geq \int_\Omega \chi'(\varphi)(\lambda_1 + \cdots + \lambda_k)|f|^2 \mathrm{e}^{-\chi(\varphi)}$$

$$+\int_{\Omega} C_{\Omega}(z)|f|^2 \mathrm{e}^{-\chi(\varphi)}. \tag{3.12}$$

Thanks to (3.12), the proof can be finished by choosing χ appropriately such that

$$\chi'(\varphi)(\lambda_1 + \cdots + \lambda_k) \geq \frac{3}{2}(|C_{\Omega}(z)| + C(z) + 1) \quad \text{in } \Omega. \tag{3.13}$$

Now the proof of the rest part is analogous to that of (3.7). Since $M_t := \{\varphi < t\} \subset\subset M$ for any $t \in \mathbb{R}$, by shrinking Ω a little bit, we can introduce

$$\lambda(t) := \sup_{M_{t+1}\cap\Omega} \frac{\frac{3}{2}(|C_{\Omega}(z)| + C(z) + 1)}{\lambda_1 + \cdots + \lambda_k}, \quad t \geq t_0, \tag{3.14}$$

where t_0 is the largest number such that $M_t \cap \Omega = \emptyset$. Thus, (3.13) holds true if we select a convex increasing $\chi \in C^{\infty}(\mathbb{R})$ satisfying

$$\chi'(t) \geq \lambda(t), \quad t \geq t_0. \tag{3.15}$$

The proof is thus complete.

Remark 3.1 If φ satisfies the condition G_q^- for $q \leq n-1$ on the closed set E, according to Proposition 2.1 again, one can find a complete metric such that for any $z_0 \in E$,

$$-\lambda_n - \cdots - \lambda_q - \lambda_{q+1} - \cdots - \lambda_{k+1} > 0 \quad \text{at } z_0. \tag{3.16}$$

Then there exists a coordinate patch U such that (3.16) holds. Using the Hodge star operator, we can deduce the following estimate for any $g \in \mathcal{D}_{p,k}(U)$ $(k \leq q)$ with support in a fixed compact subset of U,

$$\frac{3}{2}(\|T^*g\|_{\varphi}^2 + \|Sg\|_{\varphi}^2) \geq \int_U -(\lambda_n + \cdots + \lambda_{k+1})|g|^2 \mathrm{e}^{-\varphi} + \int_U C_U(z)|g|^2 \mathrm{e}^{-\varphi}.$$

However, it seems that we are unable to derive the estimate (3.9) in a neighborhood of E, since (3.12) breaks down when we replace φ by $\chi(\varphi)$ where χ is a convex increasing function with $\chi''(t) \neq 0$. This is the reason why we cannot deal with the case where φ satisfies the condition G_q^-.

Proof of Theorem 1.1 Employing Lemma 3.2 to the exhaustion function φ and the closed set $\mathcal{C}_{\varphi}$ (see (1.4)), there is a complete metric ω and an open neighborhood Ω_1 of $\mathcal{C}_{\varphi}$ such that (3.9) holds for ω. We fix the metric ω in the proof. Let Ω' be an open subset of Ω_1 containing $\mathcal{C}_{\varphi}$ so that Ω_1 and $\Omega_2 := M \setminus \overline{\Omega'}$ cover M. Choose a smooth partition of unity $\{\psi_1, \psi_2\}$ subordinate to $\{\Omega_1, \Omega_2\}$ with $\psi_1^2 + \psi_2^2 = 1$. For any $f \in \mathcal{D}_{p,q}(M)$, applying Lemma 3.1 and Lemma 3.2 to $(\Omega_2, \psi_2 f)$ and $(\Omega_1, \psi_1 f)$ respectively yields

$$\begin{cases} \displaystyle\int_{\Omega_1} |T^*(\psi_1 f)|^2 \mathrm{e}^{-\chi(\varphi)} + \int_{\Omega_1} |S(\psi_1 f)|^2 \mathrm{e}^{-\chi(\varphi)} \geq \int_{\Omega_1} (C(z)+1)|\psi_1 f|^2 \mathrm{e}^{-\chi(\varphi)}, \\ \displaystyle\int_{\Omega_2} |T^*(\psi_2 f)|^2 \mathrm{e}^{-\chi(\varphi)} + \int_{\Omega_2} |S(\psi_2 f)|^2 \mathrm{e}^{-\chi(\varphi)} \geq \int_{\Omega_2} (C(z)+1)|\psi_2 f|^2 \mathrm{e}^{-\chi(\varphi)}, \end{cases}$$

where

$$C(z) := |T^*\psi_1|^2 + |S\psi_1|^2 + |T^*\psi_2|^2 + |S\psi_2|^2$$

and the convex function $\chi \in C^\infty(\mathbb{R})$ is so rapidly increasing that (3.8) and (3.15) are fulfilled. It follows that

$$\|T^* f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)} \geq \|f\|^2_{\chi(\varphi)}, \quad \forall f \in \mathcal{D}_{p,q}(M).$$

Since any $f \in L^2_{p,q}(M, \mathrm{loc})$ belongs to $L^2_{p,q}(M, \chi(\varphi))$ for some choice of χ satisfying (3.8) and (3.15), now the conclusion follows from Hörmander's density lemma in [10] and Lemma 2.1.

To prove Theorem 1.2, we begin with the following lemma.

Lemma 3.3 *Let $0 \leq q \leq n$ and E be a compact set of M. Suppose that the function $\varphi \in C^\infty(M)$ satisfies the condition G_q on E. Then there exists an open neighborhood $\Omega \subset\subset M$ of E such that, for each function $C(z) \in C^0(M)$, there is a positive constant τ_0 satisfying, for any $f \in \mathcal{D}_{p,q}(\Omega)$ with support in a fixed compact subset K of Ω,*

$$\|T^* f\|^2_{\tau\varphi} + \|Sf\|^2_{\tau\varphi} \geq \int_\Omega (C(z) + 1)|f|^2 \mathrm{e}^{-\tau\varphi}, \quad \forall \tau > \tau_0. \tag{3.17}$$

Proof According to [9, Theorem 3.3.1], for any point $z_0 \in E$, there is a coordinate patch U around z_0 and constants C, τ_0 depending on U, such that for any $g \in \mathcal{D}_{p,q}(U)$,

$$C(\|T^* g\|^2_{\tau\varphi} + \|Sg\|^2_{\tau\varphi}) \geq \int_U \tau |g|^2 \mathrm{e}^{-\tau\varphi}, \quad \forall \tau > \tau_0. \tag{3.18}$$

Since E is compact, we can choose finite coordinate patches $U_1, \cdots, U_s$ in M to cover E where (3.18) is applicable and set $\Omega := \bigcup\limits_{\nu=1}^{s} U_\nu \subset\subset M$. Let $\{\psi_\mu\}_{\mu=1}^{s'}$ be a partition of unity subordinate to a finite refinement $\{V_\mu\}_{\mu=1}^{s'}$ of the covering $\{U_\nu \cap K\}_{\nu=1}^{s}$ of K, so that $\psi_\mu \in C_c^\infty(V_\mu)$ and $\sum\limits_{\mu=1}^{s'} \psi_\mu^2 = 1$ in Ω. Applying (3.18) to $\psi_\mu g$ and adding over μ gives

$$\sum_{\mu=1}^{s'} C_\mu(\|T^* g\|^2_{\tau\varphi} + \|Sg\|^2_{\tau\varphi}) \geq \int_\Omega s'\tau |g|^2 \mathrm{e}^{-\tau\varphi}, \quad \forall \tau > \max_\mu \{\tau_0^{(\mu)}\}, \tag{3.19}$$

where C_μ and $\tau_0^{(\mu)}$ are the constants depending on V_μ in (3.18). Put

$$\tau_0 := \max\Big\{\tau_0^{(1)}, \cdots, \tau_0^{(s')}, \frac{\sum\limits_{\mu=1}^{s'} C_\mu\Big(\sup\limits_\Omega |C(z)| + 1\Big)}{s'}\Big\}, \tag{3.20}$$

thus the desired estimate can be deduced from (3.19).

Proof of Theorem 1.2 We apply Lemma 3.3 to φ and $\overline{M}_{t_1}$, then there exists an open neighborhood Ω_1 such that (3.17) is valid for any metric. By shrinking Ω_1 if necessary, we can assume that

$$\mathcal{C}_\varphi \subseteq M_{t_1} \subset\subset M_{t_2} \subset\subset \Omega_1 \subset\subset M_{t_3} \subset\subset M_{t_4} \subset\subset M_{t_0}.$$

According to Lemma 3.2, one can find a complete Hermitian metric ω and an open neighborhood $\Omega_2 \subseteq M_{t_0} \setminus \overline{M}_{t_1}$ of $\overline{M}_{t_4} \setminus M_{t_2}$ such that (3.9) holds for ω. We fix the metric in the proof. Clearly, the open sets Ω_1, Ω_2 and $\Omega_3 := M \setminus \overline{M}_{t_4}$ form a finite covering of M. As usual we choose a smooth partition of unity $\{\psi_1, \psi_2, \psi_3\}$ subordinate to $\{\Omega_1, \Omega_2, \Omega_3\}$ satisfying $\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$ and $\psi_1 \in C_c^\infty(\Omega_1)$. Then for any $f \in \mathcal{D}_{p,q}(M)$, employing Lemma 3.1, Lemma 3.2 and Lemma 3.3 to $(\Omega_3, \psi_3 f)$, $(\Omega_2, \psi_2 f)$ and $(\Omega_1, \psi_1 f)$ respectively, we have

$$\begin{cases} \displaystyle\int_{\Omega_1} |T^*(\psi_1 f)|^2 \mathrm{e}^{-\chi(\varphi)} + \int_{\Omega_1} |S(\psi_1 f)|^2 \mathrm{e}^{-\chi(\varphi)} \geq \int_{\Omega_1} (C(z)+1)|\psi_1 f|^2 \mathrm{e}^{-\chi(\varphi)}, \\ \displaystyle\int_{\Omega_2} |T^*(\psi_2 f)|^2 \mathrm{e}^{-\chi(\varphi)} + \int_{\Omega_2} |S(\psi_2 f)|^2 \mathrm{e}^{-\chi(\varphi)} \geq \int_{\Omega_2} (C(z)+1)|\psi_2 f|^2 \mathrm{e}^{-\chi(\varphi)}, \\ \displaystyle\int_{\Omega_3} |T^*(\psi_3 f)|^2 \mathrm{e}^{-\chi(\varphi)} + \int_{\Omega_3} |S(\psi_3 f)|^2 \mathrm{e}^{-\chi(\varphi)} \geq \int_{\Omega_3} (C(z)+1)|\psi_3 f|^2 \mathrm{e}^{-\chi(\varphi)}, \end{cases}$$

where

$$C(z) := |T^*\psi_1|^2 + |S\psi_1|^2 + |T^*\psi_2|^2 + |S\psi_2|^2 + |T^*\psi_3|^2 + |S\psi_3|^2,$$

and the convex increasing function $\chi \in C^\infty(\mathbb{R})$ is selected to satisfy (3.8) on $t \geq t_0$ and (3.15) on $t \geq t_3$ respectively, and we further require that χ is linear for $t \leq t_3$ with a large slope τ, the constant τ is chosen such that

$$\tau > \tau_0 + \lambda(t_3),$$

where τ_0 and $\lambda(t_3)$ are defined by (3.20) and (3.14). Consequently,

$$\|T^* f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)} \geq \|f\|^2_{\chi(\varphi)}, \quad \forall f \in \mathcal{D}_{p,q}(M).$$

Since any $f \in L^2_{p,q}(M, \mathrm{loc})$ belongs to $L^2_{p,q}(M, \chi(\varphi))$ for some choice of χ increasing rapidly at infinity such that (3.8) and (3.15) hold true for $t \geq t_0$, the conclusion is derived from Hörmander's density lemma in [10] and Lemma 2.1.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Andreotti, A. and Grauert, H., Théorème de finitude pour la cohomologie des espaces complexes, *Bull. Soc. Math. France*, **90**, 1962, 193–259.

[2] Andreotti, A. and Vesentini, E., Carleman estimates for the Laplace-Beltrami equation on complex manifolds, *Publ. Math. Inst. Hautes Etudes Sci.*, **25**, 1965, 81–130.

[3] Demailly, J.-P., Complex Analytic and Differential Geometry, Université de Grenoble I, Institut Fourier, Saint-Martin d'Hères, 2012.

[4] Folland, G. B. and Kohn, J. J., The Neumann Problem for the Cauchy-Riemann Complex, Annals of Mathematics Studies, Vol. 75, Princeton University Press, University of Tokyo Press, Tokyo, 1972.

[5] Grauert, H., On Levi's problem and the imbedding of real-analytic manifolds, *Ann. Math.* (2), **68**, 1958, 460–472.

[6] Grauert, H., Theory of q-convexity and q-concavity, Several Complex Variables, VII, Encyclopaedia Math. Sci., Vol. 74, Springer-Verlag, Berlin, 1994.

[7] Gong, X. H., On regularity of $\overline{\partial}$-solutions on a_q domains with C^2 boundary in complex manifolds, *Trans. Amer. Math. Soc.*, **378**, 2025, 1771–1829.

[8] Ho, L.-H., $\overline{\partial}$-problem on weakly q-convex domains, *Math. Ann.*, **290**(1), 1991, 3–18.

[9] Hörmander, L. V., L^2 estimates and existence theorems for the $\overline{\partial}$ operator, *Acta Math.*, **113**, 1965, 89–152.

[10] Hörmander, L. V., An Introduction to Complex Analysis in Several Variables, Third edition, North-Holland Mathematical Library, **7**, North-Holland Publishing Co., Amsterdam, 1990.

[11] Kohn, J. J., Harmonic integrals on strongly pseudo-convex manifolds, I, *Ann. of Math.* (2), **78**, 1963, 112–148.

[12] Kohn, J. J., Harmonic integrals on strongly pseudo-convex manifolds, II, *Ann. of Math.* (2), **79**, 1964, 450–472.

[13] Morrey, C. B., The analytic embedding of abstract real-analytic manifolds, *Ann. of Math.* (2), **68**, 1958, 159–201.

[14] Ohsawa, T. and Pawlaschyk, T., Analytic Continuation and Q-convexity, SpringerBriefs in Mathematics, Springer-Verlag, Singapore, 2022.

[15] Rothstein, W., Zur Theorie der analytischen Mannigfaltigkeiten im Raume von n komplexen Veränderlichen, *Math. Ann.*, **129**, 1955, 96–138.

[16] Siu, Y. T., The $\overline{\partial}$ problem with uniform bounds on derivatives, *Math. Ann.*, **207**, 1974, 163–176.

[17] Zampieri, G., q-Pseudoconvexity and regularity at the boundary for solutions of the $\overline{\partial}$-problem, *Compositio Math.*, **121**(2), 2000, 155–162.

[18] Zampieri, G., Solvability of the $\overline{\partial}$ problem with C^∞ regularity up to the boundary on wedges of $\mathbb{C}^N$, *Israel J. Math.*, **115**, 2000, 321–331.