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On $\overline{\partial}_b$-Harmonic Maps from Pseudo-Hermitian Manifolds to Kähler Manifolds*

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Abstract This paper considers maps from pseudo-Hermitian manifolds to Kähler manifolds and introduces partial energy functionals for these maps. First, the authors obtain a foliated Lichnerowicz type result on general pseudo-Hermitian manifolds, which generalizes a related result on Sasakian manifolds by Shen–Shen–Zhang (2013). Next, the authors investigate critical maps of the partial energy functionals, which are referred to as $\overline{\partial}_b$ -harmonic maps and ∂_b-harmonic maps. The authors give a foliated result for both $\overline{\partial}_b$- and ∂_b-harmonic maps, generalizing a foliated result of Petit (2002) for harmonic maps. Then the authors are able to generalize Siu's holomorphicity result for harmonic maps by Siu (1980) to the case for $\overline{\partial}_b$- and ∂_b-harmonic maps.

Keywords Pseudo-Hermitian manifold, $\overline{\partial}_b$-Harmonic maps, Foliated CR map
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1 Introduction

In [19], Siu proved the following theorem.

Theorem A *Let* $f : M \to N$ *be a harmonic map between compact Kähler manifolds. If* (N, g) *has strongly negative curvature and* $\operatorname{rank}_{\mathbb{R}}(\mathrm{d}f_x) \geq 4$ *at some point* $x \in M$*, then* f *is holomorphic or anti-holomorphic.*

The above theorem, combined with Eells-Sampson's existence theorem (cf. [7]), implies Siu's celebrated strong rigidity for compact Kähler manifolds with strongly negative curvature. Subsequently, there have been some research efforts to generalize Siu's theorem to the case of non-Kähler Hermitian manifolds. In [11], Jost and Yau used Hermitian harmonic maps to generalize Siu's rigidity theorem to the case where the domain manifold is astheno-Kähler. In [14], Liu and Yang considered the critical points of partial energies for maps from Hermitian manifolds, and discussed related holomorphicity results for these critical maps.

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A pseudo-Hermitian manifold (M^{2m+1}, H, J, θ) is a strictly pseudoconvex CR manifold (M, H, J) endowed with a pseudo-Hermitian 1-form θ. It can be regarded as an odd dimensional analogue of a Hermitian manifold. Harmonic maps and their generalizations have also been used to study pseudo-Hermitian manifolds. In [15], Petit established some rigidity results for harmonic maps from pseudo-Hermitian manifolds. First, he proved that any harmonic map from a compact Sasakian manifold to a Riemannian manifold with non-positive sectional curvature is trivial on the Reeb field of the pseudo-Hermitian structure. A map with this property is said to be foliated. Next he proved that under a similar rank condition as above, the harmonic map from a compact Sasakian manifold to a Kähler manifold with strongly negative curvature is CR-holomorphic or CR-antiholomorphic. In [2], among other results, the authors generalized Petit's results to the case of pseudoharmonic maps. Besides, Li and Son [12] defined the following $\overline{\partial}_b$-energy functional for maps from a pseudo-Hermitian manifold to a Kähler manifold:

$$E_{\overline{\partial}_b}(f) = \frac{1}{2}\int_M |\overline{\partial}_b f|^2 \mathrm{d}v_\theta.$$

The ∂_b-energy functional $E_{\partial_b}(f)$ can be defined similarly. A critical point of $E_{\overline{\partial}_b}(\cdot)$ was called pseudo-Hermitian harmonic. Then they proved a "Siu-type holomorphicity" result for a pseudo-Hermitian harmonic map under a rank condition on a dense subset of M.

In this paper, we consider maps from a pseudo-Hermitian manifold M to a Kähler manifold $(N, \widetilde{J}, \widetilde{g})$, and introduce the following partial energy functionals:

$$E_{\overline{\partial}_b,\xi}(f) = \frac{1}{2}\int_M \Big\{|\overline{\partial}_b f|^2 + \frac{1}{4}|\mathrm{d}f(\xi)|^2\Big\}\mathrm{d}v_\theta \tag{1.1}$$

and

$$E_{\partial_b,\xi}(f) = \frac{1}{2}\int_M \Big\{|\partial_b f|^2 + \frac{1}{4}|\mathrm{d}f(\xi)|^2\Big\}\mathrm{d}v_\theta, \tag{1.2}$$

where ξ denotes the Reeb vector field of (M, θ). Note that the usual energy $E(f) = E_{\overline{\partial}_b,\xi}(f) + E_{\partial_b,\xi}(f)$. A critical point of $E_{\overline{\partial}_b,\xi}(f)$ (resp. $E_{\partial_b,\xi}(f)$) will be referred to as a $\overline{\partial}_b$-harmonic map (resp. ∂_b-harmonic map). Clearly $E_{\overline{\partial}_b,\xi}(f) = 0$ (resp. $E_{\partial_b,\xi}(f) = 0$) if and only if f is a foliated CR map (resp. foliated anti-CR map).

For a map $f : (M^{2m+1}, H, J, \theta) \to (N, \widetilde{J}, \widetilde{g})$, we set

$$K_b(f) = E_{\partial_b,\xi}(f) - E_{\overline{\partial}_b,\xi}(f) = E_{\partial_b}(f) - E_{\overline{\partial}_b}(f).$$

The authors in [18] proved that if M is a compact Sasakian manifold, then $K_b(f)$ is invariant under a foliated deformation. First, we want to generalize their result to the case that the domain manifold is a general pseudo-Hermitian manifold.

Theorem 1.1 *Let (M^{2m+1}, H, J, θ) be a compact pseudo-Hermitian manifold, and $(N, \widetilde{J}, \widetilde{g})$ be a Kähler manifold. Then $K_b(f)$ is a smooth foliated homotopy invariant, that is, $K_b(f_t)$ is constant for any family $\{f_t\}$ of foliated maps.*

This is a foliated Lichnerowicz type result, which implies that the $E_{\overline{\partial}_b,\xi}$-, $E_{\partial_b,\xi}$- and E-critical points through foliated maps coincide. Furthermore, in a given foliated homotopy class, the $E_{\overline{\partial}_b,\xi}$-, $E_{\partial_b,\xi}$- and E-minima coincide.

Next, we try to generalize Petit's foliated rigidity theorem and get the following result.

Theorem 1.2 *Let (M^{2m+1}, H, J, θ) be a compact Sasakian manifold with $m \geq 2$, and $(N, \widetilde{J}, \widetilde{g})$ be a Kähler manifold with strongly semi-negative curvature. If $f : M \to N$ is a $\overline{\partial}_b$-harmonic map or a ∂_b-harmonic map, then f is foliated. Furthermore, f must be $\overline{\partial}_b$-pluriharmonic (that is, $f^{\alpha}_{i\bar{j}} = f^{\alpha}_{\bar{j}i} = 0$), and*

$$\widetilde{R}_{\beta\overline{\alpha}\gamma\overline{\sigma}}(f^{\overline{\alpha}}_{\overline{i}} f^{\beta}_{\overline{j}} - f^{\overline{\alpha}}_{\overline{j}} f^{\beta}_{\overline{i}})\overline{(f^{\overline{\gamma}}_{\overline{i}} f^{\sigma}_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^{\sigma}_{\overline{i}})} = 0.$$

Subsequently, by a similar argument as in [2, 10, 19], we obtain the following CR rigidity result for $\overline{\partial}_b$-harmonic maps.

Theorem 1.3 *Let (M^{2m+1}, H, J, θ) be a compact Sasakian manifold with $m \geq 2$, and $(N, \widetilde{J}, \widetilde{g})$ be a Kähler manifold with strongly negative curvature. Suppose that $f : M \to N$ is a $\overline{\partial}_b$-harmonic map, and $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}f_p) \geq 3$ at some point $p \in M$. Then f is a foliated CR map or foliated anti-CR map.*

2 Preliminaries

Let M^{2m+1} be a $(2m+1)$-dimensional smooth orientable manifold. A CR structure on M^{2m+1} is a complex rank-m subbundle $H^{1,0}$ of $T(M) \otimes \mathbb{C}$ with the following properties

$$\begin{aligned} &H^{1,0} \cap H^{0,1} = \{0\}, \quad H^{0,1} = \overline{H^{1,0}}, \\ &[\Gamma(H^{1,0}), \Gamma(H^{1,0})] \subseteq \Gamma(H^{1,0}). \end{aligned} \tag{2.1}$$

The complex subbundle $H^{1,0}$ corresponds to a real rank-$2m$ subbundle $H := \Re\{H^{1,0} \oplus H^{0,1}\}$ of $T(M)$, which carries a complex structure J_b defined by

$$J_b(V + \overline{V}) = \mathrm{i}(V - \overline{V})$$

for any $V \in H^{1,0}$. The synthetic object $(M, H^{1,0})$ or (M, H, J_b) is called a CR manifold.

Let E be a real line bundle of T^*M, whose fiber at each point $x \in M$ is given by

$$E_x = \{\omega \in T^*_x M : \ker \omega \supseteq H_x\}.$$

Since both TM and H are orientable vector bundles on M, the real line bundle E is orientable, E has globally defined nowhere vanishing sections. Any such a section $\theta \in \Gamma(E \backslash \{0\})$ is referred to as a pseudo-Hermitian 1-form on M.

Given a pseudo-Hermitian 1-form θ on M, we have the Levi form L_θ corresponding to θ, which is defined by

$$L_\theta(X, Y) = \mathrm{d}\theta(X, J_b Y) \tag{2.2}$$

for any $X, Y \in H$. The second condition in (2.1) implies that L_θ is J_b-invariant, and thus symmetric. If L_θ is positive definite on H for some θ, then $(M, H^{1,0})$ is said to be strictly pseudoconvex. From now on, we will always assume that $(M, H^{1,0})$ is a strictly pseudoconvex CR manifold endowed with a pseudo-Hermitian 1-form θ, such that its Levi form L_θ is positive definite. In this case the synthetic object $(M, H^{1,0}, \theta)$ is referred to as a pseudo-Hermitian manifold.

Let $(M^{2m+1}, H^{1,0}, \theta)$ be a pseudo-Hermitian manifold. Clearly θ is a contact form. Thus there is a unique vector field $\xi \in \Gamma(T(M))$, called the Reeb vector field, such that

$$\theta(\xi) = 1, \quad i_\xi \mathrm{d}\theta = 0, \tag{2.3}$$

where i_ξ denotes the interior product with respect to ξ. The collection of all its integral curves forms an oriented one-dimensional foliation $\mathcal{F}_\xi$ on M, which is called the Reeb foliation. The first condition in (2.3) implies that ξ is transversal to H. Therefore, $T(M)$ admits a decomposition

$$T(M) = H \oplus V_\xi, \tag{2.4}$$

where $V_\xi := \mathrm{span}\{\xi\}$ is a trivial line bundle on M. In terms of terminology from foliation theory, H and V_ξ are called the horizontal and vertical distributions, respectively. Let $\pi_H : TM \to H$ and $\pi_V : TM \to V_\xi$ be the natural projections associated with the direct sum decomposition (2.4). In terms of θ, the Levi form L_θ can be extended to a Riemannian metric

$$g_\theta = L_\theta(\pi_H, \pi_H) + \theta \otimes \theta, \tag{2.5}$$

which is called the Webster metric. It is convenient to extend the complex structure J_b on H to an endomorphism J of $T(M)$ by requiring that

$$J|_H = J_b \quad \text{and} \quad J|_{V_\xi} = 0, \tag{2.6}$$

where $|$ denotes the fiberwise restriction.

It is known that there exists a unique linear connection ∇ on $(M^{2m+1}, H^{1,0}, \theta)$, called the Tanaka-Webster connection, such that (cf. [5, 20–21])

(1) $\nabla_X \Gamma(H) \subseteq \Gamma(H)$ and $\nabla_X J = 0$ for any $X \in \Gamma(TM)$;

(2) $\nabla g_\theta = 0$;

(3) $T_\nabla(X, Y) = 2\mathrm{d}\theta(X, Y)\xi$ and $T_\nabla(\xi, JX) + JT_\nabla(\xi, X) = 0$ for any $X, Y \in H$, where $T_\nabla(\cdot, \cdot)$ denotes the torsion of the connection ∇.

One important partial component of T_∇ is the pseudo-Hermitian torsion τ given by

$$\tau(X) = T_\nabla(\xi, X) \tag{2.7}$$

for any $X \in TM$. Then $(M, H^{1,0}, \theta)$ is said to be Sasakian if $\tau = 0$.

For the pseudo-Hermitian manifold $(M, H^{1,0}, \theta)$, we choose a local orthonormal frame field $\{e_A\}_{A=0}^{2m} = \{\xi, e_1, \cdots, e_m, e_{m+1}, \cdots, e_{2m}\}$ with respect to g_θ such that

$$\{e_{m+1}, \cdots, e_{2m}\} = \{Je_1, \cdots, Je_m\}.$$

Such a frame field $\{e_A\}_{A=0}^{2m}$ is referred to as an adapted frame field M. Set

$$\eta_j = \frac{1}{\sqrt{2}}(e_j - \sqrt{-1}Je_j), \quad \eta_{\bar{j}} = \overline{\eta_j}, \quad j = 1, \cdots, m. \tag{2.8}$$

Let $\{\theta^j\}_{j=1}^m$ be the dual frame field of $\{\eta_j\}_{j=1}^m$. By the properties of the Tanaka-Webster connection ∇, we have (cf. [5])

$$\nabla\xi = 0, \quad \nabla\eta_j = \theta_j^i \otimes \eta_i, \quad \nabla\eta_{\bar{j}} = \theta_{\bar{j}}^{\bar{i}} \otimes \eta_{\bar{i}}, \tag{2.9}$$

where $\{\theta_j^i\}$ denotes the connection 1-forms with respect to the frame field. Since $\tau(H^{1,0}) \subset H^{0,1}$, one may write

$$\begin{aligned} \tau &= \tau^i\eta_i + \tau^{\bar{i}}\eta_{\bar{i}} \\ &= A_{\bar{j}}^i\theta^{\bar{j}} \otimes \eta_i + A_j^{\bar{i}}\theta^j \otimes \eta_{\bar{i}}. \end{aligned} \tag{2.10}$$

From [21], we know that $\{\theta, \theta^i, \theta_j^i\}$ satisfies the following structure equations (cf. also [5, §1.4]):

$$\begin{aligned} \mathrm{d}\theta &= 2\sqrt{-1}\theta^i \wedge \theta^{\bar{i}}, \\ \mathrm{d}\theta^i &= -\theta_j^i \wedge \theta^j + A_{\bar{j}}^i\theta \wedge \theta^{\bar{j}}, \\ \mathrm{d}\theta_j^i &= -\theta_k^i \wedge \theta_j^k + \Pi_j^i \end{aligned} \tag{2.11}$$

with

$$\begin{aligned} \Pi_j^i = {} & 2\sqrt{-1}(\theta^i \wedge \tau^{\bar{j}} - \tau^i \wedge \theta^{\bar{j}}) + R_{jk\bar{l}}^i\theta^k \wedge \theta^{\bar{l}} \\ & + W_{j\bar{k}}^i\theta \wedge \theta^{\bar{k}} - W_{jk}^{\bar{i}}\theta \wedge \theta^k, \end{aligned} \tag{2.12}$$

where $W_{j\bar{k}}^i = A_{\bar{k},j}^i$, $W_{jk}^{\bar{i}} = A_{j,\bar{i}}^{\bar{k}}$ are the covariant derivatives of A and $R_{jk\bar{l}}^i$ are the components of curvature tensor of the Tanaka-Webster connection.

Lemma 2.1 (cf. [2]) *Let (M^{2m+1}, H, J, θ) be a pseudo-Hermitian manifold with Tanaka-Webster connection ∇. Let X and ρ be a vector field and 1-form on M, respectively. Then*

$$\operatorname{div} X = \sum_{A=0}^{2m} g_\theta(\nabla_{e_A}X, e_A) \quad \text{and} \quad \delta\rho = -\sum_{A=0}^{2m}(\nabla_{e_A}\rho)(e_A),$$

where $\{e_A\}_{A=0}^{2m} = \{\xi, e_1, \cdots, e_{2m}\}$ is an orthonormal frame field on M. Here $\operatorname{div}(\cdot)$ and $\delta(\cdot)$ denote the divergence and codifferential, respectively.

Definition 2.1 *A map $f : (M, H, J) \to (N, \widetilde{J})$ from a CR manifold to a complex manifold is called a CR map (resp. anti-CR map) if $\mathrm{d}f(H^{1,0}) \subset T^{1,0}(N)$ (resp. $\mathrm{d}f(H^{0,1}) \subset T^{1,0}(N)$), equivalently, $\mathrm{d}f_H \circ J = \widetilde{J} \circ \mathrm{d}f_H$ (resp. $\mathrm{d}f_H \circ J = -\widetilde{J} \circ \mathrm{d}f_H$), where $\mathrm{d}f_H = \mathrm{d}f|_H$. In particular, if $N = \mathbb{C}$, then f is called a CR function (resp. anti-CR function).*

A map $f : (M, H, J, \theta) \to N$ from a pseudo-Hermitian manifold to a smooth manifold is said to be foliated if $\mathrm{d}f(\xi) = 0$. Here the target manifold is regarded as a trivial foliation by points. In [2, 8], the following type of generalized holomorphic maps was investigated.

Definition 2.2 (cf. [8]) *A smooth map* $f:(M,H,J,\theta)\to(N,\widetilde{J})$ *from a pseudo-Hermitian manifold to a complex manifold is called* $(J,\widetilde{J})$*-holomorphic* (*resp. anti-*$(J,\widetilde{J})$*-holomorphic*) *if it satisfies* $\mathrm{d}f\circ J=\widetilde{J}\circ\mathrm{d}f$ (*resp.* $\mathrm{d}f\circ J=-\widetilde{J}\circ\mathrm{d}f$).

Remark 2.1 Clearly $f:(M,H,J,\theta)\to(N,\widetilde{J})$ is a $(J,\widetilde{J})$-holomorphic map if and only if it is a foliated CR map. Note that $(J,\widetilde{J})$-holomorphic map is also called CR-holomorphic map in [15].

Let $f:(M^{2m+1},H,J,\theta)\to(N,\widetilde{J},\widetilde{g})$ be a map from a pseudo-Hermitian manifold to a Kähler manifold. We have the partial differentials

$$\overline{\partial}_b f: H^{0,1}\to T^{1,0}N,\quad \partial_b f: H^{1,0}\to T^{1,0}N$$

defined by

$$\overline{\partial}_b f=\pi^{1,0}(\mathrm{d}f|_{H^{0,1}}),\quad \partial_b f=\pi^{1,0}(\mathrm{d}f|_{H^{1,0}}),$$

where $\pi^{1,0}:T^{\mathbb{C}}N\to T^{1,0}N$ is the natural projection morphism. Let $\{e_0,e_1,\cdots,e_{2m}\}$ be the adapted frame field on M as given above. Similarly, let $\{\widetilde{e}_1,\cdots,\widetilde{e}_{2n}\}$ be a local orthonormal frame field on $(N,\widetilde{J},\widetilde{g})$ with $\widetilde{e}_{n+1}=\widetilde{J}\widetilde{e}_1,\cdots,\widetilde{e}_{2n}=\widetilde{J}\widetilde{e}_n$. Set

$$\widetilde{\eta}_\alpha=\frac{1}{\sqrt{2}}(\widetilde{e}_\alpha-\sqrt{-1}\widetilde{J}\widetilde{e}_\alpha),\quad \alpha=1,\cdots,n. \tag{2.13}$$

Let $\{\widetilde{\theta}^\alpha\}_{\alpha=1}^n$ be the dual frame field of $\{\widetilde{\eta}_\alpha\}_{\alpha=1}^n$. In terms of the frame fields, we can write

$$\overline{\partial}_b f=f^\alpha_{\overline{j}}\theta^{\overline{j}}\otimes\widetilde{\eta}_\alpha,\quad \partial_b f=f^\alpha_j\theta^j\otimes\widetilde{\eta}_\alpha. \tag{2.14}$$

Then

$$|\overline{\partial}_b f|^2=\sum_{j,\alpha}f^\alpha_{\overline{j}}f^{\overline{\alpha}}_{j},\quad |\partial_b f|^2=\sum_{j,\alpha}f^\alpha_j f^{\overline{\alpha}}_{\overline{j}} \tag{2.15}$$

or

$$\begin{aligned}|\overline{\partial}_b f|^2&=\frac14\{\langle\mathrm{d}f(e_j),\mathrm{d}f(e_j)\rangle+\langle\mathrm{d}f(Je_j),\mathrm{d}f(Je_j)\rangle\\&\quad-2\langle\mathrm{d}f(Je_j),\widetilde{J}\mathrm{d}f(e_j)\rangle\}\\&=\frac14\sum_{A=1}^{2m}\{\langle\mathrm{d}f(e_A),\mathrm{d}f(e_A)\rangle-\langle\widetilde{J}\mathrm{d}f(e_A),\mathrm{d}f(Je_A)\rangle\},\end{aligned} \tag{2.16}$$

$$\begin{aligned}|\partial_b f|^2&=\frac14\{\langle\mathrm{d}f(e_j),\mathrm{d}f(e_j)\rangle+\langle\mathrm{d}f(Je_j),\mathrm{d}f(Je_j)\rangle\\&\quad+2\langle\mathrm{d}f(Je_j),\widetilde{J}\mathrm{d}f(e_j)\rangle\}\\&=\frac14\sum_{A=1}^{2m}\{\langle\mathrm{d}f(e_A),\mathrm{d}f(e_A)\rangle+\langle\widetilde{J}\mathrm{d}f(e_A),\mathrm{d}f(Je_A)\rangle\}.\end{aligned} \tag{2.17}$$

Then we can introduce the following two energy functionals:

$$E_{\overline{\partial}_b,\xi}(f)=\int_M\Big\{|\overline{\partial}_b f|^2+\frac14|\mathrm{d}f(\xi)|^2\Big\}\mathrm{d}v_\theta \tag{2.18}$$

and

$$E_{\partial_b,\xi}(f) = \int_M \Big\{|\partial_b f|^2 + \frac{1}{4}|\mathrm{d}f(\xi)|^2\Big\}\mathrm{d}v_\theta, \tag{2.19}$$

where ξ is the Reeb vector field of (M,θ). Clearly $E_{\overline{\partial}_b,\xi}(f)\equiv 0$ (resp. $E_{\partial_b,\xi}(f)\equiv 0$) if and only if f is a foliated CR map (resp. foliated anti-CR map).

Definition 2.3 *A critical point of $E_{\overline{\partial}_b,\xi}(f)$ (resp. $E_{\partial_b,\xi}(f)$) is called a $\overline{\partial}_b$-harmonic map (resp. ∂_b-harmonic map).*

Remark 2.2 In [12], Li and Son introduced the $\overline{\partial}_b$-energy functional $E_{\overline{\partial}_b}(f)$ of f. Compared with their definition, we include the term $\frac{1}{4}|\mathrm{d}f(\xi)|^2$ in (2.18).

For a map $f:(M,H^{1,0},\theta)\to(N,\widetilde{J},\widetilde{g})$, we define its second fundamental form by

$$\beta(X,Y) = \widetilde{\nabla}_Y \mathrm{d}f(X) - \mathrm{d}f(\nabla_Y X)$$

for any $X,Y\in\Gamma(TM)$, where ∇ and $\widetilde{\nabla}$ denote the Tanaka-Webster connection of M and the Levi-Civita connection of N, respectively. The notion of the above second fundamental form has appeared in literature in various special cases (cf. [4, 6, 15–16], etc.).

Lemma 2.2 (cf. [3]) *Let $f:(M,\nabla)\to(N,\widetilde{\nabla})$ be a map between manifolds with the linear connections. Then*

$$\widetilde{\nabla}_X \mathrm{d}f(Y) - \widetilde{\nabla}_Y \mathrm{d}f(X) - \mathrm{d}f([X,Y]) = T_{\widetilde{\nabla}}(\mathrm{d}f(X),\mathrm{d}f(Y))$$

for any $X,Y\in\Gamma(TM)$, where $T_{\widetilde{\nabla}}$ denotes the torsion of $\widetilde{\nabla}$. Equivalently, we have

$$\beta(X,Y)-\beta(Y,X) = \mathrm{d}f(T_\nabla(X,Y)) - T_{\widetilde{\nabla}}(\mathrm{d}f(X),\mathrm{d}f(Y)).$$

Now we want to derive the variation formulas of the energy functionals defined by (2.18) and (2.19).

Lemma 2.3 *Let (M^{2m+1},H,J,θ) be a pseudo-Hermitian manifold and $(N,\widetilde{J},\widetilde{g})$ be a Kähler manifold. Suppose that $\{f_t\}_{|t|<\varepsilon}$ is a family of maps from M to N with $f_0=f$ and $v=\big(\frac{\partial f_t}{\partial t}\big)\big|_{t=0}\in\Gamma(f^{-1}TN)$. Then*

$$\frac{\mathrm{d}E_{\overline{\partial}_b,\xi}(f_t)}{\mathrm{d}t}\Big|_{t=0} = -\frac{1}{2}\int_M \langle v, \mathrm{tr}_{g_\theta}\beta - 2m\widetilde{J}\mathrm{d}f(\xi)\rangle$$

and

$$\frac{\mathrm{d}E_{\partial_b,\xi}(f_t)}{\mathrm{d}t}\Big|_{t=0} = -\frac{1}{2}\int_M \langle v, \mathrm{tr}_{g_\theta}\beta + 2m\widetilde{J}\mathrm{d}f(\xi)\rangle.$$

Proof Set $F:M\times(-\varepsilon,\varepsilon)\to N$ by $F(x,t)=f_t(x)$ for any $x\in M$ and $t\in(-\varepsilon,\varepsilon)$. Then

$$\frac{\mathrm{d}E_{\overline{\partial}_b,\xi}(f_t)}{\mathrm{d}t}\Big|_{t=0}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_M \sum_{A=1}^{2m}\{2\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \mathrm{d}F(e_A), \mathrm{d}F(e_A)\rangle - \langle \widetilde{J}\widetilde{\nabla}_{\frac{\partial}{\partial t}} \mathrm{d}F(e_A), \mathrm{d}F(Je_A)\rangle \\
&\quad - \langle \widetilde{J}\mathrm{d}F(e_A), \widetilde{\nabla}_{\frac{\partial}{\partial t}} \mathrm{d}F(Je_A)\rangle\}\mathrm{d}v_\theta \ + \frac{1}{2}\int_M \langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \mathrm{d}F(\xi), \mathrm{d}F(\xi)\rangle \mathrm{d}v_\theta \\
&= \sum_{A=1}^{2m}\int_M \frac{1}{4}\{2\langle \widetilde{\nabla}_{e_A} v, \mathrm{d}f(e_A)\rangle - \langle \widetilde{J}\widetilde{\nabla}_{e_A} v, \mathrm{d}f(Je_A)\rangle \\
&\quad - \langle \widetilde{J}\mathrm{d}f(e_A), \widetilde{\nabla}_{Je_A} v\rangle\}\mathrm{d}v_\theta + \frac{1}{2}\int_M \langle \widetilde{\nabla}_\xi v, \mathrm{d}f(\xi)\rangle \mathrm{d}v_\theta \\
&= \frac{1}{2}\sum_{A=0}^{2m}\int_M \{\langle \widetilde{\nabla}_{e_A} v, \mathrm{d}f(e_A)\rangle + \langle \widetilde{\nabla}_{e_A} v, \widetilde{J}\mathrm{d}f(Je_A)\rangle\}\mathrm{d}v_\theta \\
&= \frac{1}{2}\sum_{A=0}^{2m}\int_M \{e_A\langle v, \mathrm{d}f(e_A)\rangle - \langle v, \mathrm{d}f(\nabla_{e_A} e_A)\rangle - \langle v, (\widetilde{\nabla}_{e_A}\mathrm{d}f)(e_A)\rangle \\
&\quad + e_A\langle v, \widetilde{J}\mathrm{d}f(Je_A)\rangle - \langle v, \widetilde{J}\mathrm{d}fJ(\nabla_{e_A} e_A)\rangle - \langle v, (\widetilde{\nabla}_{e_A}\widetilde{J}\mathrm{d}fJ)(e_A)\rangle\}.
\end{aligned} \tag{2.20}
$$

Define a 1-form ρ by $\rho(X) = \langle v, \mathrm{d}f(X)\rangle + \langle v, \widetilde{J}\circ \mathrm{d}f \circ J(X)\rangle$ for any $X \in TM$. By Lemma 2.1, we deduce that

$$
\delta\rho = -\sum_{A=0}^{2m}(\nabla_{e_A}\rho)(e_A). \tag{2.21}
$$

It follows from (2.20)–(2.21) that

$$
\frac{\mathrm{d}E_{\overline{\partial}_b,\xi}(f_t)}{\mathrm{d}t}\Big|_{t=0} = -\frac{1}{2}\int_M \Big\langle v, \sum_{A=0}^{2m}(\widetilde{\nabla}_{e_A}\mathrm{d}f)(e_A) + [\widetilde{\nabla}_{e_A}(\widetilde{J}\circ \mathrm{d}f\circ J)](e_A)\Big\rangle. \tag{2.22}
$$

Next,

$$
\begin{aligned}
\sum_{A=1}^{2m}[\widetilde{\nabla}_{e_A}(\widetilde{J}\circ \mathrm{d}f\circ J)](e_A) &= \sum_{A=1}^{2m}\widetilde{\nabla}_{e_A}(\widetilde{J}\circ \mathrm{d}f\circ Je_A) - \widetilde{J}\circ \mathrm{d}f\circ J(\nabla_{e_A} e_A) \\
&= \sum_{A=1}^{2m}\widetilde{J}[\widetilde{\nabla}_{e_A}\mathrm{d}f(Je_A) - \mathrm{d}f(\nabla_{e_A} Je_A)] \\
&= \sum_{A=1}^{2m}\widetilde{J}\beta(Je_A, e_A) \\
&= \sum_{j=1}^{m}\widetilde{J}[\beta(Je_j, e_j) - \beta(e_j, Je_j)] \\
&= \sum_{j=1}^{m}\widetilde{J}\mathrm{d}f(T_\nabla(Je_j, e_j)) \\
&= -2m\widetilde{J}\mathrm{d}f(\xi).
\end{aligned}
$$

Then we get the variation formula for $E_{\overline{\partial}_b,\xi}(f)$ from (2.22). The variation formula for $E_{\partial_b,\xi}(f)$ may be derived in a similar way. Hence we complete the proof of this lemma.

Define the tension field $\tau_{\overline{\partial}_b,\xi}(f)$ of f with respect to the functional $E_{\overline{\partial}_b,\xi}$ by

$$\tau_{\overline{\partial}_b,\xi}(f) := \mathrm{tr}_{g_\theta}\,\beta - 2m\widetilde{J}\mathrm{d}f(\xi).$$

Then, according to Lemma 2.3, f is $\overline{\partial}_b$-harmonic if and only if $\tau_{\overline{\partial}_b,\xi}(f) = 0$.

Note that $\tau_{\overline{\partial}_b,\xi}(f) = 0$ (or $\tau_{\partial_b,\xi}(f) = 0$) is a system of elliptic differential equations that differ from the harmonic map equation by a linear first-order term. By a similar argument as in [17], we have the following theorem.

Theorem 2.1 (Unique continuation) *Let $f : (M^{2m+1}, H, J, \theta) \to (N^{2n}, \widetilde{J}, \widetilde{g})$ be a $\overline{\partial}_b$-harmonic map or ∂_b-harmonic map. If f is constant on a non-empty open subset U of M, then f is constant on M.*

Let us recall some definitions of generalized harmonic maps from pseudo-Hermitian manifolds.

Definition 2.4 *Let (M^{2m+1}, H, J, θ) be a pseudo-Hermitian manifold and $(N^{2n}, \widetilde{J}, \widetilde{g})$ be a Kähler manifold. Suppose that $f : M \to N$ is a smooth map. We say f is*

(i) *pseudo-harmonic, if* $\mathrm{tr}_{g_\theta}(\pi_H\beta) = 0$ (cf. [1]);

(ii) *pseudo-Hermitian harmonic, if it is a critical point of* $E_{\overline{\partial}_b}(\cdot)$ (cf. [12]);

(iii) *$\overline{\partial}_b$-pluriharmonic, if $\beta(X, Y) + \beta(JX, JY) = 0$ for all $X, Y \in H$* (cf. [4]).

Remark 2.3 Clearly, we have the following results:

(a) If f is $\overline{\partial}_b$-pluriharmonic, then it must be pseudoharmonic (cf. [3]);

(b) if f is a CR map, then f is pseudo-Hermitian harmonic;

(c) if f is a CR map (resp. anti-CR map), then f is $\overline{\partial}_b$-harmonic (resp. ∂_b-harmonic) if and only if $\beta(\xi, \xi) = 0$ (cf. (5.3));

(d) if f is foliated, then notions of $\overline{\partial}_b$-harmonic, ∂_b-harmonic, pseudoharmonic, pseudo-Hermitian harmonic and harmonic maps coincide.

Besides, as proved in [2], if f is $\overline{\partial}_b$-pluriharmonic, then it is foliated; if f is $\pm(J, \widetilde{J})$-holomorphic, then it is $\overline{\partial}_b$-pluriharmonic.

3 Lichnerowicz Type Results

In this section, we generalize the Lichnerowicz type result in [18] to the case that the domain manifold is a general pseudo-Hermitian manifold.

Let $f : (M^{2m+1}, H, J, \theta) \to (N, \widetilde{J}, \omega^N)$ be a smooth map from a pseudo-Hermitian manifold to a Kähler manifold, where ω^N is the Kähler form of N, given by $\omega^N(X, Y) = \widetilde{g}(JX, Y)$ for all $X, Y \in TN$. Set

$$k_b(f) = |\partial_b f|^2 - |\overline{\partial}_b f|^2 \tag{3.1}$$

and

$$K_b(f) = E_{\partial_b,\xi}(f) - E_{\overline{\partial}_b,\xi}(f). \tag{3.2}$$

Lemma 3.1 *Under the above notations, we have*

$$k_b(f) = \langle \mathrm{d}\theta, f^*\omega^N \rangle.$$

Proof Let $\{\xi, e_1, \cdots, e_m, Je_1, \cdots, Je_m\}$ be an adapted frame on M. Using (2.2) and (2.16)–(2.17), we deduce that

$$\begin{aligned}\langle \mathrm{d}\theta, f^*\omega^N \rangle &= \sum_{i<j}\{(f^*\omega^N)(e_i, e_j)\mathrm{d}\theta(e_i, e_j) + (f^*\omega^N)(Je_i, Je_j)\mathrm{d}\theta(Je_i, Je_j)\}\\ &\quad + \sum_{i,j}(f^*\omega^N)(e_i, Je_j)\mathrm{d}\theta(e_i, Je_j)\\ &= \sum_i \langle \widetilde{J}\mathrm{d}f(e_i), \mathrm{d}f(Je_i)\rangle\\ &= k_b(f).\end{aligned}$$

The following lemma is useful.

Lemma 3.2 (Homotopy Lemma) (cf. [6, 13]) *Let $f_t : M \to N$ be a family of smooth maps between smooth manifolds, parameterized by real number t, and let ω be a closed two-form on N. Then*

$$\frac{\partial}{\partial t}(f_t^*\omega) = \mathrm{d}\Big(f_t^* i\Big(\frac{\partial f_t}{\partial t}\Big)\omega\Big),$$

where the notation $i(X)$ denotes the interior product with respect to the vector X.

Lemma 3.3 *Let $f_t : (M^{2m+1}, H, J, \theta) \to (N, \widetilde{J}, \omega^N)$ be a family of smooth maps from a compact pseudo-Hermitian manifold to a Kähler manifold. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}K_b(f_t) = 2m\int_M \omega^N(v_t, \mathrm{d}f_t(\xi))\mathrm{d}v_\theta,$$

where $v_t = \frac{\partial f_t}{\partial t}$.

Proof In terms of Lemmas 3.1–3.2, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K_b(f_t) &= \int_M \Big\langle \frac{\partial}{\partial t} f_t^*\omega^N, \mathrm{d}\theta\Big\rangle \mathrm{d}v_\theta\\ &= \int_M \Big\langle \mathrm{d}\Big(f_t^* i\Big(\frac{\partial f_t}{\partial t}\Big)\omega^N\Big), \mathrm{d}\theta\Big\rangle \mathrm{d}v_\theta\\ &= \int_M \Big\langle f_t^* i\Big(\frac{\partial f_t}{\partial t}\Big)\omega^N, \delta\mathrm{d}\theta\Big\rangle \mathrm{d}v_\theta.\end{aligned}$$

Recall that (cf. [5])

$$\nabla^\theta_X Y = \nabla_X Y - (\mathrm{d}\theta(X,Y) + A(X,Y))\xi + \theta(Y)\tau(X) + \theta(X)JY + \theta(Y)JX$$

for any $X, Y \in \Gamma(TM)$, where ∇^θ denotes the Levi-Civita connection of g_θ. Let $\{e_A\}_{A=0}^{2m} = \{\xi, e_1, \cdots, e_{2m}\}$ be an adapted frame field in M. For $X \in HM$, we compute

$$(\delta\mathrm{d}\theta)(X) = -\sum_{A=0}^{2m}(\nabla^\theta_{e_A}\mathrm{d}\theta)(e_A, X)$$

$$
\begin{aligned}
&= -\sum_{A=0}^{2m}\{e_A \mathrm{d}\theta(e_A, X) - \mathrm{d}\theta(\nabla^{\theta}_{e_A} e_A, X) - \mathrm{d}\theta(e_A, \nabla^{\theta}_{e_A} X)\}\\
&= -\sum_{A=1}^{2m}\{e_A \mathrm{d}\theta(e_A, X) - \mathrm{d}\theta(\nabla_{e_A} e_A, X) - \mathrm{d}\theta(e_A, \nabla_{e_A} X)\}\\
&= -\sum_{A=1}^{2m}(\nabla_{e_A}\mathrm{d}\theta)(e_A, X)\\
&= 0,
\end{aligned}
$$

where the last equality is due to $\nabla \mathrm{d}\theta = 0$. Next,

$$
\begin{aligned}
(\delta \mathrm{d}\theta)(\xi) &= \sum_{A=1}^{2m} \mathrm{d}\theta(e_A, \nabla^{\theta}_{e_A}\xi)\\
&= \sum_{A=1}^{2m} \mathrm{d}\theta(e_A, \tau(e_A) + Je_A)\\
&= 2m,
\end{aligned}
$$

since

$$
\begin{aligned}
&\mathrm{d}\theta(e_i, \tau(e_i)) + \mathrm{d}\theta(Je_i, \tau Je_i)\\
&= \mathrm{d}\theta(e_i, \tau(e_i)) - \mathrm{d}\theta(e_i, \tau(e_i))\\
&= 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} K_b(f_t) &= \int_M \Big\langle f_t^* i\Big(\frac{\partial f_t}{\partial t}\Big)\omega^N, \delta \mathrm{d}\theta\Big\rangle \mathrm{d}v_\theta\\
&= \int_M \langle f_t^*[\omega^N(v_t, \cdot)], \delta \mathrm{d}\theta\rangle \mathrm{d}v_\theta\\
&= \int_M \omega^N(v_t, \mathrm{d}f_t(\xi))\delta \mathrm{d}\theta(\xi)\mathrm{d}v_\theta\\
&= 2m\int_M \omega^N(v_t, \mathrm{d}f_t(\xi))\mathrm{d}v_\theta.
\end{aligned}
$$

Corollary 3.1 *Let $f_t : (M^{2m+1}, H, J, \theta) \to (N, \widetilde{J}, \omega^N)$ be a family of smooth maps from a compact pseudo-Hermitian manifold to a Kähler manifold, such that $\mathrm{d}f_t(\xi) = 0$ for every t. We refer to such $\{f_t\}$ as a family of foliated maps. Then $K_b(f_t)$ is a constant.*

Thus, if $f_t : M \to N$ is a family of foliated maps, then

$$
\frac{\mathrm{d}}{\mathrm{d}t} E_{\overline{\partial}_b,\xi}(f_t) = \frac{\mathrm{d}}{\mathrm{d}t} E_{\partial_b,\xi}(f_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E(f_t),
$$

where $E(f) = E_{\overline{\partial}_b,\xi}(f) + E_{\partial_b,\xi}(f)$ is the usual energy functional of f. Then, the following theorems are evident.

Theorem 3.1 (i) *The $E_{\overline{\partial}_b,\xi}$-, $E_{\partial_b,\xi}$- and E-critical points through foliated maps coincide. Moreover, in a given foliated homotopy class, the $E_{\overline{\partial}_b,\xi}$-, $E_{\partial_b,\xi}$- and E-minima coincide.*

(ii) *If f is $\pm(J,\widetilde{J})$-holomorphic, then it is an absolute minimum of E in its foliated class.*

Proof (i) For any f, f_0 in the same foliated homotopy class, the following equality holds:

$$E_{\overline{\partial}_b,\xi}(f) - E_{\overline{\partial}_b,\xi}(f_0) = E_{\partial_b,\xi}(f) - E_{\partial_b,\xi}(f_0).$$

Consequently, if $E_{\overline{\partial}_b,\xi}(f_0) \leq E_{\overline{\partial}_b,\xi}(f)$ for all f, then $E_{\partial_b,\xi}(f_0) \leq E_{\partial_b,\xi}(f)$ for all f. Similarly, from the equality

$$E(f) - E(f_0) = 2E_{\overline{\partial}_b,\xi}(f) - 2E_{\overline{\partial}_b,\xi}(f_0),$$

we conclude that $E_{\overline{\partial}_b,\xi}$ and E-minima coincide.

(ii) A $(J,\widetilde{J})$-holomorphic map (resp. anti-$(J,\widetilde{J})$-holomorphic map) satisfies $E_{\overline{\partial}_b,\xi}(f) = 0$ (resp. $E_{\partial_b,\xi}(f) = 0$) and is therefore an absolute minimum of E in its foliated class.

Theorem 3.2 *Let $f_t : (M^{2m+1}, H, J, \theta) \to (N, \widetilde{J}, \omega^N)$ be a family of foliated maps from a pseudo-Hermitian manifold to a Kähler manifold with $0 \leq t \leq 1$. Suppose that f_0 is $(J,\widetilde{J})$-holomorphic and f_1 is anti-$(J,\widetilde{J})$-holomorphic, then f_0 and f_1 are constant. In particular, any $\pm(J,\widetilde{J})$-holomorphic map in a trivial foliated homotopy class is constant.*

Proof Since $E_{\overline{\partial}_b,\xi}(f_0) = E_{\partial_b,\xi}(f_1) = 0$, $0 \leq E_{\partial_b,\xi}(f_0) = -E_{\overline{\partial}_b,\xi}(f_1) \leq 0$, which leads to $E_{\partial_b,\xi}(f_0) = E_{\overline{\partial}_b,\xi}(f_1) = 0$. Thus, $E(f_0) = E(f_1) = 0$.

4 Commutation Relations

In this section, we derive the commutation relations for maps from a pseudo-Hermitian manifold to a Kähler manifold. While the case of a map from a pseudo-Hermitian manifold to a general Riemannian manifold has been addressed in [2], we present it here using our notation for the sake of clarity and convenience.

Let $f : (M^{2m+1}, H, J, \theta) \to (N^{2n}, \widetilde{J}, \widetilde{g})$ be a smooth map, where (M^{2m+1}, H, J, θ) is a pseudo-Hermitian manifold and $(N^{2n}, \widetilde{J}, \widetilde{g})$ is a Kähler manifold. Let $\{\theta^i\}$ be a local adapted coframe on M, and let $\{\widetilde{\omega}^\alpha\}$ be a local orthonormal coframe on N as aforementioned. Unless otherwise stated, we adhere to the following index conventions:

$$A, B, C, D = 0, 1, \cdots, m, \overline{1}, \cdots, \overline{m};$$
$$i, j, k, l, s = 1, \cdots, m;$$
$$I, J, K, L, P = 1, \cdots, n, \overline{1}, \cdots, \overline{n};$$
$$\alpha, \beta, \gamma, \sigma = 1, \cdots n,$$

and employ the summation convention on repeated indices. The structure equations for Levi-Civita connection $\widetilde{\nabla}$ on $(N, \widetilde{J})$ can be expressed by

$$\mathrm{d}\widetilde{\omega}^\alpha = -\widetilde{\omega}^\alpha_\beta \wedge \widetilde{\omega}^\beta, \quad \widetilde{\omega}^\alpha_\beta + \widetilde{\omega}^{\overline{\beta}}_{\overline{\alpha}} = 0,$$

$$\mathrm{d}\widetilde{\omega}^\alpha_\beta = -\widetilde{\omega}^\alpha_\gamma \wedge \widetilde{\omega}^\gamma_\beta + \widetilde{\Omega}^\alpha_\beta,$$

where $\widetilde{\Omega}^\alpha_\beta = \widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}\widetilde{\omega}^\gamma \wedge \widetilde{\omega}^{\overline{\sigma}}$. Since N is Kähler, the only possibly non-zero components of $\widetilde{R}^L_{IJK}$ are

$$\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}, \quad \widetilde{R}^{\overline{\alpha}}_{\overline{\beta}\gamma\overline{\sigma}}, \quad \widetilde{R}^\alpha_{\beta\overline{\gamma}\sigma}, \quad \widetilde{R}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}\sigma}.$$

Set

$$\widetilde{R}_{IJKL} = \widetilde{g}(\widetilde{R}(\widetilde{\eta}_K, \widetilde{\eta}_L)\widetilde{\eta}_J, \widetilde{\eta}_I) = \widetilde{g}_{PI}\widetilde{R}^P_{JKL}.$$

Let

$$\begin{aligned} \mathrm{d}f &= f^I_A\theta^A \otimes \widetilde{\eta}_I, \\ \beta &= f^I_{AB}\theta^A \otimes \theta^B \otimes \widetilde{\eta}_I, \\ \widetilde{\nabla}\beta &= f^I_{ABC}\theta^A \otimes \theta^B \otimes \theta^C \otimes \widetilde{\eta}_I, \end{aligned} \tag{4.1}$$

where $\widetilde{\nabla}\beta$ is the covariant derivative of β with respect to $(\nabla, \widetilde{\nabla})$. Here, β denotes the second fundamental form of f. Thus we have

$$f^*\widetilde{\omega}^\alpha = f^\alpha_j\theta^j + f^\alpha_{\overline{j}}\theta^{\overline{j}} + f^\alpha_0\theta. \tag{4.2}$$

Differentiating (4.2), we have

$$\begin{aligned} f^*\mathrm{d}\widetilde{\omega}^\alpha = {} & f^\alpha_j\mathrm{d}\theta^j + f^\alpha_{\overline{j}}\mathrm{d}\theta^{\overline{j}} + f^\alpha_0\mathrm{d}\theta \\ & + \mathrm{d}f^\alpha_j \wedge \theta^j + \mathrm{d}f^\alpha_{\overline{j}} \wedge \theta^{\overline{j}} + \mathrm{d}f^\alpha_0 \wedge \theta. \end{aligned}$$

By structure equations on M and N, we have

$$\begin{aligned} -f^*\widetilde{\omega}^\alpha_\beta \wedge f^*\widetilde{\omega}^\beta &= -f^*\widetilde{\omega}^\alpha_\beta \wedge (f^\beta_j\theta^j + f^\beta_{\overline{j}}\theta^{\overline{j}} + f^\beta_0\theta) \\ &= f^\alpha_j(\theta^k \wedge \theta^j_k + \theta \wedge \tau^j) + f^\alpha_{\overline{j}}(\theta^{\overline{k}} \wedge \theta^{\overline{j}}_{\overline{k}} + \theta \wedge \tau^{\overline{j}}) + f^\alpha_0(2\sqrt{-1}h_{j\overline{k}}\theta^j \wedge \theta^{\overline{k}}) \\ &\quad + \mathrm{d}f^\alpha_j \wedge \theta^j + \mathrm{d}f^\alpha_{\overline{j}} \wedge \theta^{\overline{j}} + \mathrm{d}f^\alpha_0 \wedge \theta. \end{aligned}$$

After rearranging the above formula, we get

$$Df^\alpha_B \wedge \theta^B + 2\sqrt{-1}f^\alpha_0 h_{k\overline{l}}\,\theta^k \wedge \theta^{\overline{l}} - f^\alpha_k A^k_{\overline{l}}\theta^{\overline{l}} \wedge \theta - f^\alpha_{\overline{k}} A^{\overline{k}}_l\theta^l \wedge \theta = 0, \tag{4.3}$$

where

$$Df^\alpha_k \equiv \mathrm{d}f^\alpha_k - f^\alpha_l\theta^l_k + f^\beta_k\widetilde{\omega}^\alpha_\beta = f^\alpha_{kB}\theta^B, \tag{4.4}$$

$$Df^\alpha_{\overline{k}} \equiv \mathrm{d}f^\alpha_{\overline{k}} - f^\alpha_{\overline{l}}\theta^{\overline{l}}_{\overline{k}} + f^\beta_{\overline{k}}\widetilde{\omega}^\alpha_\beta = f^\alpha_{\overline{k}B}\theta^B, \tag{4.5}$$

$$Df^\alpha_0 \equiv \mathrm{d}f^\alpha_0 + f^\beta_0\widetilde{\omega}^\alpha_\beta = f^\alpha_{0B}\theta^B. \tag{4.6}$$

Here, for simplicity, we write $f^*(\widetilde{\omega}^\alpha_\beta)$ as $\widetilde{\omega}^\alpha_\beta$ on the right hand side of the above formulas. Then (4.3) gives

$$\begin{aligned} & f^\alpha_{jk} = f^\alpha_{kj}, \quad f^\alpha_{\overline{j}\overline{k}} = f^\alpha_{\overline{k}\overline{j}}, \quad f^\alpha_{j\overline{k}} - f^\alpha_{\overline{k}j} = 2\sqrt{-1}f^\alpha_0 h_{j\overline{k}}, \\ & f^\alpha_{0j} - f^\alpha_{j0} = f^\alpha_{\overline{k}}A^{\overline{k}}_j, \quad f^\alpha_{0\overline{j}} - f^\alpha_{\overline{j}0} = f^\alpha_k A^k_{\overline{j}}. \end{aligned} \tag{4.7}$$

Since we have adopted a unitary frame here and in the following, we have $h_{j\overline{k}} = \delta_{jk}$.

Differentiating (4.4), we have

$$-f_l^\alpha \mathrm{d}\theta_k^l + f_k^\beta \mathrm{d}\widetilde{\omega}_\beta^\alpha - \mathrm{d}f_l^\alpha \wedge \theta_k^l + \mathrm{d}f_k^\beta \wedge \widetilde{\omega}_\beta^\alpha = f_{kB}^\alpha \mathrm{d}\theta^B + \mathrm{d}f_{kB}^\alpha \wedge \theta^B.$$

Using structure equations again, we have

$$\begin{aligned} 0 &= f_j^\alpha(-\theta_l^j \wedge \theta_k^l + \Pi_k^j) - f_k^\beta(-\widetilde{\omega}_\gamma^\alpha \wedge \widetilde{\omega}_\beta^\gamma + \widetilde{\Omega}_\beta^\alpha) \\ &\quad + f_{kj}^\alpha(\theta^l \wedge \theta_l^j + \theta \wedge \tau^j) + f_{k\overline{j}}^\alpha(\theta^{\overline{l}} \wedge \theta_{\overline{l}}^{\overline{j}} + \theta \wedge \tau^{\overline{j}}) + 2\sqrt{-1}h_{j\overline{k}} f_{k0}^\alpha \theta^j \wedge \theta^{\overline{k}} \\ &\quad + \mathrm{d}f_l^\alpha \wedge \theta_k^l - \mathrm{d}f_k^\beta \wedge \widetilde{\omega}_\beta^\alpha + \mathrm{d}f_{kB}^\alpha \wedge \theta^B. \end{aligned}$$

It follows that

$$Df_{kB}^\alpha \wedge \theta^B + 2\sqrt{-1}f_{k0}^\alpha h_{j\overline{l}}\theta^j \wedge \theta^{\overline{l}} - f_{kl}^\alpha A_{\overline{j}}^l \theta^{\overline{j}} \wedge \theta - f_{k\overline{l}}^\alpha A_j^{\overline{l}} \theta^j \wedge \theta = -f_l^\alpha \Pi_k^l + f_k^\beta \widetilde{\Omega}_\beta^\alpha, \tag{4.8}$$

where

$$Df_{jk}^\alpha \equiv \mathrm{d}f_{jk}^\alpha - f_{jl}^\alpha \theta_k^l - f_{lk}^\alpha \theta_j^l + f_{jk}^\beta \widetilde{\omega}_\beta^\alpha = f_{jkB}^\alpha \theta^B, \tag{4.9}$$

$$Df_{j\overline{k}}^\alpha \equiv \mathrm{d}f_{j\overline{k}}^\alpha - f_{j\overline{l}}^\alpha \theta_{\overline{k}}^{\overline{l}} - f_{l\overline{k}}^\alpha \theta_j^l + f_{j\overline{k}}^\beta \widetilde{\omega}_\beta^\alpha = f_{j\overline{k}B}^\alpha \theta^B, \tag{4.10}$$

$$Df_{j0}^\alpha \equiv \mathrm{d}f_{j0}^\alpha - f_{l0}^\alpha \theta_j^l + f_{j0}^\beta \widetilde{\omega}_\beta^\alpha = f_{j0B}^\alpha \theta^B. \tag{4.11}$$

From (4.8), we have

$$\begin{aligned} f_{ijk}^\alpha &= f_{ikj}^\alpha - f_i^\beta f_j^\gamma f_k^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_i^\beta f_k^\gamma f_j^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + 2\sqrt{-1}f_j^\alpha A_{ik} - 2\sqrt{-1}f_k^\alpha A_{ij}, \\ f_{i\overline{j}\overline{k}}^\alpha &= f_{i\overline{k}\overline{j}}^\alpha - f_i^\beta f_{\overline{j}}^\gamma f_{\overline{k}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_i^\beta f_{\overline{k}}^\gamma f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + 2\sqrt{-1}f_l^\alpha h_{i\overline{j}} A_{\overline{k}}^l - 2\sqrt{-1}f_l^\alpha h_{i\overline{k}} A_{\overline{j}}^l, \\ f_{ij\overline{k}}^\alpha &= f_{i\overline{k}j}^\alpha - f_i^\beta f_j^\gamma f_{\overline{k}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_i^\beta f_{\overline{k}}^\gamma f_j^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_l^\alpha R_{ij\overline{k}}^l + 2\sqrt{-1}f_{i0}^\alpha h_{j\overline{k}}, \\ f_{ij0}^\alpha &= f_{i0j}^\alpha - f_i^\beta f_j^\gamma f_0^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_i^\beta f_0^\gamma f_j^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_l^\alpha h^{l\overline{k}} A_{ij,\overline{k}} - f_{i\overline{k}}^\alpha A_j^{\overline{k}}, \\ f_{i\overline{j}0}^\alpha &= f_{i0\overline{j}}^\alpha - f_i^\beta f_{\overline{j}}^\gamma f_0^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_i^\beta f_0^\gamma f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha - f_l^\alpha h^{l\overline{k}} A_{\overline{j}\overline{k},i} - f_{ik}^\alpha A_{\overline{j}}^k. \end{aligned} \tag{4.12}$$

Similarly, differentiating (4.5), we have

$$Df_{\overline{k}B}^\alpha \wedge \theta^B + 2\sqrt{-1}f_{\overline{k}0}^\alpha h_{j\overline{l}}\theta^j \wedge \theta^{\overline{l}} - f_{\overline{k}l}^\alpha A_{\overline{j}}^l \theta^{\overline{j}} \wedge \theta - f_{\overline{k}\overline{l}}^\alpha A_j^{\overline{l}} \theta^j \wedge \theta = -f_{\overline{l}}^\alpha \Pi_{\overline{k}}^{\overline{l}} + f_{\overline{k}}^\beta \widetilde{\Omega}_\beta^\alpha, \tag{4.13}$$

where

$$Df_{\overline{j}k}^\alpha \equiv \mathrm{d}f_{\overline{j}k}^\alpha - f_{\overline{j}l}^\alpha \theta_k^l - f_{\overline{l}k}^\alpha \theta_{\overline{j}}^{\overline{l}} + f_{\overline{j}k}^\beta \widetilde{\omega}_\beta^\alpha = f_{\overline{j}kB}^\alpha \theta^B, \tag{4.14}$$

$$Df_{\overline{j}\overline{k}}^\alpha \equiv \mathrm{d}f_{\overline{j}\overline{k}}^\alpha - f_{\overline{j}\overline{l}}^\alpha \theta_{\overline{k}}^{\overline{l}} - f_{\overline{l}\overline{k}}^\alpha \theta_{\overline{j}}^{\overline{l}} + f_{\overline{j}\overline{k}}^\beta \widetilde{\omega}_\beta^\alpha = f_{\overline{j}\overline{k}B}^\alpha \theta^B, \tag{4.15}$$

$$Df_{\overline{j}0}^\alpha \equiv \mathrm{d}f_{\overline{j}0}^\alpha - f_{\overline{l}0}^\alpha \theta_{\overline{j}}^{\overline{l}} + f_{\overline{j}0}^\beta \widetilde{\omega}_\beta^\alpha = f_{\overline{j}0B}^\alpha \theta^B. \tag{4.16}$$

From (4.13), we have

$$\begin{aligned} f_{\overline{i}jk}^\alpha &= f_{\overline{i}kj}^\alpha - f_{\overline{i}}^\beta f_j^\gamma f_k^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{i}}^\beta f_k^\gamma f_j^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + 2\sqrt{-1}f_{\overline{l}}^\alpha h_{\overline{i}k} A_j^{\overline{l}} - 2\sqrt{-1}f_{\overline{l}}^\alpha h_{\overline{i}j} A_k^{\overline{l}}, \\ f_{\overline{i}\overline{j}\overline{k}}^\alpha &= f_{\overline{i}\overline{k}\overline{j}}^\alpha - f_{\overline{i}}^\beta f_{\overline{j}}^\gamma f_{\overline{k}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{i}}^\beta f_{\overline{k}}^\gamma f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + 2\sqrt{-1}f_{\overline{k}}^\alpha A_{\overline{i}\overline{j}} - 2\sqrt{-1}f_{\overline{j}}^\alpha A_{\overline{i}\overline{k}}, \\ f_{\overline{i}j\overline{k}}^\alpha &= f_{\overline{i}\overline{k}j}^\alpha - f_{\overline{i}}^\beta f_j^\gamma f_{\overline{k}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{i}}^\beta f_{\overline{k}}^\gamma f_j^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{l}}^\alpha R_{\overline{i}j\overline{k}}^{\overline{l}} + 2\sqrt{-1}f_{\overline{i}0}^\alpha h_{j\overline{k}}, \\ f_{\overline{i}j0}^\alpha &= f_{\overline{i}0j}^\alpha - f_{\overline{i}}^\beta f_j^\gamma f_0^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{i}}^\beta f_0^\gamma f_j^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha - f_{\overline{l}}^\alpha h^{\overline{l}k} A_{jk,\overline{i}} - f_{\overline{i}\overline{k}}^\alpha A_j^{\overline{k}}, \\ f_{\overline{i}\overline{j}0}^\alpha &= f_{\overline{i}0\overline{j}}^\alpha - f_{\overline{i}}^\beta f_{\overline{j}}^\gamma f_0^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{i}}^\beta f_0^\gamma f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\beta\gamma\overline{\sigma}}^\alpha + f_{\overline{l}}^\alpha h^{\overline{l}k} A_{\overline{i}\overline{j},k} - f_{\overline{i}k}^\alpha A_{\overline{j}}^k. \end{aligned} \tag{4.17}$$

Differentiating (4.6) using the same argument yields

$$Df^\alpha_{0B}\wedge\theta^B+2\sqrt{-1}f^\alpha_{00}h_{j\overline{k}}\theta^j\wedge\theta^{\overline{k}}-f^\alpha_{0j}A^j_{\overline{k}}\theta^{\overline{k}}\wedge\theta-f^\alpha_{0\overline{j}}A^{\overline{j}}_k\theta^k\wedge\theta=f^\beta_0\widetilde{\Omega}^\alpha_\beta, \tag{4.18}$$

where

$$Df^\alpha_{0k}\equiv \mathrm{d}f^\alpha_{0k}-f^\alpha_{0j}\theta^j_k+f^\beta_{0k}\widetilde{\omega}^\alpha_\beta=f^\alpha_{0kB}\theta^B, \tag{4.19}$$

$$Df^\alpha_{0\overline{k}}\equiv \mathrm{d}f^\alpha_{0\overline{k}}-f^\alpha_{0\overline{j}}\theta^{\overline{j}}_{\overline{k}}+f^\beta_{0\overline{k}}\widetilde{\omega}^\alpha_\beta=f^\alpha_{0\overline{k}B}\theta^B, \tag{4.20}$$

$$Df^\alpha_{00}\equiv \mathrm{d}f^\alpha_{00}+f^\beta_{00}\widetilde{\omega}^\alpha_\beta=f^\alpha_{00B}\theta^B. \tag{4.21}$$

From (4.18), we have

$$\begin{aligned}
f^\alpha_{0jk}&=f^\alpha_{0kj}-f^\beta_0f^\gamma_jf^{\overline{\sigma}}_k\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+f^\beta_0f^\gamma_kf^{\overline{\sigma}}_j\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}},\\
f^\alpha_{0j\overline{k}}&=f^\alpha_{0\overline{k}j}-f^\beta_0f^\gamma_jf^{\overline{\sigma}}_{\overline{k}}\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+f^\beta_0f^\gamma_{\overline{k}}f^{\overline{\sigma}}_j\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+2\sqrt{-1}f^\alpha_{00}h_{j\overline{k}},\\
f^\alpha_{00k}&=f^\alpha_{0k0}-f^\beta_0f^\gamma_0f^{\overline{\sigma}}_k\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+f^\beta_0f^\gamma_kf^{\overline{\sigma}}_0\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+f^\alpha_{0\overline{j}}A^{\overline{j}}_k,\\
f^\alpha_{00\overline{k}}&=f^\alpha_{0\overline{k}0}-f^\beta_0f^\gamma_0f^{\overline{\sigma}}_{\overline{k}}\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+f^\beta_0f^\gamma_{\overline{k}}f^{\overline{\sigma}}_0\widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}+f^\alpha_{0j}A^j_{\overline{k}}.
\end{aligned} \tag{4.22}$$

Last, from (4.7), we have

$$\begin{aligned}
f^\alpha_{i\overline{j}k}&=f^\alpha_{\overline{j}ik}+2\sqrt{-1}h_{i\overline{j}}f^\alpha_{0k},\\
f^\alpha_{i\overline{j}\overline{k}}&=f^\alpha_{\overline{j}i\overline{k}}+2\sqrt{-1}h_{i\overline{j}}f^\alpha_{0\overline{k}},\\
f^\alpha_{0jk}&=f^\alpha_{j0k}+f^\alpha_{\overline{l}k}A^{\overline{l}}_j+f^\alpha_{\overline{l}}A^{\overline{l}}_{j,k},\\
f^\alpha_{0j\overline{k}}&=f^\alpha_{j0\overline{k}}+f^\alpha_{\overline{l}\overline{k}}A^{\overline{l}}_j+f^\alpha_{\overline{l}}A^{\overline{l}}_{j,\overline{k}},\\
f^\alpha_{0\overline{j}k}&=f^\alpha_{\overline{j}0k}+f^\alpha_{lk}A^l_{\overline{j}}+f^\alpha_lA^l_{\overline{j},k},\\
f^\alpha_{0\overline{j}\overline{k}}&=f^\alpha_{\overline{j}0\overline{k}}+f^\alpha_{l\overline{k}}A^l_{\overline{j}}+f^\alpha_lA^l_{\overline{j},\overline{k}}.
\end{aligned} \tag{4.23}$$

5 Foliated and $(J,\widetilde{J})$-Holomorphicity Results

A divergence of a vector field X on (M,H,θ) is defined by

$$L_X\Psi=\mathrm{div}(X)\Psi,$$

where $\Psi=\theta\wedge(\mathrm{d}\theta)^m$ is the volume form. One has (cf. Lemma 2.1)

$$\mathrm{div}(X)=\mathrm{tr}_{g_\theta}(Y\in TM\to\nabla_YX). \tag{5.1}$$

Also note that div is a real operator:

$$\overline{\mathrm{div}(X)}=\mathrm{div}(\overline{X}). \tag{5.2}$$

If u is a function on (M,H,θ), then its sub-Laplacian Δ_b is defined by, under an adapted frame,

$$\Delta_bu:=\mathrm{div}(\nabla^Hu)=u_{i\overline{i}}+u_{\overline{i}i},$$

where $\nabla^H u$ is the horizontal component of the gradient of u. Note that the usual Laplacian of u is

$$\Delta u = u_{i\bar{i}} + u_{\bar{i}i} + u_{00}.$$

Using an adapted frame, we can express $\tau_{\overline{\partial}_b,\xi}(f)$ as follows:

$$\tau_{\overline{\partial}_b,\xi}(f) = (f^\alpha_{j\bar{j}} + f^\alpha_{\bar{j}j} + f^\alpha_{00} - 2m\sqrt{-1}f^\alpha_0)\widetilde{\eta}_\alpha + (f^{\overline{\alpha}}_{j\bar{j}} + f^{\overline{\alpha}}_{\bar{j}j} + f^{\overline{\alpha}}_{00} + 2m\sqrt{-1}f^{\overline{\alpha}}_0)\widetilde{\eta}_{\overline{\alpha}}.$$

Besides, it follows from the third equation of (4.7) that

$$f^\alpha_{j\bar{j}} + f^\alpha_{\bar{j}j} + f^\alpha_{00} - 2m\sqrt{-1}f^\alpha_0 = 2f^\alpha_{\bar{j}j} + f^\alpha_{00}.$$

Therefore, defining $(Lf)^\alpha := 2f^\alpha_{\bar{j}j} + f^\alpha_{00}$, we may express $\tau_{\overline{\partial}_b,\xi}(f)$ as

$$\tau_{\overline{\partial}_b,\xi}(f) = (Lf)^\alpha\widetilde{\eta}_\alpha + \overline{(Lf)^\alpha}\widetilde{\eta}_{\overline{\alpha}}. \tag{5.3}$$

By applying the commutation relations in §4, we have the following lemma.

Lemma 5.1

$$\begin{aligned}\frac{1}{2}\Delta|\mathrm{d}f(\xi)|^2 = {}& 2(|f^\alpha_{0j}|^2 + |f^\alpha_{0\bar{j}}|^2 + |f^\alpha_{00}|^2) + f^{\overline{\alpha}}_0(Lf)^\alpha_0 + f^\alpha_0\overline{(Lf)^\alpha_0} + 2\sqrt{-1}m(f^{\overline{\alpha}}_0 f^\alpha_{00} - f^\alpha_0 f^{\bar{\alpha}}_{00})\\ &+ 2f^{\overline{\alpha}}_0 f^\beta_{\bar{j}} f^\gamma_j f^{\overline{\sigma}}_0 \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} + 2f^{\overline{\alpha}}_{\bar{j}} f^\beta_0 f^\gamma_0 f^{\overline{\sigma}}_j \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}\\ &- 2f^{\overline{\alpha}}_0 f^\beta_{\bar{j}} f^\gamma_0 f^{\overline{\sigma}}_j \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} - 2f^{\overline{\alpha}}_0 f^\beta_j f^\gamma_0 f^{\overline{\sigma}}_{\bar{j}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}\\ &+ 2(f^{\overline{\alpha}}_0 f^\alpha_{\bar{l}} + f^\alpha_0 f^{\overline{\alpha}}_{\bar{l}})A^{\bar{l}}_{j,\bar{j}} + 2(f^{\overline{\alpha}}_0 f^\alpha_l + f^\alpha_0 f^{\overline{\alpha}}_l)A^l_{\bar{j},j}\\ &+ 2(f^{\overline{\alpha}}_0 f^\alpha_{\bar{j}k} + f^\alpha_0 f^{\overline{\alpha}}_{\bar{j}k})A^{\bar{k}}_j + 2(f^{\overline{\alpha}}_0 f^\alpha_{lj} + f^\alpha_0 f^{\overline{\alpha}}_{lj})A^l_{\bar{j}}.\end{aligned} \tag{5.4}$$

Proof First,

$$\begin{aligned}\frac{1}{2}\Delta|\mathrm{d}f(\xi)|^2 &= (f^\alpha_0 f^{\overline{\alpha}}_0)_{j\bar{j}} + (f^\alpha_0 f^{\overline{\alpha}}_0)_{\bar{j}j} + (f^\alpha_0 f^{\overline{\alpha}}_0)_{00}\\ &= 2(f^\alpha_{0j} f^{\overline{\alpha}}_{0\bar{j}} + f^\alpha_{0\bar{j}} f^{\overline{\alpha}}_{0j} + f^\alpha_{00} f^{\overline{\alpha}}_{00}) + f^{\overline{\alpha}}_0(f^\alpha_{0\bar{j}j} + f^\alpha_{0j\bar{j}}\\ &\quad + f^\alpha_{000}) + f^\alpha_0(f^{\overline{\alpha}}_{0j\bar{j}} + f^{\overline{\alpha}}_{0\bar{j}j} + f^{\overline{\alpha}}_{000}).\end{aligned} \tag{5.5}$$

From (4.17) and (4.23), we have

$$\begin{aligned}f^\alpha_{0\bar{j}j} &= f^\alpha_{\bar{j}0j} + f^\alpha_{lj}A^l_{\bar{j}} + f^\alpha_l A^l_{\bar{j},j}\\ &= f^\alpha_{\bar{j}j0} + f^\beta_{\bar{j}} f^\gamma_j f^{\overline{\sigma}}_0 \widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}} - f^\beta_{\bar{j}} f^\gamma_0 f^{\overline{\sigma}}_j \widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}\\ &\quad + f^\alpha_{\bar{l}} h^{\bar{l}k} A_{jk,\bar{j}} + f^\alpha_{\bar{j}k} A^{\bar{k}}_j + f^\alpha_{lj} A^l_{\bar{j}} + f^\alpha_l A^l_{\bar{j},j}.\end{aligned} \tag{5.6}$$

From (4.12) and (4.23), we have

$$\begin{aligned}f^\alpha_{0j\bar{j}} &= f^\alpha_{j0\bar{j}} + f^\alpha_{\bar{l}\bar{j}} A^{\bar{l}}_j + f^\alpha_{\bar{l}} A^{\bar{l}}_{j,\bar{j}}\\ &= f^\alpha_{j\bar{j}0} + f^\beta_j f^\gamma_{\bar{j}} f^{\overline{\sigma}}_0 \widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}} - f^\beta_j f^\gamma_0 f^{\overline{\sigma}}_{\bar{j}} \widetilde{R}^\alpha_{\beta\gamma\overline{\sigma}}\\ &\quad + f^\alpha_l h^{l\bar{k}} A_{\overline{jk},j} + f^\alpha_{jk} A^k_{\bar{j}} + f^\alpha_{\bar{l}\bar{j}} A^{\bar{l}}_j + f^\alpha_{\bar{l}} A^{\bar{l}}_{j,\bar{j}}.\end{aligned} \tag{5.7}$$

Note that

$$
\begin{aligned}
&f_0^{\overline{\alpha}}(f_{\overline{j}}^{\beta} f_j^{\gamma} f_0^{\overline{\sigma}} \widetilde{R}^{\alpha}_{\beta\gamma\overline{\sigma}} - f_{\overline{j}}^{\beta} f_0^{\gamma} f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}^{\alpha}_{\beta\gamma\overline{\sigma}} + f_j^{\beta} f_{\overline{j}}^{\gamma} f_0^{\overline{\sigma}} \widetilde{R}^{\alpha}_{\beta\gamma\overline{\sigma}} - f_j^{\beta} f_0^{\gamma} f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}^{\alpha}_{\beta\gamma\overline{\sigma}}) \\
&= 2 f_0^{\overline{\alpha}} f_{\overline{j}}^{\beta} f_j^{\gamma} f_0^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} - f_0^{\overline{\alpha}} f_{\overline{j}}^{\beta} f_0^{\gamma} f_j^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} - f_0^{\overline{\alpha}} f_j^{\beta} f_0^{\gamma} f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}
\end{aligned} \tag{5.8}
$$

and, by (4.7),

$$
\begin{aligned}
&f_0^{\overline{\alpha}}(f_{\overline{l}}^{\alpha} h^{\overline{l}k} A_{jk,\overline{j}} + f_{\overline{j}\overline{k}}^{\alpha} A_j^{\overline{k}} + f_{lj}^{\alpha} A_{\overline{j}}^{l} + f_l^{\alpha} A^{l}_{\overline{j},j}) \\
&+ f_0^{\overline{\alpha}}(f_l^{\alpha} h^{l\overline{k}} A_{\overline{j}\overline{k},j} + f_{jk}^{\alpha} A_{\overline{j}}^{k} + f_{\overline{l}\overline{j}}^{\alpha} A_j^{\overline{l}} + f_{\overline{l}}^{\alpha} A^{\overline{l}}_{j,\overline{j}}) \\
&= 2 f_0^{\overline{\alpha}} f_{\overline{l}}^{\alpha} A^{\overline{l}}_{j,\overline{j}} + 2 f_0^{\overline{\alpha}} f_l^{\alpha} A^{l}_{\overline{j},j} + 2 f_0^{\overline{\alpha}} f_{\overline{j}\overline{k}}^{\alpha} A_j^{\overline{k}} + 2 f_0^{\overline{\alpha}} f_{lj}^{\alpha} A_{\overline{j}}^{l}.
\end{aligned} \tag{5.9}
$$

Therefore, substituting (5.6)–(5.9) into (5.5), we get

$$
\begin{aligned}
\frac{1}{2}\Delta|\mathrm{d}f(\xi)|^2 &= 2(|f_{0j}^{\alpha}|^2 + |f_{0\overline{j}}^{\alpha}|^2 + |f_{00}^{\alpha}|^2) + f_0^{\overline{\alpha}}(f_{\overline{j}j0}^{\alpha} + f_{j\overline{j}0}^{\alpha} + f_{000}^{\alpha}) + f_0^{\alpha}(f_{\overline{j}j0}^{\overline{\alpha}} + f_{j\overline{j}0}^{\overline{\alpha}} + f_{000}^{\overline{\alpha}}) \\
&+ 2 f_0^{\overline{\alpha}} f_{\overline{j}}^{\beta} f_j^{\gamma} f_0^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} + 2 f_{\overline{j}}^{\overline{\alpha}} f_0^{\beta} f_0^{\gamma} f_j^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} - 2 f_0^{\overline{\alpha}} f_{\overline{j}}^{\beta} f_0^{\gamma} f_j^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} \\
&- 2 f_0^{\overline{\alpha}} f_j^{\beta} f_0^{\gamma} f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} + 2(f_0^{\overline{\alpha}} f_{\overline{l}}^{\alpha} + f_0^{\alpha} f_{\overline{l}}^{\overline{\alpha}}) A^{\overline{l}}_{j,\overline{j}} + 2(f_0^{\overline{\alpha}} f_l^{\alpha} + f_0^{\alpha} f_l^{\overline{\alpha}}) A^{l}_{\overline{j},j} \\
&+ 2(f_0^{\overline{\alpha}} f_{\overline{j}\overline{k}}^{\alpha} + f_0^{\alpha} f_{\overline{j}\overline{k}}^{\overline{\alpha}}) A_j^{\overline{k}} + 2(f_0^{\overline{\alpha}} f_{lj}^{\alpha} + f_0^{\alpha} f_{lj}^{\overline{\alpha}}) A_{\overline{j}}^{l}.
\end{aligned}
$$

Taking into account the identity

$$
(Lf)_0^{\alpha} = f_{\overline{j}j0}^{\alpha} + f_{j\overline{j}0}^{\alpha} + f_{000}^{\alpha} - 2m\sqrt{-1} f_{00}^{\alpha},
$$

we obtain (5.4).

Remark 5.1 One can check that

$$
\begin{aligned}
&\widetilde{g}(\widetilde{R}(\mathrm{d}f(\eta_j), \mathrm{d}f(\xi))\overline{\mathrm{d}f(\eta_j)}, \mathrm{d}f(\xi)) \\
&= \widetilde{g}(\widetilde{R}(f_j^{\beta}\widetilde{\eta}_{\beta} + f_j^{\overline{\alpha}}\widetilde{\eta}_{\overline{\alpha}}, f_0^{\gamma}\widetilde{\eta}_{\gamma} + f_0^{\overline{\sigma}}\widetilde{\eta}_{\overline{\sigma}})(f_{\overline{j}}^{\overline{\alpha}}\widetilde{\eta}_{\overline{\alpha}} + f_{\overline{j}}^{\beta}\widetilde{\eta}_{\beta}), f_0^{\gamma}\widetilde{\eta}_{\gamma} + f_0^{\overline{\sigma}}\widetilde{\eta}_{\overline{\sigma}}) \\
&= f_0^{\overline{\alpha}} f_{\overline{j}}^{\beta} f_j^{\gamma} f_0^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} + f_{\overline{j}}^{\overline{\alpha}} f_0^{\beta} f_0^{\gamma} f_j^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} \\
&\quad - f_0^{\overline{\alpha}} f_{\overline{j}}^{\beta} f_0^{\gamma} f_j^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} - f_0^{\overline{\alpha}} f_j^{\beta} f_0^{\gamma} f_{\overline{j}}^{\overline{\sigma}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}.
\end{aligned}
$$

If N has non-positive sectional curvature, then

$$
\widetilde{g}(\widetilde{R}(Z, X)\overline{Z}, X) \geq 0
$$

for any complex vector Z and any real vector X on N. Thus, if this is the case, the curvature terms on the right-hand side of (5.4) combine to yield a non-negative quantity.

Lemma 5.2 *Let (M^{2m+1}, H, J, θ) be a compact pseudo-Hermitian manifold. Let $f : M^{2m+1} \to (N^{2n}, \widetilde{J}, \widetilde{g})$ be a smooth map. If the second fundamental form satisfies*

$$
\beta(\xi, X) = 0 \quad \forall\, X \in H,
$$

then f is foliated.

Proof Since N is a Riemannian manifold, the claim follows directly from [2]. We present the proof for readers' convenience.

By the integration by parts and the third formula in (4.7), we have

$$\begin{aligned} 0 = \sqrt{-1}\int_M (f_j^\alpha f_{0\bar{j}}^{\overline{\alpha}} - f_{\bar{j}}^\alpha f_{0j}^{\overline{\alpha}})\mathrm{d}V_g &= -\sqrt{-1}\int_M (f_{j\bar{j}}^\alpha f_0^{\overline{\alpha}} - f_{\bar{j}j}^\alpha f_0^{\overline{\alpha}})\mathrm{d}V_g \\ &= 2m\int_M |f_0^\alpha|^2\mathrm{d}V_g. \end{aligned}$$

Therefore, $f_0^\alpha = 0$.

The main difficulty in applying Lemma 5.1 arises from the mixed term

$$2\sqrt{-1}m(f_0^{\overline{\alpha}} f_{00}^\alpha - f_0^\alpha f_{00}^{\overline{\alpha}})$$

and the terms related to torsion. To address the mixed term, we need to add an extra term $|f_{00}^\alpha|^2$ (see below for details). Inspired by [2], we define the following generalized Paneitz operator acting on maps:

$$Pf := \underbrace{\Big(f_{\bar{j}jk}^\alpha + \frac{1}{2}f_{00k}^\alpha + 2m\sqrt{-1}A_{kj}f_{\bar{j}}^\alpha\Big)}_{:=(Pf)_k^\alpha}\theta^k\otimes\widetilde{\eta}_\alpha.$$

In [12] (cf. also [9]), Li and Son defined the following tensors

$$Bf = B_{i\bar{j}}f^\alpha\theta^i\otimes\theta^{\bar{j}}\otimes\widetilde{\eta}_\alpha$$

and

$$E = E_{\bar{j}}\theta^{\bar{j}},$$

where

$$B_{i\bar{j}}f^\alpha := f_{i\bar{j}}^\alpha - \frac{1}{m}f_{k\bar{k}}^\alpha h_{i\bar{j}}$$

and

$$E_{\bar{j}} := (B_{i\bar{j}}f^\alpha)f_{\bar{i}}^{\overline{\alpha}}.$$

Then $-\delta E$ is given by

$$\begin{aligned} E_{\bar{j},j} &= \Big(f_{i\bar{j}j}^\alpha - \frac{1}{m}f_{k\bar{k}j}^\alpha h_{i\bar{j}}\Big)f_{\bar{i}}^{\overline{\alpha}} + (B_{i\bar{j}}f^\alpha)f_{\bar{i}j}^{\overline{\alpha}} \\ &= |B_{i\bar{j}}f^\alpha|^2 + \frac{m-1}{m}\langle Pf, \overline{\partial}_b\overline{f}\rangle - \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}f_{\bar{i}}^{\overline{\sigma}}f_{\bar{j}}^\beta(f_i^\gamma f_j^{\overline{\alpha}} - f_j^\gamma f_i^{\overline{\alpha}}) - \frac{m-1}{2m}f_{00k}^\alpha f_{\bar{k}}^{\overline{\alpha}}. \end{aligned}$$

Taking integration of δE over M gives

$$\begin{aligned} -\frac{m-1}{m}\int_M \langle Pf, \overline{\partial}_b\overline{f}\rangle\mathrm{d}V_g = \int_M |B_{i\bar{j}}f^\alpha|^2\mathrm{d}V_g - \int_M \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}f_{\bar{i}}^{\overline{\sigma}}f_{\bar{j}}^\beta(f_i^\gamma f_j^{\overline{\alpha}} - f_j^\gamma f_i^{\overline{\alpha}})\mathrm{d}V_g \\ - \frac{m-1}{2m}\int_M f_{00k}^\alpha f_{\bar{k}}^{\overline{\alpha}}\mathrm{d}V_g. \end{aligned}$$

Note that

$$f_{\bar{k}k}^{\overline{\alpha}} - f_{k\bar{k}}^{\overline{\alpha}} = -2\sqrt{-1}mf_0^{\overline{\alpha}},$$

thus,

$$\begin{aligned}\int_M f^\alpha_{00k} f^{\overline{\alpha}}_{\overline{k}} \mathrm{d}V_g &= -\int_M f^\alpha_{00} f^{\overline{\alpha}}_{\overline{k}k} \mathrm{d}V_g \\ &= -\int_M f^\alpha_{00}(f^{\overline{\alpha}}_{k\overline{k}} - 2m\sqrt{-1} f^{\overline{\alpha}}_0) \mathrm{d}V_g \\ &= -\frac{1}{2}\int_M f^\alpha_{00}(\overline{(Lf)^\alpha} - f^{\overline{\alpha}}_{00}) \mathrm{d}V_g + 2m\sqrt{-1}\int_M f^\alpha_{00} f^{\overline{\alpha}}_0 \mathrm{d}V_g \\ &= \frac{1}{2}\int_M |f^\alpha_{00}|^2 \mathrm{d}V_g - \frac{1}{2}\int_M f^\alpha_{00}\overline{(Lf)^\alpha} \mathrm{d}V_g + 2m\sqrt{-1}\int_M f^\alpha_{00} f^{\overline{\alpha}}_0 \mathrm{d}V_g.\end{aligned}$$

Therefore,

$$\begin{aligned}-\frac{m-1}{m}\int_M \langle Pf, \overline{\partial}_b \overline{f}\rangle \mathrm{d}V_g = &\int_M |B_{i\overline{j}} f^\alpha|^2 \mathrm{d}V_g - \int_M \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} f^{\overline{\sigma}}_{\overline{i}} f^\beta_{\overline{j}} (f^\gamma_i f^{\overline{\alpha}}_j - f^\gamma_j f^{\overline{\alpha}}_i) \mathrm{d}V_g \\ &- \frac{m-1}{4m}\int_M |f^\alpha_{00}|^2 \mathrm{d}V_g + \frac{m-1}{4m}\int_M f^\alpha_{00}\overline{(Lf)^\alpha} \mathrm{d}V_g \\ &- (m-1)\sqrt{-1}\int_M f^\alpha_{00} f^{\overline{\alpha}}_0 \mathrm{d}V_g. \end{aligned} \tag{5.10}$$

Recall that the curvature tensor $\widetilde{R}_{\beta\overline{\alpha}\gamma\overline{\sigma}}$ is said to be strongly negative (resp. strongly semi-negative) if

$$\widetilde{R}_{\beta\overline{\alpha}\gamma\overline{\sigma}}(A^\beta \overline{B^\alpha} - C^\beta \overline{D^\alpha})\overline{(A^\sigma \overline{B^\gamma} - C^\sigma \overline{D^\gamma})}$$

is positive (resp. non-negative) for any complex numbers $A^\alpha, B^\alpha, C^\alpha, D^\alpha$ whenever there exists at least one pair of indices (α, β) such that $A^\beta \overline{B^\alpha} - C^\beta \overline{D^\alpha} \neq 0$ (cf. [19]). Evidently, strongly negative curvature (resp. strongly semi-negative curvature) implies negative sectional curvature (resp. semi-negative sectional curvature). If N has strongly semi-negative curvature, then

$$-\widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} f^{\overline{\sigma}}_{\overline{i}} f^\beta_{\overline{j}} (f^\gamma_i f^{\overline{\alpha}}_j - f^\gamma_j f^{\overline{\alpha}}_i) = \frac{1}{2}\widetilde{R}_{\beta\overline{\alpha}\gamma\overline{\sigma}}(f^{\overline{\alpha}}_{\overline{i}} f^\beta_{\overline{j}} - f^{\overline{\alpha}}_{\overline{j}} f^\beta_{\overline{i}})\overline{(f^{\overline{\gamma}}_{\overline{i}} f^\sigma_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^\sigma_{\overline{i}})} \geq 0.$$

Next, we introduce the 1-form $F = F_{\overline{k}}\theta^{\overline{k}}$ with

$$F_{\overline{k}} := \Big(f^\alpha_{\overline{j}j} + \frac{1}{2} f^\alpha_{00}\Big) f^{\overline{\alpha}}_{\overline{k}}.$$

Then

$$\begin{aligned}F_{\overline{k},k} &= \Big(f^\alpha_{\overline{j}jk} + \frac{1}{2} f^\alpha_{00k}\Big) f^{\overline{\alpha}}_{\overline{k}} + \Big(f^\alpha_{\overline{j}j} + \frac{1}{2} f^\alpha_{00}\Big) f^{\overline{\alpha}}_{\overline{k}k} \\ &= ((Pf)^\alpha_k - 2m\sqrt{-1} A_{kj} f^\alpha_{\overline{j}}) f^{\overline{\alpha}}_{\overline{k}} + \frac{1}{2}(Lf)^\alpha f^{\overline{\alpha}}_{\overline{k}k}.\end{aligned}$$

Integrating δF on M yields

$$\int_M \langle Pf, \overline{\partial}_b \overline{f}\rangle \mathrm{d}V_g = -\frac{1}{2}\int_M (Lf)^\alpha f^{\overline{\alpha}}_{\overline{k}k} \mathrm{d}V_g + 2m\sqrt{-1}\int_M A_{kj} f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{\overline{k}} \mathrm{d}V_g. \tag{5.11}$$

Theorem 5.1 *Let (M^{2m+1}, H, J, θ) be a compact Sasakian manifold with $m \geq 2$, and $(N^{2n}, \widetilde{J}, \widetilde{g})$ be a Kähler manifold with strongly semi-negative curvature. If $f : M \to N$ is a $\overline{\partial}_b$-harmonic map or a ∂_b-harmonic map, then f is foliated. Therefore, f must be $\overline{\partial}_b$-pluriharmonic (that is, $f^\alpha_{i\overline{j}} = f^\alpha_{\overline{j}i} = 0$) and*

$$\widetilde{R}_{\beta\overline{\alpha}\gamma\overline{\sigma}}(f^{\overline{\alpha}}_{\overline{i}} f^\beta_{\overline{j}} - f^{\overline{\alpha}}_{\overline{j}} f^\beta_{\overline{i}})\overline{(f^{\overline{\gamma}}_{\overline{i}} f^\sigma_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^\sigma_{\overline{i}})} = 0. \tag{5.12}$$

Proof Suppose that f is $\overline{\partial}_b$-harmonic (the case for ∂_b-harmonic map is similar). Then $(Lf)^\alpha = 0$, or equivalently,

$$f^\alpha_{j\bar{j}} + f^\alpha_{\bar{j}j} + f^\alpha_{00} - 2m\sqrt{-1}f^\alpha_0 = 0. \tag{5.13}$$

Since M is Sasakian, we have $A_{ij} = 0$, and hence, (5.4) simplifies to

$$\begin{aligned}\frac{1}{2}\Delta|\mathrm{d}f(\xi)|^2 = {} & 2\sum_j(|f^\alpha_{0j}|^2 + |f^\alpha_{0\bar{j}}|^2) + 2|f^\alpha_{00}|^2 + 2m\sqrt{-1}(f^{\overline{\alpha}}_0 f^\alpha_{00} - f^\alpha_0 f^{\bar{\alpha}}_{00}) \\ & + 2f^{\overline{\alpha}}_0 f^\beta_{\bar{j}} f^\gamma_j f^{\overline{\sigma}}_0 \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} + 2f^{\overline{\alpha}}_{\bar{j}} f^\beta_0 f^\gamma_0 f^{\overline{\sigma}}_j \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} \\ & - 2f^{\overline{\alpha}}_0 f^\beta_{\bar{j}} f^\gamma_0 f^{\overline{\sigma}}_j \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} - 2f^{\overline{\alpha}}_0 f^\beta_j f^\gamma_0 f^{\overline{\sigma}}_{\bar{j}} \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}}.\end{aligned} \tag{5.14}$$

Therefore, by Remark 5.1, integrating (5.14) over M and applying integrating by parts, we have

$$4m\sqrt{-1}\int_M f^{\overline{\alpha}}_0 f^\alpha_{00}\mathrm{d}V_g + 2\int_M |f^\alpha_{00}|^2\mathrm{d}V_g \leq 0. \tag{5.15}$$

On the other hand, since f is $\overline{\partial}_b$-harmonic, we get from (5.11) that

$$\int_M \langle Pf, \overline{\partial}_b\overline{f}\rangle\mathrm{d}V_g = 0. \tag{5.16}$$

From (5.10) and the curvature condition, we obtain

$$-\int_M |f^\alpha_{00}|^2\mathrm{d}V_g - 4m\sqrt{-1}\int_M f^\alpha_{00}f^{\overline{\alpha}}_0\mathrm{d}V_g \leq 0. \tag{5.17}$$

Then (5.15) and (5.17) imply that $f^\alpha_{00} = 0$. Substituting it into (5.14), we get

$$\frac{1}{2}\Delta|\mathrm{d}f(\xi)|^2 \geq 2\sum_j(|f^\alpha_{0j}|^2 + |f^\alpha_{0\bar{j}}|^2) \geq 0.$$

Thus, $\mathrm{d}f(\xi) = 0$ by utilizing the divergence theorem and Lemma 5.2.

Furthermore, by substituting (5.16) and $f^\alpha_0 = 0$ into (5.10), we obtain

$$\int_M |B_{i\bar{j}}f^\alpha|^2\mathrm{d}V_g - \int_M \widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} f^{\overline{\sigma}}_{\bar{i}} f^\beta_{\bar{j}}(f^\gamma_i f^{\overline{\alpha}}_j - f^\gamma_j f^{\overline{\alpha}}_i)\mathrm{d}V_g = 0.$$

Note that

$$-\widetilde{R}_{\overline{\alpha}\beta\gamma\overline{\sigma}} f^{\overline{\sigma}}_{\bar{i}} f^\beta_{\bar{j}}(f^\gamma_i f^{\overline{\alpha}}_j - f^\gamma_j f^{\overline{\alpha}}_i) = \frac{1}{2}\widetilde{R}_{\beta\overline{\alpha}\gamma\overline{\sigma}}(f^{\overline{\alpha}}_{\bar{i}} f^\beta_{\bar{j}} - f^{\overline{\alpha}}_{\bar{j}} f^\beta_{\bar{i}})\overline{(f^{\overline{\gamma}}_{\bar{i}} f^\sigma_{\bar{j}} - f^{\overline{\gamma}}_{\bar{j}} f^\sigma_{\bar{i}})} \geq 0.$$

Thus, we get (5.12) and $B_{i\bar{j}}f^\alpha = 0$. Clearly, $f^\alpha_0 = 0$ and $A_{ij} = 0$ imply that $f^\alpha_{j\bar{j}} = f^\alpha_{\bar{j}j} = 0$. Consequently, from the definition of $B_{i\bar{j}}f^\alpha$, we have

$$f^\alpha_{\bar{j}i} = f^\alpha_{i\bar{j}} = \frac{1}{m}f^\alpha_{k\bar{k}}h_{i\bar{j}} = 0.$$

This completes the proof.

Note that the rank condition in Siu's theorem mentioned in the introduction can be improved as $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}f_x) \geq 3$ at some point x (cf. [10]). By a similar argument as [2, 19], we get immediately from (5.12) the following theorem.

Theorem 5.2 *Let* (M^{2m+1}, H, J, θ) *be a compact Sasakian manifold with* $m \geq 2$ *and* $(N^{2n}, \widetilde{J}, \widetilde{g})$ *be a Kähler manifold with strongly negative curvature. Suppose that* $f : M \to N$ *is a* $\overline{\partial}_b$*-harmonic map and* $\mathrm{d}f$ *has real rank at least* 3 *at some point* $p \in M$*. Then* f *is either* $(J, \widetilde{J})$*-holomorphic or anti-*$(J, \widetilde{J})$*-holomorphic.*

Remark 5.2 If f is ∂_b-harmonic (with the other assumptions unchanged), then the conclusion remains valid.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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