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Deformations of Hermitian Yang-Mills Metrics on the Iwasawa and Nakamura Threefolds*

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Abstract The authors show the stability of Hermitian Yang-Mills metrics under deformations of complex structures of either the Iwasawa or the Nakamura threefolds.

Keywords Deformations, Balanced metrics, Hermitian Yang-Mills metrics
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1 Introduction

Let X_0 be a compact complex manifold with $\dim_{\mathbb{C}} X_0 = n$ and $c_1^{BC}(X_0) = 0$, where

$$c_1^{BC}(X) \in H_{BC}^{1,1}(X,\mathbb{R}) = \frac{\{\phi \in \mathcal{A}_{\mathbb{R}}^{1,1}(X) \mid \mathrm{d}\phi = 0\}}{\{\sqrt{-1}\partial\overline{\partial}u \mid u \in \mathcal{A}_{\mathbb{R}}^{0}(X)\}}.$$

Such X_0 is named as (non-Kähler) Calabi-Yau (see [25]). In particular, if the canonical bundle K_{X_0} is holomorphically trivial, then X_0 is Calabi-Yau.

Let ω_0 be a Hermitian metric on X_0. If $\partial\overline{\partial}\omega_0^{n-1} = 0$, ω_0 is called Gauduchon. If $\mathrm{d}\omega_0^{n-1} = 0$, ω_0 is called balanced. If $\mathrm{d}\omega_0 = 0$, ω_0 is called Kähler.

Let ∇_0 be the Chern connection of an arbitrary Hermitian metric ω_0 and $R_{\omega_0} \in \mathcal{A}^{1,1}(\mathrm{End}\,(T^{1,0}X_0))$ be the curvature of ∇_0. If the mean curvature form $\sqrt{-1}\Lambda_{\omega_0}R_{\omega_0}$ satisfies the Hermitian Yang-Mills equation (see Lemma 3.2)

$$\sqrt{-1}\Lambda_{\omega_0}R_{\omega_0} = 0,$$

then ω_0 is said to be Hermitian Yang-Mills with respect to itself (see [27]).

Obviously, the above equation coincides with the Calabi-Yau equation if ω_0 is Kähler (see [30]). Thus Hermitian Yang-Mills metrics are candidates for canonical metrics on non-Kähler manifolds. For other analogous of Calabi-Yau metrics, we refer to [2–3, 8–10, 13, 17, 21–22, 25–26] and the references therein.

The first problem is the existence of Hermitian Yang-Mills metrics on non-Kähler Calabi-Yau manifolds. A typical example is the Iwasawa or the Nakamura threefold. There is a natural metric ω_0 on X_0 which is balanced with flat Chern connection ∇_0. Thus X_0 is a non-Kähler

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Calabi-Yau threefold and ω_0 is Hermitian Yang-Mills. Besides, there is no general conclusion on the existence.

A well-known example of non-Kähler Calabi-Yau threefolds is the complex structures on $\#_{k\geq 2}S^3 \times S^3$ given by the conifold transitions. Since the existence of balanced metrics on $\#_{k\geq 2}S^3 \times S^3$ was proved in [8], an essential question is whether there exists on such manifolds a balanced metric ω which is also Hermitian Yang-Mills with respect to itself (see [7]).

Assumed the existence of Hermitian Yang-Mills metrics on non-Kähler Calabi-Yau manifolds, one of the further questions is the stability of Hermitian Yang-Mills metrics under deformations of complex structures. It would be fundamental for generalizations of Calabi-Yau moduli spaces.

Deformations of complex structures of Kähler (Calabi-Yau) manifolds were studied extensively (see e.g., [23–24]). However, deformations of non-Kähler manifolds are far from well-understood, see [1, 4, 8, 11, 18–20, 29] and references therein for some results.

In particular, in contrast to the Kähler case (see e.g., [15, Chapter 4]), the property of being balanced is not stable under deformations (see [1]). Certain topological conditions were imposed to guarantee the stability of balanced metrics (see e.g., [4, 11, 19, 29]).

Denote $\Delta_\epsilon = \{t \in \mathbb{C}^m : |t| < \epsilon\}$ for some $m \in \mathbb{Z}^+$ and $\epsilon \ll 1$. Let $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0. As far as we are concerned, one of the key ingredients in [19] (or [20]) is the definition of a natural Hermitian metric ω_t on X_t (see (2.2)). Such a metric ω_t is our starting point for deformations of Hermitian Yang-Mills metrics.

Here we only consider some special compact complex threefolds and leave the general case to further study.

Let X_0 be either the Iwasawa or the Nakamura manifold (see [16]). As stated above, there exists a Hermitian Yang-Mills metric ω_0 on X_0. The Kuranishi family $\{X_t\}_{t\in\Delta_\epsilon}$ of X_0 was explicitly constructed (see [16, Section 3]). For the natural metric ω_t on X_t (given by (2.2)), we calculate its curvatures R_{ω_t} and K_{ω_t}, which enable us to obtain a Hermitian Yang-Mills metric $\widetilde{\omega}_t$ after perturbing ω_t by a contraction mapping argument (see e.g., [5–6]). However, we do not know whether the metric $\widetilde{\omega}_t$ is balanced or not.

Our main result is as follows.

Theorem 1.1 *Let X_0 be either the Iwasawa or the Nakamura manifold and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0. For $\epsilon \ll 1$, each X_t admits a Hermitian Yang-Mills metric with respect to itself.*

In Section 2, we present basic facts on deformation theory and study the geometry of (X_t, ω_t). In Section 3, we prove Theorem 1.1.

2 Preliminaries

Let X_0 be a compact complex manifold and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0. We refer to [15, Chapter 4] for the deformation theory of Kodaira-Spencer.

We first recall the extension map defined in [20] (see also [19]). The crucial thing of Kodaira-

Spencer's theory is that the complex structure on X_t is described by a vector $(0,1)$-form $\varphi(t) \in \mathcal{A}^{0,1}(T^{1,0}X_0)$ satisfying the integrability condition. More precisely, there is a complex manifold $\mathcal{X}$ and a holomorphic map $\varpi : \mathcal{X} \to \Delta_\epsilon$ satisfying the following conditions:

(1) $\varpi^{-1}(t) = X_t$ is a compact complex manifold for each $t \in \Delta_\epsilon$.

(2) The rank of the Jacobian of ϖ is m at each point of $\mathcal{X}$.

Then the complex structure on X_t is obtained by deforming the complex structure on X_0 via $\varphi(t)$ such that $\varphi(0) = 0$ and

$$\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)].$$

Let $\mathrm{d} = \partial_t + \overline{\partial}_t$ be the decomposition of d with respect to the complex structure on X_t.

Let $(z^1, \cdots, z^n)$ be local holomorphic coordinates of X_0. Let $(\zeta^1(z,t), \cdots, \zeta^n(z,t))$ be local holomorphic coordinates of X_t. For $t = 0$, both

$$(z^1, \cdots, z^n) \quad \text{and} \quad (\zeta^1(z,0), \cdots, \zeta^n(z,0))$$

are holomorphic coordinates on X_0. While for $t \in \Delta_\epsilon \setminus \{0\}$ and $k \in \{1, \cdots, n\}$, $\zeta^k(z,t)$ is only smooth on t. Denote

$$\mathcal{A}_t = \Big(\frac{\partial \zeta}{\partial z}\Big)^{-1} = \begin{pmatrix} \mathcal{A}^1_{t,1} & \cdots & \mathcal{A}^1_{t,n} \\ \vdots & & \vdots \\ \mathcal{A}^n_{t,1} & \cdots & \mathcal{A}^n_{t,n} \end{pmatrix}. \tag{2.1}$$

The vector $(0,1)$-form φ is given by (see e.g., [15, 20])

$$\varphi = \Big(\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\Big)\mathcal{A}_t\Big(\frac{\partial \zeta}{\partial \overline{z}}\Big)\begin{pmatrix} \mathrm{d}\overline{z}^1 \\ \vdots \\ \mathrm{d}\overline{z}^n \end{pmatrix} = \varphi^i_{\overline{j}} \mathrm{d}\overline{z}^j \otimes \frac{\partial}{\partial z^i},$$

where

$$\varphi^i_{\overline{j}} = \mathcal{A}^i_{t,k}\frac{\partial \zeta^k(z,t)}{\partial \overline{z}^j}.$$

We may also use φ to denote the above $n \times n$ matrix.

The extended contraction operator is

$$i_\varphi : \mathcal{A}^{p,q}(X_0) \to \mathcal{A}^{p-1,q+1}(X_0).$$

Define two operators (see e.g., [20])

$$\mathrm{e}^{i_\varphi} = \sum_{k \geq 0}\frac{1}{k!}i^k_\varphi \quad \text{and} \quad \mathrm{e}^{i_{\overline{\varphi}}} = \sum_{k \geq 0}\frac{1}{k!}i^k_{\overline{\varphi}},$$

where $i^k_\varphi = i_\varphi \circ \cdots \circ i_\varphi$. Then for $\epsilon \ll 1$,

$$\{\mathrm{e}^{i_\varphi}(\mathrm{d}z^i) = \mathrm{d}z^i + \varphi^i_{\overline{j}}\mathrm{d}\overline{z}^j\}^n_{i=1} \quad \text{and} \quad \{\mathrm{e}^{i_{\overline{\varphi}}}(\mathrm{d}\overline{z}^j) = \mathrm{d}\overline{z}^j + \overline{\varphi}^j_{\overline{i}}\mathrm{d}z^i\}^n_{j=1}$$

are local smooth frames of $\mathcal{A}^{1,0}(X_t)$ and $\mathcal{A}^{0,1}(X_t)$, respectively.

By [20, Lemma 2.5], the local smooth frame $\{\mathrm{e}^{i_\varphi}(\mathrm{d}z^i)\}^n_{i=1}$ and the local holomorphic frame $\{\mathrm{d}\zeta^i\}^n_{i=1}$ are related by

$$\mathrm{e}^{i_\varphi}(\mathrm{d}z^i) = \mathcal{A}^i_{t,k}\mathrm{d}\zeta^k.$$

Moreover, for

$$\sigma = \sigma_{i_1\cdots i_p\overline{j}_1\cdots\overline{j}_q}(z)\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q} \in \mathcal{A}^{p,q}(X_0),$$

the extension map (see [20, Definition 2.8])

$$\mathrm{e}^{i_\varphi|i_{\overline{\varphi}}} : \mathcal{A}^{p,q}(X_0) \to \mathcal{A}^{p,q}(X_t)$$

is given by

$$\mathrm{e}^{i_\varphi|i_{\overline{\varphi}}}(\sigma) = \sigma_{i_1\cdots i_p\overline{j}_1\cdots\overline{j}_q}(z)\mathrm{e}^{i_\varphi}(\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p}) \wedge \mathrm{e}^{i_{\overline{\varphi}}}(\mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q}).$$

It is proved in [20, Lemma 9] that $\mathrm{e}^{i_\varphi|i_{\overline{\varphi}}}$ is a linear isomorphism for $\epsilon \ll 1$.

Let

$$\omega_0 = \sqrt{-1}g_{0,i\overline{j}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$$

be a Hermitian metric on X_0. Then for $\epsilon \ll 1$,

$$\begin{aligned}\omega_t = \mathrm{e}^{i_\varphi|i_{\overline{\varphi}}}(\omega_0) &= \sqrt{-1}g_{0,i\overline{j}}\mathrm{e}^{i_\varphi}(\mathrm{d}z^i) \wedge \mathrm{e}^{i_{\overline{\varphi}}}(\mathrm{d}\overline{z}^j) \\ &= \sqrt{-1}g_{t,i\overline{j}}\mathrm{d}\zeta^i \wedge \mathrm{d}\overline{\zeta}^j \end{aligned} \tag{2.2}$$

is a Hermitian metric on X_t, where $g_{t,i\overline{j}} = g_{0,k\overline{l}}\mathcal{A}^k_{t,i}\overline{\mathcal{A}}^l_{t,j}$. The metric ω_t is our starting point for the proof of Theorem 1.1.

Let X_0 be either the Iwasawa or the Nakamura manifold and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0 (see [16]).

There is a natural metric ω_0 on X_0 which is balanced with flat Chern connection ∇_0. For $\epsilon \ll 1$, the natural metric ω_t on X_t is given by (2.2), and the Chern connection of ω_t is ∇_t. Then the curvature and mean curvature of ∇_t are R_{ω_t} and K_{ω_t} (see e.g., [12, Section 1.7]), respectively.

Our main result in this subsection is as follows.

Proposition 2.1 *Under the above setting, the norms of the curvature and the mean curvature of the Chern connection ∇_t satisfy*

$$|R_{\omega_t}|_{\omega_t} = |K_{\omega_t}|_{\omega_t} = o(|t|^2)$$

for $\epsilon \ll 1$.

In the rest of this part, we prove Proposition 2.1 for the Iwasawa manifold. While the proof for the Nakamura manifold is given in the appendix.

Denote

$$G = \left\{ \begin{pmatrix} 1 & z^2 & z^3 \\ 0 & 1 & z^1 \\ 0 & 0 & 1 \end{pmatrix} ; z^k \in \mathbb{C} \right\} \simeq \mathbb{C}^3$$

and

$$\Gamma = \left\{ \begin{pmatrix} 1 & w^2 & w^3 \\ 0 & 1 & w^1 \\ 0 & 0 & 1 \end{pmatrix} ; w^k \in \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z} \right\}.$$

The Iwasawa manifold $X_0 = G/\Gamma$ is the quotient by the action of Γ on G:

$$\begin{pmatrix} 1 & z^2 & z^3 \\ 0 & 1 & z^1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & w^2 & w^3 \\ 0 & 1 & w^1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z^2 + w^2 & z^3 + z^2 w^1 + w^3 \\ 0 & 1 & z^1 + w^1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote

$$t = (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}) \in \Delta_\epsilon = \{t \in \mathbb{C}^6 : |t| < \epsilon\}.$$

Let $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0 for $\epsilon \ll 1$ (see [16, p. 95]).

Let $\{z^1, z^2, z^3\}$ be holomorphic coordinates on X_0. The integrable vector $(0,1)$-form $\varphi \in \mathcal{A}^{0,1}(T^{1,0}X_0)$ is (see [16, p. 95])

$$\varphi = \Big(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}\Big) \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ \eta_1 & \eta_2 + D\overline{z}^1 & -D \end{pmatrix} \begin{pmatrix} \mathrm{d}\overline{z}^1 \\ \mathrm{d}\overline{z}^2 \\ \mathrm{d}\overline{z}^3 \end{pmatrix}, \tag{2.3}$$

where

$$\eta_1 = t_{31} + t_{21} z^1, \quad \eta_2 = t_{32} + t_{22} z^1, \quad D = t_{11} t_{22} - t_{12} t_{21}.$$

The holomorphic coordinates $\{\zeta^1, \zeta^2, \zeta^3\}$ on X_t are given by (see [16, p. 95]):

$$\zeta^1 = z^1 + \sum_{l=1}^{2} t_{1l}\overline{z}^l, \qquad \zeta^2 = z^2 + \sum_{l=1}^{2} t_{2l}\overline{z}^l, \quad \zeta^3 = z^3 + \sum_{l=1}^{2} \eta_l \overline{z}^l + A - D\overline{z}^3, \tag{2.4}$$

where

$$2A = t_{11} t_{21} (\overline{z}^1)^2 + 2 t_{11} t_{22} \overline{z}^1 \overline{z}^2 + t_{12} t_{22} (\overline{z}^2)^2.$$

Then we obtain from (2.1) that

$$\begin{pmatrix} \mathcal{A}^1_{t,1} & \cdots & \mathcal{A}^1_{t,3} \\ \vdots & & \vdots \\ \mathcal{A}^3_{t,1} & \cdots & \mathcal{A}^3_{t,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(\zeta^2 - z^2) & 0 & 1 \end{pmatrix}. \tag{2.5}$$

Since

$$\{\phi_0^1 = \mathrm{d}z^1,\ \phi_0^2 = \mathrm{d}z^2,\ \phi_0^3 = \mathrm{d}z^3 - z^1 \mathrm{d}z^2\}$$

is a global basis of the space of holomorphic 1-forms on X_0, the three $(1,0)$-forms

$$\phi_t^1 = \mathrm{e}^{i_\varphi}(\phi_0^1) = \mathrm{d}\zeta^1, \quad \phi_t^2 = \mathrm{e}^{i_\varphi}(\phi_0^2) = \mathrm{d}\zeta^2$$

and

$$\phi_t^3 = \mathrm{e}^{i_\varphi}(\phi_0^3) = \mathrm{d}\zeta^3 - z^1 \mathrm{d}\zeta^2 - (\zeta^2 - z^2)\mathrm{d}\zeta^1$$

form a basis of the space of smooth $(1,0)$-forms on X_t. Since

$$\phi_t^1 \wedge \phi_t^2 \wedge \phi_t^3 = \mathrm{d}\zeta^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\zeta^3,$$

the canonical bundle K_{X_t} of X_t is holomorphically trivial.

It is well-known that the natural metric

$$\omega_0=\sqrt{-1}\sum_{l=1}^{3}\phi_0^l\wedge\overline{\phi}_0^l=\sqrt{-1}(\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1+(1+|z^1|^2)\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^2+\mathrm{d}z^3\wedge\mathrm{d}\overline{z}^3 - z^1\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^3-\overline{z}^1\mathrm{d}z^3\wedge\mathrm{d}\overline{z}^2)$$

on X_0 is balanced with flat Chern connection ∇_0. Thus X_0 is a non-Kähler Calabi-Yau threefold and ω_0 is Hermitian Yang-Mills.

Moreover, for $\epsilon \ll 1$, there is a natural Hermitian metric on X_t (given by (2.2)):

$$\omega_t=\mathrm{e}^{i_\varphi|i_{\overline{\varphi}}}(\omega_0)=\sqrt{-1}g_{t,i\overline{j}}\mathrm{d}\zeta^i\wedge\mathrm{d}\overline{\zeta}^j,$$

where

$$\begin{pmatrix} g_{t,1\overline{1}} & \cdots & g_{t,1\overline{3}}\\ \vdots & & \vdots\\ g_{t,3\overline{1}} & \cdots & g_{t,3\overline{3}}\end{pmatrix}=\begin{pmatrix} 1+|\zeta^2-z^2|^2 & \overline{z}^1(\zeta^2-z^2) & -(\zeta^2-z^2)\\ z^1(\overline{\zeta^2-z^2}) & 1+|z^1|^2 & -z^1\\ -(\overline{\zeta^2-z^2}) & -\overline{z}^1 & 1\end{pmatrix}. \tag{2.6}$$

It is direct to check

$$\omega_t=\sqrt{-1}\sum_{l=1}^{3}\phi_t^l\wedge\overline{\phi}_t^l.$$

As mentioned above, the property of being balanced is not stable under deformations of the Iwasawa manifold (see [1]). As for the natural metric ω_t, we have the following observation.

Proposition 2.2 *Let X_0 be the Iwasawa manifold and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0. The natural metric ω_t on X_t is Gauduchon and Chern-Ricci flat. Moreover, it is balanced if and only if*

$$t_{21}|t_{12}|^2-t_{12}|t_{21}|^2=t_{11}t_{22}(\overline{t_{12}-t_{21}})+t_{12}-t_{21}.$$

Proof We obtain from (2.6) that $\det\omega_t=1$. Then the Chern-Ricci curvature of ω_t is

$$\rho_{\omega_t}=\sqrt{-1}\,\overline{\partial}_t\partial_t\log\det\omega_t=0.$$

Direct calculation yields

$$\begin{aligned}\omega_t^2=2\sqrt{-1}^2(&(1+|z^1|^2+|\zeta^2-z^2|^2)\mathrm{d}\zeta^1\wedge\mathrm{d}\overline{\zeta}^1\wedge\mathrm{d}\zeta^2\wedge\mathrm{d}\overline{\zeta}^2\\ &-z^1\mathrm{d}\zeta^1\wedge\mathrm{d}\overline{\zeta}^1\wedge\mathrm{d}\zeta^2\wedge\mathrm{d}\overline{\zeta}^3-(\zeta^2-z^2)\mathrm{d}\zeta^1\wedge\mathrm{d}\zeta^2\wedge\mathrm{d}\overline{\zeta}^2\wedge\mathrm{d}\overline{\zeta}^3\\ &+\overline{z}^1\mathrm{d}\zeta^1\wedge\mathrm{d}\overline{\zeta}^1\wedge\mathrm{d}\overline{\zeta}^2\wedge\mathrm{d}\zeta^3+(\overline{\zeta^2-z^2})\mathrm{d}\overline{\zeta}^1\wedge\mathrm{d}\zeta^2\wedge\mathrm{d}\overline{\zeta}^2\wedge\mathrm{d}\zeta^3\\ &+\mathrm{d}\zeta^1\wedge\mathrm{d}\overline{\zeta}^1\wedge\mathrm{d}\zeta^3\wedge\mathrm{d}\overline{\zeta}^3+\mathrm{d}\zeta^2\wedge\mathrm{d}\overline{\zeta}^2\wedge\mathrm{d}\zeta^3\wedge\mathrm{d}\overline{\zeta}^3).\end{aligned} \tag{2.7}$$

To calculate $\overline{\partial}_t\widehat{\omega}_t^2$, we use (2.3) to show

$$(I-\varphi\overline{\varphi})=\begin{pmatrix}\overline{\alpha} & -\overline{\beta} & 0\\ -\overline{\gamma} & \overline{\delta} & 0\\ D\overline{t}_{31}-\eta_1\overline{t}_{11}-\eta_2\overline{t}_{21} & D(\overline{t}_{31}+\overline{D}z^1)-\eta_1\overline{t}_{12}-\eta_2\overline{t}_{22} & 1-|D|^2\end{pmatrix},$$

where

$$\alpha = 1 - |t_{11}|^2 - t_{21}\overline{t}_{12}, \quad \delta = 1 - |t_{22}|^2 - t_{12}\overline{t}_{21}$$

and

$$\beta = t_{12}\overline{t}_{11} + t_{22}\overline{t}_{12}, \quad \gamma = t_{11}\overline{t}_{21} + t_{21}\overline{t}_{22}.$$

Then we get

$$(I - \overline{\varphi}\varphi)^{-1} = \begin{pmatrix} \dfrac{\beta\gamma}{\alpha^2\sigma} + \dfrac{1}{\alpha} & \dfrac{\beta}{\alpha\sigma} & 0 \\ \dfrac{\gamma}{\alpha\sigma} & \dfrac{1}{\sigma} & 0 \\ * & * & \dfrac{1}{1-|D|^2} \end{pmatrix}, \tag{2.8}$$

where

$$\sigma = \delta - \frac{\beta\gamma}{\alpha}$$

and elements denoted by $*$ are irrelevant for us.

By [20, Lemma 2.4], we obtain from (2.5) and (2.8) that

$$\Big(\frac{\partial \overline{z}}{\partial \overline{\zeta}}\Big) = (I - \overline{\varphi}\varphi)^{-1}\overline{\mathcal{A}}_t = \begin{pmatrix} \dfrac{\beta\gamma}{\alpha^2\sigma} + \dfrac{1}{\alpha} & \dfrac{\beta}{\alpha\sigma} & 0 \\ \dfrac{\gamma}{\alpha\sigma} & \dfrac{1}{\sigma} & 0 \\ * & * & \dfrac{1}{1-|D|^2} \end{pmatrix},$$

which implies

$$\begin{aligned} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} &:= \begin{pmatrix} \dfrac{\partial z^1}{\zeta^1} & \dfrac{\partial z^1}{\zeta^2} \\ \dfrac{\partial z^2}{\zeta^1} & \dfrac{\partial z^2}{\zeta^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 + \dfrac{|t_{11}|^2}{\alpha} + \dfrac{\lambda_1\mu_1}{\sigma} & \dfrac{t_{11}\overline{t}_{12}}{\alpha} + \dfrac{\lambda_2\mu_1}{\sigma} \\ \dfrac{t_{21}\overline{t}_{11}}{\alpha} + \dfrac{\lambda_1\mu_2}{\sigma} & 1 + \dfrac{t_{21}\overline{t}_{12}}{\alpha} + \dfrac{\lambda_2\mu_2}{\sigma} \end{pmatrix}. \end{aligned} \tag{2.9}$$

Here

$$\lambda_1 = \overline{t}_{21} + \frac{\gamma}{\alpha}\overline{t}_{11}, \quad \lambda_2 = \overline{t}_{22} + \frac{\gamma}{\alpha}\overline{t}_{12}$$

and

$$\mu_1 = t_{12} + \frac{\beta}{\alpha}t_{11}, \quad \mu_2 = t_{22} + \frac{\beta}{\alpha}t_{21}.$$

Similarly, we obtain from (2.3), (2.5) and (2.8) that

$$-\Big(\frac{\partial z}{\partial \overline{\zeta}}\Big) = \varphi(I - \overline{\varphi}\varphi)^{-1}\overline{\mathcal{A}}_t = \begin{pmatrix} \Big(1 + \dfrac{\beta\gamma}{\alpha\sigma}\Big)\dfrac{t_{11}}{\alpha} + \dfrac{t_{12}\gamma}{\alpha\sigma} & \dfrac{t_{11}\beta}{\alpha\sigma} + \dfrac{t_{12}}{\sigma} & 0 \\ \Big(1 + \dfrac{\beta\gamma}{\alpha\sigma}\Big)\dfrac{t_{21}}{\alpha} + \dfrac{t_{22}\gamma}{\alpha\sigma} & \dfrac{t_{21}\beta}{\alpha\sigma} + \dfrac{t_{22}}{\sigma} & 0 \\ * & * & \dfrac{D}{|D|^2 - 1} \end{pmatrix},$$

which implies

$$\begin{pmatrix} \tau_{31} & \tau_{32} \\ \tau_{41} & \tau_{42} \end{pmatrix} := \begin{pmatrix} \dfrac{\partial z^1}{\overline{\zeta}^1} & \dfrac{\partial z^1}{\overline{\zeta}^2} \\ \dfrac{\partial z^2}{\overline{\zeta}^1} & \dfrac{\partial z^2}{\overline{\zeta}^2} \end{pmatrix} = - \begin{pmatrix} \dfrac{t_{11}}{\alpha} + \dfrac{\gamma\mu_1}{\alpha\sigma} & \dfrac{\mu_1}{\sigma} \\ \dfrac{t_{21}}{\alpha} + \dfrac{\gamma\mu_2}{\alpha\sigma} & \dfrac{\mu_2}{\sigma} \end{pmatrix}. \tag{2.10}$$

By (2.7) and (2.9)–(2.10), we get

$$\overline{\partial}_t \omega_t^2 = 2\sqrt{-1}^2 (\tau_{32} - \tau_{41}) \mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\overline{\zeta}^3,$$

which implies

$$\partial_t \overline{\partial}_t \omega_t^2 = 0,$$

i.e., ω_t is Gauduchon. While ω_t is balanced if and only if $\tau_{32} = \tau_{41}$, which is exactly the desired identity.

Proof of Proposition 2.1 Let X_0 be the Iwasawa manifold and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0.

By (2.6), we get

$$\begin{pmatrix} g_t^{1\overline{1}} & \cdots & g_t^{3\overline{1}} \\ \vdots & & \vdots \\ g_t^{1\overline{3}} & \cdots & g_t^{3\overline{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \zeta^2 - z^2 \\ 0 & 1 & z^1 \\ (\overline{\zeta^2 - z^2}) & \overline{z}^1 & 1 + |z^1|^2 + |\zeta^2 - z^2|^2 \end{pmatrix} \tag{2.11}$$

and

$$\partial_t g_t g_t^{-1} = \begin{pmatrix} -(\zeta^2 - z^2)\partial_t \overline{z}^2 & (\zeta^2 - z^2)\partial_t \overline{z}^1 & \theta_1^3 \\ -z^1 \partial_t \overline{z}^2 & z^1 \partial_t \overline{z}^1 & \theta_2^3 \\ \partial_t \overline{z}^2 & -\partial_t \overline{z}^1 & (\zeta^2 - z^2)\partial_t \overline{z}^2 - z^1 \partial_t \overline{z}^1 \end{pmatrix},$$

where

$$\theta_1^3 = -(\zeta^2 - z^2)^2 \partial_t \overline{z}^2 + z^1(\zeta^2 - z^2)\partial_t \overline{z}^1 - \partial_t(\zeta^2 - z^2)$$

and

$$\theta_2^3 = (z^1)^2 \partial_t \overline{z}^1 - z^1(\zeta^2 - z^2)\partial_t \overline{z}^2 - \partial_t z^1.$$

By definition of the Chern curvature (see e.g., [12, 14]), we obtain from (2.10) that

$$\begin{aligned} R_{\omega_t}^{\mathrm{T}} &= \overline{\partial}_t(\partial_t g_t g_t^{-1}) \\ &= \begin{pmatrix} \overline{\partial}_t z^2 \wedge \partial_t \overline{z}^2 & -\overline{\partial}_t z^2 \wedge \partial_t \overline{z}^1 & \Omega_1^3 \\ -\overline{\partial}_t z^1 \wedge \partial_t \overline{z}^2 & \overline{\partial}_t z^1 \wedge \partial_t \overline{z}^1 & \Omega_2^3 \\ 0 & 0 & -\overline{\partial}_t z^1 \wedge \partial_t \overline{z}^1 - \overline{\partial}_t z^2 \wedge \partial_t \overline{z}^2 \end{pmatrix}, \end{aligned}$$

where

$$\Omega_1^3 = (\zeta^2 - z^2)(2\overline{\partial}_t z^2 \wedge \partial_t \overline{z}^2 + \overline{\partial}_t z^1 \wedge \partial_t \overline{z}^1) - z^1 \overline{\partial}_t z^2 \wedge \partial_t \overline{z}^1$$

and

$$\Omega_2^3 = z^1(2\overline{\partial}_t z^1 \wedge \partial_t \overline{z}^1 + \overline{\partial}_t z^2 \wedge \partial_t \overline{z}^2) - (\zeta^2 - z^2)\overline{\partial}_t z^1 \wedge \partial_t \overline{z}^2.$$

Since R_{ω_t} is skew-symmetric with respect to ω_t: $R^*_{\omega_t} = -R_{\omega_t}$, we get (see e.g., [14])

$$
\begin{aligned}
|R_{\omega_t}|^2_{\omega_t}\frac{\omega_t^3}{3!} &= \operatorname{tr}(R_{\omega_t}\wedge *_t R^*_{\omega_t})\\
&= -2(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2\wedge *_t(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2) + \overline{\partial}_t z^1\wedge\partial_t\overline{z}^1\wedge *_t(\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1))\\
&\quad - (\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1\wedge *_t(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2) + \overline{\partial}_t z^2\wedge\partial_t\overline{z}^2\wedge *_t(\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1)\\
&\quad + \overline{\partial}_t z^1\wedge\partial_t\overline{z}^2\wedge *_t(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^1) + \overline{\partial}_t z^2\wedge\partial_t\overline{z}^1\wedge *_t(\overline{\partial}_t z^1\wedge\partial_t\overline{z}^2)),
\end{aligned}
$$

where $*_t$ is the Hodge star operator with respect to ω_t.

By (2.9)–(2.10), we have

$$
\begin{aligned}
\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1 &= |\tau_{31}|^2 d\overline{\zeta}^1\wedge d\zeta^1 + |\tau_{32}|^2 d\overline{\zeta}^2\wedge d\zeta^2\\
&\quad + \tau_{31}\overline{\tau}_{32} d\overline{\zeta}^1\wedge d\zeta^2 + \tau_{32}\overline{\tau}_{31} d\overline{\zeta}^2\wedge d\zeta^1,\\
\overline{\partial}_t z^1\wedge\partial_t\overline{z}^2 &= \tau_{31}\overline{\tau}_{41} d\overline{\zeta}^1\wedge d\zeta^1 + \tau_{32}\overline{\tau}_{42} d\overline{\zeta}^2\wedge d\zeta^2\\
&\quad + \tau_{31}\overline{\tau}_{42} d\overline{\zeta}^1\wedge d\zeta^2 + \tau_{32}\overline{\tau}_{41} d\overline{\zeta}^2\wedge d\zeta^1,\\
\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2 &= |\tau_{41}|^2 d\overline{\zeta}^1\wedge d\zeta^1 + |\tau_{42}|^2 d\overline{\zeta}^2\wedge d\zeta^2\\
&\quad + \tau_{41}\overline{\tau}_{42} d\overline{\zeta}^1\wedge d\zeta^2 + \tau_{42}\overline{\tau}_{41} d\overline{\zeta}^2\wedge d\zeta^1,
\end{aligned}
\tag{2.12}
$$

which together with (2.11) imply

$$
\begin{aligned}
\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1\wedge *_t(\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1) &= -(|\tau_{31}|^2+|\tau_{32}|^2)^2\frac{\omega_t^3}{3!},\\
\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2\wedge *_t(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2) &= -(|\tau_{41}|^2+|\tau_{42}|^2)^2\frac{\omega_t^3}{3!},\\
\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1\wedge *_t(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^2) &= \overline{\partial}_t z^2\wedge\partial_t\overline{z}^2\wedge *_t(\overline{\partial}_t z^1\wedge\partial_t\overline{z}^1)\\
&= -|\tau_{31}\overline{\tau}_{41}+\tau_{42}\overline{\tau}_{32}|^2\frac{\omega_t^3}{3!}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\partial}_t z^1\wedge\partial_t\overline{z}^2\wedge *_t(\overline{\partial}_t z^2\wedge\partial_t\overline{z}^1) &= \overline{\partial}_t z^2\wedge\partial_t\overline{z}^1\wedge *_t(\overline{\partial}_t z^1\wedge\partial_t\overline{z}^2)\\
&= -(|\tau_{31}|^2+|\tau_{32}|^2)(|\tau_{41}|^2+|\tau_{42}|^2)\frac{\omega_t^3}{3!}.
\end{aligned}
$$

Combining the above yields

$$
\begin{aligned}
|R_{\omega_t}|^2_{\omega_t} &= 2(|\tau_{31}\overline{\tau}_{41}+\tau_{42}\overline{\tau}_{32}|^2 + (|\tau_{31}|^2+|\tau_{32}|^2)(|\tau_{41}|^2+|\tau_{42}|^2)\\
&\quad + (|\tau_{31}|^2+|\tau_{32}|^2)^2 + (|\tau_{41}|^2+|\tau_{42}|^2)^2).
\end{aligned}
$$

By (2.11)–(2.12), the mean curvature $K_{\omega_t} = \sqrt{-1}\Lambda_{\omega_t}R_{\omega_t}$ is

$$
K^{\mathrm{T}}_{\omega_t} = \begin{pmatrix} -|\tau_{41}|^2-|\tau_{42}|^2 & \tau_{41}\overline{\tau}_{31}+\tau_{42}\overline{\tau}_{32} & K_1^3\\ \tau_{31}\overline{\tau}_{41}+\tau_{32}\overline{\tau}_{42} & -|\tau_{31}|^2-|\tau_{32}|^2 & K_2^3\\ 0 & 0 & |\tau_{31}|^2+|\tau_{32}|^2+|\tau_{41}|^2+|\tau_{42}|^2 \end{pmatrix},
$$

where

$$
K_1^3 = -(\zeta^2-z^2)(|\tau_{31}|^2+|\tau_{32}|^2+2|\tau_{41}|^2+2|\tau_{42}|^2) + z^1(\overline{\tau}_{31}\tau_{41}+\overline{\tau}_{32}\tau_{42})
$$

and

$$K_2^3 = -z^1(2|\tau_{31}|^2 + 2|\tau_{32}|^2 + |\tau_{41}|^2 + |\tau_{42}|^2) + (\zeta^2 - z^2)(\tau_{31}\overline{\tau}_{41} + \tau_{32}\overline{\tau}_{42}).$$

Since K_{ω_t} is skew-symmetric with respect to ω_t: $K_{\omega_t}^* = K_{\omega_t}$, we get (see e.g., [14])

$$|K_{\omega_t}|_{\omega_t}^2 = \mathrm{tr}(K_{\omega_t} \circ K_{\omega_t}^*) = |R_{\omega_t}|_{\omega_t}^2.$$

The conclusion for the Iwasawa manifold is valid.

3 Proof of Theorem 1.1

Let X_0 be either the Iwasawa or the Nakamura manifold and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0. In this section, we will perturb the natural metric ω_t (given by (2.2)) on X_t via a contraction mapping to show the existence of a Hermitian Yang-Mills metric $\widetilde{\omega}_t$ on X_t. We refer to [5–6] and references therein for gluing constructions of Hermitian Yang-Mills metrics. We refer to [2, 13] and references therein for constructions of Hermitian Yang-Mills metrics by the implicit function theorem.

Fix $t \in \Delta_\epsilon$ for sufficiently small ϵ (to be determined) and a diffeomorphism $\Psi_t : X_t \to X_0$ such that $\Psi_0 = \mathrm{id}$. Let ω_0 be the natural balanced metric on X_0. For $\epsilon \ll 1$, the $(1,1)$-part

$$\widehat{\omega}_t = (\Psi_t^*\omega_0)^{1,1}$$

of the 2-form $\Psi_t^*\omega_0$ on X_t is a Hermitian metric (see e.g., [8, 11]). We introduce the extra metric $\widehat{\omega}_t$ for the uniqueness in the contraction mapping argument (see (3.17)).

Let ω_t be the natural Hermitian metric (given by (2.2)) on X_t. Let D_t be the Chern connection of ω_t and ∇_t be its covariant derivatives. Hereafter we will use ω_t, $\widehat{\omega}_t$, g_t and $\widehat{g}_t$ interchangeably to denote the two metrics.

By calculations in Proposition 2.1, we have the following Lemma.

Lemma 3.1 *For $\epsilon \ll 1$ and $l = 0, 1, 2, \cdots$, there exist uniform constants $C_l > 0$ such that*

$$|\widehat{g}_t^{-1} - g_t^{-1}|_{\widehat{g}_t} \le C_0|t|, \quad |\widehat{\nabla}_{\widehat{g}_t}^l(\widehat{g}_t - g_t)|_{\widehat{g}_t} \le C_l|t|,$$

where $\widehat{\nabla}_{\widehat{g}_t}$ is the covariant derivative of $\widehat{g}_t$.

Denote

$$W_t = \{\eta \in \Gamma(\mathrm{End}(T^{1,0}X_t)) : \eta^* = \eta\},$$

where the adjoint $*$ is with respect to g_t.

Recall the definition of Hölder spaces (see e.g., [5–6, 14]). For $\eta \in \Gamma(\mathrm{End}(T^{1,0}X_t))$, we define

$$\|\eta\|_{C^k(g_t,\widehat{g}_t)} = \sum_{l=0}^k \sup |\nabla_t^l \eta|_{\widehat{g}_t}.$$

For $\Phi \in \mathcal{A}^{p,q}(X_t)$, we define

$$[\Phi]_{C^{0,\alpha}(g_t)} = \sup_{x\neq y}\Big(\frac{|\Phi(x) - \Phi(y)|_{\widehat{g}_t}}{d(x,y)^\alpha}\Big),$$

where the sup is taken over points x and y with distance less than the injectivity radius and $\Phi(x)-\Phi(y)$ is $\widehat{\nabla}_{\widehat{g}_t}$-parallel transport along the minimal $\widehat{g}_t$ geodesic connecting x and y. Then we define

$$\|\eta\|_{C^{k,\alpha}(X_t)} = \|\eta\|_{C^k(g_t,\widehat{g}_t)} + [\nabla_t^k \eta]_{C^{0,\alpha}(\widehat{g}_t)}.$$

We need one more auxiliary result.

Lemma 3.2 *Let X_0 be a compact (non-Kähler) Calabi-Yau manifold. If ω is a Hermitian metric with $K_\omega = \varphi I$ for some $\varphi \in \mathcal{A}^0_{\mathbb{R}}(X_0)$ of constant sign, then $\varphi = 0$ and ω is Chern-Ricci flat.*

Proof Since $c_1^{BC}(X_0) = 0$, there exists $u \in \mathcal{A}^0_{\mathbb{R}}(X_0)$ such that the Chern-Ricci curvature of ω is

$$\rho_\omega = \sqrt{-1}\partial\overline{\partial}u,$$

which implies the scalar curvature is

$$s_\omega = \Lambda_\omega \rho_\omega = \sqrt{-1}\Lambda_\omega \partial\overline{\partial}u.$$

On the other hand, since $s_\omega = \mathrm{tr}K_\omega = n\varphi$ is of constant sign, u is a constant by the maximum principle (see e.g., [14, Lemma 7.2.7]).

Then we have $\varphi = 0$ and $\rho_\omega = 0$.

Let $\mathcal{G}^{\mathbb{C}} = \Gamma(\mathrm{GL}(T^{1,0}X_t, \mathbb{C}))$ be the complex gauge group of $T^{1,0}X_t$. Denote

$$\mathcal{H}_t = \mathcal{G}^{\mathbb{C}} \cap W_t.$$

For $f \in W_t$ with

$$S = f + I \in \mathcal{H}_t,$$

we define

$$D_f = D'_f + D''_f = S \circ D'_t \circ \widetilde{S} + \widetilde{S} \circ D''_t \circ S \tag{3.1}$$

and

$$\widetilde{D}_t = \widetilde{D}'_t + \widetilde{D}''_t = S \circ D_f \circ \widetilde{S}, \tag{3.2}$$

where we have denoted $\widetilde{S} = S^{-1}$ for convenience later.

For $u, v \in \Gamma(T^{1,0}X_t)$, we define $\widetilde{g}_t = (\widetilde{S}^2)^{\mathrm{T}} g_t$ by

$$\widetilde{g}_t(u,v) = g_t(\widetilde{S}(u), \widetilde{S}(v)) = g_t(\widetilde{S}^2(u), v), \tag{3.3}$$

where the last identity follows from that $\widetilde{S}$ is symmetric with respect to g_t. Then S and $\widetilde{S}$ are also symmetric with respect to $\widetilde{g}_t$.

Lemma 3.3 *The connection D_f is the Chern connection of (g_t, D''_f). The connection $\widetilde{D}_t$ is the Chern connection of $(\widetilde{g}_t, D''_t)$.*

Proof Since

$$D_f'' \circ D_f'' = \widetilde{S} \circ D_t'' \circ D_t'' \circ S = 0,$$

to show the first statement, we only need to check

$$g_t(D_f'(u), v) + g_t(u, D_f''(v)) = \partial_t g_t(u, v).$$

By definition (3.1), we get

$$\begin{aligned} g_t(D_f'(u), v) + g_t(u, D_f''(v)) &= g_t(D_t'(\widetilde{S}(u)), S(v)) + g_t(\widetilde{S}(u), D_t''(S(v))) \\ &= \partial_t g_t(\widetilde{S}(u), S(v)) = \partial_t g_t(u, v). \end{aligned}$$

Since

$$\widetilde{D}_f'' = \widetilde{S} \circ D_f'' \circ S = D_t'',$$

to show the second statement, we only need to check

$$\widetilde{g}_t(\widetilde{D}_t'(u), v) + \widetilde{g}_t(u, \widetilde{D}_t''(v)) = \partial_t \widetilde{g}_t(u, v).$$

By definitions (3.2)–(3.3), we get

$$\begin{aligned} \widetilde{g}_t(\widetilde{D}_t'(u), v) + \widetilde{g}_t(u, \widetilde{D}_t''(v)) &= g_t(D_t'(\widetilde{S}^2(u)), v) + g_t(\widetilde{S}^2(u), D_t''(v)) \\ &= \partial_t g_t(\widetilde{S}^2(u), v) = \partial_t \widetilde{g}_t(u, v), \end{aligned}$$

which completes the proof.

Indeed, one can deform either a metric or a connection for the convenience of their respective problems, e.g., [2, 5–6, 13].

Denote the curvatures of D_t, $\widetilde{D}_t$ and D_f by R_{ω_t}, $R_{\widetilde{\omega}_t}$ and F_{D_f}, respectively. Then we obtain from (3.1)–(3.2) that

$$R_{\widetilde{\omega}_t} = S \circ F_{D_f} \circ \widetilde{S},$$

which implies that

$$\sqrt{-1}\Lambda_{\widetilde{\omega}_t} R_{\widetilde{\omega}_t} = 0 \tag{3.4}$$

if and only if

$$\sqrt{-1}\Lambda_{\widetilde{\omega}_t} F_{D_f} = 0. \tag{3.5}$$

Here we have used Lemma 3.2.

We will solve the equation (3.5) in this part. To begin with, we calculate the linearization of the mean curvature $\sqrt{-1}\Lambda_{\widetilde{\omega}_t} F_{D_f}$ at $f = 0$.

Let $\{\frac{\partial}{\partial \zeta^l}\}_{l=1}^3$ be a local holomorphic frame of D_t''. Then we can write

$$\omega_t = \sqrt{-1} g_{t, i\overline{j}} \mathrm{d}\zeta^i \wedge \mathrm{d}\overline{\zeta}^j$$

and

$$f = f_i^p \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i \quad \text{and} \quad S = S_i^p \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i$$

for $S_i^p = f_i^p + \delta_i^p$.

Denote the Chern connection of (g_t, D_t'') and its curvature by

$$\Gamma_{t,ik}^p = g_t^{p\bar{j}} \frac{\partial g_{t,i\bar{j}}}{\partial \zeta^k} \quad \text{and} \quad R_{t,ik\bar{l}}^p = -\frac{\partial \Gamma_{t,ik}^p}{\partial \overline{\zeta}^l}.$$

Then we can express R_{ω_t} and F_{D_f} by

$$R_{\omega_t} = R_{t,ik\bar{l}}^p \mathrm{d}\zeta^k \wedge \mathrm{d}\overline{\zeta}^l \otimes \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i$$

and

$$F_{D_f} = F_{t,ik\bar{l}}^p \mathrm{d}\zeta^k \wedge \mathrm{d}\overline{\zeta}^l \otimes \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i.$$

Thus the mean curvature is

$$\sqrt{-1}\Lambda_{\widetilde{\omega}_t} F_{D_f} = \widetilde{g}_t^{k\bar{l}} F_{t,ik\bar{l}}^p \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i.$$

Let $\tau \in \mathbb{R}$ and $\{S(\tau) : S(0) = I\} \subseteq \mathcal{H}_t$. Denote

$$\psi = \partial_\tau S(\tau)|_{\tau=0} = \psi_i^p \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i \in W_t.$$

Proposition 3.1 *The linearization of the mean curvature* $\sqrt{-1}\Lambda_{\widetilde{\omega}_t} F_{D_f}$ *is*

$$\partial_\tau(\widetilde{g}_t^{k\bar{l}} F_{t,ik\bar{l}}^p)|_{\tau=0} = g_t^{k\bar{l}}(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})\psi_i^p + 2g_t^{m\bar{l}}\psi_m^k R_{t,ik\bar{l}}^p. \tag{3.6}$$

Proof By definition (3.3), we get

$$\widetilde{\omega}_t = \sqrt{-1}\widetilde{g}_{t,i\bar{j}} \mathrm{d}\zeta^i \wedge \mathrm{d}\overline{\zeta}^j,$$

where

$$\widetilde{g}_{t,i\bar{j}} = g_{t,k\bar{l}} \widetilde{S}_i^k \overline{\widetilde{S}}_j^l = g_{t,k\bar{j}} \widetilde{S}_m^k \widetilde{S}_i^m, \quad \widetilde{g}_t^{i\bar{j}} = g_t^{m\bar{j}} S_m^k S_k^i,$$

since

$$\widetilde{S} = \widetilde{S}_i^p \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i$$

is symmetric with respect to g_t. Then we get

$$\partial_\tau \widetilde{g}_t^{k\bar{l}}|_{\tau=0} = 2g_t^{i\bar{l}}\psi_i^k. \tag{3.7}$$

While we obtain from definition (3.1) that

$$\begin{aligned} D_f' D_f''\Big(\frac{\partial}{\partial \zeta^i}\Big) &= D_f'\Big(\widetilde{S}_m^p \nabla_{t,\bar{l}} S_i^m \mathrm{d}\overline{\zeta}^l \otimes \frac{\partial}{\partial \zeta^p}\Big) \\ &= (S_\lambda^p \nabla_{t,k}(\widetilde{S}_r^\lambda \widetilde{S}_m^r \nabla_{t,\bar{l}} S_i^m) + \widetilde{S}_r^p \nabla_{t,\bar{l}} S_m^r \Gamma_{t,ik}^m) \mathrm{d}\zeta^k \wedge \mathrm{d}\overline{\zeta}^l \otimes \frac{\partial}{\partial \zeta^p} \end{aligned}$$

and

$$D_f'' D_f'\Big(\frac{\partial}{\partial \zeta^i}\Big) = D_f''\Big((\Gamma_{ik}^p + S_m^p \nabla_{t,k} \widetilde{S}_i^m) \mathrm{d}\zeta^k \otimes \frac{\partial}{\partial \zeta^p}\Big)$$

$$= (R^p_{t,ik\bar{l}} - \widetilde{S}^p_\lambda \nabla_{t,\bar{l}}(S^\lambda_r S^r_m \nabla_{t,k} \widetilde{S}^m_i) - \widetilde{S}^p_r \nabla_{t,\bar{l}} S^r_m \Gamma^m_{t,ik}) \mathrm{d}\zeta^k \wedge \mathrm{d}\overline{\zeta}^l \otimes \frac{\partial}{\partial \zeta^p}.$$

Since

$$F_{D_f} = D'_f D''_f + D''_f D'_f,$$

we have

$$F^p_{t,ik\bar{l}} = R^p_{t,ik\bar{l}} + S^p_r \nabla_{t,k}(\widetilde{S}^r_m \widetilde{S}^m_\lambda \nabla_{t,\bar{l}} S^\lambda_i) + \widetilde{S}^p_r \nabla_{t,\bar{l}}(S^r_m \nabla_{t,k} S^m_\lambda \widetilde{S}^\lambda_i), \tag{3.8}$$

which together with $S(0) = I$ implies

$$\partial_\tau F^p_{t,ik\bar{l}}|_{\tau=0} = (\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})\psi^p_i.$$

Then (3.6) follows from

$$\partial_\tau(\widetilde{g}^{k\bar{l}}_t F^p_{t,ik\bar{l}})|_{\tau=0} = \partial_\tau \widetilde{g}^{k\bar{l}}_t|_{\tau=0} R^p_{t,ik\bar{l}} + g^{k\bar{l}}_t \partial_\tau F^p_{t,ik\bar{l}}|_{\tau=0}$$

and (3.7).

Fix an arbitrary point $q \in X_t$. For

$$\eta = \eta^p_i \frac{\partial}{\partial \zeta^p} \otimes \mathrm{d}\zeta^i \in W_t,$$

we define

$$L_t \eta = g^{k\bar{l}}_t (\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})\eta - \mathrm{tr}_q(\eta) I.$$

For any $S \in \mathcal{H}_t$, we define a connection (see (3.1))

$$\check{D} = S \circ D'_t \circ \widetilde{S} + \widetilde{S} \circ D''_t \circ S \tag{3.9}$$

and a Hermitian metric (see (3.3))

$$\check{g} = (\widetilde{S}^2)^T g_t.$$

Then $\check{D}$ is the Chern connection of $(g_t, \check{D}'')$ (see Lemma 3.3).

Let $\check{\nabla}$ be the covariant derivative of $\check{D}$. For $\eta \in W_t$, we define

$$L_{t,S}, \eta = \check{g}^{k\bar{l}} (\check{\nabla}_k \check{\nabla}_{\bar{l}} + \check{\nabla}_{\bar{l}} \check{\nabla}_k)\eta - \mathrm{tr}_q(\eta) I.$$

Then $L_{t,I} = L_t$. Moreover, $L_{t,S}, \eta \in W_t$ due to Lemma 3.3.

Lemma 3.4 *Suppose that $S \in \mathcal{H}_t$ with $\|S - I\|_{C^{2,\alpha}} \le c$ for some $c \ll 1$. Then*

$$\|L_{t,S} - L_t\| \le C\|S - I\|_{C^{2,\alpha}}$$

for some uniform constant $C > 0$.

Proof For $\eta \in C^{2,\alpha}(W_t)$, by definition (3.9), direct calculation (see e.g., [6, p. 539]) yields:

$$\begin{aligned}(L_{t,S} - L_t)\eta_i^p = {} & (\check{g}^{k\bar{l}} - g^{k\bar{l}})(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})\eta_i^p \\ & + \check{g}^{k\bar{l}} S_a^p(\nabla_{t,k}\widetilde{S}_m^a \nabla_{t,\bar{l}}\eta_i^m + \nabla_{t,k}(\widetilde{S}_m^a \widetilde{S}_r^m \nabla_{t,\bar{l}} S_\lambda^r \eta_i^\lambda)) \\ & + \check{g}^{k\bar{l}} \widetilde{S}_a^p(\nabla_{t,\bar{l}} S_m^a \nabla_{t,k}\eta_i^m + \nabla_{t,\bar{l}}(S_m^a S_r^m \nabla_{t,k}\widetilde{S}_\lambda^r \eta_i^\lambda)).\end{aligned} \tag{3.10}$$

Since

$$\check{g}^{k\bar{l}} - g_t^{k\bar{l}} = g_t^{m\bar{l}}(S_m^r S_r^k - \delta_m^k) = g_t^{m\bar{l}}((S_m^r - \delta_m^r)(S_r^k - \delta_r^k) + 2(S_m^k - \delta_m^k)), \tag{3.11}$$

there exists a uniform constant $C > 0$ such that

$$\|(\check{g}^{k\bar{l}} - g_t^{k\bar{l}})(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})\eta_i^p\|_{C^{0,\alpha}} \le C\|S - I\|_{C^{2,\alpha}}\|\eta\|_{C^{2,\alpha}}.$$

For $c \ll 1$, there exists a uniform constant $C' > 0$ such that

$$\|\widetilde{S} - I\|_{C^{2,\alpha}} \le C'\|S - I\|_{C^{2,\alpha}}.$$

Then we have

$$\begin{aligned} & \|\check{g}^{k\bar{l}} S_a^p \nabla_k \widetilde{S}_m^a \nabla_{\bar{l}}\eta_i^m\|_{C^{0,\alpha}} \\ \le {} & \|(\check{g}^{k\bar{l}} - g_t^{k\bar{l}})\nabla_{t,k}\widetilde{S}_m^p \nabla_{t,\bar{l}}\eta_i^m\|_{C^{0,\alpha}} + \|g_t^{k\bar{l}}\nabla_{t,k}\widetilde{S}_m^p \nabla_{t,\bar{l}}\eta_i^m\|_{C^{0,\alpha}} \\ & + \|(\check{g}^{k\bar{l}} - g_t^{k\bar{l}})(S_a^p - \delta_a^p)\nabla_{t,k}\widetilde{S}_m^a \nabla_{t,\bar{l}}\eta_i^m\|_{C^{0,\alpha}} \\ & + \|g_t^{k\bar{l}}(S_r^p - \delta_r^p)\nabla_{t,k}\widetilde{S}_m^r \nabla_{t,\bar{l}}\eta_i^m\|_{C^{0,\alpha}} \\ \le {} & C\|S - I\|_{C^{2,\alpha}}\|\eta\|_{C^{2,\alpha}}.\end{aligned}$$

We can estimate other terms on the right hand side of (3.10) similarly. Then we obtain

$$\|(L_{t,S} - L_t)\eta\|_{C^{0,\alpha}} \le C\|S - I\|_{C^{2,\alpha}}\|\eta\|_{C^{2,\alpha}},$$

which implies the conclusion.

Proposition 3.2 *There exists a uniform constant $C > 0$ and some $\alpha \in (0, 1)$ such that*

$$L_t : C^{2,\alpha}(W_t) \to C^{0,\alpha}(W_t)$$

is invertible, and the inverse $P_t = L_t^{-1}$ satisfies $\|P_t\| \le C$.

Proof For $\eta \in C^{2,\alpha}(W_t)$, we obtain from Lemma 3.1 and the Schauder estimate that

$$\|\eta\|_{C^{2,\alpha}} \le C(\|\eta\|_{C^0} + \|L_t\eta\|_{C^{0,\alpha}}),$$

where $C > 0$ is uniform in t and $\alpha \in (0, 1)$. We will show that the term $\|\eta\|_{C^0}$ is superfluous.

Otherwise there exist sequences $\{t_l\}$ and $\{\eta_l\} \subset C^{2,\alpha}(W_{t_l})$ such that for $t_l \to 0$,

$$\|\eta_l\|_{C^{2,\alpha}(W_{t_l})} = 1 \quad \text{and} \quad \|L_{t_l}\eta_l\|_{C^{0,\alpha}(W_{t_l})} \le \frac{1}{l}.$$

By the definition of ω_t and Lemma 3.1, there exists $\eta_0 \in C^{2,\alpha}(W_0)$ such that

$$\|\eta_0\|_{C^{2,\alpha}(W_0)} = 1 \tag{3.12}$$

and

$$0 = L_0\eta_0 = g_0^{k\bar{l}}(\nabla_{0,k}\nabla_{0,\bar{l}} + \nabla_{0,\bar{l}}\nabla_{0,k})\eta_0 - \mathrm{tr}_q(\eta_0)I, \tag{3.13}$$

where g_0 is the natural balanced metric on X_0 and ∇_0 is the covariant derivative of the Chern connection of g_0.

Since ω_t is Gauduchon (see Propositions 2.2, 4.1–4.2), there exists a uniform constant $C > 0$ such that

$$3|\mathrm{tr}_q(\eta_l)|\int_{X_{t_l}} \frac{\omega_{t_l}^3}{3!} \le \Big|\int_{X_{t_l}} (L_{t_l}\eta_l, I)_{g_{t_l}} \frac{\omega_{t_l}^3}{3!}\Big| \le \frac{\sqrt{3}}{l}\int_{X_{t_l}} \frac{\omega_{t_l}^3}{3!} \le \frac{C}{l},$$

which implies $\mathrm{tr}_q(\eta_0) = 0$.

Since ω_0 is a balanced metric on X_0 with flat Chern connection ∇_0, the holomorphic tangent bundle $T^{1,0}X_0$ is stable with respect to ω_0 (see e.g., [27]). By [14, Lemma 1.2.5], we have

$$\ker(g_0^{k\bar{l}}(\nabla_{0,k}\nabla_{0,\bar{l}} + \nabla_{0,\bar{l}}\nabla_{0,k})) = \mathbb{C}I. \tag{3.14}$$

Then we obtain from (3.13) that

$$\eta_0 \in H^0(X_0, \mathrm{End}(T^{1,0}X_0)) = \mathbb{C}I.$$

Thus we get $\eta_0 = 0$ due to $\mathrm{tr}_q(\eta_0) = 0$, which contradicts (3.12).

Hence, we have

$$\|\eta\|_{C^{2,\alpha}} \le C\|L_t\eta\|_{C^{0,\alpha}}, \tag{3.15}$$

which implies L_t is injective.

Since ϕ_t^1, ϕ_t^2 and ϕ_t^3 are global $(1,0)$-forms on X_t (see Section 2.2), $T^{1,0}X_t$ is trivial smoothly (but maybe not holomorphically). We obtain from the elliptic semicontinuity (see e.g., [28, Theorem 4.13]), Lemma 3.1 and (3.14) that

$$\ker(g_t^{k\bar{l}}(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})) = \mathbb{C}I. \tag{3.16}$$

Denote

$$W_t^\circ = W_t \cap \Big\{\eta \in \Gamma(\mathrm{End}(T^{1,0}X_t)), \int_{X_t} \mathrm{tr}(\eta)\omega_t^3 = 0\Big\}.$$

Then we obtain from (3.16) and the standard theory of elliptic operator that

$$L_t(\cdot) + \mathrm{tr}_q(\cdot)I = g_t^{k\bar{l}}(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k}) : C^{2,\alpha}(W_t^\circ) \to C^{0,\alpha}(W_t^\circ)$$

is isomorphic. For any $\sigma \in C^{0,\alpha}(W_t)$, we have

$$\sigma - \frac{\displaystyle\int_{X_t} \mathrm{tr}(\sigma)\omega_t^3}{3\displaystyle\int_{X_t} \omega_t^3} I \in C^{0,\alpha}(W_t^\circ).$$

Then there exists a unique $\eta \in C^{2,\alpha}(W_t^\circ)$ such that

$$\widehat{L}_t\eta + \mathrm{tr}_q(\eta)I = \sigma - \frac{\displaystyle\int_{X_t} \mathrm{tr}(\sigma)\omega_t^3}{3\displaystyle\int_{X_t}\omega_t^3} I.$$

Taking

$$\lambda = -\frac{\displaystyle\int_{X_t} \mathrm{tr}(\sigma)\omega_t^3}{9\displaystyle\int_{X_t}\omega_t^3} - \frac{1}{3}\mathrm{tr}_q(\eta)$$

yields

$$L_t(\eta + \lambda I) = \sigma.$$

Since $\eta + \lambda I \in C^{2,\alpha}(W_t)$, L_t is surjective.

Then L_t is isomorphic with a uniform bound of $P_t = L_t^{-1}$ given by (3.15).

For $f \in W_t$ with $f + I \in \mathcal{H}_t$, we define

$$Q_t(f) = \sqrt{-1}\Lambda_{\widetilde{\omega}_t} F_{D_f} - \sqrt{-1}\Lambda_{\omega_t} R_{\omega_t} - g_t^{k\bar{l}}(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})f.$$

We consider the equation

$$L_t f = c_t I - \sqrt{-1}\Lambda_{\omega_t} R_{\omega_t} - Q_t(f), \tag{3.17}$$

where

$$c_t = \frac{\displaystyle\int_{X_t} \rho_{\widehat{\omega}_t} \wedge \widehat{\omega}_t^2}{\displaystyle\int_{X_t} \widehat{\omega}_t^3}.$$

If (3.17) admits a solution f, then f is the unique solution to (3.5) after forcing $c_t + \mathrm{tr}_q(f) = 0$. However, if we replace $\widehat{\omega}_t$ by ω_t, then

$$\frac{\displaystyle\int_{X_t} \rho_{\omega_t} \wedge \omega_t^2}{\displaystyle\int_{X_t} \omega_t^3} = 0,$$

since ω_t is Gauduchon (see Propositions 2.2 and 4.1–4.2) and X_t is Calabi-Yau. Thus the above trick for the uniqueness fails. It is why we have to introduce the metric $\widehat{\omega}_t$.

Once the equation (3.5) admits a unique solution, the Hermitian metric $\widetilde{\omega}_t$ (given by (3.3)) satisfies (3.4), i.e., $\widetilde{\omega}_t$ is a Hermitian Yang-Mills metric with respect to itself.

To solve the equation (3.17), with Proposition 3.2 in hand, we define

$$\mathcal{N}_t : C^{2,\alpha}(W_t) \to C^{2,\alpha}(W_t)$$

by

$$\mathcal{N}_t(f) = P_t(c_t I - \sqrt{-1}\Lambda_{\omega_t} R_{\omega_t} - Q_t(f)).$$

We want to show that $\mathcal{N}_t$ restricted to some subset $\mathcal{U}_t$ is a contraction mapping.

Lemma 3.5 *There exists a uniform constant $C > 0$ such that*

$$\|\mathcal{N}_t(0)\|_{C^{2,\alpha}} \le C|t|.$$

Proof Since $Q_t(0) = 0$, we have

$$\mathcal{N}_t(0) = P_t(c_t I - \sqrt{-1}\Lambda_{\omega_t} R_{\omega_t}).$$

According to Proposition 3.2, we have to estimate $\|c_t I - \sqrt{-1}\Lambda_{\omega_t} R_{\omega_t}\|_{C^{0,\alpha}}$.

Since ω_0 is a balanced metric on X_0 with flat Chern connection ∇_0,

$$c_0 = \frac{\displaystyle\int_{X_0} \rho_{\omega_0} \wedge \omega_0^2}{\displaystyle\int_{X_0} \omega_0^3} = 0.$$

For $\widehat{\omega}_t = (\Psi_t^* \omega_0)^{1,1}$, there exists a uniform constant $C > 0$ such that $|c_t| \le C|t|$.

By Proposition 2.1 and Lemma 3.1, there exists a uniform constant $C > 0$ such that

$$\|\sqrt{-1}\Lambda_{\omega_t} R_{\omega_t}\|_{C^{0,\alpha}} \le C|t|^2.$$

Then we have

$$\|\mathcal{N}_t(0)\|_{C^{2,\alpha}} \le C(\|c_t I\|_{C^{0,\alpha}} + \|\sqrt{-1}\Lambda_{\omega_t} R_{\omega_t}\|_{C^{0,\alpha}}) \le C|t|,$$

which implies the conclusion for sufficiently small ϵ.

Lemma 3.6 *If $f_1, f_2 \in W_t$ satisfy*

$$\|f_1\|_{C^{2,\alpha}} \le c, \quad \|f_2\|_{C^{2,\alpha}} \le c$$

for some $c \ll 1$, then

$$\|\mathcal{N}_t(f_1) - \mathcal{N}_t(f_2)\|_{C^{2,\alpha}} \le \frac{1}{2}\|f_1 - f_2\|_{C^{2,\alpha}}.$$

Proof Since

$$\mathcal{N}_t(f_1) - \mathcal{N}_t(f_2) = P_t(Q_t(f_2) - Q_t(f_1)),$$

by Proposition 3.2, we have to estimate $\|Q_t(f_2) - Q_t(f_1)\|_{C^{0,\alpha}}$.

Denote

$$S_1 = f_1 + I \in \mathcal{H}_t, \quad S_2 = f_2 + I \in \mathcal{H}_t$$

and $\widetilde{S}_1 = S_1^{-1}$ and $\widetilde{S}_2 = S_2^{-1}$.

By the definition of Q_t and the mean value theorem, there exists a $\lambda \in [0,1]$ such that

$$S := \lambda S_1 + (1-\lambda) S_2 \in \mathcal{H}_t$$

satisfies

$$\begin{aligned} Q_t(f_2) - Q_t(f_1) = {} & \sqrt{-1}\Lambda_{(\widetilde{S}_2^2)^{\mathrm{T}} g_t} F_{D_{f_2}} - \sqrt{-1}\Lambda_{(\widetilde{S}_1^2)^{\mathrm{T}} g_t} F_{D_{f_1}} \\ & - g_t^{k\bar{l}}(\nabla_{t,k}\nabla_{t,\bar{l}} + \nabla_{t,\bar{l}}\nabla_{t,k})(f_2 - f_1) \end{aligned}$$

$$= (L_{t,S} - L_t)(f_2 - f_1) + 2\check{g}^{m\bar{l}}(f_2 - f_1)^k_m \check{F}^p_{ik\bar{l}},$$

where

$$\check{g} = (\widetilde{S}^2)^{\mathrm{T}} g_t$$

and $\check{F}^p_{ik\bar{l}}$ is the curvature of $\check{D}$ (see (3.9)). Here we have used (3.6) for the second equality. Moreover, we have

$$\|S - I\|_{C^{2,\alpha}} \le \lambda \|f_1\|_{C^{2,\alpha}} + (1-\lambda)\|f_2\|_{C^{2,\alpha}} \le c.$$

By Lemma 3.4, there exists a uniform constant $C > 0$ such that

$$\begin{aligned}\|(L_{t,S} - L_t)(f_2 - f_1)\|_{C^{0,\alpha}} &\le C\|S - I\|_{C^{2,\alpha}} \|f_2 - f_1\|_{C^{2,\alpha}} \\ &\le cC\|f_2 - f_1\|_{C^{2,\alpha}}.\end{aligned}$$

While we obtain from (3.8) that

$$\begin{aligned}\check{g}^{m\bar{l}}(f_2 - f_1)^k_m \check{F}^p_{ik\bar{l}} &= (\check{g}^{m\bar{l}} - g_t^{m\bar{l}} + g_t^{m\bar{l}})(f_2 - f_1)^k_m (R^p_{t,ik\bar{l}} \\ &\quad + S^p_r \nabla_{t,k}(\widetilde{S}^r_m \widetilde{S}^m_\lambda \nabla_{t,\bar{l}} S^\lambda_i) + \widetilde{S}^p_r \nabla_{t,\bar{l}}(S^r_m \nabla_{t,k} S^m_\lambda \widetilde{S}^\lambda_i)).\end{aligned}$$

Similar to the estimations for (3.10), combining Proposition 2.1, Lemma 3.1 and (3.11) yields

$$\begin{aligned}\|\check{g}^{m\bar{l}}(f_2 - f_1)^k_m \check{F}^p_{ik\bar{l}}\|_{C^{0,\alpha}} &\le C(\|S - I\|_{C^{2,\alpha}} + |t|^2)\|f_2 - f_1\|_{C^{2,\alpha}} \\ &\le C(c + |t|^2)\|f_2 - f_1\|_{C^{2,\alpha}}.\end{aligned}$$

Then we get

$$\|\mathcal{N}_t(f_1) - \mathcal{N}_t(f_2)\|_{C^{2,\alpha}} \le C(c + |t|^2)\|f_2 - f_1\|_{C^{2,\alpha}} \le \frac{1}{2}\|f_1 - f_2\|_{C^{2,\alpha}}$$

for c and ϵ sufficiently small.

Choose $\epsilon > 0$ sufficiently small such that all the above propositions and lemmas hold. Choose $\delta \in (0,1)$ sufficiently small such that $|t|^\delta$ is smaller than c (given by Lemma 3.6).

Denote

$$\mathcal{U}_t = \{f \in C^{2,\alpha}(W_t) : \|f\|_{C^{2,\alpha}} \le |t|^\delta\}.$$

We may assume $S = f + I \in \mathcal{H}_t$ for all $f \in \mathcal{U}_t$.

Proposition 3.3 *The $\mathcal{N}_t$ restricted to $\mathcal{U}_t$ is a contraction mapping.*

Proof For $f \in \mathcal{U}_t$, by Lemmas 3.5–3.6, we have

$$\|\mathcal{N}_t(f)\|_{C^{2,\alpha}} \le \|\mathcal{N}_t(f) - \mathcal{N}_t(0)\|_{C^{2,\alpha}} + \|\mathcal{N}_t(0)\|_{C^{2,\alpha}} \le \frac{1}{2}|t|^\delta + C|t|.$$

Since $\delta \in (0,1)$, we get $\mathcal{N}_t \in \mathcal{U}_t$.

By the Banach fixed point theorem, there exists $f \in \mathcal{U}_t$ such that $\mathcal{N}_t(f) = f$. Moreover, f is smooth by the standard theory of elliptic equations. Then the Hermitian metric $\widetilde{\omega}_t$ (given by (3.3)) is a solution to (3.4).

For the solution to (3.17), we have the following observation.

Since ω_0 is a balanced metric on X_0 with flat Chern connection ∇_0, we obtain from (3.5)–(3.6) that for $m \in \{1,2,3\}$ and $n \in \{1,2\}$,

$$0 = \partial_{t_{mn}} (\widetilde{g}_t^{k\bar{l}} F^p_{t,ik\bar{l}})|_{t=0} = g_0^{k\bar{l}} (\nabla_{0,k}\nabla_{0,\bar{l}} + \nabla_{0,\bar{l}}\nabla_{0,k})\psi_i^p,$$

where $\psi_i^p = \partial_{t_{mn}} f_i^p|_{t=0} \in W_0$. Then we obtain from (3.14) that

$$\psi = \lambda_{mn} I$$

for some $\lambda_{mn} \in \mathbb{R}$.

On the other hand, the Chern-Ricci curvature of $\widetilde{\omega}_t$ is $\rho_{\widetilde{\omega}_t} = 0$ due to Lemma 3.2. While the Chern-Ricci curvature of ω_t is $\rho_{\omega_t} = 0$ by Proposition 2.1. We obtain from definition (3.3) that

$$0 = \rho_{\omega_t} - \rho_{\widetilde{\omega}_t} = \sqrt{-1}\, \overline{\partial}_t \partial_t \log \frac{\det\, \omega_t}{\det\, \widetilde{\omega}_t} = \sqrt{-1}\, \overline{\partial}_t \partial_t \log \det(f + I)^2.$$

Then the function

$$\mu_t := \frac{1}{3} \log \det(f + I)$$

depends only on t by the maximum principle (see e.g., [14, Lemma 7.2.7]). Thus we obtain

$$\partial_{t_{mn}} \mu_t|_{t=0} = \frac{1}{3} \partial_{t_{mn}} \log \det(f + I)|_{t=0} = \lambda_{mn},$$

which implies

$$f = \sum_{m,n} (\partial_{t_{mn}} \mu_t|_{t=0} t_{mn}) I + o(|t|^2).$$

Here we have used $f|_{t=0} = 0$ since ω_0 is already Hermitian Yang-Mills.

4 Appendix

Let X_0 be the Nakamura manifold and $\{X_t\}_{t \in \Delta_\epsilon}$ be the Kuranishi family of X_0. We refer to [16] for details on the construction of X_0 and X_t. In this section, we will prove Proposition 2.1 for the current case.

Let $\{z^1, z^2, z^3\}$ be holomorphic coordinates on X_0. A global basis of the space of holomorphic $(1,0)$-forms on X_0 is

$$\{\phi_0^1 = \mathrm{d}z^1,\ \phi_0^2 = \mathrm{e}^{z^1} \mathrm{d}z^2,\ \phi_0^3 = \mathrm{e}^{-z^1} \mathrm{d}z^3\}.$$

It is direct to check the metric

$$\omega_0 = \sqrt{-1} \sum_{l=1}^{3} \phi_0^l \wedge \overline{\phi_0^l} = \sqrt{-1}(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + \mathrm{e}^{z^1+\overline{z}^1} \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 + \mathrm{e}^{-(z^1+\overline{z}^1)} \mathrm{d}z^3 \wedge \mathrm{d}\overline{z}^3)$$

on X_0 is balanced with flat Chern connection ∇_0.

As in [16, Theorem 1], we will distinguish two cases according to $h^{0,1}(X_0)$, and prove Proposition 2.1 for each case.

4.1 Type III-(3a)

The integrable vector $(0,1)$-form $\varphi \in \mathcal{A}^{0,1}(T^{1,0}X_0)$ is (see [16, p. 99])

$$\varphi = \Big(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}\Big) \begin{pmatrix} t_1 & 0 & 0 \\ t_2 e^{-z^1} & 0 & 0 \\ t_3 e^{z^1} & 0 & 0 \end{pmatrix} \begin{pmatrix} d\overline{z}^1 \\ d\overline{z}^2 \\ d\overline{z}^3 \end{pmatrix}. \tag{4.1}$$

For the two subcases in [16, p. 99], here we only consider the case 1: $t_1 \neq 0$. While the following calculations for the case 2: $t_1 = 0$ are similar.

The holomorphic coordinates $\{\zeta^1, \zeta^2, \zeta^3\}$ on X_t are given by (see [16, p. 99])

$$\zeta^1 = z^1 + t_1 \overline{z}^1, \quad \zeta^1 = z^2 - \mu e^{-z^1}, \quad \zeta^3 = z^3 + \lambda e^{z^1},$$

where

$$\lambda = (e^{t_1 \overline{z}^1} - 1)\frac{t_3}{t_1} \quad \text{and} \quad \mu = (e^{-t_1 \overline{z}^1} - 1)\frac{t_2}{t_1}.$$

Then we obtain from (2.1) that

$$\begin{pmatrix} \mathcal{A}^1_{t,1} & \cdots & \mathcal{A}^1_{t,3} \\ \vdots & & \vdots \\ \mathcal{A}^3_{t,1} & \cdots & \mathcal{A}^3_{t,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\mu e^{-z^1} & 1 & 0 \\ -\lambda e^{z^1} & 0 & 1 \end{pmatrix}. \tag{4.2}$$

The three $(1,0)$-forms

$$\phi^1_t = e^{i_\varphi}(\phi^1_0) = d\zeta^1, \quad \phi^2_t = e^{i_\varphi}(\phi^2_0) = e^{z^1} d\zeta^2 - \mu d\zeta^1,$$
$$\phi^3_t = e^{i_\varphi}(\phi^3_0) = e^{-z^1} d\zeta^3 - \lambda d\zeta^1$$

form a basis of the space of smooth $(1,0)$-forms on X_t. Since

$$\phi^1_t \wedge \phi^2_t \wedge \phi^3_t = d\zeta^1 \wedge d\zeta^2 \wedge d\zeta^3,$$

the canonical bundle K_{X_t} of X_t is holomorphically trivial.

For $\epsilon \ll 1$, there is a natural Hermitian metric on X_t:

$$\omega_t = e^{i_\varphi | i_{\overline{\varphi}}}(\omega_0) = \sqrt{-1} g_{t,i\overline{j}} d\zeta^i \wedge d\overline{\zeta}^j,$$

where

$$\begin{pmatrix} g_{t,1\overline{1}} & \cdots & g_{t,1\overline{3}} \\ \vdots & & \vdots \\ g_{t,3\overline{1}} & \cdots & g_{t,3\overline{3}} \end{pmatrix} = \begin{pmatrix} 1 + |\lambda|^2 + |\mu|^2 & -\mu e^{\overline{z}^1} & -\lambda e^{-\overline{z}^1} \\ -\overline{\mu} e^{z^1} & e^{z^1 + \overline{z}^1} & 0 \\ -\overline{\lambda} e^{-z^1} & 0 & e^{-(z^1 + \overline{z}^1)} \end{pmatrix}. \tag{4.3}$$

It is direct to check

$$\omega_t = \sqrt{-1} \sum_{l=1}^{3} \phi^l_t \wedge \overline{\phi}^l_t.$$

Proposition 4.1 *Let X_0 be the Nakamura manifold for this case and $\{X_t\}_{t \in \Delta_\epsilon}$ be the Kuranishi family of X_0. The natural metric ω_t on X_t is Gauduchon and Chern-Ricci flat. Moreover, it is balanced if and only if $t_2 = t_3 = 0$.*

Proof We obtain from (4.3) that $\det \omega_t = 1$, which implies that ω_t is Chern-Ricci flat. While we obtain from (4.1) that

$$(I - \varphi\overline{\varphi})^{-1} = \begin{pmatrix} \frac{1}{1-|t_1|^2} & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}, \tag{4.4}$$

where elements denoted by $*$ are irrelevant for us.

By [20, Lemma 2.4], we obtain from (4.2) and (4.4) that

$$\Big(\frac{\partial z}{\partial \zeta}\Big) = (I - \varphi\overline{\varphi})^{-1}\mathcal{A}_t = \begin{pmatrix} \frac{1}{1-|t_1|^2} & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}.$$

Similarly, we obtain from (4.1)–(4.2) and (4.4) that

$$-\Big(\frac{\partial \overline{z}}{\partial \zeta}\Big) = \overline{\varphi}(I - \varphi\overline{\varphi})^{-1}\mathcal{A}_t = \begin{pmatrix} \frac{\overline{t}_1}{1-|t_1|^2} & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Then we get

$$\frac{\partial z^1}{\partial \zeta^1} = \frac{1}{1-|t_1|^2} \quad \text{and} \quad \frac{\partial z^1}{\partial \overline{\zeta}^1} = -\frac{t_1}{1-|t_1|^2}. \tag{4.5}$$

By (4.5), we calculate

$$\begin{aligned} \overline{\partial}_t \omega_t^2 &= 2\sqrt{-1}^2(-\overline{\partial}_t(\mu \mathrm{e}^{-z^1}) \wedge \mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\zeta^3 \wedge \mathrm{d}\overline{\zeta}^3 \\ &\quad - \overline{\partial}_t(\lambda \mathrm{e}^{z^1}) \wedge \mathrm{d}\zeta^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\overline{\zeta}^3) \\ &= \frac{2\sqrt{-1}^2}{1-|t_1|^2}(-t_2\mathrm{e}^{-z^1}\mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\zeta^3 \wedge \mathrm{d}\overline{\zeta}^3 \\ &\quad + t_3\mathrm{e}^{z^1}\mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\overline{\zeta}^3), \end{aligned}$$

which implies

$$\partial_t \overline{\partial}_t \omega_t^2 = 0,$$

i.e., ω_t is Gauduchon. While ω_t is balanced if and only if $t_2 = t_3 = 0$.

Proof of Proposition 2.1 Let X_0 be the Nakamura manifold for this case and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0.

By (4.3) and (4.5), we get

$$\begin{pmatrix} g_t^{1\overline{1}} & \cdots & g_t^{3\overline{1}} \\ \vdots & & \vdots \\ g_t^{1\overline{3}} & \cdots & g_t^{3\overline{3}} \end{pmatrix} = \begin{pmatrix} 1 & \mu\mathrm{e}^{-z^1} & \lambda\mathrm{e}^{z^1} \\ \overline{\mu}\mathrm{e}^{-\overline{z}^1} & (1+|\mu|^2)\mathrm{e}^{-(z^1+\overline{z}^1)} & \lambda\overline{\mu}\mathrm{e}^{z^1-\overline{z}^1} \\ \overline{\lambda}\mathrm{e}^{\overline{z}^1} & \mu\overline{\lambda}\mathrm{e}^{\overline{z}^1-z^1} & (1+|\lambda|^2)\mathrm{e}^{z^1+\overline{z}^1} \end{pmatrix} \tag{4.6}$$

and

$$\partial_t g_t g_t^{-1} = \begin{pmatrix} -\mu\overline{t}_2 + \lambda\overline{t}_3 & \Big(-q\mu + \frac{t_2\overline{t}_1}{\eta}\Big)\mathrm{e}^{-z^1} & (p\lambda + \eta t_3\overline{t}_1)\mathrm{e}^{z^1} \\ \overline{t}_2\mathrm{e}^{z^1} & \mu\overline{t}_2 - \overline{t}_1 + 1 & \lambda\overline{t}_2\mathrm{e}^{2z^1} \\ -\overline{t}_3\mathrm{e}^{-z^1} & -\mu\overline{t}_3\mathrm{e}^{-2z^1} & -\lambda\overline{t}_3 + \overline{t}_1 - 1 \end{pmatrix} \frac{\mathrm{d}\zeta^1}{1-|t_1|^2},$$

where

$$\eta = \mathrm{e}^{t_1\overline{z}^1}, \quad p = \lambda\overline{t}_3 - \mu\overline{t}_2 - \overline{t}_1, \quad q = \mu\overline{t}_2 - \lambda\overline{t}_3 - \overline{t}_1.$$

Since $R^{\mathrm{T}}_{\omega_t} = \overline{\partial}_t(\partial_t g_t g_t^{-1})$, we obtain from (4.5) that

$$R^{\mathrm{T}}_{\omega_t} = \begin{pmatrix} \nu & (t_2q + \nu\mu)\mathrm{e}^{-z^1} & (t_3p + \nu\lambda)\mathrm{e}^{z^1} \\ -t_1\overline{t}_2\mathrm{e}^{z^1} & -\frac{|t_2|^2}{\eta} & t_3\overline{t}_2(2-\eta)\mathrm{e}^{2z^1} \\ -t_1\overline{t}_3\mathrm{e}^{-z^1} & t_2\overline{t}_3\Big(2 - \frac{1}{\eta}\Big)\mathrm{e}^{-2z^1} & -|t_3|^2\eta \end{pmatrix} \frac{\mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\zeta^1}{(1-|t_1|^2)^2},$$

where

$$\nu = \frac{|t_2|^2}{\eta} + |t_3|^2\eta.$$

By (4.6), the mean curvature $K_{\omega_t} = \sqrt{-1}\Lambda_{\omega_t}R_{\omega_t}$ is

$$K^{\mathrm{T}}_{\omega_t} = \begin{pmatrix} \nu & (t_2q + \nu\mu)\mathrm{e}^{-z^1} & (t_3p + \nu\lambda)\mathrm{e}^{z^1} \\ -t_1\overline{t}_2\mathrm{e}^{z^1} & -\frac{|t_2|^2}{\eta} & t_3\overline{t}_2(2-\eta)\mathrm{e}^{2z^1} \\ -t_1\overline{t}_3\mathrm{e}^{-z^1} & t_2\overline{t}_3\Big(2 - \frac{1}{\eta}\Big)\mathrm{e}^{-2z^1} & -|t_3|^2\eta \end{pmatrix} \frac{-1}{(1-|t_1|^2)^2}.$$

Since $K^*_{\omega_t} = K_{\omega_t}$, direct calculation yields

$$|K_{\omega_t}|^2_{\omega_t} = \mathrm{tr}(K_{\omega_t} \circ K^*_{\omega_t}) = \frac{2(|t_2|^2 + |t_3|^2)(|t_1|^2 + |t_2|^2 + |t_3|^2)}{(1-|t_1|^2)^4}.$$

Since R_{ω_t} is skew-symmetric with respect to ω_t: $R^*_{\omega_t} = -R_{\omega_t}$, we get

$$|R_{\omega_t}|^2_{\omega_t} = -\frac{\mathrm{tr}(R_{\omega_t} \wedge *_t R_{\omega_t})}{\frac{\omega_t^3}{3!}} = |K_{\omega_t}|^2_{\omega_t}.$$

The conclusion for this case is valid.

4.2 Type III-(3b)

The integrable vector $(0,1)$-form $\varphi \in \mathcal{A}^{0,1}(T^{1,0}X_0)$ is (see [16, p. 97])

$$\varphi = \Big(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}\Big) \begin{pmatrix} t_{11} & t_{12}\mathrm{e}^{z^1} & t_{13}\mathrm{e}^{-z^1} \\ t_{21}\mathrm{e}^{-z^1} & t_{22} & t_{23}\mathrm{e}^{-2z^1} \\ t_{31}\mathrm{e}^{z^1} & t_{32}\mathrm{e}^{2z^1} & t_{33} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}\overline{z}^1 \\ \mathrm{d}\overline{z}^2 \\ \mathrm{d}\overline{z}^3 \end{pmatrix}. \tag{4.7}$$

For the four subcases in [16, p. 98], here we only consider the case 1:

$$t_{11} \neq 0 \quad \text{and} \quad t_{12} = t_{13} = t_{23} = t_{32} = 0.$$

The calculations for cases 2 to 4 are similar and are omitted.

The holomorphic coordinates $\{\zeta^1, \zeta^2, \zeta^3\}$ on X_t are given by (see [16, p. 98])

$$\zeta^1 = z^1 + t_{11}\overline{z}^1, \quad \zeta^1 = z^2 + t_{22}\overline{z}^2 + \lambda \mathrm{e}^{-z^1}, \quad \zeta^3 = z^3 + t_{33}\overline{z}^3 - \mu \mathrm{e}^{z^1},$$

where

$$\lambda = (1 - \mathrm{e}^{-t_{11}\overline{z}^1})\frac{t_{21}}{t_{11}} \quad \text{and} \quad \mu = (1 - \mathrm{e}^{t_{11}\overline{z}^1})\frac{t_{31}}{t_{11}}.$$

Then we obtain from (2.1) that

$$\begin{pmatrix} \mathcal{A}^1_{t,1} & \cdots & \mathcal{A}^1_{t,3} \\ \vdots & & \vdots \\ \mathcal{A}^3_{t,1} & \cdots & \mathcal{A}^3_{t,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda \mathrm{e}^{-z^1} & 1 & 0 \\ \mu \mathrm{e}^{z^1} & 0 & 1 \end{pmatrix}. \tag{4.8}$$

The three $(1,0)$-forms

$$\phi^1_t = \mathrm{e}^{i_\varphi}(\phi^1_0) = \mathrm{d}\zeta^1, \quad \phi^2_t = \mathrm{e}^{i_\varphi}(\phi^2_0) = \mathrm{e}^{z^1}\mathrm{d}\zeta^2 + \lambda \mathrm{d}\zeta^1,$$
$$\phi^3_t = \mathrm{e}^{i_\varphi}(\phi^3_0) = \mathrm{e}^{-z^1}\mathrm{d}\zeta^3 + \mu \mathrm{d}\zeta^1$$

form a basis of the space of smooth $(1,0)$-forms on X_t. Since

$$\phi^1_t \wedge \phi^2_t \wedge \phi^3_t = \mathrm{d}\zeta^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\zeta^3,$$

the canonical bundle K_{X_t} of X_t is holomorphically trivial.

For $\epsilon \ll 1$, there is a natural Hermitian metric on X_t:

$$\omega_t = \mathrm{e}^{i_\varphi | i_{\overline{\varphi}}}(\omega_0) = \sqrt{-1} g_{t,i\overline{j}} \mathrm{d}\zeta^i \wedge \mathrm{d}\overline{\zeta}^j,$$

where

$$\begin{pmatrix} g_{t,1\overline{1}} & \cdots & g_{t,1\overline{3}} \\ \vdots & & \vdots \\ g_{t,3\overline{1}} & \cdots & g_{t,3\overline{3}} \end{pmatrix} = \begin{pmatrix} 1 + |\lambda|^2 + |\mu|^2 & \lambda \mathrm{e}^{\overline{z}^1} & \mu \mathrm{e}^{-\overline{z}^1} \\ \overline{\lambda} \mathrm{e}^{z^1} & \mathrm{e}^{z^1 + \overline{z}^1} & 0 \\ \overline{\mu} \mathrm{e}^{-z^1} & 0 & \mathrm{e}^{-(z^1 + \overline{z}^1)} \end{pmatrix}. \tag{4.9}$$

It is direct to check

$$\omega_t = \sqrt{-1} \sum_{l=1}^{3} \phi^l_t \wedge \overline{\phi}^l_t.$$

Proposition 4.2 *Let X_0 be the Nakamura manifold for this case and $\{X_t\}_{t \in \Delta_\epsilon}$ be the Kuranishi family of X_0. The natural metric ω_t on X_t is Gauduchon and Chern-Ricci flat. Moreover, it is balanced if and only if $t_{21} = t_{31} = 0$.*

Proof We obtain from (4.9) that $\det \omega_t = 1$, which implies that ω_t is Chern-Ricci flat. While we obtain from (4.1) that

$$(I - \varphi\overline{\varphi})^{-1} = \begin{pmatrix} \dfrac{1}{1-|t_{11}|^2} & 0 & 0 \\ * & \dfrac{1}{1-|t_{22}|^2} & 0 \\ * & 0 & \dfrac{1}{1-|t_{33}|^2} \end{pmatrix}. \tag{4.10}$$

By [20, Lemma 2.4], we obtain from (4.8) and (4.10) that

$$\Big(\frac{\partial z}{\partial \zeta}\Big) = (I - \varphi\overline{\varphi})^{-1}\mathcal{A}_t = \begin{pmatrix} \dfrac{1}{1-|t_{11}|^2} & 0 & 0 \\ * & \dfrac{1}{1-|t_{22}|^2} & 0 \\ * & 0 & \dfrac{1}{1-|t_{33}|^2} \end{pmatrix}.$$

Similarly, we obtain from (4.7)–(4.8) and (4.10) that

$$-\Big(\frac{\partial \overline{z}}{\partial \zeta}\Big) = \overline{\varphi}(I - \varphi\overline{\varphi})^{-1}\mathcal{A}_t = \begin{pmatrix} \dfrac{\overline{t}_{11}}{1-|t_{11}|^2} & 0 & 0 \\ * & \dfrac{\overline{t}_{22}}{1-|t_{22}|^2} & 0 \\ * & 0 & \dfrac{\overline{t}_{33}}{1-|t_{33}|^2} \end{pmatrix}.$$

Then we get

$$\frac{\partial z^1}{\partial \zeta^1} = \frac{1}{1-|t_{11}|^2} \quad \text{and} \quad \frac{\partial z^1}{\partial \overline{\zeta}^1} = -\frac{t_{11}}{1-|t_{11}|^2}. \tag{4.11}$$

By (4.11), we calculate

$$\begin{aligned} \overline{\partial}_t \omega_t^2 &= 2\sqrt{-1}^2 (\overline{\partial}_t(\mu \mathrm{e}^{z^1}) \wedge \mathrm{d}\zeta^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\overline{\zeta}^3 \\ &\quad + \overline{\partial}_t(\lambda \mathrm{e}^{-z^1}) \wedge \mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\zeta^3 \wedge \mathrm{d}\overline{\zeta}^3) \\ &= \frac{2\sqrt{-1}^2}{1-|t_{11}|^2} (t_{31}\mathrm{e}^{z^1} \mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\zeta^2 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\overline{\zeta}^3 \\ &\quad - t_{21}\mathrm{e}^{-z^1} \mathrm{d}\zeta^1 \wedge \mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\overline{\zeta}^2 \wedge \mathrm{d}\zeta^3 \wedge \mathrm{d}\overline{\zeta}^3), \end{aligned}$$

which implies

$$\partial_t \overline{\partial}_t \omega_t^2 = 0,$$

i.e., ω_t is Gauduchon. While ω_t is balanced if and only if $t_{21} = t_{31} = 0$.

Proof of Proposition 2.1 Let X_0 be the Nakamura manifold for this case and $\{X_t\}_{t\in\Delta_\epsilon}$ be the Kuranishi family of X_0.

By (4.9) and (4.11), we get

$$\begin{pmatrix} g_t^{1\overline{1}} & \cdots & g_t^{3\overline{1}} \\ \vdots & & \vdots \\ g_t^{1\overline{3}} & \cdots & g_t^{3\overline{3}} \end{pmatrix} = \begin{pmatrix} 1 & -\lambda \mathrm{e}^{-z^1} & -\mu \mathrm{e}^{z^1} \\ -\overline{\lambda}\mathrm{e}^{-\overline{z}^1} & (1+|\lambda|^2)\mathrm{e}^{-(z^1+\overline{z}^1)} & \mu\overline{\lambda}\mathrm{e}^{z^1-\overline{z}^1} \\ -\overline{\mu}\mathrm{e}^{\overline{z}^1} & \lambda\overline{\mu}\mathrm{e}^{\overline{z}^1-z^1} & (1+|\mu|^2)\mathrm{e}^{z^1+\overline{z}^1} \end{pmatrix} \tag{4.12}$$

and

$$\partial_t g_t g_t^{-1} = \begin{pmatrix} -\eta & (\lambda\eta - \alpha\overline{t}_{11})\mathrm{e}^{-z^1} & (\mu\eta + \beta\overline{t}_{11})\mathrm{e}^{z^1} \\ \overline{t}_{21}\mathrm{e}^{z^1} & 1-\overline{t}_{11}-\lambda\overline{t}_{21} & -\mu\overline{t}_{21}\mathrm{e}^{2z^1} \\ -\overline{t}_{31}\mathrm{e}^{-z^1} & \lambda\overline{t}_{31}\mathrm{e}^{-2z^1} & -(1-\overline{t}_{11}-\mu\overline{t}_{31}) \end{pmatrix} \frac{\mathrm{d}\zeta^1}{1-|t_{11}|^2},$$

where

$$\eta = -\lambda\overline{t}_{21} + \mu\overline{t}_{31}, \quad \alpha = t_{21} + (1-t_{11})\lambda, \quad \beta = t_{31} + (1-t_{11})\mu.$$

Since $R^{\mathrm{T}}_{\omega_t} = \overline{\partial}_t(\partial_t g_t g_t^{-1})$, we obtain from (4.11) that

$$\begin{aligned} &R^{\mathrm{T}}_{\widehat{\omega}_t} \\ &= \begin{pmatrix} \nu & (t_{21}\eta - \lambda\nu - t_{21}\overline{t}_{11})\mathrm{e}^{-z^1} & -(t_{31}\eta + \mu\nu + t_{31}\overline{t}_{11})\mathrm{e}^{z^1} \\ -t_{11}\overline{t}_{21}\mathrm{e}^{z^1} & -|t_{21}|^2\mathrm{e}^{-t_{11}\overline{z}^1} & (t_{31}+t_{11}\mu)\overline{t}_{21}\mathrm{e}^{2z^1} \\ -t_{11}\overline{t}_{31}\mathrm{e}^{-z^1} & (t_{21}+t_{11}\lambda)\overline{t}_{31}\mathrm{e}^{-2z^1} & -|t_{31}|^2\mathrm{e}^{t_{11}\overline{z}^1} \end{pmatrix} \frac{\mathrm{d}\overline{\zeta}^1 \wedge \mathrm{d}\zeta^1}{(1-|t_{11}|^2)^2}, \end{aligned}$$

where

$$\nu = |t_{21}|^2\mathrm{e}^{-t_{11}\overline{z}^1} + |t_{31}|^2\mathrm{e}^{t_{11}\overline{z}^1}.$$

By (4.12), the mean curvature $K_{\omega_t} = \sqrt{-1}\Lambda_{\omega_t} R_{\omega_t}$ is

$$\begin{aligned} &K^{\mathrm{T}}_{\omega_t} \\ &= \begin{pmatrix} \nu & (t_{21}\eta - \lambda\nu - t_{21}\overline{t}_{11})\mathrm{e}^{-z^1} & -(t_{31}\eta + \mu\nu + t_{31}\overline{t}_{11})\mathrm{e}^{z^1} \\ -t_{11}\overline{t}_{21}\mathrm{e}^{z^1} & -|t_{21}|^2\mathrm{e}^{-t_{11}\overline{z}^1} & (t_{31}+t_{11}\mu)\overline{t}_{21}\mathrm{e}^{2z^1} \\ -t_{11}\overline{t}_{31}\mathrm{e}^{-z^1} & (t_{21}+t_{11}\lambda)\overline{t}_{31}\mathrm{e}^{-2z^1} & -|t_{31}|^2\mathrm{e}^{t_{11}\overline{z}^1} \end{pmatrix} \frac{-1}{(1-|t_{11}|^2)^2}. \end{aligned}$$

Since $K^*_{\omega_t} = K_{\omega_t}$, direct calculation yields

$$|K_{\omega_t}|^2_{\omega_t} = \mathrm{tr}(K_{\omega_t} \circ K^*_{\omega_t}) = \frac{2(|t_{21}|^2+|t_{31}|^2)(|t_{11}|^2+|t_{21}|^2+|t_{31}|^2)}{(1-|t_{11}|^2)^4}.$$

Since R_{ω_t} is skew-symmetric with respect to ω_t: $R^*_{\omega_t} = -R_{\omega_t}$, we get

$$|R_{\omega_t}|^2_{\omega_t} = -\frac{\mathrm{tr}(R_{\omega_t} \wedge *_t R_{\omega_t})}{\frac{\omega_t^3}{3!}} = |K_{\omega_t}|^2_{\omega_t}.$$

The conclusion for this case is valid.

Declarations

Conflicts of interest Jixiang Fu is an editorial board member for Chinese Annals of Mathematics Series B and was not involved in the editorial review or the decision to publish this article. The authors declare no conflicts of interest.

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