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The New Gap Theorem for Certain Riemannian Manifolds*

Juan LI[1] Hongwei XU[2] Entao ZHAO[1]

In memory of Professor Hesheng HU

Abstract In this paper, the authors investigate the geometric rigidity of Riemannian manifolds under suitable curvature restrictions. The authors first prove a new gap theorem for the Ricci curvature of compact locally conformally flat Riemannian manifolds. Subsequently, the authors consider the Riemannian manifolds with the Cotton tensor C satisfying $\operatorname{div} C = 0$ and prove some integral curvature pinching theorems.

Keywords Gap theorems, Locally conformally flat manifolds, Ricci curvature, Constant scalar curvature, Cotton tensor
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1 Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. The Riemannian curvature tensor Rm of M can be orthogonally decomposed as $\mathrm{Rm} = W + V + U$, where W is the Weyl curvature tensor, V and U correspond to the traceless Ricci part and the scalar curvature part, respectively. A manifold has constant sectional curvature if and only if $\overset{\circ}{\mathrm{Rm}}$ vanishes, where $\overset{\circ}{\mathrm{Rm}} = \mathrm{Rm} - U$, or equivalently if and only if both W and V vanish. A manifold with vanishing V is called Einstein. It is well-known that for $n \geq 4$, a manifold is locally conformally flat if and only if $W = 0$, and for $n = 3$, one always has $W = 0$, and a manifold is locally conformally flat if and only if $C = 0$, where C denotes the Cotton tensor (see e.g., [17] for the proof).

Based on the seminal work of Schoen [35] on the Yamabe problem (see [1, 27, 40, 46], etc.), a compact $n(\geq 3)$-dimensional locally conformally flat Riemannian manifold is conformal to a manifold with constant scalar curvature. There are many results on the rigidity and classification of locally conformally flat Riemannian manifolds with constant scalar curvature. Tani [39] showed that the universal cover of a compact oriented locally conformally flat Riemannian manifold with positive Ricci curvature and constant scalar curvature is isometrically a sphere.

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[1]Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China.
E-mail: juanli@zju.edu.cn zhaoet@zju.edu.cn
[2]Corresponding author. Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China.
E-mail: xuhw@zju.edu.cn
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Goldberg and Okumura [22] proved that for a compact locally conformally flat Riemannian manifold M of dimension $n \geq 3$, if the scalar curvature R is a positive constant and the Ricci curvature tensor Ric satisfies $\frac{|\mathrm{Ric}|^2}{R^2} < \frac{1}{n-1}$, then M is of constant curvature. This pinching theorem was later generalized to the case of complete locally conformally flat Riemannian manifolds by Goldberg [21], Hasanis [24], Pigola, Rigoli and Setti [34], etc. In particular, Hasanis [24] proved that for a complete locally conformally flat Riemannian manifold M of dimension $n \geq 3$, if the scalar curvature R is a positive constant and the Ricci curvature tensor Ric satisfies $\frac{|\mathrm{Ric}|^2}{R^2} \leq \frac{1}{n-1}$, then either M is of constant curvature or $\frac{|\mathrm{Ric}|^2}{R^2} \equiv \frac{1}{n-1}$. Cheng [10] proved that for an n-dimensional compact locally conformally flat Riemannian manifold M with constant scalar curvature, if the Ricci curvature is nonnegative, then M is isometric to a space form or a product space $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. When $n = 3$, Cheng, Ishikawa and Shiohama [11] classified complete locally conformally flat three-dimensional Riemannian manifolds with positive constant scalar curvature and constant squared norm of the Ricci curvature tensor. From the theorems mentioned above, we know that an $n(\geq 4)$-dimensional compact locally conformally flat Riemannian manifold with positive constant scalar curvature satisfying $\frac{|\mathrm{Ric}|^2}{R^2} \leq \frac{1}{n-1}$ is either a space form or a product space $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. As it always holds that $\frac{|\mathrm{Ric}|^2}{R^2} \geq \frac{1}{n}$, this result can be viewed as a pinching theorem that if $\frac{1}{n} \leq \frac{|\mathrm{Ric}|^2}{R^2} \leq \frac{1}{n-1}$, then either $\frac{|\mathrm{Ric}|^2}{R^2} \equiv \frac{1}{n}$ or $\frac{|\mathrm{Ric}|^2}{R^2} \equiv \frac{1}{n-1}$ and all the manifolds satisfying the equalities are determined. There are also other versions of pinching theorems for locally conformally flat Riemannian manifolds, see e.g., [10, 12].

The first purpose of the present paper is to investigate a new gap phenomenon for compact locally conformally flat Riemannian manifolds with positive constant scalar curvature and constant squared norm of the Ricci curvature tensor. Motivated by the geometric rigidity theorem due to Peng and Terng [32] on the Chern conjecture for closed minimal hypersurfaces with constant scalar curvature in the sphere and the new gap theorem due to Gu, Lei and Xu [23] on the generalized Chern conjecture for closed hypersurfaces with constant mean curvature and constant scalar curvature in the sphere, we prove the following new gap theorem.

Theorem 1.1 *Let (M^n, g) be an $n(\geq 4)$-dimensional compact locally conformally flat Riemannian manifold with positive constant scalar curvature R and constant squared norm of the Ricci curvature tensor $|\mathrm{Ric}|^2$. If*

$$\frac{1}{n-1} \leq \frac{|\mathrm{Ric}|^2}{R^2} \leq \frac{1+\varepsilon(n)}{n-1},$$

then $\frac{|\mathrm{Ric}|^2}{R^2} = \frac{1}{n-1}$ and M^n is isometric to a product space $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ of round spheres.

Remark 1.1 For the progress on the Chern conjecture, the generalized Chern conjecture and related problems, see [8–9, 13, 15, 28–29, 32–33, 41–44, 47] and the references therein.

Based on Theorem 1.1 and the pinching result for compact locally conformally flat Riemannian manifolds with positive constant scalar curvature under the condition $\frac{1}{n} \leq \frac{|\mathrm{Ric}|^2}{R^2} \leq \frac{1}{n-1}$, we have the following general version of the gap theorem.

Theorem 1.2 *Let (M^n, g) be an $n(\geq 4)$-dimensional compact locally conformally flat Rie-*

mannian manifold with positive constant scalar curvature R and constant squared norm of the Ricci curvature tensor $|\mathrm{Ric}|^2$. There exists an explicit constant $\varepsilon(n) > 0$ depending only on n such that if

$$\frac{1}{n} \leq \frac{|\mathrm{Ric}|^2}{R^2} \leq \frac{1+\varepsilon(n)}{n-1},$$

then M^n is isometric to a sphere or a product space $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ of round spheres.

Next, we consider the manifolds satisfying integral curvature conditions. Hebey and Vaugon [25] classified compact conformally flat manifolds that satisfy certain integral curvature pinching conditions. Chang, Gursky and Yang [6] proved a sharp theorem that if a smooth closed four-manifold M^4 with positive Yamabe invariant satisfies $\int_M |W|^2 \mathrm{d}\mu < 16\pi^2\chi(M)$, where $\chi(M)$ is the Euler characteristic of M, then M is diffeomorphic to either $\mathbb{S}^4$ or $\mathbb{RP}^4$. They also characterized the manifolds for the case of equality. There are many results of this type, under optimal or non-optimal integral pinching conditions (see [2–4, 7, 16, 18–20, 30, 34, 36–38, 45], etc.). Recently, Catino [5] showed that an n-dimensional closed locally conformally flat Riemannian manifold with positive constant scalar curvature R satisfies

$$\int_M |\mathring{\mathrm{Ric}}|^{\frac{n-2}{n}} (R - \sqrt{n(n-1)}|\mathring{\mathrm{Ric}}|)\mathrm{d}\mu \leq 0.$$

He also classified the manifolds that satisfy the equality. We investigate a general case in the present paper. Recall that the Cotton tensor C and the Weyl tensor W are related by the equation $\operatorname{div} W = \frac{n-3}{n-2}C$, $n \geq 3$, where W is considered as a (1, 3) tensor. Mastrolia, Monticelli and Rigoli [31] provided some sufficient conditions for Riemannian manifolds to be Einstein by employing the weak maximum principle at infinity, assuming that the Cotton tensor satisfies $\operatorname{div} C = 0$. Inspired by the results mentioned above, we prove the following theorem.

Theorem 1.3 *Let (M^n, g) be an $n(\geq 3)$-dimensional closed Riemannian manifold. Suppose that $\operatorname{div} C = 0$ and the scalar curvature R is a positive constant. Then*

$$\int_M |\mathring{\mathrm{Ric}}|^\alpha (\sqrt{2}R - \sqrt{2n(n-1)}|\mathring{\mathrm{Ric}}| - \sqrt{(n-1)(n-2)}|W|)\mathrm{d}\mu \leq 0 \tag{1.1}$$

for any $\alpha \geq 1$. Moreover, if g is real analytic, then the equality holds if and only if M is Einstein or M is $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric or with a rotationally symmetric Derdzinski metric.

As a consequence, we obtain the following result.

Corollary 1.1 *Let (M^n, g) be an $n(\geq 3)$-dimensional closed Riemannian manifold. Suppose that $\operatorname{div} C = 0$, the scalar curvature R is a positive constant. If*

$$|\mathring{\mathrm{Ric}}| + \sqrt{\frac{n-2}{2n}}|W| < \frac{1}{\sqrt{n(n-1)}}R,$$

then M is Einstein.

Motivated by the L^p Ricci curvature pinching theorems due to Xu and Zhao [45] for complete locally conformally flat Riemannian manifolds, we prove the following theorem for complete Riemannian manifolds satisfying $\operatorname{div} C = 0$ under $L^{\frac{n}{2}}$ curvature pinching condition.

Theorem 1.4 *Let (M^n, g) be an n-dimensional complete Riemannian manifold. Suppose that $\operatorname{div} C = 0$, the scalar curvature R is constant and the Yamabe constant $Y(M, [g])$ is positive. If*

(i) $R = 0$ *and* $n \geq 5$, *or* $R > 0$ *and* $n = 5, 6$, *or* $R < 0$ *and* $n \geq 7$, *and*

$$\|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|W\|_{\frac{n}{2}} < \frac{8(n-2)}{n^2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M, [g]),$$

or if

(ii) $R > 0$ *and* $n \geq 7$, *and*

$$\|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|W\|_{\frac{n}{2}} < \frac{4}{n-2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M, [g]),$$

then M is Einstein.

Remark 1.2 On $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric, the Weyl curvature tensor $W = 0$ and the scalar curvature $R = \sqrt{n(n-1)}|\mathring{\mathrm{Ric}}| \neq 0$. Through direct computation, we find that $\|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}} = \frac{4}{n-2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M, [g])$ on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Hence in Theorem 1.4, the pinching constant $\frac{4}{n-2}\sqrt{\frac{n-1}{n}}$ for $R > 0$ and $n \geq 7$ is optimal.

The paper is organized as follows. In Section 2, we set our notations and recall the fundamental formulas, including the Simons-type equation. In Section 3, we provide the proof of the gap phenomenon for locally conformally flat Riemannian manifolds. In Section 4, we derive integral inequalities and characterize metrics that satisfy equality on more general manifolds where the Cotton tensor C satisfies $\operatorname{div} C = 0$ and the integral pinching conditions are met. We also prove that the manifold is Einstein if the $L^{\frac{n}{2}}$-norm of certain curvatures satisfy suitable pinching conditions.

2 Preliminaries

Let (M^n, g) be an $n(\geq 3)$-dimensional connected Riemannian manifold. Choose a local orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ and the dual coframe field $\{\omega_1, \omega_2, \cdots, \omega_n\}$ adapted to the Riemannian metric of (M^n, g). The connection 1-forms $\{\omega_{ij}\}$ of (M^n, g) are characterized by the structure equations

$$\begin{aligned} \mathrm{d}\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ \mathrm{d}\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \end{aligned}$$

where R_{ijkl} are the components of the Riemannian curvature tensor Rm of (M^n, g).

Let W_{ijkl} denote the components of the Weyl curvature tensor W of (M^n, g). We have

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(R_{ik}\delta_{jl} + R_{jl}\delta_{ik} - R_{il}\delta_{jk} - R_{jk}\delta_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \end{aligned} \tag{2.1}$$

where $R_{ij} = \sum_{k,l} R_{ikjl} g^{kl}$ and $R = \sum_{i,j} R_{ij} g^{ij}$ are the components of the Ricci curvature tensor Ric and the scalar curvature R of (M^n, g), respectively.

Let ∇ denote the covariant differentiation on M^n. For simplicity of presentation, let $R_{,i} = \nabla_i R$, $R_{ij,k} = \nabla_k R_{ij}$ and $R_{ij,kl} = \nabla_l \nabla_k R_{ij}$, etc. Denote by C_{ijk} the components of Cotton tensor C of (M^n, g). We have

$$\begin{aligned} C_{ijk} &= R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}(R_{,k}\delta_{ij} - R_{,j}\delta_{ik}), \\ \sum_l W_{ijkl,l} &= -\frac{n-3}{n-2} C_{ijk}. \end{aligned}$$

The traceless Ricci curvature tensor $\mathring{\mathrm{Ric}}$ is defined as $\mathring{\mathrm{Ric}} = \sum_{i,j} \mathring{R}_{ij}\omega_i \otimes \omega_j$ with $\mathring{R}_{ij} = R_{ij} - \frac{R}{n}$. From the definition, we have $|\mathring{\mathrm{Ric}}|^2 = \sum_{i,j}(\mathring{R}_{ij})^2 = |\mathrm{Ric}|^2 - \frac{R^2}{n}$. Moreover, we have the following Simons-type equation as referenced in [31]:

$$\begin{aligned} \frac{1}{2}\Delta|\mathring{\mathrm{Ric}}|^2 = |\nabla\mathring{\mathrm{Ric}}|^2 &+ \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{n}{n-2}\mathrm{tr}(\mathring{\mathrm{Ric}}^3) + \sum_{i,j,k}\mathring{R}_{ij}C_{ijk,k} \\ &+ \sum_{i,j,k,l} W_{kijl}\mathring{R}_{kl}\mathring{R}_{ij} + \frac{n-2}{2(n-1)}\mathrm{tr}(\mathring{\mathrm{Ric}} \circ \mathrm{Hess}(R)). \end{aligned} \tag{2.2}$$

From [26, Lemmas 2.4 and 3.4], we have the following inequalities

$$\sum_{i,j,k,l} W_{kijl}\mathring{R}_{kl}\mathring{R}_{ij} \le \sqrt{\frac{n-2}{2(n-1)}}|W||\mathring{\mathrm{Ric}}|^2, \tag{2.3}$$

$$|\mathrm{tr}(\mathring{\mathrm{Ric}}^3)| \le \frac{n-2}{\sqrt{n(n-1)}}|\mathring{\mathrm{Ric}}|^3. \tag{2.4}$$

To analyze the equality case of (2.3), we briefly recall the proof that was given in [26]. For the given local orthonormal frame, the traceless Ricci part V of Rm is given by

$$V_{ijkl} = \frac{1}{n-2}(\mathring{R}_{ik}\delta_{jl} + \mathring{R}_{jl}\delta_{ik} - \mathring{R}_{il}\delta_{jk} - \mathring{R}_{jk}\delta_{il}).$$

Hence

$$\sum_{i,j,k,l} W_{kijl}\mathring{R}_{kl}\mathring{R}_{ij} = \frac{(n-2)^2}{8}\langle W, V \circ V\rangle = \frac{(n-2)^2}{8}\langle W, T\rangle,$$

where T is the Weyl part of $V \circ V$. Therefore,

$$\sum_{i,j,k,l} W_{kijl}\mathring{R}_{kl}\mathring{R}_{ij} \le \frac{(n-2)^2}{8}|W||T|.$$

The equality holds if and only if $|W||T| = 0$, or $|W||T| \neq 0$ and $W = c_1 T$ for a positive constant c_1. Based on the computation in [26], one has

$$|T|^2 = \frac{32}{(n-2)^5}\Big(- nZ + \frac{n^2 - 3n + 3}{n-1}|\mathring{\mathrm{Ric}}|^4\Big),$$

where $Z = \sum\limits_{i,j,k,l} \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{kl}\mathring{R}_{li}$. Choose the local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that the traceless Ricci tensor $\mathring{R}_{ij}$ is diagonalized, i.e., $\mathring{R}_{ij} = \lambda_i \delta_{ij}$. Within this orthonormal frame, we have $Z = \sum\limits_i \lambda_i^4$. By the Cauchy-Schwarz inequality, one has

$$\Big(\sum_i \lambda_i^2\Big)^2 \leq \sum_i \lambda_i^4 \cdot \sum_i 1 = nZ.$$

Hence,

$$|T|^2 \leq \frac{32}{(n-1)(n-2)^3}|\mathring{\mathrm{Ric}}|^4,$$

and the equality holds if and only if $\mathring{\mathrm{Ric}}^2 = c_2 g$ for a nonnegative constant c_2. Here $\mathring{\mathrm{Ric}}^2 = \sum\limits_{i,j,k} \mathring{R}_{ij}\mathring{R}_{jk}\omega_i \otimes \omega_k$. This inequality implies (2.3).

From the brief proof above we see that if (2.3) is an equality at a point, then either $W = 0$, or $W \neq 0$ and $\mathring{\mathrm{Ric}}^2 = cg$ for a nonnegative constant c at that point. Additionally, the equality in (2.4) holds at a point if and only if at least $n-1$ numbers of the eigenvalues of $\mathring{\mathrm{Ric}}$ are the same with each other at this point.

Now we consider a locally conformally flat Riemannian manifold (M^n, g) of dimension $n(\geq 4)$. Using the Ricci identity, we have

$$R_{ij,kl} - R_{ij,lk} = \sum_t R_{tj}R_{tikl} + \sum_t R_{it}R_{tjkl}. \tag{2.5}$$

For $n \geq 4$, we know that M^n is locally conformally flat if and only if $W_{ijkl} = 0$ on M^n. We assume that the scalar curvature R is constant. From the second Bianchi identities and (2.1), we know that the Ricci curvature tensor is a Codazzi tensor, i.e., the following identities hold

$$R_{ij,k} = R_{ik,j}. \tag{2.6}$$

Combining (2.5)–(2.6) with the Ricci identity, we calculate that

$$\begin{aligned}\Delta R_{ij} &= \sum_l R_{ij,ll} = \sum_{t,l} R_{tl}R_{tijl} + \sum_t R_{it}R_{tj} \\ &= \frac{1}{n-2}\Big(n\sum_t R_{it}R_{tj} - \sum_{t,l} R_{tl}^2\delta_{ij} - \frac{n}{n-1}RR_{ij} + \frac{R^2}{n-1}\delta_{ij}\Big).\end{aligned} \tag{2.7}$$

Therefore,

$$\frac{1}{2}\Delta|\mathring{\mathrm{Ric}}|^2 = |\nabla\mathring{\mathrm{Ric}}|^2 + \frac{n}{n-2}\sum_{i,j,k}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2. \tag{2.8}$$

(2.8) can also be derived from (2.2).

For a Riemannian manifold (M^n, g), the Yamabe constant $\mathrm{Y}(M, [g])$ is defined as

$$\mathrm{Y}(M,[g]) = \inf_{u\in C_0^\infty(M)\setminus\{0\}} \frac{\displaystyle\int_M |\nabla u|^2 \mathrm{d}\mu + \frac{n-2}{4(n-1)}\int_M R u^2 \mathrm{d}\mu}{\Big(\displaystyle\int_M |u|^{\frac{2n}{n-2}} \mathrm{d}\mu\Big)^{\frac{n-2}{n}}},$$

where $\mathrm{d}\mu$ is the volume form with respect to the metric g. The Yamabe constant $\mathrm{Y}(M, [g])$ is invariant under the conformal change of g. As a result, it is also referred to as the Yamabe invariant.

3 A New Gap Theorem for Locally Conformally Flat Riemannian Manifolds

Let (M^n, g) be an $n(\geq 4)$-dimensional locally conformally flat Riemannian manifold. Suppose that λ is an eigenvalue of the Ricci curvature whose multiplicity is 1 at some point $x \in M$. Then λ is smooth in a neighborhood U of x. Let u be the unit eigenvector corresponding to λ, i.e.,

$$\widetilde{\mathrm{Ric}}(u) = \lambda u,$$

where $\widetilde{\mathrm{Ric}}$ is given by $\langle \widetilde{\mathrm{Ric}}(u), v\rangle = \mathrm{Ric}(u, v) = \langle u, \widetilde{\mathrm{Ric}}(v)\rangle$. Then for any vector field X on U,

$$\nabla_X(\widetilde{\mathrm{Ric}}(u)) = X(\lambda)u + \lambda\nabla_X u.$$

On the other hand,

$$(\nabla_X\widetilde{\mathrm{Ric}})(u) = \nabla_X(\widetilde{\mathrm{Ric}}(u)) - \widetilde{\mathrm{Ric}}(\nabla_X u).$$

Combining the two equations above yields

$$X(\lambda)u + \lambda\nabla_X u = (\nabla_X\widetilde{\mathrm{Ric}})(u) + \widetilde{\mathrm{Ric}}(\nabla_X u). \tag{3.1}$$

Let V be the linear subspace of T_xM that is orthogonal to u. Then V is an $(n-1)$-dimensional $\widetilde{\mathrm{Ric}}$-invariant subspace. Given that $|u| = 1$, it follows that $\nabla_X u \in \mathrm{V}$, and consequently, $\widetilde{\mathrm{Ric}}(\nabla_X u) \in \mathrm{V}$. Let $(\cdot)^{\mathrm{V}}$ denote the projection onto V. From (3.1), we obtain

$$\begin{aligned} X(\lambda) &= \langle \nabla_X(\widetilde{\mathrm{Ric}}(u)) - \lambda\nabla_X u, u\rangle \\ &= \langle (\nabla_X\widetilde{\mathrm{Ric}})(u), u\rangle + \langle \nabla_X u, \widetilde{\mathrm{Ric}}(u)\rangle \\ &= (\nabla\mathrm{Ric})(u, u, X) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} (\lambda\mathrm{Id} - \widetilde{\mathrm{Ric}})(\nabla_X u) &= [(\nabla_X\widetilde{\mathrm{Ric}})(u) - X(\lambda)u]^{\mathrm{V}} \\ &= (\nabla_X\widetilde{\mathrm{Ric}})(u) - \langle (\nabla_X\widetilde{\mathrm{Ric}})(u), u\rangle u. \end{aligned} \tag{3.3}$$

From the above analysis, we know that $\lambda\mathrm{Id}-\widetilde{\mathrm{Ric}}$ is invertible. Therefore, (3.3) can be rewritten as

$$\nabla_X u = (\lambda\mathrm{Id} - \widetilde{\mathrm{Ric}})^{-1}[(\nabla_X\widetilde{\mathrm{Ric}})(u)]^{\mathrm{V}}. \tag{3.4}$$

Furthermore, for the vector field Y on U,

$$\begin{aligned} Y(X(\lambda)) &= \nabla_Y[(\nabla\mathrm{Ric})(u,u,X)] \\ &= \nabla_Y(\nabla\mathrm{Ric})(u,u,X) + 2(\nabla\mathrm{Ric})(\nabla_Y u,u,X) + (\nabla\mathrm{Ric})(u,u,\nabla_Y X) \\ &= (\nabla^2\mathrm{Ric})(u,u,X,Y) + 2(\nabla\mathrm{Ric})(\nabla_Y u,u,X) + (\nabla\mathrm{Ric})(u,u,\nabla_Y X). \end{aligned}$$

Choose the local orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ on U. Combining the above identity, we derive that

$$e_j(e_i(\lambda)) = (\nabla^2\mathrm{Ric})(u,u,e_i,e_j) + 2(\nabla\mathrm{Ric})(\nabla_{e_j}u,u,e_i) + (\nabla\mathrm{Ric})(u,u,\nabla_{e_j}e_i).$$

Similarly, (3.2) yields

$$\nabla_{e_j}e_i(\lambda) = (\nabla\mathrm{Ric})(u,u,\nabla_{e_j}e_i).$$

Taking the trace of the above equation, we obtain

$$\begin{aligned} \Delta\lambda &= \sum_i [e_i(e_i(\lambda)) - \nabla_{e_i}e_i(\lambda)] \\ &= \sum_i [(\nabla^2\mathrm{Ric})(u,u,e_i,e_i) + 2(\nabla\mathrm{Ric})(\nabla_{e_i}u,u,e_i)]. \end{aligned} \tag{3.5}$$

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of the Ricci curvature tensor, with corresponding eigenvectors $u_1, u_2, \cdots, u_n$. At the point x under consideration, $\{u_1, u_2, \cdots, u_n\}$ forms an orthonormal basis of T_xM. Without loss of generality, let $\lambda_1 = \lambda$ and $u_1 = u$ denote the first eigenvalue and its corresponding eigenvector, respectively.

Assume $[(\nabla_X\widetilde{\mathrm{Ric}})(u)]^{\mathrm{V}} = \sum\limits_{j=2}^{n} a^j u_j$. Then

$$a^j = \langle[(\nabla_X\widetilde{\mathrm{Ric}})(u)]^{\mathrm{V}}, u_j\rangle = (\nabla\mathrm{Ric})(u,u_j,X).$$

Hence

$$\begin{aligned} &[(\nabla_X\widetilde{\mathrm{Ric}})(u)]^{\mathrm{V}} \\ =& \begin{pmatrix} (\nabla\mathrm{Ric})(u,u_2,X) & 0 & \cdots & 0 \\ 0 & (\nabla\mathrm{Ric})(u,u_3,X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\nabla\mathrm{Ric})(u,u_n,X) \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}. \end{aligned}$$

Substituting the above identity into (3.4), we obtain

$$\nabla_X u = \begin{pmatrix} \frac{(\nabla \mathrm{Ric})(u,u_2,X)}{\lambda_1-\lambda_2} & 0 & \cdots & 0 \\ 0 & \frac{(\nabla \mathrm{Ric})(u,u_3,X)}{\lambda_1-\lambda_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(\nabla \mathrm{Ric})(u,u_n,X)}{\lambda_1-\lambda_n} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$$
$$= \sum_{j=2}^{n} \frac{(\nabla \mathrm{Ric})(u,u_j,X)}{\lambda_1-\lambda_j} u_j. \tag{3.6}$$

Combining (3.5)–(3.6), we get

$$\begin{aligned} \Delta\lambda_1 &= \sum_{i=1}^{n} (\nabla^2 \mathrm{Ric})(u_1,u_1,u_i,u_i) \\ &\quad + 2\sum_{i=1}^{n}\sum_{j=2}^{n} (\nabla \mathrm{Ric})\Big(\frac{(\nabla \mathrm{Ric})(u_1,u_j,u_i)}{\lambda_1-\lambda_j}u_j, u_1, u_i\Big) \\ &= \Delta R_{11} + 2\sum_{i=1}^{n}\sum_{j=2}^{n} \frac{R_{1i,j}^2}{\lambda_1-\lambda_j}. \end{aligned}$$

From (2.7), we deduce the following identity

$$\begin{aligned} \Delta R_{11} &= \frac{1}{n-2}\Big(n\sum_t R_{t1}^2 - |\mathrm{Ric}|^2 - \frac{n}{n-1}RR_{11} + \frac{R^2}{n-1}\Big) \\ &= \frac{1}{n-2}\Big(n\lambda_1^2 - |\mathrm{Ric}|^2 - \frac{n}{n-1}R\lambda_1 + \frac{R^2}{n-1}\Big). \end{aligned}$$

Rearranging the above equation, we obtain

$$\begin{aligned} \Delta\lambda_1 &= \frac{1}{n-2}\Big(n\lambda_1^2 - |\mathrm{Ric}|^2 - \frac{n}{n-1}R\lambda_1 + \frac{R^2}{n-1}\Big) + 2\sum_{i=1}^{n}\sum_{j=2}^{n} \frac{R_{1i,j}^2}{\lambda_1-\lambda_j} \\ &= \frac{1}{n-2}\Big(\frac{R^2}{n-1} - |\mathrm{Ric}|^2\Big) + \frac{n}{n-2}\Big(\lambda_1^2 - \frac{1}{n-1}R\lambda_1\Big) + 2\sum_{i=1}^{n}\sum_{j=2}^{n} \frac{R_{1i,j}^2}{\lambda_1-\lambda_j} \\ &= \frac{1}{n-2}\Big(\frac{R^2}{n(n-1)} - |\mathring{\mathrm{Ric}}|^2\Big) + \frac{n}{n-2}\Big(\lambda_1^2 - \frac{1}{n-1}R\lambda_1\Big) + 2\sum_{i=1}^{n}\sum_{j=2}^{n} \frac{R_{1i,j}^2}{\lambda_1-\lambda_j}. \end{aligned} \tag{3.7}$$

Letting $\mathring{\lambda}_i = \lambda_i - \frac{R}{n}$ and $\mu_i = \frac{\mathring{\lambda}_i}{|\mathring{\mathrm{Ric}}|}$, we have

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = 1.$$

We define some new quantities as

$$\phi = \sum_i \mu_i^3 + \frac{n-2}{\sqrt{n(n-1)}}, \quad \eta = \sqrt{\frac{n}{n-1}}\mu_1 + 1, \quad \sigma = \Big[\sum_{i\geq 2}\Big(\mu_i + \frac{\mu_1}{n-1}\Big)^2\Big]^{\frac{1}{2}}.$$

From Gu, Lei and Xu [23], we derive the following algebraic inequalities.

Lemma 3.1 (see [23]) (1) *As previously defined, ϕ, η and σ satisfy*

$$\frac{\sqrt{n(n-1)}}{n-2}\phi \geq \eta \geq \frac{\sigma^2}{2}.$$

(2) *Let $n \geq 4$. If $\phi \leq \frac{1}{6}\sqrt{\frac{n}{n-1}}$, then*

$$\frac{11}{6}\eta \leq \sqrt{\frac{n-1}{n}}\phi, \quad \mu_2 - \mu_1 > \frac{2}{3}\sqrt{\frac{n}{n-1}}.$$

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1 Given that $|\mathrm{Ric}|^2 =$constant and $R =$constant, it follows that $\Delta|\mathring{\mathrm{Ric}}|^2 = 0$ and $|\nabla\mathring{\mathrm{Ric}}| = |\nabla\mathrm{Ric}|$. Consequently, the formula (2.8) implies

$$\begin{aligned} 0 &= |\nabla\mathrm{Ric}|^2 + \frac{n}{n-2}\sum_i \mathring{\lambda}_i^3 + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 \\ &= |\nabla\mathrm{Ric}|^2 + \frac{n}{n-2}|\mathring{\mathrm{Ric}}|^3\sum_i \mu_i^3 + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 \\ &= |\nabla\mathrm{Ric}|^2 + \frac{n}{n-2}|\mathring{\mathrm{Ric}}|^3\phi - \sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3 + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2. \end{aligned} \tag{3.8}$$

Based on the above equation, we obtain the identity concerning $|\mathring{\mathrm{Ric}}|^2$,

$$|\mathring{\mathrm{Ric}}|^2 = \frac{1}{|\mathring{\mathrm{Ric}}|}\sqrt{\frac{n-1}{n}}\Big(|\nabla\,\mathrm{Ric}\,|^2 + \frac{n}{n-2}|\mathring{\mathrm{Ric}}|^3\phi + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2\Big).$$

Assuming $|\mathring{\mathrm{Ric}}|^2 \leq \frac{1+\varepsilon}{n(n-1)}R^2$, where $\varepsilon > 0$ is to be determined, we have

$$\begin{aligned} |\nabla\mathrm{Ric}|^2 + \frac{n}{n-2}|\mathring{\mathrm{Ric}}|^3\phi &= |\mathring{\mathrm{Ric}}|^3\Big(\sqrt{\frac{n}{n-1}} - \frac{1}{n-1}\frac{R}{|\mathring{\mathrm{Ric}}|}\Big) \\ &\leq |\mathring{\mathrm{Ric}}|^3\Big(\sqrt{\frac{n}{n-1}} - \frac{1}{n-1}\frac{1}{\sqrt{\frac{1+\varepsilon}{n(n-1)}}}\Big) \\ &= \sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3\Big(1 - \frac{1}{\sqrt{1+\varepsilon}}\Big). \end{aligned} \tag{3.9}$$

Then we obtain the boundedness of ϕ that

$$\phi \leq \frac{n-2}{n}\sqrt{\frac{n}{n-1}}\Big(1 - \frac{1}{\sqrt{1+\varepsilon}}\Big).$$

Choose an appropriate $\varepsilon_0 > 0$ such that the following equality is satisfied

$$\frac{n-2}{n}\sqrt{\frac{n}{n-1}}\Big(1 - \frac{1}{\sqrt{1+\varepsilon_0}}\Big) = \frac{1}{6}\sqrt{\frac{n}{n-1}}.$$

Through a direct calculation, we obtain $\varepsilon_0 = \big(\frac{6n-12}{5n-12}\big)^2 - 1$. Hence, if

$$|\mathring{\mathrm{Ric}}|^2 \leq \frac{1+\varepsilon_0}{n(n-1)}R^2,$$

or equivalently,

$$|\operatorname{Ric}|^2 \le \frac{1+\varepsilon(n)}{n-1} R^2,$$

where $\varepsilon(n) = \frac{\varepsilon_0}{n}$, then $\phi \le \frac{1}{6}\sqrt{\frac{n}{n-1}}$. Under this condition, based on Lemma 3.1, we obtain $\mu_2 - \mu_1 > \frac{2}{3}\sqrt{\frac{n}{n-1}}$.

Since $\mu_2 > \mu_1$, it follows that $\lambda_1 < \lambda_2 \le \cdots \le \lambda_n$. This implies that λ_1 is of multiplicity 1 at every point and hence is a smooth function on M. By the definitions of λ_1 and μ_1, we have

$$\lambda_1 = \mathring{\lambda}_1 + \frac{R}{n} = (\eta - 1)\sqrt{\frac{n-1}{n}}|\mathring{\operatorname{Ric}}| + \frac{R}{n}.$$

Substituting this into (3.7), one has

$$\begin{aligned}
\Delta\lambda_1 &= 2\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{R_{1i,j}^2}{\lambda_1 - \lambda_j} + \frac{1}{n-2}\Big(\frac{R^2}{n(n-1)} - |\mathring{\operatorname{Ric}}|^2\Big) + \frac{n}{n-2}\Big(\lambda_1^2 - \frac{1}{n-1}R\lambda_1\Big)\\
&= 2\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{R_{1i,j}^2}{\lambda_1 - \lambda_j} + \frac{1}{n-2}\Big(\frac{R^2}{n(n-1)} - |\mathring{\operatorname{Ric}}|^2\Big)\\
&\quad + \frac{n}{n-2}\Big(\frac{R}{n} + (\eta-1)\sqrt{\frac{n-1}{n}}|\mathring{\operatorname{Ric}}|\Big)\Big(-\frac{R}{n(n-1)} + (\eta-1)\sqrt{\frac{n-1}{n}}|\mathring{\operatorname{Ric}}|\Big)\\
&= 2\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{R_{1i,j}^2}{\lambda_1 - \lambda_j} + |\mathring{\operatorname{Ric}}|^2 - \frac{1}{\sqrt{n(n-1)}}R|\mathring{\operatorname{Ric}}|\\
&\quad + \frac{\eta}{n-2}|\mathring{\operatorname{Ric}}|\Big(\frac{n-2}{\sqrt{n(n-1)}}R + (\eta-2)(n-1)|\mathring{\operatorname{Ric}}|\Big).
\end{aligned} \tag{3.10}$$

For the first terms on the right-hand side, based on the assumption $\phi \le \frac{1}{6}\sqrt{\frac{n}{n-1}}$, by using Lemma 3.1, we estimate

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{R_{1i,j}^2}{\lambda_1 - \lambda_j} &= \frac{1}{|\mathring{\operatorname{Ric}}|}\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{R_{1i,j}^2}{\mu_1 - \mu_j}\\
&\ge -\frac{3}{2|\mathring{\operatorname{Ric}}|}\sqrt{\frac{n-1}{n}}\sum_{i=1}^{n}\sum_{j=2}^{n}R_{1i,j}^2\\
&\ge -\frac{1}{2|\mathring{\operatorname{Ric}}|}\sqrt{\frac{n-1}{n}}|\nabla\operatorname{Ric}|^2.
\end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11), we derive the following estimate for $\Delta\lambda_1$,

$$\begin{aligned}
\Delta\lambda_1 \ge& -\frac{1}{|\mathring{\operatorname{Ric}}|}\sqrt{\frac{n-1}{n}}|\nabla\operatorname{Ric}|^2 + |\mathring{\operatorname{Ric}}|^2 - \frac{1}{\sqrt{n(n-1)}}R|\mathring{\operatorname{Ric}}|\\
&+ \frac{\eta}{n-2}|\mathring{\operatorname{Ric}}|\Big(\frac{n-2}{\sqrt{n(n-1)}}R + (\eta-2)(n-1)|\mathring{\operatorname{Ric}}|\Big).
\end{aligned}$$

Based on the identity concerning $|\mathring{\operatorname{Ric}}|^2$, we can further refine the above estimate

$$\Delta\lambda_1 \ge -\frac{1}{|\mathring{\operatorname{Ric}}|}\sqrt{\frac{n-1}{n}}|\nabla\operatorname{Ric}|^2 - \frac{1}{\sqrt{n(n-1)}}R|\mathring{\operatorname{Ric}}|$$

$$
\begin{aligned}
&+\frac{1}{|\mathring{\mathrm{Ric}}|}\sqrt{\frac{n-1}{n}}\Big(|\nabla\mathrm{Ric}|^2+\frac{n}{n-2}|\mathring{\mathrm{Ric}}|^3\phi+\frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2\Big)\\
&+\frac{\eta}{n-2}|\mathring{\mathrm{Ric}}|\Big(\frac{n-2}{\sqrt{n(n-1)}}R+(\eta-2)(n-1)|\mathring{\mathrm{Ric}}|\Big)\\
=&\frac{\sqrt{n(n-1)}}{n-2}|\mathring{\mathrm{Ric}}|^2\phi+\frac{\eta}{n-2}|\mathring{\mathrm{Ric}}|\Big(\frac{n-2}{\sqrt{n(n-1)}}R+(\eta-2)(n-1)|\mathring{\mathrm{Ric}}|\Big). \qquad (3.12)
\end{aligned}
$$

From the assumption of Theorem 1.1, we estimate the term within the brackets in the last line of the above equation

$$
\begin{aligned}
&\frac{n-2}{\sqrt{n(n-1)}}R+(\eta-2)(n-1)|\mathring{\mathrm{Ric}}|\\
\geq&\frac{n-2}{\sqrt{1+\varepsilon_0}}|\mathring{\mathrm{Ric}}|-2(n-1)|\mathring{\mathrm{Ric}}|\geq-\frac{3n}{2}|\mathring{\mathrm{Ric}}|. \qquad (3.13)
\end{aligned}
$$

Based on (3.13), we simplify (3.12) as follows

$$
\Delta\lambda_1\geq\frac{\sqrt{n(n-1)}}{n-2}|\mathring{\mathrm{Ric}}|^2\phi-\frac{3n\eta}{2(n-2)}|\mathring{\mathrm{Ric}}|^2. \qquad (3.14)
$$

Since we previously chose an appropriate ε_0 so that Lemma 3.1 is applicable, we obtain the following estimate

$$
\sqrt{n(n-1)}\phi\geq\frac{11}{6}n\eta.
$$

Substituting the above inequality into (3.14), we have

$$
\Delta\lambda_1\geq\frac{2}{11}\frac{\sqrt{n(n-1)}}{n-2}|\mathring{\mathrm{Ric}}|^2\phi. \qquad (3.15)
$$

Since M is compact, by the maximum principle, we obtain that λ_1 is a constant function. Consequently, (3.15) implies that $\phi\equiv 0$. By Lemma 3.1, it follows that $\eta\equiv 0$ and $\sigma\equiv 0$. Hence

$$
\mu_1=-\sqrt{\frac{n-1}{n}},\quad \mu_2=\cdots=\mu_n=\frac{1}{\sqrt{n(n-1)}}.
$$

From the definition of μ_i, one has

$$
\lambda_1=-\sqrt{\frac{n-1}{n}}|\mathring{\mathrm{Ric}}|+\frac{R}{n},\quad \lambda_2=\cdots=\lambda_n=\frac{1}{\sqrt{n(n-1)}}|\mathring{\mathrm{Ric}}|+\frac{R}{n}.
$$

Therefore, one has $\nabla\mathrm{Ric}\equiv 0$. Substituting this into (3.8), we obtain

$$
\sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3=\frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2.
$$

This implies that $|\mathring{\mathrm{Ric}}|^2=\frac{R^2}{n-1}$ on M. Consequently, M^n is isometric to the product space $\mathbb{S}^1\times\mathbb{S}^{n-1}$.

4 Integral Curvature Pinching Theorems for Manifolds with div $C = 0$

Let (M^n, g) be an $n(\geq 3)$-dimensional Riemannian manifold. Since $\operatorname{div} C = 0$ and the scalar curvature R is a positive constant, the Simons-type equation (2.2) implies that

$$\begin{aligned}\frac{1}{2}\Delta|\mathring{\mathrm{Ric}}|^2 &= |\nabla\mathring{\mathrm{Ric}}|^2 + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \sum_{i,j,k,l} W_{kijl}\mathring{R}_{kl}\mathring{R}_{ij} + \frac{n}{n-2}\mathrm{tr}(\mathring{\mathrm{Ric}}^3) \\ &\geq |\nabla\mathring{\mathrm{Ric}}|^2 + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n-2}{2(n-1)}}|W||\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3, \qquad (4.1)\end{aligned}$$

where for the inequality we have used (2.3)–(2.4). We first give the proof of Theorem 1.3.

Proof of Theorem 1.3 Set $f_\tau = \sqrt{|\mathring{\mathrm{Ric}}|^2 + \tau^2}$ for $\tau > 0$. By the Kato inequality, one has

$$|\nabla\mathring{\mathrm{Ric}}|^2 \geq |\nabla f_\tau|^2.$$

Note that we cannot obtain the stronger inequality $|\nabla\mathring{\mathrm{Ric}}|^2 \geq \frac{n+2}{n}|\nabla f_\tau|^2$, as in this case $\mathring{\mathrm{Ric}}$ is not necessarily a Codazzi tensor. From the definition of f_τ and (4.1), we get

$$\frac{1}{2}\Delta f_\tau^2 \geq |\nabla f_\tau|^2 + \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n-2}{2(n-1)}}|W||\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3. \qquad (4.2)$$

Multiplying both sides of (4.2) by $f_\tau^{\alpha-2}$ for $\alpha \geq 1$ and integrating by parts, we get

$$\begin{aligned}0 \geq &-\frac{1}{2}\int_M f_\tau^{\alpha-2}\Delta f_\tau^2 \mathrm{d}\mu + \int_M |\nabla f_\tau|^2 f_\tau^{\alpha-2}\mathrm{d}\mu + \frac{1}{n-1}\int_M R|\mathring{\mathrm{Ric}}|^2 f_\tau^{\alpha-2}\mathrm{d}\mu \\ &- \sqrt{\frac{n-2}{2(n-1)}}\int_M |W||\mathring{\mathrm{Ric}}|^2 f_\tau^{\alpha-2}\mathrm{d}\mu - \sqrt{\frac{n}{n-1}}\int_M |\mathring{\mathrm{Ric}}|^3 f_\tau^{\alpha-2}\mathrm{d}\mu \\ = &(\alpha-1)\int_M |\nabla f_\tau|^2 f_\tau^{\alpha-2}\mathrm{d}\mu + \frac{1}{n-1}\int_M R|\mathring{\mathrm{Ric}}|^2 f_\tau^{\alpha-2}\mathrm{d}\mu \\ &- \sqrt{\frac{n-2}{2(n-1)}}\int_M W|\mathring{\mathrm{Ric}}|^2 f_\tau^{\alpha-2}\mathrm{d}\mu - \sqrt{\frac{n}{n-1}}\int_M |\mathring{\mathrm{Ric}}|^3 f_\tau^{\alpha-2}\mathrm{d}\mu.\end{aligned}$$

Letting $\tau \to 0$, the above inequality implies

$$\int_M |\mathring{\mathrm{Ric}}|^\alpha(\sqrt{2}R - \sqrt{(n-1)(n-2)}|W| - \sqrt{2n(n-1)}|\mathring{\mathrm{Ric}}|)\mathrm{d}\mu \leq 0. \qquad (4.3)$$

If the equality in (4.3) holds, the proof shows that either at every point $\mathring{\mathrm{Ric}} = 0$, or at some point $\mathring{\mathrm{Ric}} \neq 0$, where $\mathring{\mathrm{Ric}}$ has an eigenvalue of multiplicity $n-1$ and another of multiplicity 1. Now we consider the latter case. Assume $\mathring{\mathrm{Ric}} \neq 0$ at the point $x \in M$. Since the equality in (2.3) also holds at x, we have that either $W = 0$, or $W \neq 0$ and $\mathring{\mathrm{Ric}}^2 = cg$ at x for a nonnegative constant c. We can choose an orthonormal basis of the tangent space at x such that $\mathring{\mathrm{Ric}} = \mathrm{diag}(\lambda, \cdots, \lambda, -(n-1)\lambda)$ for some $\lambda \neq 0$. It is evident that $\mathring{\mathrm{Ric}}^2 = cg$ cannot hold for a nonnegative constant c. So we have $W = 0$ at x. Due to the smoothness of $\mathring{\mathrm{Ric}}$, there exists a neighborhood U_x of x such that $\mathring{\mathrm{Ric}} \neq 0$ and $W = 0$ on U_x. If we assume that g is real analytic, then the function $|W|$ is also real analytic. Hence $W = 0$ on M. Therefore, either M

is Einstein, or M is locally conformally flat but not Einstein. According to the argument in [5], we see that for the second case, M is $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric or with a rotationally symmetric Derdzinski metric. This completes the proof.

Next, we give the proof of Theorem 1.4.

Proof of Theorem 1.4 We choose a cut-off function $\phi_r \in C^\infty(M)$ with the following properties:

$$\phi_r(x) = \begin{cases} 1, & \text{if } x \in B_r(q), \\ \phi_r(x) \in [0,1], \quad |\nabla \phi_r| \leq \dfrac{2}{r}, & \text{if } x \in B_{2r}(q) \setminus B_r(q), \\ 0, & \text{if } x \in M \setminus B_{2r}(q), \end{cases}$$

where $B_r(q)$ is the geodesic ball in M with radius r centered at $q \in M$. In particular, if M is compact, and if $r \geq d$, where d is the diameter of M, then $\phi_r \equiv 1$ on M. From (4.2), we get the inequality that

$$0 \geq -f_\tau \Delta f_\tau + \frac{1}{n-1} R|\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n-2}{2(n-1)}}|W||\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3. \tag{4.4}$$

We set $f = |\mathring{\mathrm{Ric}}|$ and $w = |W|$. Multiplying both sides of (4.4) by $\phi_r^2 f_\tau^{\frac{n}{2}-2}$ and integrating by parts, we get

$$\begin{aligned} 0 \geq\ & 2\int_M \phi_r f_\tau^{\frac{n}{2}-1} \langle \nabla \phi_r, \nabla f_\tau \rangle \mathrm{d}\mu + \frac{8(n-2)}{n^2} \int_M \phi_r^2 |\nabla f_\tau^{\frac{n}{4}}|^2 \mathrm{d}\mu \\ & + \frac{R}{n-1} \int_M \phi_r^2 f_\tau^{\frac{n}{2}-2} f^2 \mathrm{d}\mu - \sqrt{\frac{n-2}{2(n-1)}} \int_M \phi_r^2 w f_\tau^{\frac{n}{2}-2} f^2 \mathrm{d}\mu \\ & - \sqrt{\frac{n}{n-1}} \int_M \phi_r^2 f_\tau^{\frac{n}{2}-2} f^3 \mathrm{d}\mu \\ =\ & (\sigma+2) \int_M \phi_r f_\tau^{\frac{n}{2}-1} \langle \nabla \phi_r, \nabla f_\tau \rangle \mathrm{d}\mu - \sigma \int_M \phi_r f_\tau^{\frac{n}{2}-1} \langle \nabla \phi_r, \nabla f_\tau \rangle \mathrm{d}\mu \\ & + \frac{8(n-2)}{n^2} \int_M \phi_r^2 |\nabla f_\tau^{\frac{n}{4}}|^2 \mathrm{d}\mu + \frac{R}{n-1} \int_M \phi_r^2 f_\tau^{\frac{n}{2}-2} f^2 \mathrm{d}\mu \\ & - \sqrt{\frac{n-2}{2(n-1)}} \int_M \phi_r^2 w f_\tau^{\frac{n}{2}-2} f^2 \mathrm{d}\mu - \sqrt{\frac{n}{n-1}} \int_M \phi_r^2 f_\tau^{\frac{n}{2}-2} f^3 \mathrm{d}\mu, \end{aligned} \tag{4.5}$$

where σ is an arbitrary positive constant. To estimate the second term in the first line on the right-hand side, we apply Young's inequality as follows:

$$\begin{aligned} \int_M \phi_r f_\tau^{\frac{n}{2}-1} \langle \nabla \phi_r, \nabla f_\tau \rangle \mathrm{d}\mu &\leq \frac{\rho}{2} \int_M \phi_r^2 f_\tau^{\frac{n}{2}-2} |\nabla f_\tau|^2 \mathrm{d}\mu + \frac{1}{2\rho} \int_M f_\tau^{\frac{n}{2}} |\nabla \phi_r|^2 \mathrm{d}\mu \\ &= \frac{8\rho}{n^2} \int_M \phi_r^2 |\nabla f_\tau^{\frac{n}{4}}|^2 \mathrm{d}\mu + \frac{1}{2\rho} \int_M f_\tau^{\frac{n}{2}} |\nabla \phi_r|^2 \mathrm{d}\mu, \end{aligned}$$

where ρ is also an arbitrary positive constant. Substituting the above inequality into (4.5), we get

$$0 \geq \frac{2(\sigma+2)}{n} \int_M \frac{n}{2} \phi_r f_\tau^{\frac{n}{2}-1} \langle \nabla \phi_r, \nabla f_\tau \rangle \mathrm{d}\mu - \frac{\sigma}{2\rho} \int_M f_\tau^{\frac{n}{2}} |\nabla \phi_r|^2 \mathrm{d}\mu$$

$$
\begin{aligned}
&+\Big(\frac{8(n-2)}{n^2}-\frac{8\sigma\rho}{n^2}\Big)\int_M \phi_r^2|\nabla f_\tau^{\frac{n}{4}}|^2\mathrm{d}\mu+\frac{R}{n-1}\int_M \phi_r^2 f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu\\
&-\sqrt{\frac{n-2}{2(n-1)}}\int_M \phi_r^2 w f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu-\sqrt{\frac{n}{n-1}}\int_M \phi_r^2 f_\tau^{\frac{n}{2}-2}f^3\mathrm{d}\mu. \qquad (4.6)
\end{aligned}
$$

By a direct computation, we have

$$
|\nabla(\phi_r f_\tau^{\frac{n}{4}})|^2=f_\tau^{\frac{n}{2}}|\nabla\phi_r|^2+\frac{n}{2}\phi_r f_\tau^{\frac{n}{2}-1}\langle\nabla\phi_r,\nabla f_\tau\rangle+\phi_r^2|\nabla f_\tau^{\frac{n}{4}}|^2. \qquad (4.7)
$$

Choose $\rho>0$ such that $\frac{8(n-2)}{n^2}-\frac{8\sigma\rho}{n^2}=\frac{2(\sigma+2)}{n}$, i.e., $\rho=\frac{2n-8-n\sigma}{4\sigma}$. For $\rho>0$, it is necessary that $\sigma<2-\frac{8}{n}$. This inequality implies that $n\geq 5$. From (4.6)–(4.7), we obtain

$$
\begin{aligned}
0\geq{}&\frac{2(\sigma+2)}{n}\int_M\Big(\frac{n}{2}\phi_r f_\tau^{\frac{n}{2}-1}\langle\nabla\phi_r,\nabla f_\tau\rangle+\phi_r^2|\nabla f_\tau^{\frac{n}{4}}|^2\Big)\mathrm{d}\mu\\
&-\frac{\sigma}{2\rho}\int_M f_\tau^{\frac{n}{2}}|\nabla\phi_r|^2\mathrm{d}\mu+\frac{R}{n-1}\int_M\phi_r^2 f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu\\
&-\sqrt{\frac{n-2}{2(n-1)}}\int_M\phi_r^2 w f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu-\sqrt{\frac{n}{n-1}}\int_M\phi_r^2 f_\tau^{\frac{n}{2}-2}f^3\mathrm{d}\mu\\
={}&\frac{2(\sigma+2)}{n}\int_M|\nabla(\phi_r f_\tau^{\frac{n}{4}})|^2\mathrm{d}\mu-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f_\tau^{\frac{n}{2}}|\nabla\phi_r|^2\mathrm{d}\mu\\
&+\frac{R}{n-1}\int_M\phi_r^2 f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu-\sqrt{\frac{n-2}{2(n-1)}}\int_M\phi_r^2 w f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu\\
&-\sqrt{\frac{n}{n-1}}\int_M\phi_r^2 f_\tau^{\frac{n}{2}-2}f^3\mathrm{d}\mu.
\end{aligned}
$$

Combining the definition of the Yamabe constant and the positivity condition, we obtain

$$
\mathrm{Y}(M,[g])\Big(\int_M u^{\frac{2n}{n-2}}\mathrm{d}\mu\Big)^{\frac{n-2}{n}}\leq\int_M|\nabla u|^2\mathrm{d}\mu+\frac{n-2}{4(n-1)}\int_M Ru^2\mathrm{d}\mu
$$

for all $u\in C_0^\infty(M)$. Hence we get

$$
\begin{aligned}
0\geq{}&\frac{2(\sigma+2)}{n}\Big(\mathrm{Y}(M,[g])\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}}-\frac{(n-2)R}{4(n-1)}\int_M\phi_r^2 f_\tau^{\frac{n}{2}}\mathrm{d}\mu\Big)\\
&-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f_\tau^{\frac{n}{2}}|\nabla\phi_r|^2\mathrm{d}\mu+\frac{R}{n-1}\int_M\phi_r^2 f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu\\
&-\sqrt{\frac{n-2}{2(n-1)}}\int_M\phi_r^2 w f_\tau^{\frac{n}{2}-2}f^2\mathrm{d}\mu-\sqrt{\frac{n}{n-1}}\int_M\phi_r^2 f_\tau^{\frac{n}{2}-2}f^3\mathrm{d}\mu.
\end{aligned}
$$

Now letting $\tau\to 0$, the above inequality implies

$$
\begin{aligned}
0\geq{}&\frac{2(\sigma+2)}{n}\mathrm{Y}(M,[g])\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}}+\Big(\frac{R}{n-1}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{\frac{n}{2}}\|_1\\
&-\sqrt{\frac{n-2}{2(n-1)}}\int_M\phi_r^2 w f^{\frac{n}{2}}\mathrm{d}\mu-\sqrt{\frac{n}{n-1}}\int_M\phi_r^2 f^{\frac{n}{2}+1}\mathrm{d}\mu\\
&-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}|\nabla\phi_r|^2\mathrm{d}\mu. \qquad (4.8)
\end{aligned}
$$

(i) When $R=0$, (4.8) implies

$$\begin{aligned}0 \geq\ & \frac{2(\sigma+2)}{n}\mathrm{Y}(M,[g])\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}|\nabla\phi_r|^2 \mathrm{d}\mu \\ & - \sqrt{\frac{n-2}{2(n-1)}}\int_M \phi_r^2 w f^{\frac{n}{2}}\mathrm{d}\mu - \sqrt{\frac{n}{n-1}}\int_M \phi_r^2 f^{\frac{n}{2}+1}\mathrm{d}\mu \\ \geq\ & \frac{2(\sigma+2)}{n}\mathrm{Y}(M,[g])\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}|\nabla\phi_r|^2 \mathrm{d}\mu \\ & - \sqrt{\frac{n-2}{2(n-1)}}\|w\|_{\frac{n}{2}}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} - \sqrt{\frac{n}{n-1}}\|f\|_{\frac{n}{2}}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} \\ \geq\ & \sqrt{\frac{n}{n-1}}\Big(\sqrt{\frac{n-1}{n}}\frac{2(\sigma+2)}{n}\mathrm{Y}(M,[g]) - \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} - \|f\|_{\frac{n}{2}}\Big)\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} \\ & - \frac{4}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}\mathrm{d}\mu. \end{aligned} \tag{4.9}$$

Put $\sigma = 2-\frac{8}{n}-\varepsilon$, where ε is a small positive constant. It follows from the assumption $\|f\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} < \infty$ that

$$\lim_{r\to+\infty}\frac{4}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}\mathrm{d}\mu = 0. \tag{4.10}$$

Combining (4.9)–(4.10), we get

$$0 \geq \Big(\sqrt{\frac{n-1}{n}}\frac{2(4-\frac{8}{n}-\varepsilon)}{n}\mathrm{Y}(M,[g]) - \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} - \|f\|_{\frac{n}{2}}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}}$$

for any small $\varepsilon>0$. As $\varepsilon\to 0$, we have

$$0 \geq \Big(\frac{8(n-2)}{n^2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M,[g]) - \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} - \|f\|_{\frac{n}{2}}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}}.$$

If we assume

$$\|f\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} < \frac{8(n-2)}{n^2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M,[g]),$$

then $\lim_{r\to+\infty}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} = 0$, which implies $f\equiv 0$. This means that M is a Ricci flat manifold.

(ii) When $R>0$ and $n\geq 7$, we have $\frac{4}{n-2} < 2-\frac{8}{n}$. We pick $\sigma = \frac{4}{n-2}$, then $\frac{R}{n-1} - \frac{(\sigma+2)(n-2)R}{2n(n-1)} = 0$. Then (4.8) implies

$$\begin{aligned}0 \geq\ & \frac{4}{n-2}\mathrm{Y}(M,[g])\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}|\nabla\phi_r|^2\mathrm{d}\mu \\ & - \sqrt{\frac{n-2}{2(n-1)}}\int_M \phi^2 w f_r^{\frac{n}{2}}\mathrm{d}\mu - \sqrt{\frac{n}{n-1}}\int_M \phi_r^2 f^{\frac{n}{2}+1}\mathrm{d}\mu. \end{aligned}\tag{4.11}$$

It follows from the assumption $\|f\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} < \infty$ that

$$\lim_{r\to+\infty}\frac{4}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{\frac{n}{2}}\mathrm{d}\mu = 0. \tag{4.12}$$

Combining (4.11)–(4.12), we get

$$0 \geq \Big(\frac{4}{n-2}\mathrm{Y}(M,[g]) - \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} - \|f\|_{\frac{n}{2}}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}}.$$

If we assume

$$\|f\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} < \frac{4}{n-2}\mathrm{Y}(M,[g]),$$

then $\lim_{r\to+\infty}\|\phi_r^2 f^{\frac{n}{2}}\|_{\frac{n}{n-2}} = 0$, which implies $f \equiv 0$. This means that M is an Einstein manifold.

When $R > 0$ and $n < 7$, we have $\frac{4}{n-2} > 2 - \frac{8}{n}$. We pick $\sigma = 2 - \frac{8}{n} - \varepsilon$ for small $\varepsilon > 0$. Then we also have $\frac{R}{n-1} - \frac{(\sigma+2)(n-2)R}{2n(n-1)} \geq 0$. By a similar argument, we get that if

$$\|f\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} < \frac{8(n-2)}{n^2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M,[g]),$$

then $f \equiv 0$ and M is an Einstein manifold.

(iii) When $R < 0$ and $n \geq 7$, we pick $\sigma = 2 - \frac{8}{n} - \varepsilon$ for small $\varepsilon > 0$ such that $\frac{4}{n-2} \leq 2 - \frac{8}{n} - \varepsilon$. Then $\frac{R}{n-1} - \frac{(\sigma+2)(n-2)R}{2n(n-1)} \geq 0$. Using a similar argument, we find that if

$$\|f\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|w\|_{\frac{n}{2}} < \frac{8(n-2)}{n^2}\sqrt{\frac{n-1}{n}}\mathrm{Y}(M,[g]),$$

then $f \equiv 0$ and M is an Einstein manifold.

At last, we consider a special case that the manifold has harmonic curvature, i.e., M satisfies $\sum_i \nabla^i R_{ijkl} = 0$, or equivalently $\nabla_k R_{jl} = \nabla_l R_{jk}$. According to [14], a metric with harmonic curvature is real analytic. Furthermore, by the Bianchi identity, the scalar curvature R is constant. As $\mathring{\mathrm{Ric}}$ is a Codazzi tensor, one has $|\nabla\mathring{\mathrm{Ric}}|^2 \geq \frac{n+2}{n}|\nabla f_\tau|^2$. Then

$$\frac{1}{2}\Delta f_\tau^2 \geq \frac{n+2}{n}|\nabla f_\tau|^2 + \frac{R}{n-1}|\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n-2}{2(n-1)}}|W||\mathring{\mathrm{Ric}}|^2 - \sqrt{\frac{n}{n-1}}|\mathring{\mathrm{Ric}}|^3.$$

With the aid of this improved inequality, the following theorems can be proved by using a similar argument as in the proofs of Theorems 1.3–1.4.

Theorem 4.1 *Let (M^n, g) be an $n(\geq 3)$-dimensional closed Riemannian manifold with harmonic curvature. Suppose that the scalar curvature R is positive. Then*

$$\int_M |\mathring{\mathrm{Ric}}|^\alpha(\sqrt{2}R - \sqrt{2n(n-1)}|\mathring{\mathrm{Ric}}| - \sqrt{(n-1)(n-2)}|W|)\mathrm{d}\mu \leq 0$$

for all $\alpha \geq \frac{n-2}{n}$, and the equality holds if and only if M is Einstein or M is $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric or with a rotationally symmetric Derdzinski metric. In particular, if

$$|\mathring{\mathrm{Ric}}| + \sqrt{\frac{n-2}{2n}}|W| \leq \frac{1}{\sqrt{n(n-1)}}R,$$

then M is either Einstein or isometric to $\mathbb{S}^1\times\mathbb{S}^{n-1}$ with the product metric or with a rotationally symmetric Derdzinski metric.

Theorem 4.2 *Let (M^n, g) be an $n(\geq 3)$-dimensional complete Riemannian manifold with harmonic curvature. Suppose that the Yamabe constant $\Upsilon(M,[g])$ of M is positive. If*

(i) $R = 0$ *and* $n \geq 3$, *or* $R > 0$ *and* $n = 3, 4, 5$, *or* $R < 0$ *and* $n \geq 6$, *and*

$$\|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|W\|_{\frac{n}{2}} < \frac{8(n^2-2n+4)}{n^2}\sqrt{\frac{n-1}{n}}\Upsilon(M,[g]),$$

or if

(ii) $R > 0$ *and* $n \geq 6$, *and*

$$\|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}} + \sqrt{\frac{n-2}{2n}}\|W\|_{\frac{n}{2}} < \frac{4}{n-2}\sqrt{\frac{n-1}{n}}\Upsilon(M,[g]),$$

then M is Einstein.

Similar to Theorem 1.4, the product manifold $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ shows that the pinching inequality in Theorem 4.2 is optimal for $R > 0$ and $n \geq 6$.

Remark 4.1 Theorem 4.1 for $\alpha = \frac{n-2}{n}$ and a similar result as Theorem 4.2 for positive scalar curvature were proved previously by Fu [18]. Fu and his collaborators also proved several other interesting rigidity theorems (see [18–20], etc.).

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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