

Regularity for Weakly H-surfaces into Static Lorentzian Manifolds*

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Abstract The authors investigate H-surfaces into static Lorentzian manifolds and show the Hölder continuity of weak solutions.

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1 Introduction

The theory of H-surfaces (surfaces of prescribed mean curvature) in Riemannian manifolds is an important and classical conformally invariant geometric variational problem. The Euler-Lagrange equations are critical nonlinear elliptic systems. Extensive work has been devoted to studying the regularity issue of weak solutions of such systems under various situations, e.g. [2, 4–6, 8–12, 16, 18–19, 23–24, 31]. A similar problem is the regularity theory for weakly harmonic maps from surfaces into Riemannian manifolds, e.g. [3, 7, 13–15, 17]. Vast techniques developed for the regularity problems of harmonic maps into Riemannian manifolds and H-surfaces into Riemannian manifolds have led to a regularity theory for critical elliptic systems with an L^2 -antisymmetric potential, e.g. [6, 17, 23–30, 32].

When the target manifold becomes non-compact or non-Riemannian, however, then the L^2 -antisymmetric structure for harmonic map systems may not hold anymore. Therefore, we need to develop a regularity theory for critical elliptic systems with the structure of potentials being more general than L^2 -antisymmetric. This is partially achieved in [1, 20, 33] which can be applied to harmonic map type systems into certain pseudo-Riemannian manifolds.

In this article, we shall explore this scheme further and apply it to investigate the regularity of weakly H-surfaces into static Lorentzian manifolds.

To describe the problem, we shall first recall the notion of static Lorentzian manifolds. Let (M, g_M) be a closed Riemannian manifold of dimension d and (\mathbb{R}, dt^2) be the 1-dimensional Euclidean space. A static Lorentzian manifold is a product manifold $\mathbb{R} \times M$ equipped with the following metric:

$$g = -(dt)^2 + g_M. \tag{1.1}$$

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By the Nash-Moser’s embedding theorem, we can embed (M, g_M) isometrically into a Euclidean space \mathbb{R}^n for some integer n . For more about Lorentzian manifolds, we refer to, e.g. [21–22].

We consider the case that the domain is the unit disk D in \mathbb{R}^2 . Let ω be a 2-form on $(\mathbb{R} \times M, g)$. Without loss of generality, we extend ω to some open neighborhood $\mathbb{R} \times M_\delta$ of $\mathbb{R} \times M$ in $\mathbb{R} \times \mathbb{R}^n$. Here M_δ is a tubular neighborhood of M in \mathbb{R}^n for some $\delta > 0$, and let Π be the projection from M_δ to M .

Let $\{y^I\}_{I=0,1,\dots,n}$ be canonical coordinates of $\mathbb{R} \times \mathbb{R}^n$ and write

$$\omega = \omega_{ml} dy^m \wedge dy^l, \quad (\omega_{ml}) \in C^1(\mathbb{R} \times M_\delta, so(n + 1)),$$

where $m, l = 0, 1, \dots, n$. Moreover, we define

$$H_{ml}^0 := \partial_0 \omega_{ml}, \quad H_{ml}^j := \partial_j \omega_{ml}, \tag{1.2}$$

$j = 1, 2, \dots, n$. In this paper, we shall make an assumption as follows:

$$H_{ml}^0, H_{ml}^j \in W^{1,2} \cap L^\infty(D). \tag{1.3}$$

To define the notion of weakly H-surfaces into static Lorentzian manifolds $(\mathbb{R} \times M, g)$, we consider the following Sobolev space of maps:

$$W^{1,2}(D, \mathbb{R} \times M) := \{(t, u) \in W^{1,2}(D, \mathbb{R}) \times W^{1,2}(D, \mathbb{R}^n) \mid u(p) \in M \text{ a.e. } p \in D\},$$

and we introduce the following action functional:

$$E^\omega(t, u) = -\frac{1}{2} \int_D |\nabla t|^2 + \frac{1}{2} \int_D |\nabla u|^2 + \frac{1}{2} \int_D (t, u)^* \omega. \tag{1.4}$$

Definition 1.1 We call a map $U = (t, u) \in W^{1,2}(D, \mathbb{R} \times M)$ a weakly H-surface¹ from D into a static Lorentzian manifold $(\mathbb{R} \times M, g)$, if

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E^\omega(t_\varepsilon, u_\varepsilon) = 0, \tag{1.5}$$

where $t_\varepsilon := t + \varepsilon s$, $u_\varepsilon := \Pi(u + \varepsilon v)$ for any $s \in W_0^{1,2} \cap L^\infty(D, \mathbb{R})$, $v \in W_0^{1,2} \cap L^\infty(D, \mathbb{R}^n)$.

If the given 2-form is vanishing, $\omega \equiv 0$, then the functional becomes

$$E(t, u) = -\frac{1}{2} \int_D |\nabla t|^2 + \frac{1}{2} \int_D |\nabla u|^2, \tag{1.6}$$

and its critical points are called weakly harmonic maps from D into $(\mathbb{R} \times M, g)$. The interior Hölder continuity of such weak solutions was proved again in a different and more general setting in [1, 20, 33].

By straightforward calculations, we can derive the Euler-Lagrange equations of a weakly H-surface $U = (t, u)$ into the static Lorentzian manifolds as follows:

$$-\operatorname{div}(\nabla t + \omega_{j0}(U)\nabla^\perp u^j) = -H_{ml}^0(U)\nabla^\perp U^m \cdot \nabla U^l,$$

¹The classical notion of H-surfaces in Riemannian manifolds requires the conformality condition, which is satisfied when considering stationary critical points of the action functional. However, it turns out that the conformality condition is not necessary for the study of the regularity of weak solutions. Here, for the case of static Lorentzian manifolds, we still use “H-surface” to define critical points of the action functional without imposing the extra conformality condition.

$$-\operatorname{div}(\nabla u^j + \omega_{0j}(U)\nabla^\perp t + 2\omega_{kj}\nabla^\perp u^k) = \nu_z \nabla \nu_z \cdot \nabla u^j - H_{ml}^j(U)\nabla^\perp U^m \cdot \nabla U^l,$$

where the indices $m, l = 0, 1, \dots, n$, $j, k = 1, 2, \dots, n$ and $z = d+1, d+2, \dots, n$; see Proposition 2.1 for more details about the notations.

However, regularity results for weak solutions cannot be directly derived from the critical elliptic system in the above preliminary form by applying classical regularity theory. Therefore, we need to make some new observations. The main ingredient of the present paper is to explore the structure of the above Euler-Lagrange equations for a weak solution $U = (t, u)$ and rewrite it into a critical elliptic system of the following special form:

$$-\operatorname{div}(Q\nabla U) = \Theta \cdot Q\nabla U + \sum_{K=1}^2 F_K \operatorname{curl} \xi_K \cdot G_K \nabla U. \quad (1.7)$$

This is achieved in Theorem 2.1; see Section 2 for more details about the notations.

Now we state the main result of this paper.

Theorem 1.1 *Any weakly H-surface $(t, u) \in W^{1,2}(D, \mathbb{R} \times M)$ into a static Lorentzian manifold $(\mathbb{R} \times M, g)$ with g being of the form (1.1) is Hölder continuous in D .*

The rest of this paper, which concentrates on Section 2, is to prove the main result Theorem 1.1.

2 The Euler-Lagrange Equation and the New Form

In this section, we shall first derive the Euler-Lagrange equation for a weakly H-surface into a static Lorentzian manifold $(\mathbb{R} \times M, g)$. Then, we show that this elliptic system can be rewritten into the special form of (1.7) and prove Theorem 1.1.

By Definition 1.1, we have the following result.

Proposition 2.1 *Let $U = (t, u) = (t, u_1, \dots, u_n) \in W^{1,2}(D, \mathbb{R} \times M)$ be a weakly H-surface into $(\mathbb{R} \times M, g)$ with g being of the form (1.1). Then we have the following Euler-Lagrange equations*

$$-\operatorname{div}(\nabla t + \omega_{j0}\nabla^\perp u^j) = -H_{ml}^0(U)\nabla^\perp U^m \cdot \nabla U^l \quad (2.1)$$

and

$$-\operatorname{div}(\nabla u^j + \omega_{0j}\nabla^\perp t + 2\omega_{ij}\nabla^\perp u^i) = \nu_z \nabla \nu_z \cdot \nabla u^j - H_{ml}^j(U)\nabla^\perp U^m \cdot \nabla U^l \quad (2.2)$$

hold for $m, l = 0, 1, \dots, n$, $i, j = 1, 2, \dots, n$ and $z = d+1, d+2, \dots, n$, where $\{\nu_z\}_{z=d+1}^n$ is an orthonormal frame for the normal bundle $T^\perp M$.

Proof For any $s \in W_0^{1,2} \cap L^\infty(D, \mathbb{R})$ and for any $v \in W_0^{1,2} \cap L^\infty(D, \mathbb{R}^n)$, we set $t_\varepsilon = t + \varepsilon s$, $u_\varepsilon = \Pi(u + \varepsilon v)$. Thus, we can write $U_\varepsilon = (t_\varepsilon, u_\varepsilon) = (t + \varepsilon s, \Pi(u + \varepsilon v))$ as an admissible variation for U .

Since the Euler-Lagrange equations for the functional (1.6) were already derived in [20, 33], here, we mainly focus on the additional term

$$\frac{1}{2} \int_D U^* \omega$$

in (1.4).

By direct calculations,

$$\begin{aligned} U^* \omega &= \omega_{ml}(U)(\partial_{x_1} U^m \partial_{x_2} U^l - \partial_{x_1} U^l \partial_{x_2} U^m) dx_1 \wedge dx_2 \\ &= 2\omega_{0j}(U)(\partial_{x_1} t \partial_{x_2} u^j - \partial_{x_1} u^j \partial_{x_2} t) dx_1 \wedge dx_2 \\ &\quad + 2 \sum_{i < j} \omega_{ij}(U)(\partial_{x_1} u^i \partial_{x_2} u^j - \partial_{x_1} u^j \partial_{x_2} u^i) dx_1 \wedge dx_2, \end{aligned} \quad (2.3)$$

where $m, l = 0, 1, 2, \dots, n$, $i, j = 1, 2, \dots, n$.

Define $w := d\Pi(u) \cdot v$, then we obtain $w = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} u_\varepsilon$ by

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (\Pi(u + \varepsilon v)) = d\Pi(u + \varepsilon v) \cdot v \Big|_{\varepsilon=0} = d\Pi(u) \cdot v.$$

Let the symbol w^k represent the k -th component of w , $k = 1, 2, \dots, n$. Noting that

$$\omega_{ij}(\partial_{x_1} u_\varepsilon^i \partial_{x_2} u_\varepsilon^j - \partial_{x_1} u_\varepsilon^j \partial_{x_2} u_\varepsilon^i) = 2\omega_{ij}(\partial_{x_1} u_\varepsilon^i \partial_{x_2} u_\varepsilon^j),$$

we have

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \partial_{x_1} u_\varepsilon^i \partial_{x_2} u_\varepsilon^j = \partial_{x_1} w^i \partial_{x_2} u^j + \partial_{x_1} u^i \partial_{x_2} w^j. \quad (2.4)$$

Similarly,

$$\omega_{0j}(\partial_{x_1} t_\varepsilon \partial_{x_2} u_\varepsilon^j - \partial_{x_1} u_\varepsilon^j \partial_{x_2} t_\varepsilon) = 2\omega_{0j}(\partial_{x_1} t \partial_{x_2} u_\varepsilon^j + \varepsilon \partial_{x_1} s \partial_{x_2} u_\varepsilon^j),$$

we get

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (\partial_{x_1} t \partial_{x_2} u_\varepsilon^j + \varepsilon \partial_{x_1} s \partial_{x_2} u_\varepsilon^j) = \partial_{x_1} t \partial_{x_2} w^j + \partial_{x_1} s \partial_{x_2} u^j. \quad (2.5)$$

Then, by uniting (2.3)–(2.5), we calculate that

$$\begin{aligned} &\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_D U_\varepsilon^* \omega \\ &= 2 \sum_{m < l} \int_D s \partial_0 \omega_{ml}(U)(\partial_{x_1} U^m \partial_{x_2} U^l - \partial_{x_1} U^l \partial_{x_2} U^m) \\ &\quad + 2 \sum_{m < l} \int_D w^k \partial_k \omega_{ml}(U)(\partial_{x_1} U^m \partial_{x_2} U^l - \partial_{x_1} U^l \partial_{x_2} U^m) \\ &\quad + 2 \int_D \omega_{0j}(U)(\partial_{x_1} s \partial_{x_2} u^j + \partial_{x_1} t \partial_{x_2} w^j - \partial_{x_2} s \partial_{x_1} u^j - \partial_{x_2} t \partial_{x_1} w^j) \\ &\quad + 2 \sum_{i < j} \int_D \omega_{ij}(U)(\partial_{x_1} w^i \partial_{x_2} u^j + \partial_{x_1} u^i \partial_{x_2} w^j - \partial_{x_1} w^j \partial_{x_2} u^i - \partial_{x_1} u^j \partial_{x_2} w^i) \\ &= 0, \end{aligned} \quad (2.6)$$

where $k = 1, 2, \dots, n$. In the sequel, we default $m < l$, $i < j$, and omit the summation signs.

To derive the Euler-Lagrange equations for $U = (t, u)$, we shall consider the following two cases.

Case 1 Take $v \equiv 0$, then $w \equiv 0$. Recall $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, thus, we have

$$0 = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \frac{1}{2} \int_D U_\varepsilon^* \omega$$

$$\begin{aligned}
 &= \int_D s \partial_0 \omega_{ml}(U) \nabla^\perp U^m \cdot \nabla U^l + \int_D \omega_{j0}(U) \nabla^\perp u^j \nabla s \\
 &= \int_D s H_{ml}^0(U) \nabla^\perp U^m \cdot \nabla U^l - \int_D s \cdot \operatorname{div}(\omega_{j0}(U) \nabla^\perp u^j),
 \end{aligned} \tag{2.7}$$

where we use the notation in (1.2).

For the functional (1.6), recall that

$$0 = \int_D -\nabla t \cdot \nabla s \tag{2.8}$$

in [33, p. 116]. Combining (2.7) with (2.8), we obtain (2.1).

Case 2 Taking $s \equiv 0$, it yields

$$\begin{aligned}
 0 &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \frac{1}{2} \int_D U_\varepsilon^* \omega \\
 &= \int_D w^k H_{ml}^k(U) \nabla^\perp U^m \cdot \nabla U^l + \int_D \omega_{0j}(U) \nabla^\perp t \nabla w^j + 2 \int_D \omega_{ij}(U) \nabla^\perp u^i \nabla w^j \\
 &= \int_D w^j H_{ml}^j(U) \nabla^\perp U^m \cdot \nabla U^l - \int_D \operatorname{div}\{\omega_{0j}(U) \nabla^\perp t\} w^j - 2 \int_D \operatorname{div}\{\omega_{ij}(U) \nabla^\perp u^i\} w^j,
 \end{aligned} \tag{2.9}$$

where we use the notation in (1.2) and the identity

$$\begin{aligned}
 &\omega_{ij}(\partial_{x_1} w^i \partial_{x_2} w^j + \partial_{x_1} u^i \partial_{x_2} w^j - \partial_{x_1} w^j \partial_{x_2} u^i - \partial_{x_1} u^j \partial_{x_2} w^i) \\
 &= \omega_{ij}(\nabla^\perp u^i \nabla w^j + \nabla^\perp w^i \nabla u^j) \\
 &= \omega_{ij}(\nabla^\perp w^j \nabla u^i - \nabla^\perp w^i \nabla u^j) \\
 &= 2\omega_{ij} \nabla^\perp u^i \nabla w^j,
 \end{aligned}$$

which is established due to

$$\omega_{ij} = -\omega_{ji}.$$

For any $v \in W_0^{1,2} \cap L^\infty(D, \mathbb{R}^n)$, we recall

$$0 = \int_D (-\operatorname{div} \nabla u) \cdot d\Pi(u)v \tag{2.10}$$

and

$$-\operatorname{div} \nabla u - A(u)(\nabla u, \nabla u) = 0 \tag{2.11}$$

for the functional (1.6) as shown in [33, p. 116], where A is the second fundamental form of $M \subset \mathbb{R}^n$. Moreover,

$$A(u)(\nabla u, \nabla u) = \nu \nabla \nu \cdot \nabla u,$$

where we continue to use ν_l ($l = d + 1, d + 2, \dots, n$) to represent the corresponding normal frame along the map u . Through the integration of (2.9) and (2.11), we derive (2.2).

Next, we shall rewrite the Euler-Lagrange equations into some elliptic system with a special structure.

Theorem 2.1 *Let $U \in W^{1,2}(D, \mathbb{R} \times M)$ be a weakly H-surface into a static Lorentzian manifold $(\mathbb{R} \times M, g)$ with g satisfying (1.1), then there exist $\Theta \in L^2(D, so(n + 1) \otimes \wedge^1 \mathbb{R}^2)$,*

$F_K, G_K \in W^{1,2} \cap L^\infty(D, M(n+1)), \xi_K \in W^{1,2}(D, M(n+1) \otimes \wedge^2 \mathbb{R}^2), K = 1, 2,$ and $Q \in W^{1,2} \cap L^\infty(D, GL(n+1))$ such that U is a weak solution of the following system

$$-\operatorname{div}(Q \nabla U) = \Theta \cdot Q \nabla U + F_1 \operatorname{curl} \xi_1 \cdot G_1 \nabla U + F_2 \operatorname{curl} \xi_2 \cdot G_2 \nabla U. \tag{2.12}$$

Moreover, the coefficients satisfy

$$\begin{aligned} & \|\nabla U\|_{L^2(D)} + \|\Theta\|_{L^2(D)} + \sum_{K=1}^2 \|\operatorname{curl} \xi_K\|_{L^2(D)} + \|\nabla Q\|_{L^2(D)} \\ & + \sum_{K=1}^2 \|\nabla F_K\|_{L^2(D)} + \sum_{K=1}^2 \|\nabla G_K\|_{L^2(D)} \leq \varepsilon(\Lambda) \end{aligned} \tag{2.13}$$

and

$$|Q| + |Q^{-1}| + \sum_{K=1}^2 |F_K| + \sum_{K=1}^2 |G_K| \leq \Lambda \quad \text{a.e. in } D \tag{2.14}$$

for some $\Lambda > 0$ depending on $(\mathbb{R} \times M, g)$ and for some $\varepsilon(\Lambda) > 0$ depending on Λ .

Proof We may write (2.2) as

$$-\operatorname{div}(\nabla u^j + \omega_{0j} \nabla^\perp t + 2\omega_{ij} \nabla^\perp u^i) = \Theta^{jk} \cdot \nabla u^k - H_{ml}^j \nabla^\perp U^m \cdot \nabla U^l \tag{2.15}$$

with

$$\Theta^{jk} := \nu_i^j \nabla \nu_i^k - \nu_i^k \nabla \nu_i^j.$$

Combining (2.1) with (2.15), we obtain the following system for a weak solution $U = (t, u)$ by transforming the original combination of equations into some matrix form:

$$-\operatorname{div} \left\{ Q \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix} \right\} - \operatorname{div} \left\{ N \begin{pmatrix} \nabla^\perp t \\ \nabla^\perp u \end{pmatrix} \right\} = \Theta \cdot Q \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix} - H \nabla^\perp V \cdot I_{n+1} \nabla U, \tag{2.16}$$

where I_{n+1} is an identity matrix of order $n+1$, and we define some matrices of order $n+1$ as follows. \tilde{Q} is an identity matrix:

$$\tilde{Q} := \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix}, \quad Q := \tilde{Q} \circ u,$$

then

$$Q \in W^{1,2} \cap L^\infty(D, GL(n+1)). \tag{2.17}$$

Θ is defined by

$$\Theta := \begin{pmatrix} 0 & 0 \\ 0 & (\Theta^{jk}) \end{pmatrix}, \quad j, k = 1, 2, \dots, n.$$

It is easy to verify that

$$\Theta \in L^2(D, so(n+1) \otimes \wedge^1 \mathbb{R}^2). \tag{2.18}$$

Meanwhile, we set

$$N := \begin{pmatrix} 0 & \omega_{i0} \\ \omega_{0j} & 2\omega_{ij} \end{pmatrix},$$

where

$$\omega_{0j} = (\omega_{01}, \dots, \omega_{0n})^T, \quad \omega_{i0} = (\omega_{i0}, \dots, \omega_{in}), \quad \omega_{ij} = \begin{pmatrix} \omega_{11} & \dots & \omega_{1n} \\ \vdots & \vdots & \vdots \\ \omega_{n1} & \dots & \omega_{nn} \end{pmatrix}.$$

For the last term of (2.16), to be more precise,

$$H\nabla^\perp V \cdot I_{n+1}\nabla U = \begin{cases} H_{ml}^0 \nabla^\perp U^m \cdot \nabla U^l, \\ H_{ml}^j \nabla^\perp U^m \cdot \nabla U^l, \quad j = 1, 2, \dots, n, \end{cases}$$

we set

$$H := \begin{pmatrix} H^1 & 0 & \dots & 0 \\ 0 & H^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H^{n+1} \end{pmatrix},$$

$$\nabla^\perp V := \begin{pmatrix} R & R & \dots & R \\ R & R & \dots & R \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \dots & R \end{pmatrix},$$

where

$$R := \sum_{r=0}^n \nabla^\perp U^r = \nabla^\perp t + \nabla^\perp u^1 + \dots + \nabla^\perp u^n.$$

Then for the second term of the left-hand side in (2.16), we have

$$\operatorname{div} \left\{ N \begin{pmatrix} \nabla^\perp t \\ \nabla^\perp u \end{pmatrix} \right\} = \nabla N \cdot \begin{pmatrix} \nabla^\perp t \\ \nabla^\perp u \end{pmatrix} + N \cdot \operatorname{div} \begin{pmatrix} \nabla^\perp t \\ \nabla^\perp u \end{pmatrix} = -\nabla^\perp N \cdot \nabla U. \tag{2.19}$$

Therefore, (2.16) becomes

$$-\operatorname{div} \left\{ Q \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix} \right\} = \Theta \cdot Q \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix} - \nabla^\perp N \cdot \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix} - H\nabla^\perp V \cdot I_{n+1} \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix}. \tag{2.20}$$

Take ξ_1, ξ_2 such that

$$\operatorname{curl} \xi_1 = \nabla^\perp N, \quad \operatorname{curl} \xi_2 = \nabla^\perp V.$$

Then we can rewrite the above system (2.20) as follows

$$-\operatorname{div}(Q\nabla U) = \Theta \cdot Q\nabla U + F_1 \operatorname{curl} \xi_1 \cdot G_1 \nabla U + F_2 \operatorname{curl} \xi_2 \cdot G_2 \nabla U, \tag{2.21}$$

where we set

$$F_1 = -I_{n+1}, \quad G_1 = I_{n+1}, \quad F_2 = -H, \quad G_2 = I_{n+1},$$

$$\xi_K = \operatorname{diag} (\xi_K, \xi_K, \dots, \xi_K), \quad K = 1, 2.$$

It is easy to check that

$$F_K, G_K \in W^{1,2} \cap L^\infty(D, M(n+1)), \quad \xi_K \in W^{1,2}(D, M(n+1) \otimes \wedge^2 \mathbb{R}^2). \tag{2.22}$$

There exist some positive constants

$$C_1 > 0, \quad C_2 > 0$$

depending on the target $(\mathbb{R} \times M, g)$ such that

$$|Q| + |Q^{-1}| \leq C_1 \quad \text{a.e. in } D \tag{2.23}$$

and

$$| - H | \leq C_2 \quad \text{a.e. in } D \tag{2.24}$$

with (2.24) being deduced from (1.3). Taking

$$\Lambda = 3 + C_1 + C_2 > 0$$

from (2.23)–(2.24), we have (2.14).

By (2.17)–(2.18) and (2.22), there exists positive constant $C_3 > 0$ depending on the target $(\mathbb{R} \times M, g)$ such that

$$\begin{aligned} & \|\Theta\|_{L^2(D)} + \sum_{K=1}^2 \|\text{curl } \xi_K\|_{L^2(D)} + \|\nabla Q\|_{L^2(D)} \\ & + \sum_{K=1}^2 \|\nabla F_K\|_{L^2(D)} + \sum_{K=1}^2 \|\nabla G_K\|_{L^2(D)} \leq C_3 \|\nabla U\|_{L^2(D)}. \end{aligned} \tag{2.25}$$

Then

$$\begin{aligned} & \|\nabla U\|_{L^2(D)} + \|\Theta\|_{L^2(D)} + \sum_{K=1}^2 \|\text{curl } \xi_K\|_{L^2(D)} + \|\nabla Q\|_{L^2(D)} \\ & + \sum_{K=1}^2 \|\nabla F_K\|_{L^2(D)} + \sum_{K=1}^2 \|\nabla G_K\|_{L^2(D)} \leq (1 + C_3) \|\nabla U\|_{L^2(D)}. \end{aligned} \tag{2.26}$$

If $\|\nabla U\|_{L^2(D)} \leq \varepsilon_1$ for some $\varepsilon_1 > 0$, and we take

$$\varepsilon(\Lambda) = (1 + C_3)\varepsilon_1,$$

where $\Lambda > 0$ depends on $(\mathbb{R} \times M, g)$, then (2.26) yields (2.13).

This completes the proof.

Now we recall the following regularity theorem for such elliptic systems, which is built upon the substantial works developed for a regularity theory for critical elliptic systems with an L^2 potential, e.g. [6, 17, 23–30, 32–33].

Theorem 2.2 (cf. [33, Theorem 1.2]) (for $K = 1$) *Let $M(n + 1)$ denote the set of $(n + 1) \times (n + 1)$ real matrices. For any constant $\Lambda > 0$, there exists $\varepsilon(\Lambda) > 0$, such that for any $\Theta \in L^2(D, so(n + 1) \otimes \wedge^1 \mathbb{R}^2)$, $\xi_K \in W^{1,2}(D, M(n + 1) \otimes \wedge^2 \mathbb{R}^2)$, $F_K, G_K \in W^{1,2} \cap L^\infty(D, M(n + 1))$ ($K = 1, 2$), $Q \in W^{1,2} \cap L^\infty(D, GL(n + 1))$ and for any weak solution $U \in W^{1,2}(D, \mathbb{R}^{n+1})$ of the following system*

$$-\text{div}(Q\nabla U) = \Theta \cdot Q\nabla U + \sum_{K=1}^2 F_K \text{curl } \xi_K \cdot G_K \nabla U \tag{2.27}$$

with the coefficients satisfying the following conditions

$$\begin{aligned} & \|\nabla U\|_{L^2(D)} + \|\Theta\|_{L^2(D)} + \sum_{K=1}^2 \|\operatorname{curl} \xi_K\|_{L^2(D)} + \|\nabla Q\|_{L^2(D)} \\ & + \sum_{K=1}^2 \|\nabla F_K\|_{L^2(D)} + \sum_{K=1}^2 \|\nabla G_K\|_{L^2(D)} \leq \varepsilon(\Lambda) \end{aligned} \tag{2.28}$$

and

$$|Q| + |Q^{-1}| + \sum_{K=1}^2 |F_K| + \sum_{K=1}^2 |G_K| \leq \Lambda \quad \text{a.e. in } D, \tag{2.29}$$

we have that U is Hölder continuous in D .

Compared to the statement in [33, Theorem 1.2], here in Theorem 2.2, the dimension of the domain is 2 and there are two terms in the second part of the right-hand side of the system (2.27), namely,

$$\sum_{K=1}^2 F_K \operatorname{curl} \xi_K \cdot G_K \nabla U.$$

It is easy to see that Theorem 2.2 can be proved by slightly adapting the arguments in the proof of [33, Theorem 1.2].

Proof of Theorem 1.1 By utilizing Theorem 2.2 on the elliptic system (2.12), we can deduce the conclusion.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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