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On Geometric Realization of the Discrete Manakov Equation of Mixed Type*

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In memory of Professor Hesheng Hu (1928–2024)

Abstract The authors introduce the coupled discrete 2-component nonlinear Schrödinger equation with M-solutions and prove that this type of discrete equation is an integrable discretization of the integrable Manakov equation of mixed type. Moreover, the integrable discrete equation of 1-d Schrödinger flow to the pseudo-projective 2-space $U(2,1)/U(1,1)\times U(1)$ is shown to be a geometric realization of the integrable discrete Manakov equation of mixed type.

Keywords Manakov equation, Geometric realization, Integrable discretization
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1 Introduction

It is well known that many 1+1 equations, such as the nonlinear Schrödinger equation (NLS for short), the Korteweg-de Vries equation (KdV for short), the Manakov equation and the coupled 2-component nonlinear Schrödinger equation and so on, are prototypical integrable partial differential equations in mathematical physics that model a wide range of physical phenomena, such as nonlinear optical pulse propagation, hydrodynamics, biophysics and so on (see, for example, [23–24] for a list of physical motivations). Since most work in nonlinear wave propagation involves to some extent a numerical study of the problem, the issue of the discretization of integrable equations was addressed in [1]. Among a large number of possible discretizations of an integrable equation/system, there may be one that is integrable. For example, Ablowitz and Ladik noticed in [2] that there is a discrete version of the NLS equation, usually referred to as the AL equation, which is exactly integrable. Since then, the analytic or geometric studies of integrable discrete equations have attracted considerable attention in the theory of integrable systems. It should be mentioned that there are many other discrete versions of an integrable equation considered physically (see, for example, [11, 16]) which are

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not integrable. The investigation of these discrete equations/systems is also very meaningful and remarkable (see [8]).

In this paper, we shall focus on displaying some nice mathematical properties, such as the geometric realization (for the concept of geometric realization, see [19]) of the integrable discrete Manakov equation of mixed type. The motivation of this study is as follows. Recall that the general Manakov equation, or in other words, the general 2-component nonlinear Schrödinger equation, reads (see [20])

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + (b_1|\varphi_1|^2 + b_2|\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + (c_1|\varphi_1|^2 + c_2|\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{1.1}$$

where $\varphi_1 = \varphi_1(x,t), \varphi_2 = \varphi_2(x,t)$ are unknown complex functions, subscript t and x denote differentiation with respect to time and position, respectively, and b_1, b_2, c_1, c_2 are nonzero real parameters. (1.1) has important applications in nonlinear optics, superfluids, plasmas, Bose-Einstein condensed matter physics etc. (see [4, 6–7, 10, 18, 21–22, 26–27]). Although involving 4 free real parameters, looking complicated and being generally non-integrable, the analytic properties of (1.1) have been explored deeply and summarized in [5]. Recently, some geometric properties of (1.1) have been described. For example, in [14], three models of moving curves evolving in three different symmetric Lie algebras are shown to be simultaneously the geometric realizations of (1.1) by the first author and Zhong. In this process, the following integrable equations:

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = 0, \end{cases}$$

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = 0 \end{cases}$$

and

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} \pm 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} \pm 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = 0 \end{cases}$$

play important roles. These equations are called in the literature the Manakov equations of focusing, defocusing and mixed types, respectively. Because the two systems in the mixed type are actually equivalent to each other by the change of variables $\varphi_1 \to \varphi_2$ and $\varphi_2 \to \varphi_1$, we set

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} - 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} - 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = 0 \end{cases} \tag{1.2}$$

as their representation. The above three integrable Manakov equations are respectively gauge equivalent to the equations of 1-d Schrödinger flows to the projective 2-spaces $U(3)/U(2) \times U(1)$, $U(2,1)/U(2) \times U(1)$ and $U(2,1)/U(1,1) \times U(1)$ (see [15]). The results for the equations of focusing and defocusing types are obtained respectively, as special cases, by Terng and

Uhlenbeck in [25] and Chen in [9] for the matrix nonlinear Schödinger equations of focusing and defocusing types. However, (1.2) of mixed type cannot be involved as a special case of the matrix nonlinear equations, since the third term in the left hand side of (1.2) admits a minus symbol (see [13]). It seems that (1.2) of mixed type may possess some different characters comparing to those of focusing and defocusing types. This leads us to display geometric characters of the discrete Manakov equation of mixed type.

Discrete Manakov equations have already been investigated in the literature and have important applications in physical phenomena (see, for example, [3, 17] and the references therein). However, most of such studies concentrate mainly on discretizations of focusing and defocusing types. The geometric studies of integrable discrete Manakov equations of the focusing or defocusing type can be described as special cases from those of the integrable coupled discrete matrix nonlinear Schrödinger equation by the first author in [12]. Little geometric information is known for the integrable discrete Manakov equation of mixed type. One notes that there is naturally a discrete version of the Manakov equation of mixed type:

$$\mathrm{i}\frac{\mathrm{d}r_n}{\mathrm{d}t} + (r_{n+1} + r_{n-1} - 2r_n) + (r_{n+1}r_n^\dagger r_n + r_n r_n^\dagger r_{n-1}) = 0, \tag{1.3}$$

where $r_n = (\varphi_{1n}, \varphi_{2n})$ and $r_n^\dagger = \begin{pmatrix} -\overline{\varphi}_{1n} \\ \overline{\varphi}_{2n} \end{pmatrix}$ denotes the pseudo-complex transposed conjugate of r_n, which is a parallel version of the following discrete matrix nonlinear Schrödinger equation:

$$\mathrm{i}\frac{\mathrm{d}q_n}{\mathrm{d}t} + (q_{n+1} + q_{n-1} - 2q_n) + (q_{n+1}q_n^* q_n + q_n q_n^* q_{n-1}) = 0, \tag{1.4}$$

where q_n is a $(k \times m)$ complex matrix. (1.4) reduces to the integrable AL equation when $k = m = 1$. However, in spite of the fact that the continuous limit of (1.4) (resp. (1.3)) is the classical matrix nonlinear Schrödinger equation (resp. (1.2)), to the best knowledge of the authors, the integrability of (1.4) with $k \geq 2$ or $m \geq 2$ (resp. (1.3)) is not known in literature up to now. Therefore, at this time, (1.3) cannot be regarded and treated as an integrable discrete equation of the Manakov equation (1.2) of mixed type.

The aim of this paper is to show that the following coupled discrete 2-component nonlinear Schrödinger system:

$$\begin{cases} \mathrm{i}\dfrac{\mathrm{d}q_n}{\mathrm{d}t} + (q_{n+1} + q_{n-1} - 2q_n) + (q_{n+1}r_n q_n + q_n r_n q_{n-1}) = 0, \\ -\mathrm{i}\dfrac{\mathrm{d}r_n}{\mathrm{d}t} + (r_{n+1} + r_{n-1} - 2r_n) + (r_{n+1}q_n r_n + r_n q_n r_{n-1}) = 0, \end{cases} \tag{1.5}$$

where r_n is a complex row 2-vector and q_n is a complex column 2-vector, together with a class of solutions which we call M-solutions, is an integrable discretization of the Manakov equation (1.2) of mixed type. In fact, we know that (1.5) is an integrable discretization of the coupled 2-component nonlinear Schrödinger system:

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2qrq = 0, \\ -\mathrm{i}r_t + r_{xx} + 2rqr = 0, \end{cases} \tag{1.6}$$

since the continuous limit of (1.5) is (1.6). When $r = (\overline{\varphi_1}, \overline{\varphi_2})$ and $q = -r^{\dagger}$, the coupled 2-component nonlinear Schrödinger system (1.6) reduces exactly to the Manakov equation (1.2) of mixed type. As we will see below, the continuous limit of an M-solution to (1.5) is a solution to (1.6) with $q = -r^{\dagger}$. Hence, according to the quantum correspondence principle, we have that (1.5) with the class of M-solutions is actually an integrable discretization of the Manakov equation (1.2) and we call it (i.e., (1.5) together with M-solutions) the integrable discrete Manakov equation of mixed type. Furthermore, we shall show that the integrable discrete equation of 1-d Schrödinger flow to $U(2,1)/U(1,1) \times U(1)$ is a geometric realization of the integrable discrete Manakov equation of mixed type. This exploration leads us to give a unified geometric interpretation for the integrable discrete Manakov equations of focusing, defocusing and mixed types.

The paper is organized as follows. Section 2 gives a detailed discussion of the integrability of (1.5) and calculates its continuous limit. We also give a brief description of 1-d Schrödinger flow to the pseudo-projective space $U(2,1)/U(1,1) \times U(1)$ and a detailed verification of the integrability of its integrable discrete equation. In Section 3, we show that (1.5) with the class of M-solutions is gauge equivalent to the integrable discrete equation of 1-d Schrödinger flow to $U(2,1)/U(1,1) \times U(1)$. A unified geometric interpretation for the discrete integrable Manakov equations of focusing, defocusing and mixed types is described in this section.

2 Integrable Discretization

First, we come to verify the integrability of (1.5) and calculate its continuous limit. The Lax pair of (1.5) is explicitly constructed as follows:

$$\phi_{n+1} = L_n \phi_n, \quad \frac{\mathrm{d}\phi_n}{\mathrm{d}t} = M_n \phi_n, \tag{2.1}$$

where $\{\phi_n\}$ is a column 3-vector-valued potential sequence and

$$L_n = \begin{pmatrix} z & r_n z^{-1} \\ -q_n z & I_2 z^{-1} \end{pmatrix}, \tag{2.2}$$

$$M_n = \mathrm{i} \begin{pmatrix} 1 - z^2 + z - z^{-1} - r_n q_{n-1} & -r_n + r_{n-1} z^{-2} \\ -q_n + q_{n-1} z^2 & (-1 + z^{-2} + z - z^{-1}) I_2 + q_n r_{n-1} \end{pmatrix}, \tag{2.3}$$

in which z is a spectral parameter, r_n and q_n are respectively complex row and column 2-vectors and I_k stands for the $k \times k$ unit matrix ($k \geq 2$). It is a direct verification that the integrability condition:

$$\frac{\mathrm{d}L_n}{\mathrm{d}t} + L_n M_n - M_{n+1} L_n = 0 \tag{2.4}$$

of (2.1) is equivalent to the validity of (1.5). Furthermore, by taking the continuous limit of (2.1), i.e., when $\Delta x \to 0$,

$$z \sim 1 + \lambda \Delta x, \quad r_n \sim r \Delta x, \quad q_n \sim q \Delta x, \quad n \Delta x = x (\text{fixed}), \quad t \Delta x^2 \sim t,$$

and $\phi_n \sim \phi$ with the expansion $\phi_{n\pm1} \sim \Delta x(\phi \pm \Delta x \phi_x + \frac{\Delta x^2}{2}\phi_{xx} \pm \cdots)$, we obtain

$$\phi_x = L\phi, \quad \phi_t = M\phi, \tag{2.5}$$

where

$$L = -2\mathrm{i}\lambda\sigma_3 + U, \quad M = -4\lambda^2\sigma_3 - 2\mathrm{i}\lambda U + 2(U^2 + U_x)\sigma_3, \tag{2.6}$$

in which $U = \begin{pmatrix} 0 & r \\ -q & 0 \end{pmatrix}$, $\sigma_3 = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -I_2 \end{pmatrix}$ and λ is a spectral parameter. It is also a direct verification that the integrability condition of the Lax pair (2.5) is equivalent to the validity of (1.6).

Next, we come to demonstrate the integrable discrete equation of 1-d Schrödinger flow to the pseudo-projective 2-space $U(2,1)/U(1,1) \times U(1)$. Recall that $U(2,1)$ is the set of complex linear transformations that preserve the metric $\mathrm{d}s^2 = |\mathrm{d}z_1|^2 + |\mathrm{d}z_2|^2 - |\mathrm{d}z_3|^2$ invariant on $\mathbb{C}^3$, which is a Lie group. The Lie algebra $u(2,1)$ of $U(2,1)$ admits a decomposition: $u(2,1) = \mathbf{k} \oplus \mathbf{m}$ with

$$\mathbf{k} = \mathrm{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} \mathrm{i}a & 0 & 0 \\ 0 & \mathrm{i}b & \alpha \\ 0 & \overline{\alpha} & \mathrm{i}c \end{pmatrix} \in u(2,1) \middle| a, b, c \in \mathbb{R}, \alpha \in \mathbb{C} \right\}$$

and

$$\mathbf{m} = \left\{ \begin{pmatrix} 0 & \psi & \varphi \\ -\overline{\psi} & 0 & 0 \\ \overline{\varphi} & 0 & 0 \end{pmatrix} \in u(2,1) \middle| \psi, \varphi \in \mathbb{C} \right\},$$

in which σ_3 is indicated previously. Therefore, $U(2,1)/U(1,1) \times U(1)$ can be presented as an adjoint orbit in the symmetric Lie algebra $u(2,1)$ as follows:

$$U(2,1)/U(1,1) \times U(1) = \{E^{-1}\sigma_3 E \mid \forall E \in U(2,1)\} \hookrightarrow u(2,1).$$

We would mention that there is another decomposition of $u(2,1)$ (in other words, its isomorphic algebra $u(1,2)$) that produces the adjoint orbit space $U(2,1)/U(2) \times U(1)$ in $u(2,1)$ (refer to [14–15] for details). For such $U(2,1)$ with $u(2,1) = \mathbf{k} \oplus \mathbf{m}$ fixed above, a map from $\mathbb{R}^1 \times \mathbb{R}^1$ to $U(2,1)/U(1,1) \times U(1)$ may be denoted by $S = E^{-1}\sigma_3 E$ with $E = E(t,x) \in U(2,1)$ and $E_x = PE$ with $P \in \mathbf{m}$. We have that the equation of 1-d Schrödinger flow to $U(2,1)/U(1,1) \times U(1)$ is expressed explicitly by (also refer to [15])

$$S_t = -[S, S_{xx}]. \tag{2.7}$$

This is an integrable equation, since its Lax pair is

$$\psi_x = \widetilde{L}\psi, \quad \psi_t = \widetilde{M}\psi, \tag{2.8}$$

in which

$$\widetilde{L} = -2\mathrm{i}\lambda S, \quad \widetilde{M} = -4\lambda^2 S + 2\mathrm{i}\lambda[S, S_x]$$

and λ is a spectral parameter. In order to verify this conclusion, one needs to use the constriction condition: $S^2 = -\frac{1}{4}I_3$.

We claim that the following differential-difference equation:

$$\frac{\mathrm{d}S_n}{\mathrm{d}t} = 2(I_3 - 4S_{n+1}S_n)^{-1} - 2(I_3 - 4S_nS_{n-1})^{-1}, \tag{2.9}$$

where S_n is $U(2,1)/U(1,1)\times U(1)$-valued, in other words, $S_n = E_n^{-1}\sigma_3 E_n$ with $E_n \in U(2,1)$ and the Lie algebra $u(2,1)$ of $U(2,1)$ has the decomposition indicated above, is an integrable discretization of (2.7). In fact, the integrability of (2.9) comes from that it has a Lax pair:

$$\psi_{n+1} = \widetilde{L}_n\psi_n, \quad \frac{\mathrm{d}\psi_n}{\mathrm{d}t} = \widetilde{M}_n\psi_n, \tag{2.10}$$

where $\{\psi_n\}$ is a column 3-vector-valued potential sequence and

$$\begin{aligned}\widetilde{L}_n &= \frac{z+z^{-1}}{2}I_3 - \mathrm{i}(z-z^{-1})S_n,\\ \widetilde{M}_n &= 4\Big(1-\frac{z^2+z^{-2}}{2}\Big)(I_3-4S_nS_{n-1})^{-1}S_n + \mathrm{i}(z-z^{-1})I_3\\ &\quad - \mathrm{i}(z^2-z^{-2})(I_3-4S_nS_{n-1})^{-1},\end{aligned}$$

in which z is a spectral parameter. We should point out that the verification of this conclusion is not so easy. One needs to use the following matrix identities carefully in the calculation (notice again that $S_n^2 = -\frac{1}{4}I_3$ for any n):

$$\begin{aligned}(I_3-4S_nS_{n-1})^{-1}S_nS_{n-1} &= -\frac{1}{4}(I_3-(I_3-4S_nS_{n-1})^{-1}),\\ (I_3-4S_nS_{n-1})^{-1}S_n &= (I_3-(I_3-4S_nS_{n-1})^{-1})S_{n-1} = S_{n-1}(I_3-4S_nS_{n-1})^{-1},\\ S_n(I_3-4S_nS_{n-1})^{-1} &= (I_3-(I_3-4S_nS_{n-1})^{-1})S_n,\\ S_n(I_3-4S_nS_{n-1})^{-1}S_n &= -\frac{1}{4}(I_3-(I_3-4S_nS_{n-1})^{-1}).\end{aligned}$$

Taking the continuous limit of (2.10) (when $\Delta x \to 0$, $z \sim 1+\lambda\Delta x, n\Delta x = x(\text{fixed})$, $t\Delta x^2 \sim t$, $S_n \sim S$ and $\psi_n \sim \psi$), we clearly obtain that the limit equation is just (2.8) with S being $U(2,1)/U(1,1)\times U(1)$-valued. This shows that the differential-difference equation (2.9) is actually an integrable discretization of (2.7).

Quite interestingly, we would like to point out that (2.9) with $S_n = E_n^{-1}\sigma_3E_n$ and $E_n \in GL(3,\mathbb{C})$ is also integrable, and it contains three different types of integrable discrete equations, say, (2.9) with S_n being respectively $U(3)/U(2)\times U(1)$-, $U(2,1)/U(2)\times U(1)$- and $U(2,1)/U(1,1)\times U(1)$-valued. The continuous limit of (2.9) with $S_n = E_n^{-1}\sigma_3E_n$ and $E_n \in GL(3,\mathbb{C})$ is just (2.7) with $S = E^{-1}\sigma_3E$ and $E \in GL(3,\mathbb{C})$. The latter also contains three integrable equations, say, (2.7) with S being respectively $U(3)/U(2)\times U(1)$-, $U(2,1)/U(2)\times U(1)$- and $U(2,1)/U(1,1)\times U(1)$-valued, which are respectively the equations of 1-d Schrödinger flows to the projective 2-spaces $U(3)/U(2)\times U(1)$, $U(2,1)/U(2)\times U(1)$ and $U(2,1)/U(1,1)\times U(1)$. These equations are also gauge equivalent to the integrable Manakov equations of focusing, defocusing and mixed types respectively.

3 Gauge Equivalence

In this section we come to prove that the integrable discrete equation (2.9) is gauge equivalent to the integrable discrete Manakov equation of mixed type, i.e., (1.5) together with M-solutions. First, we come to establish the following lemma.

Lemma 3.1 *Solutions to* (2.9) *are gauge equivalent to solutions to* (1.5).

Proof Let $\{S_n(t)\}$ be a solution to (2.9), and a corresponding solution to the Lax pair (2.10) is denoted by $\{\psi_n(t,z)\}$. We may now choose a sequence of 3×3-matrices $\{G_n(t)\}$ such that $\sigma_3 = G_nS_nG_n^{-1}$ and

$$G_{n+1}G_n^{-1} = \begin{pmatrix} 1 & r_n \\ -q_n & I_2 \end{pmatrix} \tag{3.1}$$

for some complex row 2-vector sequence $\{r_n\}$ and column 2-vector sequence $\{q_n\}$. In fact, because the given solution $\{S_n\}$ is expressed by $S_n = E_n^{-1}\sigma_3E_n$ $(\forall n)$ for some given sequence $\{E_n\}$ with $E_n \in U(2,1)$, we see that general solutions to $\sigma_3 = G_nS_nG_n^{-1}$ are of the form

$$G_n = \mathrm{diag}(a_n, B_n)E_n, \tag{3.2}$$

where $\{a_n\}$ is a complex-valued sequence and $\{B_n\}$ is a complex 2×2-matrix valued sequence. Now we first fix a_0 and B_0 (the choice of them is referred to Remark 3.1 below), and then come to prove that a_n and B_n $(\forall n \neq 0)$ can be chosen progressively such that (3.1) holds for some r_n and q_n. Substituting (3.2) into (3.1), we have

$$\begin{pmatrix} a_{n+1}^{-1} & 0 \\ 0 & B_{n+1}^{-1} \end{pmatrix}\begin{pmatrix} 1 & r_n \\ -q_n & I_2 \end{pmatrix}\begin{pmatrix} a_n & 0 \\ 0 & B_n \end{pmatrix} = E_{n+1}E_n^{-1}.$$

Notice that for $E \in U(2,1)$, $E^{-1} = J_3E^*J_3$, where E^* stands for the complex transposed conjugate of E and $J_3 = \mathrm{diag}(1, J_2)$ with $J_2 = \mathrm{diag}(1,-1)$. If we denote E_n by $E_n = \begin{pmatrix} E_n^1 & E_n^2 \\ E_n^3 & E_n^4 \end{pmatrix}$, then we obtain the following from the above matrix equation:

$$a_{n+1} = a_n(E_{n+1}^1E_n^{1*} + E_{n+1}^2J_2E_n^{2*})^{-1}, \tag{3.3}$$

$$B_{n+1} = B_nJ_2(E_{n+1}^3E_n^{3*} + E_{n+1}^4J_2E_n^{4*})^{-1}, \tag{3.4}$$

$$r_n = a_n(E_{n+1}^1E_n^{1*} + E_{n+1}^2J_2E_n^{2*})^{-1}(E_{n+1}^1E_n^{3*} + E_{n+1}^2J_2E_n^{4*})J_2B_n^{-1}, \tag{3.5}$$

$$q_n = B_nJ_2(E_{n+1}^3E_n^{3*} + E_{n+1}^4J_2E_n^{4*})^{-1}(E_{n+1}^3E_n^{1*} + E_{n+1}^4J_2E_n^{2*})a_n^{-1}, \tag{3.6}$$

where the invertibility of $E_{n+1}^1E_n^{1*} + E_{n+1}^2J_2E_n^{2*}$ and $E_{n+1}^3E_n^{3*} + E_{n+1}^4J_2E_n^{4*}$ is due to the fact that $I - 4S_{n+1}S_n = E_{n+1}^{-1}(E_{n+1}E_n^{-1} - 4\sigma_3E_{n+1}E_n^{-1}\sigma_3)E_n$ is invertible in (2.9) and the fact that $E_{n+1}E_n^{-1} - 4\sigma_3E_{n+1}E_n^{-1}\sigma_3 = 2\mathrm{diag}(E_{n+1}^1E_n^{1*} + E_{n+1}^2J_2E_n^{2*}, (E_{n+1}^3E_n^{3*} + E_{n+1}^4J_2E_n^{4*})J_2)$. Therefore, we may choose a_n and B_n for $n \neq 0$ progressively by relations (3.3) and (3.4) respectively and choose r_n and q_n by (3.5) and (3.6) respectively. This proves the existence of $\{G_n\}$. By using $\{G_n\}$, we make the following gauge transformation:

$$L_n^G = G_{n+1}\widetilde{L}_nG_n^{-1} = \begin{pmatrix} z & r_nz^{-1} \\ -q_nz & z^{-1}I_2 \end{pmatrix},$$

$$M_n^G = \frac{\mathrm{d}G_n}{\mathrm{d}t}G_n^{-1} + G_n\widetilde{M}_nG_n^{-1}$$
$$= \frac{\mathrm{d}G_n}{\mathrm{d}t}G_n^{-1} + \mathrm{i}\begin{pmatrix} 1 - z^2 + z - z^{-1} & r_{n-1}(z^{-2}-1) \\ -q_{n-1}(1-z^2) & (-1+z^{-2}+z-z^{-1})I_2 \end{pmatrix},$$

where $\widetilde{L}_n = \widetilde{L}_n(t,z)$ and $\widetilde{M}_n = \widetilde{M}_n(t,z)$ are the coefficients in the Lax pair (2.10). Since $\widetilde{L}_n$ and $\widetilde{M}_n$ satisfy the integrability condition, we thus have

$$\frac{\mathrm{d}L_n^G}{\mathrm{d}t} + L_n^GM_n^G - M_{n+1}^GL_n^G = 0.$$

If we set $\frac{\mathrm{d}G_n}{\mathrm{d}t}G_n^{-1} = \mathrm{i}\left(\begin{smallmatrix}\alpha_n & \beta_n \\ \mu_n & \nu_n\end{smallmatrix}\right)$, in which α_n, β_n, μ_n and ν_n do not depend on z (since $\{G_n\}$ does so) and will be determined later, then the vanishing (i.e., the coefficients of z and z^{-1}) of the diagonal part of the above identity leads to $(\forall n)$

$$\beta_n = -r_n + r_{n-1}, \quad \mu_n = -q_n + q_{n-1},$$
$$\alpha_n + r_nq_{n-1} = \alpha_{n+1} + r_{n+1}q_n, \quad \nu_n - q_nr_{n-1} = \nu_{n+1} - q_{n+1}r_n.$$

Hence,

$$\frac{\mathrm{d}G_n}{\mathrm{d}t}G_n^{-1} = \mathrm{i}\begin{pmatrix} -r_nq_{n-1} & -r_n + r_{n-1} \\ -q_n + q_{n+1} & q_nr_{n-1} \end{pmatrix} + \mathrm{i}\begin{pmatrix} \tau(t) & 0 \\ 0 & \pi(t) \end{pmatrix} \tag{3.7}$$

for some function $\tau(t)$ and 2×2-matrix valued function $\pi(t)$, which do not depend on n but may be on t, a_0 and B_0. Notice that the above restrictions on $\{G_n\}$ allow an arbitrary transformation of the form

$$G_n \to \widetilde{G}_n = \begin{pmatrix} p(t) & 0 \\ 0 & Q(t) \end{pmatrix} G_n$$

for a nonzero function $p(t)$ and a non-singular 2×2-matrix $Q(t)$. In fact, under this transformation, we have

$$\widetilde{G}_{n+1}\widetilde{G}_n^{-1} = \begin{pmatrix} 1 & \widetilde{r}_n \\ -\widetilde{q}_n & I_2 \end{pmatrix}$$

for $\widetilde{r}_n = p(t)r_nQ^{-1}(t)$ and $\widetilde{q}_n = Q(t)q_np^{-1}(t)$, and the relation $S_n = \widetilde{G}_n^{-1}\sigma_3\widetilde{G}_n$ is still preserved. A straightforward calculation shows that

$$\frac{\mathrm{d}\widetilde{G}_n}{\mathrm{d}t}\widetilde{G}_n^{-1} = \mathrm{i}\begin{pmatrix} -\widetilde{r}_n\widetilde{q}_{n-1} & -\widetilde{r}_n + \widetilde{r}_{n-1} \\ -\widetilde{q}_n + \widetilde{q}_{n+1} & \widetilde{q}_n\widetilde{r}_{n-1} \end{pmatrix} + \begin{pmatrix} p_tp^{-1} + \mathrm{i}\tau(t) & 0 \\ 0 & Q_tQ^{-1} + \mathrm{i}Q\pi(t)Q^{-1} \end{pmatrix}.$$

It is obvious that we may find a nonzero $p = p(t)$ such that $p_tp^{-1} + \mathrm{i}\tau(t) = 0$ and a non-singular $Q = Q(t)$ such that $Q_tQ^{-1} + \mathrm{i}Q\pi(t)Q^{-1} = 0$. Hence, $\{G_n\}$ can be modified so that for the new $\{G_n\}$, the second term on the right of (3.7) vanishes. This implies that L_n^G and M_n^G constructed above by the gauge transformation are respectively the coefficients L_n and M_n in the Lax pair (2.1) with $\{(r_n, q_n)\}$ given above. So the obtained $\{(r_n, q_n)\}$ is a solution to (1.5).

Now we give the following definition based on Lemma 3.1.

Definition 3.1 *A solution to* (1.5) *constructed in Lemma* 3.1 *is called an M-solution, which means that it relates to the mixed type.*

All the M-solutions consist of a class of solutions to (1.5). One notes that an M-solution $\{(r_n, q_n)\}$ to (1.5) is of the property that there exists $U(2,1)$-sequence $\{E_n\}$ such that G_n $(\forall n)$ given by (3.2) fulfills

$$\begin{cases} G_{n+1} = \begin{pmatrix} 1 & r_n \\ -q_n & I_2 \end{pmatrix} G_n, \\ \dfrac{\mathrm{d}G_n}{\mathrm{d}t} = \mathrm{i}\begin{pmatrix} -r_n q_{n-1} & -r_n + r_{n-1} \\ -q_n + q_{n+1} & q_n r_{n-1} \end{pmatrix} G_n. \end{cases} \tag{3.8}$$

One notes that (3.8) is a matrix-version of the Lax pair (2.1) at $z = 1$.

Remark 3.1 It is easy to see that, from (3.5)–(3.6), the continuous limits of a_n and B_n are $U(1)$- and $U(1,1)$-valued, respectively, is equivalent to that the continuous limits of a_0 and B_0 are $U(1)$- and $U(1,1)$-valued, respectively. Therefore, in order to guarantee that the continuous limit (r, q) of an M-solution $\{(r_n, q_n)\}$ is a solution to (1.6) satisfying $q = -r^\dagger$, and hence r is a solution to the Manakov equation (1.2) of mixed type, we require that the choices of a_0 and B_0 fulfill these restrictions. The choices of a_0 and B_0 are always assumed to fulfill the restrictions in this paper.

Next, we shall prove that the above process is reversible, that is to say, an M-solution to (1.5) is gauge equivalent to a solution to (2.9).

Lemma 3.2 *An M-solution to* (1.5) *is gauge equivalent to a solution to* (2.9)*.*

Proof Let $\{(r_n(t), q_n(t))\}$ be an M-solution to (1.5), and a corresponding solution to the Lax pair (2.1) be denoted by $\{\phi_n(t,z)\}$. Since $\{(r_n, q_n)\}$ is M-solution, from the definition, we see there is a sequence $\{G_n(t)\}$ given by (3.2) fulfills (3.8).

We now consider the following gauge transformation:

$$\phi_n(t,z) = G_n(t)\psi_n(t,z) \tag{3.9}$$

and come to prove that $\{\psi_n(t,z)\}$ determined by (3.9) is a solution to the Lax pair (2.10) of (2.9) for some $U(2,1)/U(1,1)\times U(1)$-valued sequence $\{S_n\}$. In order to do this, we put $\psi_{n+1} = \widetilde{L}_n\psi_n$ and $\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = \widetilde{M}_n\psi_n$ for some $\widetilde{L}_n$ and $\widetilde{M}_n$ which will be determined later. Applying the first equation of Lax pair (2.1), from (3.9) we have

$$\widetilde{L}_n = G_{n+1}^{-1} L_n G_n. \tag{3.10}$$

Substituting $G_{n+1} = \begin{pmatrix} 1 & r_n \\ -q_n & I_2 \end{pmatrix} G_n$ into (3.10), we obtain

$$\widetilde{L}_n = \frac{z+z^{-1}}{2} I_3 + \frac{z-z^{-1}}{2} G_n^{-1}\sigma_3 G_n := \frac{z+z^{-1}}{2} I_3 + \frac{z-z^{-1}}{2} S_n,$$

where $S_n = G_n^{-1}\sigma_3 G_n$ with $S_n^2 = -\frac{1}{4} I_3$. In what follows, we have to show that S_n is $U(2,1)/U(1,1)\times U(1)$-valued, that is to say, $S_n = E_n^{-1}\sigma_2 E_n$, $E_n \in U(2,1)$ (again, here the Lie algebra $u(2,1)=\mathbf{k}\oplus\mathbf{m}$ is indicated in §2). In fact, from the definition of M-solutions, we see that, $\forall n$,

$$G_n = G_n(t) = \mathrm{diag}(a_n(t), B_n(t))E_n(t)$$

for some $a_n(t), B_n(t)$ and $E_n = E_n(t) \in U(2,1)$ with $u(2,1)=\mathbf{k}\oplus\mathbf{m}$. Thus, we have that $S_n = G_n^{-1}\sigma_3 G_n = E_n^{-1}\sigma E_n$ is $U(2,1)/U(1,1)\times U(1)$-valued. Now, from the second equation of Lax pair (2.1) and by applying the identity: $G_{n-1}^{-1}G_n = 2(I_3 - 4S_nS_{n-1})^{-1}$, we have

$$\begin{aligned}\widetilde{M}_n &= G_n^{-1}M_nG_n - G_n^{-1}\frac{\mathrm{d}G_n}{\mathrm{d}t} = G_n^{-1}(M_n(t,z) - M_n(t,1))G_n\\ &= 4\Big(1 - \frac{z^2+z^{-2}}{2}\Big)(I_3 - 4S_nS_{n-1})^{-1}S_n + \mathrm{i}(z - z^{-1})I_3 - \mathrm{i}(z^2 - z^{-2})(I_3 - 4S_nS_{n-1})^{-1}.\end{aligned}$$

Hence, we see that $\widetilde{L}_n$ and $\widetilde{M}_n$ defined above are exactly the same coefficients as in (2.10) with S_n being $U(2,1)/U(1,1)\times U(1)$-valued. This proves that $\{S_n\}$ constructed from the M-solution $\{(r_n,q_n)\}$ to (1.5) is a solution to (2.9) with S_n being $U(2,1)/U(1,1)\times U(1)$-valued. The proof of Lemma 3.2 is completed.

Combining Lemma 3.1 with Lemma 3.2, we arrive finally at the following main result of the paper.

Theorem 3.1 *The integrable discrete equation* (2.9) *is gauge equivalent to the integrable discrete Manakov equation of mixed type.*

Theorem 3.1 indicates that the integrable discrete equation (2.9) with S_n being $U(2,1)/U(1,1)\times U(1)$-valued is a geometric realization of the integrable discrete Manakov equation of mixed type. This exhibition leads us to give a unified geometric interpretation for the integrable discrete Manakov equations of focusing, defocusing and mixed types as follows. If we call solutions to (1.5) that are constructed similarly in Lemma 3.1 from solutions $\{S_n\}$ to (2.9) with S_n being $U(2,1)/U(2)\times U(1)$-valued, as D-solutions (defocusing type), and meanwhile, call solutions to (1.5) that are constructed similarly in Lemma 3.1 from solutions $\{S_n\}$ to (2.9) with S_n being $U(3)/U(2)\times U(1)$-valued, as F-solutions (focusing type), then we see that (1.5) together with D-solutions is an integrable discretization of the Manakov equation of defocusing type, and meanwhile, (1.5) together with F-solutions is an integrable discretization of the Manakov equation of focusing type. The unified geometric interpretations are described as follows. (2.9) with S_n being $U(3)/U(2)\times U(1)$-valued, (2.9) with S_n being $U(2,1)/U(2)\times U(1)$-valued and (2.9) with S_n being $U(2,1)/U(1,1)\times U(1)$-valued are geometric realizations of the integrable discrete Manakov equation of focusing, defocusing and mixed types, respectively. Furthermore, the continuous limits of these geometric models are just respectively the Manakov equations of focusing, defocusing and mixed types.

It is obvious that Theorem 3.1 is not only a geometric interpretation for the integrable discrete Manakov equation of mixed type, but also a very useful property which may be applied in constructing solutions or in the numerical study of related problems. The applications of Theorem 3.1 deserve future investigations.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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