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On the Payne-Schaefer Conjecture About an Overdetermined Boundary Problem of Sixth Order

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Abstract This paper considers overdetermined boundary problems. Firstly, the author gives a proof of the Payne-Schaefer conjecture about an overdetermined problem of sixth order in the two-dimensional case and under an additional condition for the case of dimension no less than three. Secondly, the author proves an integral identity for an overdetermined problem of fourth order which can be used to deduce Bennett's symmetry theorem. Finally, the author proves a symmetry result for an overdetermined problem of second order by integral identities.

Keywords Overdetermined problem, Payne-Schaefer conjecture, Bennett theorem, Euclidean ball
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1 Introduction and the Main Results

In a celebrated paper in 1971, Serrin initiated the study of elliptic equations under an overdetermined boundary condition and established in particular the following seminal result.

Theorem 1.1 (see [12]) *If Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ and if the solution to the problem*

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has the property that $\frac{\partial u}{\partial \nu}$ is equal to a constant c on $\partial\Omega$, then Ω is a ball of radius $|nc|$ and $u = \frac{n^2c^2-r^2}{2n}$, where ν is the outward unit normal of $\partial\Omega$ and r is the distance from the center of the ball.

Several proofs to the above result have appeared. Serrin's proof is based on the Hopf maximum principle and a reflection-in-moving-planes argument which could be extended to more general elliptic equations and somewhat more general boundary conditions. A simple proof of Serrin's result based on a Rellich identity and a maximum principle was given by Weinberger in [14]. By the method of duality theorem, Payne and Schaefer [10] gave a proof of Theorem 1.1 which does not make explicit use of maximum principle. Choulli and Henrot

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[6] used domain derivative to prove Serrin's theorem which also does not use the maximum principle explicitly. From the need to extend Serrin overdetermined result to non-uniformly elliptic operators of Hessian type, Brandolini, Nitsch, Salani and Trombetti [4] used an integral approach via arithmetic-geometric mean inequality to prove Serrin's theorem and they also established the stability of the Serrin problem (see [5]). Serrin's theorem is a landmark in the study of overdetermined boundary value problem. The ideas and techniques in proving Serrin's theorem have been widely used and generalized to prove symmetry for more general overdetermined problems. Troy [13] used Serrin's moving planes method to prove a symmetry theorem for a system of semilinear elliptic equations, Alessandrini [1] adapted this method to condensers in a capacity problem. Farina and Kawohl [7], Garofalo and Lewis [8] extended Weinberger's method to more general second order partial differential equations. Benett [3], Philippin and Ragoub [11] considered the fourth order elliptic overdetermined problems. In [10], Payne and Schaefer studied overdetermined problems of higher orders, obtained various symmetry results and proposed the following important conjecture.

Conjecture 1.1 (see [10]) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. If u is a sufficiently smooth solution of the following overdetermined problem:

$$\begin{cases} \Delta^3 u = -1 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \Delta u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial(\Delta u)}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

then Ω is an n-ball.

In this paper, we prove Payne-Schaefer's conjecture in the case $n = 2$ and also prove the case $n \geq 3$ under an additional hypothesis.

Theorem 1.2 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^{6+\epsilon}$ boundary. Suppose that the following overdetermined problem has a solution in $C^6(\overline{\Omega})$:*

$$\Delta^3 u = -1 \qquad \textit{in } \Omega, \tag{1.3}$$

$$u = \frac{\partial u}{\partial \nu} = \Delta u = 0 \qquad \textit{on } \partial\Omega, \tag{1.4}$$

$$\frac{\partial(\Delta u)}{\partial \nu} = c \qquad \textit{on } \partial\Omega, \tag{1.5}$$

where c is a constant. When $n \geq 3$, we assume that

$$\int_\Omega (\Delta^2 u)^2 \gamma \mathrm{d}x \leq \frac{2(n+2)c^2|\Omega|}{n+6}. \tag{1.6}$$

Here $|\Omega|$ denotes the volume of Ω and γ is the torsion function of Ω given by

$$\begin{cases} \Delta\gamma = -1 & \textit{in } \Omega, \\ \gamma = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then Ω *is a ball of radius* $(|c|n(n+2)(n+4))^{\frac{1}{3}}$, *and*

$$u(x) = -\frac{1}{48n(n+2)(n+4)}r^6 + \Big(\frac{c^2}{n(n+2)(n+4)}\Big)^{\frac{1}{3}} \cdot \frac{r^4}{16} \\ -(c^4 n(n+2)(n+4))^{\frac{1}{3}} \cdot \frac{r^2}{16} + \frac{c^2 n(n+2)(n+4)}{48}, \tag{1.7}$$

where r *denotes the distance from* x *to the center of* Ω.

It should be mentioned that for a ball in $\mathbb{R}^n$, (1.6) becomes an equality. We shall explain this in the next section.

An integral dual for (1.3)–(1.5) is

$$\int_\Omega \phi \mathrm{d}x = c\int_{\partial\Omega} \Delta\phi \mathrm{d}s \tag{1.8}$$

for any triharmonic function ϕ in Ω with $\phi = \frac{\partial\phi}{\partial\nu} = 0$ on $\partial\Omega$. Thus, we have from Theorem 1.2 the following corollary.

Corollary 1.1 *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with* $C^{6+\epsilon}$ *boundary and if* (1.8) *holds for any triharmonic function* ϕ *in* Ω *with* $\phi = \frac{\partial\phi}{\partial\nu} = 0$ *on* $\partial\Omega$, *where* c *is a constant. Then* Ω *is a disk.*

In [3], Bennett established the following symmetry result.

Theorem 1.3 *If* Ω *is a bounded domain in* $\mathbb{R}^n$ *with* $C^{4+\epsilon}$ *boundary and if the following overdetermined problem has a solution in* $C^4(\overline{\Omega})$:

$$\begin{cases} \Delta^2 u = -1 & \textit{in } \Omega, \\ u = \dfrac{\partial u}{\partial\nu} = 0 & \textit{on } \partial\Omega, \\ \Delta u \equiv c & \textit{on } \partial\Omega \ (c \textit{ is a constant}), \end{cases} \tag{1.9}$$

then Ω *is a ball of radius* $(|c|n(n+2))^{\frac{1}{2}}$, *and*

$$u(x) = \frac{-1}{2n}\Big\{\frac{1}{4}(n+2)(nc)^2 + \frac{nc}{2}r^2 + \frac{1}{4(n+2)}r^4\Big\}, \tag{1.10}$$

where r *denotes the distance from* x *to the center of* Ω.

A crucial point in Bennett's proof is to use the following identity (see [9]):

$$\frac{1}{2}\Delta\Phi = \sum_{i,j,k} u_{ijk}^2 - \frac{3}{n+2}|\nabla(\Delta u)|^2 \\ = \sum_{i,j,k}\Big\{u_{ijk} - \frac{1}{n+2}((\Delta u)_i\delta_{jk} + (\Delta u)_j\delta_{ik} + (\Delta u)_k\delta_{ij})\Big\}^2, \tag{1.11}$$

where

$$\Phi = \frac{n-4}{n+2}u + \frac{n-4}{2(n+2)}(\Delta u)^2 + |\nabla^2 u|^2 - \langle\nabla u, \nabla(\Delta u)\rangle. \tag{1.12}$$

Since

$$\Phi|_{\partial\Omega} = \frac{3nc^2}{2(n+2)}, \tag{1.13}$$

it follows from the maximum principle that

$$\Phi \le \frac{3nc^2}{2(n+2)} \quad \text{in } \Omega. \tag{1.14}$$

On the other hand, from Green's theorem and Rellich identity, one has

$$\int_\Omega \Phi \mathrm{d}x = \frac{3nc^2}{2(n+2)}\ |\Omega|. \tag{1.15}$$

Thus, $\Phi \equiv \frac{3nc^2}{2(n+2)}$ in $\overline{\Omega}$ and so $\Delta\Phi \equiv 0$ in $\overline{\Omega}$. Therefore, each term of the sum on the right hand side of (1.11) vanishes which implies that

$$(\Delta u)_{ij} = -\frac{1}{n}\delta_{ij}. \tag{1.16}$$

One can then obtain the conclusions of Theorem 1.3 easily.

In this paper, we obtain an integral identity for an overdetermined problem of fourth order from which one can prove Bennett's theorem without using the subharmonicity of the function Φ.

Theorem 1.4 *Let Ω be a bounded domain in $\mathbb{R}^n, n \ge 2$, with $C^{4+\epsilon}$ boundary. Let $g : \mathbb{R} \to \mathbb{R}$ be a C^2 function and set $G(t) = \int_0^t g(s)\mathrm{d}s$. If $u \in C^4(\overline{\Omega})$ is a solution of the following overdetermined problem:*

$$\Delta^2 u = -g(u) \qquad \text{in } \ \Omega, \tag{1.17}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \ \partial\Omega, \tag{1.18}$$

$$\Delta u = c \qquad \text{on } \ \partial\Omega, \tag{1.19}$$

where c is a constant, then, we have

$$\begin{aligned}&\int_\Omega (2(n+2)(3G(u)+c^2) + (3n\Delta u - (n-4)c)(\Delta u - c))g(u)\mathrm{d}x \\ &= 4(n+2)\int_\Omega (\Delta u - c)\Big\{|\nabla^3 u|^2 - \frac{3}{n+2}|\nabla(\Delta u)|^2\Big\}\mathrm{d}x \\ &\quad + (n+2)\int_\Omega |\nabla u|^2 \Delta(g(u))\mathrm{d}x.\end{aligned} \tag{1.20}$$

Here, $\nabla^3 u = \nabla(\nabla^2 u)$ is the covariant derivative of the Hessian $\nabla^2 u$ of u.

Another proof of Theorem 1.3 We have from Rellich identity that

$$\int_\Omega u\mathrm{d}x = -\frac{nc^2|\Omega|}{n+4}. \tag{1.21}$$

Observe that

$$\int_\Omega \Delta u\mathrm{d}x = 0, \quad \int_\Omega (\Delta u)^2\mathrm{d}x = \int_\Omega u\Delta^2 u\mathrm{d}x = -\int_\Omega u\mathrm{d}x. \tag{1.22}$$

Taking $g(u) = 1,\ G(u) = u$, the left hand side of (1.20) then becomes

$$\begin{aligned}&\int_\Omega (2(n+2)(3u+c^2) + (3n\Delta u - (n-4)c)(\Delta u - c))\mathrm{d}x \\ &= \int_\Omega (3(n+4)u + 3nc^2)\mathrm{d}x = 0.\end{aligned} \tag{1.23}$$

Since $\Delta^2 u = -1$ in $\Omega,\ \Delta u|_{\partial\Omega} = c$, we know that $\Delta u - c > 0$ in the interior of Ω. Therefore, we have from (1.20) and (1.23) that

$$\sum_{i,j,k} u_{ijk}^2 - \frac{1}{n+2}|\nabla(\Delta u)|^2 = |\nabla^3 u|^2 - \frac{1}{n+2}|\nabla(\Delta u)|^2 = 0 \tag{1.24}$$

in the interior of Ω, and so on $\overline{\Omega}$ by continuity. Theorem 1.3 follows as above.

We shall also prove the following symmetry result using integral identities.

Theorem 1.5 *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$, with C^2 boundary. If the following overdetermined problem has a solution in $C^2(\overline{\Omega})$:*

$$\Delta u = -1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.25}$$

$$\frac{\partial u}{\partial \nu} = c|x| \quad \text{on } \partial\Omega, \tag{1.26}$$

where c is a constant, then Ω is a ball centered at the origin, $c = -\frac{1}{n}$ and

$$u(x) = -\frac{1}{2n}(|x|^2 - R^2), \tag{1.27}$$

where R is the radius of Ω.

When Ω contains the origin strictly in its interior, Theorem 1.5 has been proven by Amdeberhan [2] using the maximum principle.

2 Proof of the Results

In this section, we prove Theorems 1.2 and 1.4–1.5. Firstly we make some conventions about notation to be used. Let $x = (x_1, \cdots, x_n)$ and $\langle\ ,\ \rangle$ be the position vector and the standard inner product of $\mathbb{R}^n$, respectively. We shall use u_i, u_{ij}, u_{ijk}, u_{ijkl} and u_{ijklm} to denote, respectively,

$$\frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \quad \frac{\partial^4 u}{\partial x_i \partial x_j \partial x_k \partial x_l} \quad \text{and} \quad \frac{\partial^5 u}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m}.$$

Lemma 2.1 *Let u satisfy* (1.3)–(1.5) *and η be the solution of the Dirichlet problem*

$$\begin{cases} \Delta\eta = \langle \nabla(\Delta u), \nabla(\Delta^2 u)\rangle & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The function

$$F := \frac{1}{2}\sum_{i,j,k} u_{ijk}^2 - \frac{1}{2}\sum_{i,j}(\Delta u)_{ij}u_{ij} + \frac{1}{4}\langle \nabla u, \nabla(\Delta^2 u)\rangle + \frac{n-8}{4(n+4)}|\nabla(\Delta u)|^2$$

$$+\frac{3}{(n+4)(n+2)}(\Delta^2 u\Delta u+u)-\frac{3n(n-2)}{4(n+4)(n+2)}\eta \tag{2.2}$$

assumes its maximum value on $\partial\Omega$.

Proof We need only to show that $\Delta F\geq 0$ in Ω. A straightforward calculation gives

$$\begin{aligned}\Delta F=&\sum_{i,j,k,l}u_{ijkl}^2+\sum_{i,j,k}(\Delta u)_{ijk}u_{ijk}\\&-\frac{1}{2}\Big(\sum_{i,j}((\Delta^2 u)_{ij}u_{ij}+(\Delta u)_{ij}^2)+2\sum_{i,j,k}(\Delta u)_{ijk}u_{ijk}\Big)\\&+\frac{1}{4}\Big(2\sum_{i,j}(\Delta^2 u)_{ij}u_{ij}+\langle\nabla(\Delta^2 u),\nabla(\Delta u)\rangle\Big)\\&+\frac{n-8}{2(n+4)}\Big(\sum_{i,j}(\Delta u)_{ij}^2+\langle\nabla(\Delta^2 u),\nabla(\Delta u)\rangle\Big)\\&+\frac{3}{(n+4)(n+2)}((\Delta^2 u)^2+2\langle\nabla(\Delta^2 u),\nabla(\Delta u)\rangle)\\&-\frac{3n(n-2)}{4(n+4)(n+2)}\langle\nabla(\Delta u),\nabla(\Delta^2 u)\rangle\\=&\sum_{i,j,k,l}u_{ijkl}^2-\frac{6}{n+4}\sum_{i,j}(\Delta u)_{ij}^2+\frac{3}{(n+4)(n+2)}(\Delta^2 u)^2.\end{aligned}\tag{2.3}$$

To see that the right hand side of (2.3) is nonnegative, it suffices to note that

$$\begin{aligned}&\sum_{i,j,k,l}\Big\{u_{ijkl}-\frac{1}{n+4}((\Delta u)_{ij}\delta_{kl}+(\Delta u)_{il}\delta_{jk}+(\Delta u)_{ik}\delta_{jl}+(\Delta u)_{jk}\delta_{il}\\&+(\Delta u)_{jl}\delta_{ik}+(\Delta u)_{kl}\delta_{ij})+\frac{\Delta^2 u}{(n+4)(n+2)}(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl})\Big\}^2\\=&\sum_{i,j,k,l}u_{ijkl}^2-\frac{6}{n+4}\sum_{i,j}(\Delta u)_{ij}^2+\frac{3}{(n+4)(n+2)}(\Delta^2 u)^2.\end{aligned}\tag{2.4}$$

This completes the proof of Lemma 2.1.

Lemma 2.2 *Let u be a solution of* (1.3)–(1.5). *The following identities hold*:

$$\int_\Omega u\,\mathrm{d}x=\frac{nc^2|\Omega|}{n+6}, \tag{2.5}$$

$$\int_\Omega F\,\mathrm{d}x=\frac{3(n+2)nc^2|\Omega|}{2(n+4)(n+6)}-\frac{3n(n-2)}{8(n+4)(n+2)}\int_\Omega(\Delta^2 u)^2\gamma\,\mathrm{d}x. \tag{2.6}$$

Proof It follows from (1.4) that

$$\nabla^2 u=0\quad\text{on }\partial\Omega. \tag{2.7}$$

Here $\nabla^2 u$ denotes the Hessian of u and is given by

$$\nabla^2 u(\alpha,\beta)=\langle\nabla_\alpha\nabla u,\beta\rangle \tag{2.8}$$

for all $\alpha, \beta \in \mathfrak{X}(\Omega)$. From (1.3), we have

$$\Delta^3\langle x, \nabla u\rangle = 6\Delta^3 u + \langle x, \nabla(\Delta^3 u)\rangle = -6. \tag{2.9}$$

Multiplying (2.9) by u and integrating on Ω, one gets from (1.3)–(1.5), (2.7) and the divergence theorem that

$$\begin{aligned}
-6\int_\Omega u\mathrm{d}x &= \int_\Omega u\Delta^3\langle x, \nabla u\rangle \mathrm{d}x \\
&= \int_\Omega \Delta u\ \Delta^2\langle x, \nabla u\rangle \mathrm{d}x \\
&= -\int_\Omega \langle \nabla(\Delta u), \nabla(\Delta\langle x, \nabla u\rangle)\rangle \mathrm{d}x \\
&= \int_\Omega \Delta^2 u\ \Delta\langle x, \nabla u\rangle \mathrm{d}x - \int_{\partial\Omega} \Delta\langle x, \nabla u\rangle \frac{\partial(\Delta u)}{\partial\nu}\mathrm{d}s \\
&= \int_\Omega \Delta^2 u\ \Delta\langle x, \nabla u\rangle \mathrm{d}x - c\int_{\partial\Omega} (2\Delta u + \langle x, \nabla(\Delta u)\rangle)\mathrm{d}s \\
&= \int_\Omega \Delta^2 u\ \Delta\langle x, \nabla u\rangle \mathrm{d}x - c\int_{\partial\Omega} \langle x, \nabla(\Delta u)\rangle \mathrm{d}s \\
&= \int_\Omega \Delta^2 u\ \Delta\langle x, \nabla u\rangle \mathrm{d}x - c\int_{\partial\Omega} \langle x, \nu\rangle \frac{\partial(\Delta u)}{\partial\nu}\mathrm{d}s \\
&= \int_\Omega \Delta^2 u\ \Delta\langle x, \nabla u\rangle \mathrm{d}x - c^2\int_{\partial\Omega} \langle x, \nu\rangle \mathrm{d}s \\
&= -\int_\Omega \langle \nabla(\Delta^2 u), \nabla\langle x, \nabla u\rangle\rangle \mathrm{d}x + \int_{\partial\Omega} \Delta^2 u \frac{\partial\langle x, \nabla u\rangle}{\partial\nu}\mathrm{d}s - nc^2|\Omega| \\
&= \int_\Omega \Delta^3 u\langle x, \nabla u\rangle \mathrm{d}x + \int_{\partial\Omega} \Delta^2 u(\langle \nu, \nabla u\rangle + \nabla^2 u(x, \nu))\mathrm{d}s - nc^2|\Omega| \\
&= -\int_\Omega \langle x, \nabla u\rangle \mathrm{d}x - nc^2|\Omega| \\
&= n\int_\Omega u\mathrm{d}x - nc^2|\Omega|.
\end{aligned}$$

This proves (2.5). In order to obtain (2.6), we integrate

$$\frac{1}{2}\sum_{i,j}\Delta(u_{ij}^2) = \sum_{i,j,k} u_{ijk}^2 + \sum_{i,j}(\Delta u)_{ij}u_{ij} \tag{2.10}$$

on Ω and use $u_{ij}|_{\partial\Omega} = 0$ $(\forall i, j)$ to obtain

$$\sum_{i,j,k}\int_\Omega u_{ijk}^2 \mathrm{d}x = -\sum_{i,j}\int_\Omega (\Delta u)_{ij}u_{ij}\mathrm{d}x. \tag{2.11}$$

Similarly, one gets by integrating

$$\Delta\langle \nabla(\Delta u), \nabla u\rangle = 2\sum_{i,j}(\Delta u)_{ij}u_{ij} + |\nabla(\Delta u)|^2 + \langle \nabla(\Delta^2 u), \nabla u\rangle \tag{2.12}$$

on Ω that

$$-\int_\Omega (\Delta u)_{ij}u_{ij}\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla(\Delta u)|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \langle \nabla(\Delta^2 u), \nabla u\rangle \mathrm{d}x$$

$$= -\frac{1}{2}\int_\Omega \Delta u \Delta^2 u \mathrm{d}x - \frac{1}{2}\int_\Omega u\Delta^3 u \mathrm{d}x$$
$$= \int_\Omega u \mathrm{d}x. \tag{2.13}$$

Integrating F on Ω and using (2.11), (2.13), (1.3), (2.5) and the divergence theorem, one has

$$\begin{aligned}\int_\Omega F\mathrm{d}x &= \frac{3(n+2)}{2(n+4)}\int_\Omega u\mathrm{d}x - \frac{3n(n-2)}{4(n+4)(n+2)}\int_\Omega \eta\mathrm{d}x \\ &= \frac{3(n+2)nc^2|\Omega|}{2(n+4)(n+6)} - \frac{3n(n-2)}{4(n+4)(n+2)}\int_\Omega \eta\mathrm{d}x. \end{aligned}\tag{2.14}$$

To finish the proof of (2.6), we need to calculate $\int_\Omega \eta\mathrm{d}x$. Multiplying the equation

$$\Delta\eta = \langle\nabla(\Delta u), \nabla(\Delta^2 u)\rangle$$

by γ and integrating on Ω, we infer

$$\begin{aligned}-\int_\Omega \eta\mathrm{d}x &= \int_\Omega \eta\Delta\gamma\mathrm{d}x \\ &= \int_\Omega \gamma\Delta\eta\mathrm{d}x \\ &= \int_\Omega \gamma\langle\nabla(\Delta u), \nabla(\Delta^2 u)\rangle\mathrm{d}x \\ &= -\int_\Omega \Delta u(\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle + \gamma\Delta^3 u)\mathrm{d}x \\ &= -\int_\Omega \Delta u\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle\mathrm{d}x + \int_\Omega \gamma\Delta u\mathrm{d}x \\ &= -\int_\Omega \Delta u\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle\mathrm{d}x - \int_\Omega u\mathrm{d}x. \end{aligned}\tag{2.15}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \gamma(\Delta^2 u)^2\mathrm{d}x &= \int_\Omega \Delta u\Delta(\gamma\Delta^2 u)\mathrm{d}x \\ &= \int_\Omega \Delta u((\Delta\gamma)\Delta^2 u + \gamma\Delta^3 u + 2\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle)\mathrm{d}x \\ &= \int_\Omega \Delta u(-\Delta^2 u - \gamma + 2\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle)\mathrm{d}x \\ &= \int_\Omega u(-\Delta^3 u - \Delta\gamma)\mathrm{d}x + 2\int_\Omega \Delta u\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle\mathrm{d}x \\ &= 2\int_\Omega u\mathrm{d}x + 2\int_\Omega \Delta u\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle\mathrm{d}x. \end{aligned}\tag{2.16}$$

Combining the above two equalities, we arrive at

$$\int_\Omega \eta\mathrm{d}x = \frac{1}{2}\int_\Omega \gamma(\Delta^2 u)^2\mathrm{d}x. \tag{2.17}$$

Substituting (2.17) into (2.14), we obtain (2.6).

Proof of Theorem 1.2 One knows from (1.4)–(1.5) that

$$\sum_{i,j,k} u_{ijk}^2|_{\partial\Omega} = (|\nabla(\Delta u)|_{\partial\Omega})^2 = \Big(\frac{\partial(\Delta u)}{\partial\nu}\Big|_{\partial\Omega}\Big)^2 = c^2. \tag{2.18}$$

Hence,

$$F|_{\partial\Omega} = \frac{3nc^2}{4(n+4)}, \tag{2.19}$$

which, in turn implies from Lemma 2.1 that

$$F \le \frac{3nc^2}{4(n+4)} \quad \text{in } \overline{\Omega}. \tag{2.20}$$

When $n = 2$, we know from (2.6) that

$$\int_\Omega F\mathrm{d}x = \frac{3\cdot 2\cdot c^2|\Omega|}{4\cdot 6}$$

and when $n \ge 3$, we have from (1.6) and (2.6) that

$$\int_\Omega F\mathrm{d}x \ge \frac{3nc^2|\Omega|}{4(n+4)}. \tag{2.21}$$

Hence, for $n \ge 2$, $F \equiv \frac{3nc^2}{4(n+2)}$ in $\overline{\Omega}$ and so ΔF vanishes identically in $\overline{\Omega}$. Therefore, each term of the sum on the left hand side of (2.4) vanishes. Consequently, we have

$$\begin{aligned} u_{ijkl} = &\frac{1}{n+4}\{(\Delta u)_{ij}\delta_{kl} + (\Delta u)_{il}\delta_{jk} + (\Delta u)_{ik}\delta_{jl} \\ &+ (\Delta u)_{jk}\delta_{il} + (\Delta u)_{jl}\delta_{ik} + (\Delta u)_{kl}\delta_{ij}\} \\ &- \frac{\Delta^2 u}{(n+4)(n+2)}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \quad \forall i,j,k,l. \end{aligned} \tag{2.22}$$

By differentiating the above equality with respect to x_l and summing over l, we obtain

$$\sum_l u_{ijkll} = (\Delta u)_{ijk} = \frac{1}{n+2}((\Delta^2 u)_i\delta_{jk} + (\Delta^2 u)_j\delta_{ik} + (\Delta^2 u)_k\delta_{ij}). \tag{2.23}$$

Differentiating with respect to x_k and summing over k, one gets

$$(\Delta^2 u)_{ij} = -\frac{1}{n}\delta_{ij}. \tag{2.24}$$

Thus we have

$$\Delta^2 u(x) = \frac{1}{2n}(A - |x - a_0|^2), \tag{2.25}$$

where A is a constant and $\Delta^2 u(a_0) = \frac{A}{2n}$. Without loss of generality, we assume that a_0 is the origin. Substituting (2.25) into (2.23), we get

$$(\Delta u)_{ijk}(x) = -\frac{1}{n(n+2)}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}). \tag{2.26}$$

Differentiating (2.22) with respect to x_m, using (2.26) and

$$(\Delta^2 u)_m = -\frac{x_m}{n},$$

we get

$$\begin{aligned} u_{ijklm} = -\frac{1}{n(n+2)(n+4)} &\{x_i(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) \\ &+ x_j(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) \\ &+ x_k(\delta_{ji}\delta_{lm} + \delta_{jl}\delta_{im} + \delta_{jm}\delta_{il}) \\ &+ x_l(\delta_{jk}\delta_{im} + \delta_{ji}\delta_{km} + \delta_{jm}\delta_{ki}) \\ &+ x_m(\delta_{jk}\delta_{li} + \delta_{jl}\delta_{ki} + \delta_{ji}\delta_{kl})\}, \quad \forall i, j, k, l, m. \end{aligned} \tag{2.27}$$

Consider the function $q : \Omega \to \mathbb{R}$ given by

$$q(x) = u(x) + \frac{1}{48n(n+2)(n+4)}|x|^6.$$

Using a straightforward calculation and (2.27), we get

$$q_{ijklm} = 0, \quad \forall i, j, k, l, m.$$

Thus q is a polynomial of $x_1, \cdots, x_n$ of order 4 and so

$$\Delta u(x) = -\frac{1}{8n(n+2)}|x|^4 + p(x). \tag{2.28}$$

Here, p is a quadratic polynomial of $x_1, \cdots, x_n$. Now let us determine p. From (1.4) we know from the divergence theorem that

$$\int_\Omega (\Delta u) h \mathrm{d}x = 0 \quad \text{for all harmonic } h \text{ in } \Omega. \tag{2.29}$$

After some calculations by using (2.25), we see that

$$h = \Big(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\Big)(x_i(\Delta u)_j - x_j(\Delta u)_i)$$

is harmonic in Ω. Then integration by parts using $\Delta u|_{\partial\Omega} = 0$ results in

$$\begin{aligned} 0 &= \int_\Omega (\Delta u)\Big(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\Big)(x_i(\Delta u)_j - x_j(\Delta u)_i)\mathrm{d}x \\ &= -\int_\Omega (x_i(\Delta u)_j - x_j(\Delta u)_i)^2 \mathrm{d}x. \end{aligned} \tag{2.30}$$

Hence $x_i(\Delta u)_j - x_j(\Delta u)_i \equiv 0$ in Ω and so Δu is a radial function. Consequently, we have

$$\Delta u(x) = -\frac{1}{8n(n+2)}|x|^4 + \kappa_1|x|^2 + \kappa_2, \tag{2.31}$$

where κ_1, κ_2 are constants. Since $\Delta u = 0$ on $\partial\Omega$, Ω is a ball. We note from (2.31) that

$$\Delta(x_i u_j - x_j u_i) = 0 \quad \text{in } \Omega \tag{2.32}$$

and from (1.4) that

$$(x_i u_j - x_j u_i)|_{\partial\Omega} = 0. \tag{2.33}$$

Hence, $x_i u_j - x_j u_i = 0$ in Ω and so u is a radial function, which, combining with (2.31) and the fact that u is a polynomial, gives

$$u(x) = -\frac{1}{48n(n+2)(n+4)}|x|^6 + \frac{\kappa_1}{4(n+2)}|x|^4 + \frac{\kappa_2}{2n}|x|^2 + \kappa_3, \tag{2.34}$$

where κ_3 is a constant. Let us denote by ρ the radius of Ω. One deduces from (1.4)–(1.5) that

$$-\frac{1}{48n(n+2)(n+4)}\rho^6 + \frac{\kappa_1}{4(n+2)}\rho^4 + \frac{\kappa_2}{2n}\rho^2 + \kappa_3 = 0, \tag{2.35}$$

$$-\frac{1}{8n(n+2)(n+4)}\rho^4 + \frac{\kappa_1}{n+2}\rho^2 + \frac{\kappa_2}{n} = 0, \tag{2.36}$$

$$-\frac{1}{8n(n+2)}\rho^4 + \kappa_1\rho^2 + \kappa_2 = 0, \tag{2.37}$$

$$-\frac{1}{2n(n+2)}\rho^3 + 2\kappa_1\rho = c. \tag{2.38}$$

Solving (2.35)–(2.38), we obtain

$$\rho = (|c|n(n+2)(n+4))^{\frac{1}{3}}, \tag{2.39}$$

$$\frac{\kappa_1}{4(n+2)} = \Big(\frac{c^2}{n(n+2)(n+4)}\Big)^{\frac{1}{3}} \cdot \frac{1}{16}, \tag{2.40}$$

$$\frac{\kappa_2}{2n} = -(c^4 n(n+2)(n+4))^{\frac{1}{3}} \cdot \frac{1}{16}, \tag{2.41}$$

$$\kappa_3 = \frac{c^2 n(n+2)(n+4)}{48}. \tag{2.42}$$

Substituting (2.39)–(2.42) into (2.34), we get (1.7). This completes the proof of Theorem 1.2.

Remark 2.1 From (2.16), we have

$$\begin{aligned}\int_\Omega \gamma(\Delta^2 u)^2 \mathrm{d}x &= 2\int_\Omega u\mathrm{d}x + 2\int_\Omega u\Delta\langle\nabla\gamma, \nabla(\Delta^2 u)\rangle \mathrm{d}x \\ &= 2\int_\Omega u\mathrm{d}x + 4\int_\Omega u\Big\{\sum_{i,j}\gamma_{ij}(\Delta^2 u)_{ij}\Big\}\mathrm{d}x.\end{aligned} \tag{2.43}$$

In the case that Ω is a ball with center a and radius R, γ is given by

$$\gamma(x) = -\frac{|x-a|^2 - R^2}{2n}. \tag{2.44}$$

Thus

$$\gamma_{ij} = -\frac{1}{n}\delta_{ij}, \quad \forall i, j, \tag{2.45}$$

which gives

$$\int_\Omega u\Big\{\sum_{i,j}\gamma_{ij}(\Delta^2 u)_{ij}\Big\}\mathrm{d}x = -\frac{1}{n}\int_\Omega u\Delta^3 u = \frac{1}{n}\int_\Omega u\mathrm{d}x. \tag{2.46}$$

We then obtain from (2.43) and (2.5) that

$$\int_\Omega \gamma(\Delta^2 u)^2 \mathrm{d}x = \Big(2+\frac{4}{n}\Big)\int_\Omega u \mathrm{d}x = \frac{2(n+2)c^2|\Omega|}{n+6}. \tag{2.47}$$

That is, (1.6) becomes an equality when Ω is a ball.

Proof of Theorem 1.4 From (1.18)–(1.19), we know that

$$|\nabla^2 u|^2 = c^2 \quad \text{on } \partial\Omega. \tag{2.48}$$

Multiplying (1.17) by $|\nabla^2 u|^2$ and integrating on Ω, we have

$$-\int_\Omega g(u)|\nabla^2 u|^2 \mathrm{d}x = \int_\Omega (\Delta^2 u)|\nabla^2 u|^2 \mathrm{d}x. \tag{2.49}$$

Observe that

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle. \tag{2.50}$$

Using (1.18)–(1.19), (2.48) and the divergence theorem, we have

$$\begin{aligned}
-\int_\Omega g(u)|\nabla^2 u|^2 \mathrm{d}x &= -\int_\Omega g(u)\Big(\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla(\Delta u)\rangle\Big)\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega \langle \nabla(g(u)), \nabla|\nabla u|^2\rangle \mathrm{d}x + \int_\Omega \langle \nabla(G(u)), \nabla(\Delta u)\rangle \mathrm{d}x \\
&= -\frac{1}{2}\int_\Omega |\nabla u|^2 \Delta(g(u))\mathrm{d}x - \int_\Omega G(u)\Delta^2 u \mathrm{d}x \\
&= -\frac{1}{2}\int_\Omega |\nabla u|^2 \Delta(g(u))\mathrm{d}x + \int_\Omega G(u) g(u)\mathrm{d}x, \qquad (2.51)\\
\int_\Omega (\Delta^2 u)|\nabla^2 u|^2 \mathrm{d}x &= \int_{\partial\Omega} |\nabla^2 u|^2 \frac{\partial(\Delta u)}{\partial\nu}\mathrm{d}s - \int_\Omega \langle \nabla(\Delta u), \nabla|\nabla^2 u|^2\rangle \mathrm{d}x \\
&= c^2\int_\Omega \Delta^2 u \mathrm{d}x + \int_\Omega (\Delta u)\Delta|\nabla^2 u|^2 \mathrm{d}x - \int_{\partial\Omega} (\Delta u)\frac{\partial(|\nabla^2 u|^2)}{\partial\nu}\mathrm{d}s \\
&= -c^2\int_\Omega g(u)\mathrm{d}x + \int_\Omega (\Delta u - c)\Delta|\nabla^2 u|^2 \mathrm{d}x. \qquad (2.52)
\end{aligned}$$

We have

$$\begin{aligned}
\Delta|\nabla^2 u|^2 &= 2\sum_{i,j,k} u_{ijk}^2 + 2\sum_{i,j} u_{ij}(\Delta u)_{ij} \\
&= 2|\nabla^3 u|^2 + 2\sum_{i,j} u_{ij}(\Delta u)_{ij}, \qquad (2.53)\\
\Delta\langle \nabla u, \nabla(\Delta u)\rangle &= 2\sum_{i,j} u_{ij}(\Delta u)_{ij} + |\nabla(\Delta u)|^2 + \langle \nabla u, \nabla(\Delta^2 u)\rangle \\
&= 2\sum_{i,j} u_{ij}(\Delta u)_{ij} + |\nabla(\Delta u)|^2 - \langle \nabla u, \nabla(g(u))\rangle. \qquad (2.54)
\end{aligned}$$

Combining (2.52)–(2.54), we get

$$\int_\Omega (\Delta^2 u)|\nabla^2 u|^2 \mathrm{d}x$$

$$
\begin{aligned}
&= -c^2\int_\Omega g(u)\mathrm{d}x + \int_\Omega (\Delta u - c)(2|\nabla^3 u|^2 + \Delta\langle\nabla u, \nabla(\Delta u)\rangle - |\nabla(\Delta u)|^2 \\
&\quad + \langle\nabla u, \nabla(g(u))\rangle)\mathrm{d}x \\
&= -c^2\int_\Omega g(u)\mathrm{d}x + 2\int_\Omega (\Delta u - c)\Big(|\nabla^3 u|^2 - \frac{3}{n+2}|\nabla(\Delta u)|^2\Big)\mathrm{d}x \\
&\quad + \int_\Omega (\Delta u - c)\Big(\Delta\langle\nabla u, \nabla(\Delta u)\rangle + \frac{4-n}{n+2}|\nabla(\Delta u)|^2 + \langle\nabla u, \nabla(g(u))\rangle\Big)\mathrm{d}x.
\end{aligned}
$$

Observing

$$
\begin{aligned}
\int_\Omega |\nabla(\Delta u)|^2\mathrm{d}x &= \int_{\partial\Omega} \Delta u\frac{\partial(\Delta u)}{\partial\nu}\mathrm{d}s - \int_\Omega \Delta u\Delta^2 u\mathrm{d}x \\
&= c\int_\Omega \Delta^2 u\mathrm{d}x - \int_\Omega \Delta u\Delta^2 u\mathrm{d}x \\
&= \int_\Omega (\Delta u - c)g(u)\mathrm{d}x, \qquad (2.55) \\
\int_\Omega \Delta u|\nabla(\Delta u)|^2\mathrm{d}x &= \frac{1}{2}\int_\Omega \langle\nabla(\Delta u), \nabla(\Delta u)^2\rangle\mathrm{d}x \\
&= \frac{1}{2}\Big\{\int_{\partial\Omega} (\Delta u)^2\frac{\partial(\Delta u)}{\partial\nu}\mathrm{d}s - \int_\Omega (\Delta u)^2\Delta^2 u\mathrm{d}x\Big\} \\
&= \frac{1}{2}\int_\Omega ((\Delta u)^2 - c^2)g(u)\mathrm{d}x, \qquad (2.56) \\
\int_\Omega (\Delta u - c)\Delta\langle\nabla u, \nabla(\Delta u)\rangle\mathrm{d}x &= \int_\Omega \Delta(\Delta u - c)\langle\nabla u, \nabla(\Delta u)\rangle\mathrm{d}x \\
&= \int_\Omega \Delta^2 u\langle\nabla u, \nabla(\Delta u)\rangle\mathrm{d}x \\
&= -\int_\Omega G(u)g(u)\mathrm{d}x, \\
\int_\Omega (\Delta u - c)\langle\nabla u, \nabla(g(u))\rangle\mathrm{d}x &= -\int_\Omega g(u)(\langle\nabla(\Delta u), \nabla u\rangle + (\Delta u - c)\Delta u)\mathrm{d}x \\
&= -\int_\Omega ((\Delta u - c)\Delta u + G(u))g(u)\mathrm{d}x, \qquad (2.57)
\end{aligned}
$$

one arrives at

$$
\begin{aligned}
\int_\Omega (\Delta^2 u)|\nabla^2 u|^2\mathrm{d}x = 2\int_\Omega (\Delta u - c)\Big(|\nabla^3 u|^2 - \frac{3}{n+2}|\nabla(\Delta u)|^2\Big)\mathrm{d}x \\
- \int_\Omega (G(u) + c^2)g(u)\mathrm{d}x \\
+ \frac{4-n}{2(n+2)}\int_\Omega (\Delta u - c)^2 g(u)\mathrm{d}x \\
- \int_\Omega ((\Delta u - c)\Delta u + G(u))g(u)\mathrm{d}x. \qquad (2.58)
\end{aligned}
$$

Combining (2.49), (2.51) and (2.58), we obtain (1.20). This completes the proof of Theorem 1.4.

Proof of Theorem 1.5 We have

$$
\int_\Omega |\nabla u|^2\mathrm{d}x = -\int_\Omega u\Delta u\mathrm{d}x = \int_\Omega u\mathrm{d}x. \qquad (2.59)
$$

Multiplying $\Delta u = -1$ by $|\nabla u|^2$ and integrating on Ω, we get

$$\begin{aligned}
-\int_\Omega u\mathrm{d}x &= -\int_\Omega |\nabla u|^2\mathrm{d}x \\
&= \int_\Omega |\nabla u|^2\Delta u\mathrm{d}x \\
&= -\int_\Omega \langle\nabla|\nabla u|^2, \nabla u\rangle\mathrm{d}x + \int_{\partial\Omega} |\nabla u|^2\frac{\partial u}{\partial\nu}\mathrm{d}s \\
&= \int_\Omega u\Delta|\nabla u|^2\mathrm{d}x + c^3\int_{\partial\Omega} |x|^3\mathrm{d}s \\
&= 2\int_\Omega u|\nabla^2 u|^2\mathrm{d}x + c^3\int_{\partial\Omega} |x|^3\mathrm{d}s. \qquad (2.60)
\end{aligned}$$

Multiplying $\Delta u = -1$ by $\langle x, \nabla u\rangle$ and integrating on Ω, one has

$$\begin{aligned}
n\int_\Omega u\mathrm{d}x &= -\int_\Omega \langle x, \nabla u\rangle\mathrm{d}x \\
&= \int_\Omega \Delta u\langle x, \nabla u\rangle\mathrm{d}x \\
&= -\int_\Omega \langle\nabla u, \nabla\langle x, \nabla u\rangle\rangle\mathrm{d}x + \int_{\partial\Omega} \langle x, \nabla u\rangle\frac{\partial u}{\partial\nu}\mathrm{d}s \\
&= \int_\Omega u\Delta\langle x, \nabla u\rangle\mathrm{d}x + \int_{\partial\Omega} \langle x, \nu\rangle\Big(\frac{\partial u}{\partial\nu}\Big)^2\mathrm{d}s \\
&= -2\int_\Omega u\mathrm{d}x + \int_{\partial\Omega} c^2\langle x, \nu\rangle|x|^2\mathrm{d}s \\
&= -2\int_\Omega u\mathrm{d}x + \frac{c^2}{4}\int_{\partial\Omega} \frac{\partial|x|^4}{\partial\nu}\mathrm{d}s \\
&= -2\int_\Omega u\mathrm{d}x + \frac{c^2}{4}\int_\Omega \Delta|x|^4\mathrm{d}x \\
&= -2\int_\Omega u\mathrm{d}x + c^2(n+2)\int_\Omega |x|^2\mathrm{d}x. \qquad (2.61)
\end{aligned}$$

Thus, we have

$$\int_\Omega u\mathrm{d}x = c^2\int_\Omega |x|^2\mathrm{d}x. \qquad (2.62)$$

On the other hand, multiplying $\Delta u = -1$ by $|x|^2$ and integrating on Ω, one arrives at

$$\begin{aligned}
-\int_\Omega |x|^2\mathrm{d}x &= \int_\Omega |x|^2\Delta u\mathrm{d}x \\
&= -\int_\Omega \langle\nabla|x|^2, \nabla u\rangle\mathrm{d}x + \int_{\partial\Omega} |x|^2\frac{\partial u}{\partial\nu}\mathrm{d}s \\
&= \int_\Omega u\Delta|x|^2\mathrm{d}x + c\int_{\partial\Omega} |x|^3\mathrm{d}s \\
&= 2n\int_\Omega u\mathrm{d}x + c\int_{\partial\Omega} |x|^3\mathrm{d}s. \qquad (2.63)
\end{aligned}$$

Substituting (2.63) into (2.62), we infer

$$(1+2nc^2)\int_\Omega u\mathrm{d}x + c^3\int_{\partial\Omega} |x|^3\mathrm{d}s = 0. \qquad (2.64)$$

Also, we have

$$|\Omega| = -\int_\Omega \Delta u \mathrm{d}x = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\mathrm{d}s = (-c)\int_{\partial\Omega} |x|\mathrm{d}s$$
$$= |c|\int_\Omega |x|\mathrm{d}x \geq |c|\int_{\partial\Omega} \langle x, \nu\rangle \mathrm{d}s = |c|n|\Omega|, \tag{2.65}$$

which gives

$$|c| \leq \frac{1}{n}. \tag{2.66}$$

Thus, we have from (2.64)–(2.65) that

$$\Big(1+\frac{2}{n}\Big)\int_\Omega u\mathrm{d}x + c^3\int_{\partial\Omega} |x|^3\mathrm{d}s \geq 0. \tag{2.67}$$

It follows from (2.60) and (2.67) that

$$0 \geq \int_\Omega u\Big(|\nabla^2 u|^2 - \frac{1}{n}\Big)\mathrm{d}x = \int_\Omega u\Big(|\nabla^2 u|^2 - \frac{(\Delta u)^2}{n}\Big)\mathrm{d}x. \tag{2.68}$$

Since $\Delta u = -1$ in Ω, $u|_{\partial\Omega} = 0$, $u > 0$ in the interior of Ω. The Schwarz inequality implies that

$$|\nabla^2 u|^2 - \frac{(\Delta u)^2}{n} \geq 0. \tag{2.69}$$

Therefore, we conclude from (2.68) that

$$|\nabla^2 u|^2 - \frac{(\Delta u)^2}{n} = 0 \tag{2.70}$$

and that the inequality (2.65) is actually an equality. Consequently, we have

$$c = -\frac{1}{n}, \tag{2.71}$$

$$u_{ij} = -\frac{1}{n}\delta_{ij}, \quad \forall i, j \tag{2.72}$$

and

$$x = |x|\nu \quad \text{on} \quad \partial\Omega. \tag{2.73}$$

Consider the function $\beta : \partial\Omega \to \mathbb{R}$ given by $\beta(x) = |x|^2$. For any $w \in \mathfrak{X}(\partial\Omega)$, it follows from (2.73) that

$$w\beta = 2\langle x, w\rangle = 2\langle |x|\nu, w\rangle = 0, \tag{2.74}$$

which shows that β is a constant. Hence, $\partial\Omega$ is a sphere centered at the origin and so Ω is a ball centered at the origin. One then knows from (2.72) that

$$u = -\frac{1}{2n}(|x|^2 - R^2), \tag{2.75}$$

where R is the radius of the ball. This completes the proof of Theorem 1.5.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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