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The General Theory of $\mathcal{S}_T$-Graded Modules and Kernel Method*

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Abstract In this paper, the authors develop the theory of $\mathcal{S}_T$-graded modules to study some kinds of graded structures on reducing subspaces. They introduce some concepts and find an effective way to study $\mathcal{S}_T$-graded modules, which they name the kernel method. For its application, they define standard models for some multiplication operators and special Toeplitz operators, and show that these standard models can be solved by the kernel method. The authors also generalize the kernel method to arbitrary Hilbert spaces without any $\mathcal{S}_T$-graded condition. Finally, for any bounded operator T, they can find a reducing subspace that is $\mathcal{S}_T$-graded.

Keywords Reducing subspace, Graded structure, Multiplication operator, Toeplitz operator
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1 Introduction

Many results of reducing subspaces of multiplication operators with the symbol $z + w$ on various function Hilbert spaces such as weighted Hardy and weighted Bergman spaces are well known, and then Toeplitz type operators with symbols $z + \overline{w}$ and $z\overline{w}$ are also studied. For a list of classical and more recent results, we refer to [1–3, 6, 13–17, 22], for more general Toeplitz operators see [4–5, 7–12, 20–21, 23]. In [14], we have introduced a unified framework for the graded structure induced by operators. Then we successfully characterize the unitary equivalence among the reducing subspaces of M_{z^k} clearly, which was not mentioned in [19] and it turns out that a multiplication operator induced by a much wider class of polynomials always has a minimal reducing subspace.

The purpose of this paper is to present our new progress on the theory of $\mathcal{S}_T$-graded modules. We will make more precise statements on some key concepts and reformulate many propositions more clearly. We then present some more streamlined proofs for classical results. The general results about reducing subspaces of multiplication operators with the symbol $z+w$ in abstract Hilbert spaces can then be treated directly.

This paper is organized as follows. In Section 2 we give some notions about $\mathcal{S}$-calculus and state some important propositions without any graded conditions. Such concepts as $\mathcal{S}_T$-strict, $\mathcal{S}_T$-regular and $\mathcal{S}_T$-codimension one property are discussed. Then the theory of $\mathcal{S}_T$-graded modules is entirely reformulated in Section 3. We simplify this theory by using $\mathcal{S}_T$-grading

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to define $\mathcal{S}_T$-graded modules. The role of boundedness and stability in $\mathcal{S}_T$-grading is quickly surveyed. We find that a bounded $\mathcal{S}_T$-graded module can be decomposed as a direct sum of stable $\mathcal{S}_T$-graded submodules. We also make a generalization about isomorphism between $\mathcal{S}_T$-graded modules. Then a main theorem about $\mathcal{S}$-graded isomorphism is presented. The two sections establish the elementary part of our work.

Section 4 starts with a new method. We define the invariant $\ker \mathcal{S}_T^n$ and relate it to the von Neumann algebra $\mathcal{V}^*(T) := \{T, T^*\}'$. We point out that the kernel can be considered as another description for an $\mathcal{S}_T$-grading.

In Section 5, we show that this method is efficient in many examples, and these examples will be called standard models. As applications, we give a new and quick proof about the unitarily equivalence of unilateral weighted shifts. We investigate deeply the general structure of the reducing subspaces of the multiplication operator M_{z+w} on any analytic function subspaces. We also introduce the general Toeplitz operator with the symbol $z + \overline{w}$ as a kind of standard models. Finally, we explain why we need to define these standard models.

As a by-product of the kernel method, we conclude that the kernel method can be working on any bounded operator at the end of Section 4. Then we conclude that any bounded operator T has a reducing subspace which is a stable $\mathcal{S}_T$-graded module. Furthermore, we have a necessary and sufficient condition that a bounded operator T can make the Hilbert space H be bounded below $\mathcal{S}_T$-graded if and only if H is $\mathcal{S}_T$-regular!

2 The $\mathcal{S}$-Calculus

Throughout this paper, we will use $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ to denote the set of integers, non-negative integers and positive integers, respectively.

Let H be a Hilbert space and $\mathcal{B}(H)$ be the set of bounded linear operators on H. Suppose that $T \in \mathcal{B}(H)$. Let us begin with the following definition.

Definition 2.1 *By the $\mathcal{S}$-calculus for T, we mean*

$$\mathcal{S}_T^n := \Big\{ \prod_{k=1}^{N} T^{*j_k} T^{i_k}, N \in \mathbb{N}, i_k, j_k \in \mathbb{Z}_+ : \sum_{k=1}^{N} (i_k - j_k) = n \Big\}$$

for each $n \in \mathbb{Z}$. If $n = 0$, then we make the convention that $I \in \mathcal{S}_T^0$.

For any $F \subseteq H$ and $\mathcal{A} \subseteq \mathcal{B}(H)$, denote by $[F]$ the closed subspace spanned by F. Define

$$\mathcal{A}F := [\{Af : A \in \mathcal{A}, f \in F\}].$$

If $F = \{f\}$, then we write $\mathcal{A}f$ for $\mathcal{A}\{f\}$. Note that if $T \in \mathcal{B}(H)$, then $TF \subseteq \{T\}F$, and the equality holds if and only if TF is a closed subspace.

We record an important proposition about $\mathcal{S}$-calculus as follows (see [14, Proposition 2.2]).

Proposition 2.1 *For the $\mathcal{S}$-calculus for $T \in \mathcal{B}(H)$, we have the following results:*

(i) *$\mathcal{S}_T^0$ is a monoid (i.e., a semigroup with a unit).*

(ii) *$(\mathcal{S}_T^n)^* = \mathcal{S}_T^{-n}$ for any $n \in \mathbb{Z}$.*

(iii) *$\mathcal{S}_T^m \mathcal{S}_T^n \subseteq \mathcal{S}_T^{m+n}$ for any $m, n \in \mathbb{Z}$.*

(iv) *If $mn \geq 0$, then $\mathcal{S}_T^m \mathcal{S}_T^n = \mathcal{S}_T^{m+n}$.*

(v) *If $F \subseteq H$, then $(\mathcal{S}_T^m \mathcal{S}_T^n)F = \mathcal{S}_T^m(\mathcal{S}_T^n F) \subseteq \mathcal{S}_T^{m+n} F$ for any $m, n \in \mathbb{Z}$.*

Then we will inspect the relationship between reducing subspaces and $\mathcal{S}$-calculus. A closed subspace $M \subseteq H$ is called a reducing subspace of T if $TM \subseteq M$ and $T^*M \subseteq M$. It is easy to see that M is a reducing subspace of T if and only if $\mathcal{S}_T^n M \subseteq M$ for all $n \in \mathbb{Z}$. Therefore the reducing subspace of T is referred to as an $\mathcal{S}_T$-module. If $F \subseteq H$ is a subset, then the reducing subspace of T generated by F is denoted by $[F]_T$ and it is equal to $[\bigvee_{n\in\mathbb{Z}} \mathcal{S}_T^n F]$. If all $\mathcal{S}_T^n F$ are orthogonal to each other, then $[F]_T = \bigoplus_{n\in\mathbb{Z}} \mathcal{S}_T^n F$, and we have $\mathcal{S}_T^m(\mathcal{S}_T^n F) \subseteq \mathcal{S}_T^{m+n} F$ for all $m, n \in \mathbb{Z}$. This is the main motivation for the definition of $\mathcal{S}_T$-grading.

Some new notions are introduced as follows.

Definition 2.2 *Let M be a closed subspace of H.*

(a) *If for all nonzero closed subspaces $K \subseteq M$, $\mathcal{S}_T^1 K \neq K$, then M is said to be $\mathcal{S}_T$-strict.*

(b) *If there exists $k \in \mathbb{Z}$ such that $\bigcap_{n\geq k} \mathcal{S}_T^n M = 0$, then M is said to be $\mathcal{S}_T$-regular at k.*

(c) *If M reduces T and $\dim M \ominus \mathcal{S}_T^1 M = 1$, then M is said to have the $\mathcal{S}_T$-codimension one property.*

A nonzero reducing subspace M of T is called minimal if each nonzero reducing subspace $N \subseteq M$ of T is equal to M. If H is minimal, then T is called irreducible.

Proposition 2.2 *Let $T \in \mathcal{B}(H)$, M be a closed subspace of H.*

(i) *If M is $\mathcal{S}_T$-regular at some nonnegative integer, then M is $\mathcal{S}_T$-strict.*

(ii) *If M is an $\mathcal{S}_T$-strict reducing subspace of T and has $\mathcal{S}_T$-codimension one property, then M is minimal.*

Proof (i) If M is not $\mathcal{S}_T$-strict, then there is a nonzero closed subspace $K \subseteq M$ such that $\mathcal{S}_T^1 K = K$. It results that for any $k \geq 1$, $\mathcal{S}_T^k K = K$. Of course $\mathcal{S}_T^0 K \supseteq K$. Thus for any $k \geq 0$, $\bigcap_{n\geq k} \mathcal{S}_T^n M \supseteq \bigcap_{n\geq k} \mathcal{S}_T^n K = K \neq 0$ and M is not $\mathcal{S}_T$-regular at any nonnegative integer.

(ii) Suppose that M is an $\mathcal{S}_T$-strict reducing subspace of T, and $\dim M \ominus \mathcal{S}_T^1 M = 1$. Let $K \subseteq M$ be any nonzero reducing subspace of T and $L = M \ominus K$. It is easy to check that $\dim M \ominus \mathcal{S}_T^1 M = \dim K \ominus \mathcal{S}_T^1 K + \dim L \ominus \mathcal{S}_T^1 L$. Since M is $\mathcal{S}_T$-strict, we must have $\dim L \ominus \mathcal{S}_T^1 L = 0$. Then $L = \mathcal{S}_T^1 L$. By M's $\mathcal{S}_T$-strict property again, $L = 0$. Thus $M = K$ and M is minimal.

If M is $\mathcal{S}_T$-regular at any nonnegative integer, then M is $\mathcal{S}_T$-regular at any integer. In this case, M is called $\mathcal{S}_T$-regular.

3 $\mathcal{S}_T$-Graded Module

Definition 3.1 *Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. An $\mathcal{S}_T$-grading of H induced by T is a family of closed subspaces $H_n \subseteq H$ such that*

(a) $H = \bigoplus_{n=-\infty}^{+\infty} H_n$;

(b) $\mathcal{S}_T^m H_n \subseteq H_{n+m}$ *for any $m, n \in \mathbb{Z}$, where $\mathcal{S}_T$ is the $\mathcal{S}$-calculus for T.*

H is called an $\mathcal{S}_T$-graded module if H is equipped with an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$, and then $H = \bigoplus_{n=-\infty}^{+\infty} H_n$ is called the $\mathcal{S}_T$-decomposition of H.

Fix an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$ of H, we can define

$$n_l(H) := \inf\{n : H_n \neq 0\}, \quad n_r(H) := \sup\{n : H_n \neq 0\}.$$

Then we will introduce some concepts that were originated in [14].

For boundedness, we say that an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below if $n_l(H)$ is finite; it is said to be bounded above if $n_r(H)$ is finite; it is said to be bounded if it is bounded below and above; and it is said to be unbounded if $n_l(H) = -\infty$ and $n_r(H) = +\infty$.

For stability, we say that an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$ is stable if $\mathcal{S}_T^m H_n = H_{n+m}$ for all $n \geq n_l(H)$ and $m \geq 0$; it is said to be reverse stable if $\mathcal{S}_T^m H_n = H_{n+m}$ for all $n \leq n_r(H)$ and $m \leq 0$; and it is said to be double stable if it is stable and reverse stable.

Remark 3.1 When no confusion will arise, we say that H is bounded or stable in some sense when its $\mathcal{S}_T$-grading is specified.

Proposition 3.1 *If H has a bounded below $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$, then H is $\mathcal{S}_T$-regular.*

Proof Taking $n_0 = n_l(H)$ and any integer $k \geq 0$, and then

$$\bigcap_{n\geq k} \mathcal{S}_T^n H = \bigcap_{n\geq k} \bigoplus_{m\in\mathbb{Z}_+} \mathcal{S}_T^n H_{n_0+m} \subseteq \bigcap_{n\geq k} \bigoplus_{m\in\mathbb{Z}_+} H_{n_0+n+m} = 0.$$

Of course, H is $\mathcal{S}_T$-regular.

By an $\mathcal{S}_T$-submodule M of H, we refer to that M is a reducing subspace of T. Note that if M is an $\mathcal{S}_T$-submodule of H, then $H \ominus M$ is an $\mathcal{S}_T$-submodule too.

Proposition 3.2 *Suppose that H has an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$. If M is an $\mathcal{S}_T$-submodule of H, then the following statements are equivalent:*

(i) *For any $m \in M$, if $m = \sum_{n\in\mathbb{Z}} m_n$ such that $m_n \in H_n$, $n \in \mathbb{Z}$, then $m_n \in M$;*

(ii) *$\{M_n := H_n \cap M\}_{n\in\mathbb{Z}}$ is an $\mathcal{S}_T$-grading of M;*

(iii) *$\{\widetilde{M_n} := H_n \ominus M\}_{n\in\mathbb{Z}}$ is an $\mathcal{S}_T$-grading of $H \ominus M$.*

Of course, $H_n \ominus M = H_n \ominus M_n$.

We leave its proof to the reader and introduce the following definitions immediately.

Definition 3.2 *Suppose that H has an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$ and M is an $\mathcal{S}_T$-submodule of H. If M has one of the above properties, then we call M an $\mathcal{S}_T$-graded submodule of H and $\{M_n = H_n \cap M\}_{n\in\mathbb{Z}}$ the $\mathcal{S}_T$-subgrading.*

Definition 3.3 *Suppose that H has an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$, $F \subseteq H$. We say that F is $\mathcal{S}_T$-homogeneous, if for any $f \in F$, $f = \sum_{n\in\mathbb{Z}} f_n$, $f_n \in H_n$, $n \in \mathbb{Z}$, then $f_n \in F$. That is, F is $\mathcal{S}_T$-homogeneous if and only if $F \subseteq \sum_{n\in\mathbb{Z}} (F \cap H_n)$.*

Roughly speaking, an $\mathcal{S}_T$-submodule has an $\mathcal{S}_T$-subgrading if and only if it is $\mathcal{S}_T$-homogeneous. And for an $\mathcal{S}_T$-homogeneous subset we can make an $\mathcal{S}_T$-graded submodule from it in a natural way.

Proposition 3.3 *Suppose that H has an $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$, and $F \subseteq H$ is $\mathcal{S}_T$-homogeneous, then $[F]_T$ is an $\mathcal{S}_T$-graded submodule of H. Moreover, if there exists $k \in \mathbb{Z}$*

such that $F \subseteq H_k$ and $\mathcal{S}_T^{-1}F = 0$, then $[F]_T$ is stable. If there exists $l \in \mathbb{Z}$ such that $F \subseteq H_l$ and $\mathcal{S}_T^1 F = 0$, then $[F]_T$ is reverse stable.

Proof

$$[F]_T = \bigvee_{n\in\mathbb{Z}} \mathcal{S}_T^n F = \bigvee_{n\in\mathbb{Z}} \mathcal{S}_T^n \sum_{m\in\mathbb{Z}} (F \cap H_m) = \bigvee_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \mathcal{S}_T^n (F \cap H_m) = \bigoplus_{n\in\mathbb{Z}} \Big(\sum_{m\in\mathbb{Z}} \mathcal{S}_T^{n-m}(F \cap H_m)\Big).$$

If there exists $k \in \mathbb{Z}$ such that $F \subseteq H_k$ and $\mathcal{S}_T^{-1}F = 0$, then

$$[F]_T = \mathcal{S}_T^0 F \oplus \mathcal{S}_T^1 F \oplus \cdots .$$

But for any $m \geq 0$, $\mathcal{S}_T^m \mathcal{S}_T^i F = \mathcal{S}_T^{i+m} F$ for all $i \geq 0$, thus $[F]_T$ is stable.

If there exists $l \in \mathbb{Z}$ such that $F \subseteq H_l$ and $\mathcal{S}_T^1 F = 0$, then

$$[F]_T = \cdots \oplus \mathcal{S}_T^{-1} F \oplus \mathcal{S}_T^0 F.$$

But for $m \leq 0$, $\mathcal{S}_T^m \mathcal{S}_T^i F = \mathcal{S}_T^{i+m} F$ for all $i \leq 0$, thus $[F]_T$ is reverse stable.

For duality, we consider the adjoint T^*. Then we have $\mathcal{S}_{T^*}^n = \mathcal{S}_T^{-n}$ for each $n \in \mathbb{Z}$. Put $H_n^* = H_{-n}$ for each $n \in \mathbb{Z}$. Then the family $\{H_n^*\}_{n\in\mathbb{Z}}$ is an $\mathcal{S}_{T^*}$-grading of H induced by T^*. Of course, the $\mathcal{S}_T$-submodules and the $\mathcal{S}_{T^*}$-submodules are the same things. We have the following proposition for the duality.

Proposition 3.4 (i) *$n_l(H) = n_r(H^*)$, $n_r(H) = n_l(H^*)$.*

(ii) *The $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below (resp. above) if and only if the $\mathcal{S}_{T^*}$-grading $\{H_n^*\}_{n\in\mathbb{Z}}$ is bounded above (resp. below).*

(iii) *The $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$ is stable (resp. reverse stable) if and only if the $\mathcal{S}_{T^*}$-grading $\{H_n^*\}_{n\in\mathbb{Z}}$ is reverse stable (resp. stable).*

The next theorem says that stable plus bounded below will be of power.

Theorem 3.1 *Let H be an $\mathcal{S}_T$-graded module with $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$. If $\{H_n\}_{n\in\mathbb{Z}}$ is stable, then H has a bounded below and stable $\mathcal{S}_T$-grading if and only if $H \ominus \mathcal{S}_T^1 H \neq 0$. In this case H's stable grading is uniquely determined in the following sense: If $\{K_n\}_{n\in\mathbb{Z}}$ is a stable $\mathcal{S}_T$-grading of H, then there exists an integer n_0 such that $K_n = H_{n+n_0}$, $n \in \mathbb{Z}$. The dual version also holds.*

Proof If $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below, put $n_1 = n_l(H)$. By stability of $\{H_n\}$, $H \ominus \mathcal{S}_T^1 H = H_{n_1} \neq 0$.

Conversely, if $H \ominus \mathcal{S}_T^1 H \neq 0$, we consider $\{H_n\}_{n\in\mathbb{Z}}$. If $\{H_n\}_{n\in\mathbb{Z}}$ is not bounded below, then by stability of $\{H_n\}$, we have $\mathcal{S}_T^1 H = H$. Ihis is a contradiction. Thus $\{H_n\}_{n\in\mathbb{Z}}$ is already bounded below.

Now we suppose that $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below and let $\{K_n\}_{n\in\mathbb{Z}}$ be a stable $\mathcal{S}_T$-grading of H. Then by the same reason, $\{K_n\}_{n\in\mathbb{Z}}$ is bounded below too. Put $n_2 = n_l(K)$, we have $K_{n_2} = H \ominus \mathcal{S}_T^1 H = H_{n_1}$. If we take $n_0 = n_1 - n_2$, then for any $n \geq n_2$, $K_n = K_{n_2+n-n_2} = \mathcal{S}_T^{n-n_2} H \ominus \mathcal{S}_T^{n-n_2+1} H = H_{n_1+n-n_2} = H_{n+n_0}$. On the other hand, if $n < n_2$, then $n + n_0 = n + n_1 - n_2 < n_1$, thus $K_n = 0 = H_{n+n_0}$.

The proof is complete. The dual version is left to the reader.

In Section 4, we will revisit the invariant $H \ominus \mathcal{S}_T^1 H$.

We will say that an $\mathcal{S}_T$-graded submodule M is minimal if for any nonzero $\mathcal{S}_T$-graded submodule $N \subseteq M$, we have $N = M$. There is a very pleasant result about the relation between minimality and stability.

Theorem 3.2 *Every minimal $\mathcal{S}_T$-graded submodule of an $\mathcal{S}_T$-graded module is double stable.*

Proof It was proved that minimality implies stability in [14]. Using duality, since the minimal $\mathcal{S}_T$-graded submodule is the same as the minimal $\mathcal{S}_{T^*}$-graded submodule, it follows immediately that minimality implies reverse stability.

It is natural to ask whether double stability implies minimality? We will consider this problem in Section 4 and see a really interesting example in Section 5.

Proposition 3.5 *Every bounded below $\mathcal{S}_T$-graded module is a direct sum of stable $\mathcal{S}_T$-graded submodules. Every bounded above $\mathcal{S}_T$-graded module is a direct sum of reverse stable $\mathcal{S}_T$-graded submodules.*

Proof Let H be a bounded below $\mathcal{S}_T$-graded module, $\mathcal{F}$ be a collection of $\mathcal{S}_T$-graded submodules that is a direct sum of stable $\mathcal{S}_T$-graded submodules. Then we can use Zorn's lemma to get a maximal element M in $\mathcal{F}$. If $M \neq H$, then the orthogonal complement $N = H \ominus M$ is still bounded below with the $\mathcal{S}_T$-decomposition

$$N = \bigoplus_{n=n_0}^{\infty} N_n.$$

Taking $h \in N_{n_0}$, we have

$$[h]_T = \bigoplus_{n=-\infty}^{+\infty} \mathcal{S}^n h.$$

Since $S_T^{-1} N_{n_0} \subseteq N_{n_0-1} = 0$, we have

$$[h]_T = \mathcal{S}^0 h \oplus \mathcal{S}^1 h \oplus \cdots .$$

Then $[h]_T$ is surely stable, since $\mathcal{S}_T^n \mathcal{S}_T^m = \mathcal{S}_T^{n+m}$ if $nm \geq 0$. Then $M \oplus [h]_T \in \mathcal{F}$, which contradicts that M is a maximal element. Hence $M = H$ and H is a direct sum of stable $\mathcal{S}_T$-graded submodules. For the case of bounded above, we use duality.

We will give an alternative proof of this proposition after Theorem 4.1 without using Zorn's lemma.

In categorical viewpoint, the concept of stability is more important than that of subobjects. However, it has its shortcoming. For example, the direct sum of stable $\mathcal{S}_T$-graded submodules may not be stable. However, using the stable $\mathcal{S}_T$-graded submodules as basic bricks is nice enough. We will investigate more on stability in further detail.

Next, we consider the isomorphism theory in $\mathcal{S}_T$-graded modules. In operator theory of Hilbert spaces, the natural isomorphism is the unitary equivalence. Suppose that H and K are Hilbert spaces, $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$. We call A and B are unitary equivalent, if there exists a unitary operator $U \in \mathcal{B}(H, K)$ such that $UA = BU$. Then it is easy to see that for all $n \in \mathbb{Z}$, $U\mathcal{S}_A^n = \mathcal{S}_B^n U$, and U preserves the properties for $\mathcal{S}_T$-strict and $\mathcal{S}_T$-regular. In particularly, if M and N are reducing subspaces of $T \in \mathcal{B}(H)$, and there is a unitary operator $U \in \mathcal{B}(M, N)$ such that $UT = TU$, then we call M and N unitary equivalent. In this case, M and N are unitary equivalent if and only if there exists a partial isometry $P \in \mathcal{B}(H)$ such that M is its initial space and N is its final space and $PT = TP$. In other words, if we let P_M be the orthogonal projection with image M and P_N be the orthogonal projection with image N, then P_M and P_N are equivalent in the von Neumann algebra $\mathcal{V}^*(T) = \{T, T^*\}'$.

From now on, if M is an $\mathcal{S}_A$-module and N are an $\mathcal{S}_B$-module, then we call that M and N is $\mathcal{S}$-isomorphic if A and B are unitary equivalent, i.e., there exists a unitary operator $U \in \mathcal{B}(M, N)$ such that $UA = BU$, and we call U an $\mathcal{S}$-isomorphism between M and N. Then we can define the concept of $\mathcal{S}$-graded isomorphism for $\mathcal{S}$-graded modules. For the sake of simplicity, we note that the case when M and N are both zero is trivial by taking $U = 0$.

Definition 3.4 *Let M be an $\mathcal{S}_A$-graded module with $\mathcal{S}_A$-grading $\{M_n\}_{n\in\mathbb{Z}}$, N be an $\mathcal{S}_B$-graded module with $\mathcal{S}_B$-grading $\{N_n\}_{n\in\mathbb{Z}}$. M and N are said to be $\mathcal{S}$-graded isomorphic, if there exists an integer n_0 and for all integers n, a unitary operator $U_n \in \mathcal{B}(M_n, N_{n+n_0})$ such that $B|_{N_{n+n_0}} U_n = U_{n+1} A|_{M_n}$. Then $\{U_n\}_{n\in\mathbb{Z}}$ is called an $\mathcal{S}$-graded isomorphism between $\{M_n\}_{n\in\mathbb{Z}}$ and $\{N_n\}_{n\in\mathbb{Z}}$.*

It is not hard to see that an $\mathcal{S}$-graded isomorphism preserves boundedness and stability. After a long trip, we can state one of our main results now.

Theorem 3.3 *Let M be an $\mathcal{S}_A$-graded module with $\mathcal{S}_A$-grading $\{M_n\}_{n\in\mathbb{Z}}$, N be an $\mathcal{S}_B$-graded module with $\mathcal{S}_B$-grading $\{N_n\}_{n\in\mathbb{Z}}$.*

(i) *If $\{U_n\}_{n\in\mathbb{Z}}$ is an $\mathcal{S}$-graded isomorphism between $\{M_n\}_{n\in\mathbb{Z}}$ and $\{N_n\}_{n\in\mathbb{Z}}$, then $\bigoplus_{n\in\mathbb{Z}} U_n$ is an $\mathcal{S}$-isomorphism between M and N.*

(ii) *If $\{M_n\}_{n\in\mathbb{Z}}$ is stable and bounded below, $\{N_n\}_{n\in\mathbb{Z}}$ is stable and bounded below, then for any $\mathcal{S}$-isomorphism U between M and N, there exists an $\mathcal{S}$-graded isomorphism $\{U_n\}_{n\in\mathbb{Z}}$ between $\{M_n\}_{n\in\mathbb{Z}}$ and $\{N_n\}_{n\in\mathbb{Z}}$ such that $U = \bigoplus_{n\in\mathbb{Z}} U_n$.*

(iii) *If $\{M_n\}_{n\in\mathbb{Z}}$ is reverse stable and bounded above, $\{N_n\}_{n\in\mathbb{Z}}$ is reverse stable and bounded above, then for any $\mathcal{S}$-isomorphism U between M and N, there exists an $\mathcal{S}$-graded isomorphism $\{U_n\}_{n\in\mathbb{Z}}$ between $\{M_n\}_{n\in\mathbb{Z}}$ and $\{N_n\}_{n\in\mathbb{Z}}$ such that $U = \bigoplus_{n\in\mathbb{Z}} U_n$.*

Proof (i) This can be proved directly.

(ii) If U is an $\mathcal{S}$-isomorphism between M and N, then $\{UM_n\}_{n\in\mathbb{Z}}$ is a stable and bounded below $\mathcal{S}_B$-grading of N. By Theorem 3.1, there exists an integer n_0 such that $UM_n = N_{n+n_0}$, $n \in \mathbb{Z}$. Take $U_n = U|_{M_n}$, then $U_n \in \mathcal{B}(M_n, N_{n+n_0})$ and U_n is unitary. Since $UA = BU$, $\{U_n\}_{n\in\mathbb{Z}}$ is an $\mathcal{S}$-graded isomorphism between $\{M_n\}_{n\in\mathbb{Z}}$ and $\{N_n\}_{n\in\mathbb{Z}}$, and $U = \bigoplus_{n\in\mathbb{Z}} U_n$.

(iii) Use duality.

Roughly speaking, every $\mathcal{S}$-graded isomorphism is an $\mathcal{S}$-isomorphism and under some conditions every $\mathcal{S}$-isomorphism is an $\mathcal{S}$-graded isomorphism.

4 Kernel Method

Fix an $\mathcal{S}_T$-graded module H and its $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$. For all $m, n \in \mathbb{Z}$, we define the m-th kernel of $\mathcal{S}_T^n$ by

$$\ker_m \mathcal{S}_T^n := \{h \in H_m : \mathcal{S}_T^n h = 0\},$$

and the kernel of $\mathcal{S}_T^n$ by

$$\ker \mathcal{S}_T^n := \{h \in H : \mathcal{S}_T^n h = 0\}.$$

Of course, we have $\ker \mathcal{S}_T^n \cap H_m = \ker_m \mathcal{S}_T^n$. First, we have the following key proposition.

Proposition 4.1 (i) *For the* m-th *kernel* $\ker_m \mathcal{S}_T^n$, *we have*

$$\ker_m \mathcal{S}_T^n = H_m \ominus \mathcal{S}_T^{-n} H_{m+n}.$$

Hence $\ker_m \mathcal{S}_T^n$ *is a closed subspace of* H_m.

(ii) *For the kernel* $\ker \mathcal{S}_T^n$, *we have*

$$\ker \mathcal{S}_T^n = H \ominus \mathcal{S}_T^{-n} H = \bigoplus_{m=-\infty}^{+\infty} \ker_m \mathcal{S}_T^n.$$

Hence $\ker \mathcal{S}_T^n$ *is a closed subspace of* H.

Proof For $h \in H_m$,

$$\begin{aligned} h \in H_m \ominus \mathcal{S}_T^{-n} H_{m+n} &\Leftrightarrow \langle h, Af\rangle = 0, \quad \forall A \in \mathcal{S}_T^{-n},\ \forall f \in H_{m+n} \\ &\Leftrightarrow \langle A^* h, f\rangle = 0, \quad \forall A \in \mathcal{S}_T^{-n},\ \forall f \in H_{m+n} \\ &\Leftrightarrow \langle Ah, f\rangle = 0, \quad \forall A \in \mathcal{S}_T^{n},\ \forall f \in H_{m+n} \\ &\Leftrightarrow Ah = 0, \quad \forall A \in \mathcal{S}_T^{n} \\ &\Leftrightarrow h \in \ker_m \mathcal{S}_T^n. \end{aligned}$$

This proves (i).

For (ii), we still have

$$\ker \mathcal{S}_T^n = H \ominus \mathcal{S}_T^{-n} H = \bigoplus_{m=-\infty}^{+\infty} (H_m \ominus \mathcal{S}_T^{-n} H_{m+n}) = \bigoplus_{m=-\infty}^{+\infty} \ker_m \mathcal{S}_T^n.$$

We point out that in fact $\ker \mathcal{S}_T^n$ can be defined for general T, and $\ker \mathcal{S}_T^n = H \ominus \mathcal{S}_T^{-n} H$ is still true.

Although $\ker \mathcal{S}_T^n$ may not be an $\mathcal{S}_T$-submodule of H, it is $\mathcal{S}_T$-homogeneous. We state some results about kernels as follows, and omit their proofs.

Remark 4.1 Suppose that H is an $\mathcal{S}_T$-graded module with $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$.

(i) $\ker \mathcal{S}_T^0 = 0$.

(ii) For all $n \in \mathbb{N}$, $\ker S_T^n \subseteq \ker T^n$, $\ker S_T^{-n} \subseteq \ker(T^*)^n$.

(iii) For all $m, n, k \in \mathbb{Z}$, $\mathcal{S}_T^k \ker_m \mathcal{S}_T^n \subseteq \ker_{m+k} \mathcal{S}_T^{n-k}$. Moreover, $\mathcal{S}_T^0 \ker_m \mathcal{S}_T^{-1} = \ker_m \mathcal{S}_T^{-1}$.

(iv) For all $m, n \in \mathbb{Z}$, $\mathcal{S}_T^m \ker \mathcal{S}_T^n \subseteq \ker \mathcal{S}_T^{n-m}$. Moreover, $\mathcal{S}_T^0 \ker \mathcal{S}_T^{-1} = \ker \mathcal{S}_T^{-1}$.

(v) For all $m, n \in \mathbb{Z}$, if $mn > 0$ and $|m| > |n|$, then $\ker \mathcal{S}_T^n \subseteq \ker \mathcal{S}_T^m$. In other words, we have

$$\cdots \supseteq \ker \mathcal{S}_T^{-2} \supseteq \ker \mathcal{S}_T^{-1} \supseteq \ker \mathcal{S}_T^0 = 0 \subseteq \ker \mathcal{S}_T^1 \subseteq \ker \mathcal{S}_T^2 \subseteq \cdots.$$

(vi) If $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below, then $\{H_n\}_{n\in\mathbb{Z}}$ is stable if and only if $\ker \mathcal{S}_T^{-1} = H_{n_l(H)}$.

(vii) If $\{H_n\}_{n\in\mathbb{Z}}$ is not bounded below, then $\{H_n\}_{n\in\mathbb{Z}}$ is stable if and only if $\ker \mathcal{S}_T^{-1} = 0$.

(viii) If $\{H_n\}_{n\in\mathbb{Z}}$ is bounded above, then $\{H_n\}_{n\in\mathbb{Z}}$ is reverse stable if and only if $\ker \mathcal{S}_T^1 = H_{n_r(H)}$.

(ix) If $\{H_n\}_{n\in\mathbb{Z}}$ is not bounded above, then $\{H_n\}_{n\in\mathbb{Z}}$ is reverse stable if and only if $\ker \mathcal{S}_T^1 = 0$.

(x) If $\{H_n\}_{n\in\mathbb{Z}}$ is unbounded, then $\{H_n\}_{n\in\mathbb{Z}}$ is double stable if and only if $\ker \mathcal{S}_T^1 = \ker \mathcal{S}_T^{-1} = 0$.

(xi) If $\{H_n\}_{n\in\mathbb{Z}}$ is stable, then $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below if and only if $\ker \mathcal{S}_T^{-1} \neq 0$.

(xii) If $\{H_n\}_{n\in\mathbb{Z}}$ is reverse stable, then $\{H_n\}_{n\in\mathbb{Z}}$ is bounded above if and only if $\ker \mathcal{S}_T^1 \neq 0$.

(xiii) $\ker \mathcal{S}_T^{-1} = 0$ if and only if $\{H_n\}_{n\in\mathbb{Z}}$ is both stable and not bounded below.

(xiv) $\ker \mathcal{S}_T^1 = 0$ if and only if $\{H_n\}_{n\in\mathbb{Z}}$ is both reverse stable and not bounded above.

(xv) $\ker \mathcal{S}_T^{-1} = \ker \mathcal{S}_T^1 = 0$ if and only if $\{H_n\}_{n\in\mathbb{Z}}$ is double stable and unbounded.

In some sense, we may think of the kernel as another description of an $\mathcal{S}_T$-grading. It is astonishing that we can quickly get some information about the $\mathcal{S}_T$-grading through the T directly.

Corollary 4.1 *If T is injective, then $\{H_n\}_{n\in\mathbb{Z}}$ is reverse stable and not bounded above. If T^* is injective, then $\{H_n\}_{n\in\mathbb{Z}}$ is stable and not bounded below. Thus if both T and T^* are injective, then $\{H_n\}_{n\in\mathbb{Z}}$ is double stable and unbounded.*

Proof Note that $\ker \mathcal{S}_T^1 \subseteq \ker T$. Then the proof is immediate.

Let us begin to see a very trivial example. If $T = 0$, then of course H is an $\mathcal{S}_T$-graded module by setting $H_0 = H$ and $H_n = 0$ if $n \neq 0$. Then H is double stable, bounded, and $\ker \mathcal{S}_T^1 = \ker \mathcal{S}_T^{-1} = H$. This example is not useless. Firstly, it indicates that double stable does not imply $\mathcal{S}_T$-minimal unconditionally. Secondly, for general $T \in \mathcal{B}(H)$, it is easy to see that $\ker T \cap \ker T^* = \ker \mathcal{S}_T^1 \cap \ker \mathcal{S}_T^{-1}$. If H is an $\mathcal{S}_T$-graded module, then set $N = \ker T \cap \ker T^*$, we will understand that N is an $\mathcal{S}_T$-graded submodule. Moreover, $\widetilde{H} := H \ominus N$ will be an $\mathcal{S}_T$-graded module that

$$\widetilde{H}_n = H_n \ominus (\ker_n \mathcal{S}_T^1 \cap \ker_n \mathcal{S}_T^{-1}) = [\mathcal{S}_T^1 H_{n-1} + \mathcal{S}_T^{-1} H_{n+1}].$$

In this case, set $\widetilde{T} = T|_{\widetilde{H}}$, we must have $\ker \widetilde{T} \cap \ker \widetilde{T}^* = 0$. We call $(\widetilde{H}, \widetilde{T})$ the reduction of (H, T). Then we have proved the following proposition.

Proposition 4.2 *If an $\mathcal{S}_T$-graded module H is reduced, then*

$$H_n = [\mathcal{S}_T^1 H_{n-1} + \mathcal{S}_T^{-1} H_{n+1}]$$

for each $n \in \mathbb{Z}$.

Thus, for general case, we have $H_n = [\mathcal{S}_T^1 H_{n-1} + \mathcal{S}_T^{-1} H_{n+1}] \oplus (\ker_n \mathcal{S}_T^1 \cap \ker_n \mathcal{S}_T^{-1})$. In some sense, this may be the best result about the neighborhood relationship between forward and backward.

Note that in Theorem 3.1 the invariant $H \ominus \mathcal{S}_T^1 H$ is exactly $\ker \mathcal{S}_T^{-1}$. Since the definition of $\ker \mathcal{S}_T^{-1}$ is not dependedent on H's any $\mathcal{S}_T$-grading, we conclude that for an $\mathcal{S}_T$-module H, $\ker \mathcal{S}_T^{-1} = 0$ implies that if H has an $\mathcal{S}_T$-grading, then the $\mathcal{S}_T$-grading must be stable and not bounded below. And $\ker \mathcal{S}_T^{-1} \neq 0$ implies that if H has a stable and bounded below $\mathcal{S}_T$-grading, then H has only one stable and bounded below $\mathcal{S}_T$-grading essentially. We also have the dual result for $\ker \mathcal{S}_T^1$.

The next results show again that the invariant $\ker \mathcal{S}_T^{-1}$ should get more of our attention, whatever graded condition is given. First we need a lemma.

Lemma 4.1 *Suppose that H is a Hilbert space, $H = H_1 \oplus K_1$, $K_1 = H_2 \oplus K_2$, $\cdots$, $K_n = H_{n+1} \oplus K_{n+2}$, $\cdots$, where H_i and K_i are closed subspaces of H for all $i \in \mathbb{N}$.*

(i)

$$H = \Big(\bigcap_{i=1}^{+\infty} K_i\Big) \oplus H_1 \oplus H_2 \oplus \cdots .$$

(ii)

$$(H \ominus K_1) \oplus (K_1 \ominus K_2) \oplus \cdots = H \ominus \Big(\bigcap_{i=1}^{+\infty} K_i\Big).$$

Proof Of course, $H_1 \oplus H_2 \oplus \cdots = (H \ominus K_1) \oplus (K_1 \ominus K_2) \oplus \cdots$, take it as M. All we need to prove is just

$$M^{\perp} = \bigcap_{i=1}^{+\infty} K_i.$$

For any positive integer n, since $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n \oplus K_n$,

$$M^{\perp} \subseteq (H_1 \oplus H_2 \oplus \cdots \oplus H_n)^{\perp} = K_n.$$

Thus $M^{\perp} \subseteq \bigcap\limits_{i=1}^{+\infty} K_i$. For any $g \in \bigcap\limits_{i=1}^{+\infty} K_i$, $g \in K_1$ implies $g \perp H_1$. Again, $g \in K_2$ implies $g \perp H_2$, and so on. Since $g \perp H_n$ for all positive integer n, thus $g \perp M$ and $g \in M^{\perp}$.

Theorem 4.1 *Suppose that an $\mathcal{S}_T$-graded module H has a bounded below $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$.*

(i) *H is generated by $\ker \mathcal{S}_T^{-1}$: $H = [\ker \mathcal{S}_T^{-1}]_T = \bigoplus\limits_{n\in\mathbb{Z}} [\ker_n \mathcal{S}_T^{-1}]_T$.*

(ii) *$\mathcal{V}^*(T)$ is isomorphic to a closed $*$-subalgebra of $\mathcal{B}(\ker \mathcal{S}_T^{-1})$.*

(iii) *If $\dim \ker \mathcal{S}_T^{-1} = 1$, then T is irreducible.*

Proof (i) For any $n \in \mathbb{Z}$,

$$\begin{aligned}\bigoplus_{i\in\mathbb{Z}_+} \mathcal{S}_T^i \ker_{n-i} \mathcal{S}_T^{-1} &= (H_n \ominus \mathcal{S}_T^1 H_{n-1}) \oplus (\mathcal{S}_T^1 H_{n-1} \ominus \mathcal{S}_T^2 H_{n-2}) \oplus \cdots \\ &= H_n \ominus \Big(\bigcap_{i=1}^{\infty} \mathcal{S}_T^i H_{n-i}\Big) \\ &= H_n.\end{aligned}$$

The last equality holds since $\{H_n\}_{n\in\mathbb{Z}}$ is bounded below, thus H is $\mathcal{S}_T$-regular at 1. Therefore for all $n \in \mathbb{Z}$, $H_n \subseteq [\ker \mathcal{S}_T^{-1}]_T$. We get $H = [\ker \mathcal{S}_T^{-1}]_T$. Now

$$H = [\ker \mathcal{S}_T^{-1}]_T = \sum_{n\in\mathbb{Z}} [\ker_n \mathcal{S}_T^{-1}]_T,$$

since $\ker \mathcal{S}_T^{-1}$ is $\mathcal{S}_T$-homogeneous. Moreover, for any $i, j, n, m \in \mathbb{Z}$, $n \neq m$, for all $A \in \mathcal{S}_T^i$, $B \in \mathcal{S}_T^j$, $f \in \ker_n \mathcal{S}_T^{-1}$, $g \in \ker_m \mathcal{S}_T^{-1}$, if $i < j$, then we have

$$\langle Af, Bg\rangle = \langle B^* Af, g\rangle = 0,$$

since $B^* A \in \mathcal{S}_T^{i-j}$, $i - j < 0$ and $f \in \ker \mathcal{S}_T^{-1}$; if $i = j$, then $B^* Af \in \mathcal{S}^0 H_n = H_n$ and $g \in H_m$, we still have $\langle Af, Bg\rangle = 0$; if $i > j$, then

$$\langle Af, Bg\rangle = \langle f, A^* Bg\rangle = 0,$$

since $A^*B \in \mathcal{S}_T^{j-i}$, $j-i<0$ and $g \in \ker \mathcal{S}_T^{-1}$. Therefore we prove that when $n \neq m$, $[\ker_n \mathcal{S}_T^{-1}]_T \perp [\ker_m \mathcal{S}_T^{-1}]_T$, thus $H = [\ker \mathcal{S}_T^{-1}]_T = \bigoplus_{n\in\mathbb{Z}} [\ker_n \mathcal{S}_T^{-1}]_T$.

(ii) If $A \in \mathcal{V}^*(T)$, then $AT = TA$ and $AT^* = T^*A$, we have $\mathcal{S}_T^{-1} A \ker \mathcal{S}_T^{-1} = A\mathcal{S}_T^{-1} \ker \mathcal{S}_T^{-1} = 0$. Since $A^* \in \mathcal{V}^*(T)$, we still have $\mathcal{S}_T^{-1} A^* \ker \mathcal{S}_T^{-1} = 0$. Therefore, A reduces $\ker \mathcal{S}_T^{-1}$, and the restrict map $\Phi\colon A \mapsto A|\ker \mathcal{S}_T^{-1}$ is a $*$-algebraic homomorphism from $\mathcal{V}^*(T)$ to $\mathcal{B}(\ker \mathcal{S}_T^{-1})$. Since $[\ker \mathcal{S}_T^{-1}]_T = H$, Φ is injective.

(iii) If $\dim \ker \mathcal{S}_T^{-1} = 1$, then $\mathcal{B}(\ker \mathcal{S}_T^{-1}) \cong \mathbb{C}$ and Φ must be a $*$-isomorphism. Thus T is irreducible.

It is easy to see that every nonzero $[\ker_n \mathcal{S}_T^{-1}]_T$ is bounded below and stable, thus it gives an alternative proof of Proposition 3.5. Using duality, we have the version for bounded above $\mathcal{S}_T$-graded module.

Now, we give a generalized theorem about $\ker \mathcal{S}_T^{-1}$ without graded condition as follows. Note that $\mathcal{S}_T^0 \ker \mathcal{S}_T^{-1} = \ker \mathcal{S}_T^{-1}$ still holds.

Lemma 4.2 *Let H be a Hilbert space, $T \in \mathcal{B}(H)$, then*

$$H = \Big(\bigcap_{i=1}^{+\infty} \mathcal{S}_T^i H\Big) \oplus \ker \mathcal{S}_T^{-1} \oplus \mathcal{S}_T^1 \ker \mathcal{S}_T^{-1} \oplus \mathcal{S}_T^2 \ker \mathcal{S}_T^{-1} \oplus \cdots,$$

and we have

$$[\ker \mathcal{S}_T^{-1}]_T = \ker \mathcal{S}_T^{-1} \oplus \mathcal{S}_T^1 \ker \mathcal{S}_T^{-1} \oplus \mathcal{S}_T^2 \ker \mathcal{S}_T^{-1} \oplus \cdots.$$

Thus

$$H = \Big(\bigcap_{i=1}^{+\infty} \mathcal{S}_T^i H\Big) \oplus [\ker \mathcal{S}_T^{-1}]_T$$

is a direct sum of two reducing subspaces of T. Moreover, $[\ker \mathcal{S}_T^{-1}]_T$ is an $\mathcal{S}_T$-graded module with $\mathcal{S}_T$-grading $\{\mathcal{S}_T^n \ker \mathcal{S}_T^{-1}\}_{n\in\mathbb{Z}}$ which is bounded below and stable.

Proof Since $H = \mathcal{S}_T^1 H \oplus \ker \mathcal{S}_T^{-1}$, we have $\mathcal{S}_T^1 H = \mathcal{S}_T^2 H \oplus \mathcal{S}_T^1 \ker \mathcal{S}_T^{-1}$, $\mathcal{S}_T^2 H = \mathcal{S}_T^3 H \oplus \mathcal{S}_T^2 \ker \mathcal{S}_T^{-1}$, and so on. Then use Lemma 4.1.

Thus for any general $T \in \mathcal{B}(H)$, there are two $\mathcal{S}_T$-graded reducing subspaces, $[\ker \mathcal{S}_T^{-1}]_T$ and $[\ker \mathcal{S}_T^1]_T = [\ker \mathcal{S}_{T^*}^{-1}]_T$. One is bounded below and stable and the other is bounded above and reverse stable!

Theorem 4.2 *Let H be a Hilbert space, $T \in \mathcal{B}(H)$, H be $\mathcal{S}_T$-regular.*

(i) $H = [\ker \mathcal{S}_T^{-1}]_T$.

(ii) *$\mathcal{V}^*(T)$ is isomorphic to a closed $*$-subalgebra of $\mathcal{B}(\ker \mathcal{S}_T^{-1})$.*

(iii) *If $\dim \ker \mathcal{S}_T^{-1} = 1$, then T is irreducible.*

Proof Note that $\mathcal{S}_T^1 H \supseteq \mathcal{S}_T^2 H \supseteq \cdots$ is a decreasing chain, thus saying H is $\mathcal{S}_T$-regular is equivalent to that saying H is $\mathcal{S}_T$-regular at 1.

In fact, that is to say, if H is $\mathcal{S}_T$-regular, then H can be $\mathcal{S}_T$-graded and even be stable! More precisely, H has a bounded below $\mathcal{S}_T$-grading if and only if H is $\mathcal{S}_T$-regular. Moreover, if H has a bounded below $\mathcal{S}_T$-grading, then H has a stable bounded below $\mathcal{S}_T$-grading which is unique. The reason for this difference in stability is that we do not require $n_l(H)$ to be a fixed number. There are some advantages that we insist for the freedom of index. In summary, the most useful stable $\mathcal{S}_T$-graded module is the only minimal $\mathcal{S}_T$-graded module. Also, we have the dual version. It could warn that $\mathcal{S}_T$-regular needs be redefined in the adjoint situation.

Just out of curiosity, we record a result for normal operators as follows.

Proposition 4.3 *Let H be a Hilbert space, and $T \in \mathcal{B}(H)$ be normal. Then*

$$[\ker \mathcal{S}_T^{-1}]_T = [\ker \mathcal{S}_T^{1}]_T = \ker \mathcal{S}_T^{-1} = \ker \mathcal{S}_T^{1} = \ker T = \ker T^*.$$

Maybe it can be said that $\ker \mathcal{S}_T^1$ is a noncommutative $\ker T$ or a nonnormalize of $\ker T$.

5 Standard Model

Let $\omega = \{\omega_0, \omega_1, \cdots, \omega_n, \cdots\}$ be a sequence of positive numbers. We consider the space of sequences $f = \{\widehat{f}(n)\}$ such that

$$\|f\|^2 = \|f\|_\omega^2 := \sum_{n=0}^{+\infty} |\widehat{f}(n)|^2 \omega_n < \infty.$$

Let $f(z) = \sum\limits_{n=0}^{+\infty} \widehat{f}(n) z^n$ be the formal power series in z. Then we shall denote the space as $H^2(\omega)$. It is clear that $H^2(\omega)$ is a Hilbert space with the inner product

$$\langle f, g\rangle := \sum_{n=0}^{+\infty} \widehat{f}(n)\overline{\widehat{g}(n)}\omega_n.$$

It is easy to see that $\{\frac{z^n}{\sqrt{\omega_n}}\}_{n\in\mathbb{Z}_+}$ is an orthonormal basis for $H^2(\omega)$. In fact, it is well known that $H^2(\omega)$ can become an analytic function space over an open disc (see [18]). Then we define the multiplication operator M_z by $M_z z^n = z^{n+1}$. It is also well known that M_z is bounded if and only if $\sup\limits_n \frac{\omega_{n+1}}{\omega_n} < +\infty$. From now on, we assume that $\sup\limits_n \frac{\omega_{n+1}}{\omega_n} < +\infty$.

We state some results about M_z^*. Recall that the commutator for $T \in \mathcal{B}(H)$ is defined by $[T^*, T] = T^*T - TT^*$.

Proposition 5.1 (i) *$M_z^* 1 = 0$. $M_z^* z^n = \frac{\omega_n}{\omega_{n-1}} z^{n-1}$ if $n > 0$.*
(ii) *$[M_z^*, M_z] z^n = \varphi(n) z^n$, where*

$$\varphi(n) = \begin{cases} \dfrac{\omega_1}{\omega_0}, & n = 0, \\ \dfrac{\omega_{n+1}}{\omega_n} - \dfrac{\omega_n}{\omega_{n-1}}, & n > 0. \end{cases}$$

Take $H_n = \{az^n : a \in \mathbb{C}\}$, then $\{H_n\}_{n\in\mathbb{Z}}$ is an $\mathcal{S}_{M_z}$-grading of $H^2(\omega)$. We call this $\mathcal{S}_{M_z}$-graded module the standard model with single variable. We denote it by $(M_z, H^2(\omega))$. It is obvious that the standard model of single variable is always stable, bounded below and has $\mathcal{S}_{M_z}$-codimension one property. Since every M_z is unitarily equivalent to the unilateral weighted shift with weight sequence $\alpha_n = \sqrt{\frac{\omega_{n+1}}{\omega_n}}$, it is amazing that we can get the well-known elementary results about unilateral weighted shift very quickly.

Theorem 5.1 *M_z is irreducible. If $\delta = \{\delta_0, \delta_1, \cdots, \delta_n, \cdots\}$ is another sequence of positive numbers and $\sup\limits_n \frac{\delta_{n+1}}{\delta_n} < +\infty$, then $(M_z, H^2(\omega))$ and $(M_z, H^2(\delta))$ are unitarily equivalence if and only if $\frac{\omega_{n+1}}{\omega_n} = \frac{\delta_{n+1}}{\delta_n}$, $n \in \mathbb{Z}_+$.*

Proof By $\mathcal{S}_{M_z}$-codimension one property, $H^2(\omega)$ is a minimal reducing subspace of M_z, i.e., M_z is irreducible.

Put $A = M_z$ on $H^2(\omega)$ and $B = M_z$ on $H^2(\delta)$. Since the standard model with single variable has a stable and bounded below $\mathcal{S}$-grading, A and B are unitarily equivalence if and only if $\mathcal{S}_A$-grading module $H^2(\omega)$ and $\mathcal{S}_B$-grading module $H^2(\delta)$ are $\mathcal{S}$-grading isomorphic, if and only if for all $n \geq 0$ there exists a unitary $U_n\colon H_n \to H_n$ such that $U_{n+1}A = BU_n$. Assume that $U_n z^n = a_n z^n$, and then $|a_n|^2 = \frac{\omega_n}{\delta_n}$. Taking norm's square on $U_{n+1}Az^n = BU_n z^n$, we get $\frac{\omega_{n+1}}{\omega_n} = \frac{\delta_{n+1}}{\delta_n}$. Conversely, if $\frac{\omega_{n+1}}{\omega_n} = \frac{\delta_{n+1}}{\delta_n}$ is true for all $n \in \mathbb{Z}_+$, then we can take $U_n z^n = \sqrt{\frac{\omega_n}{\delta_n}} z^n$ and it is trivial that $U_{n+1}A = BU_n$.

Remark 5.1 For the condition $\frac{\omega_{n+1}}{\omega_n} = \frac{\delta_{n+1}}{\delta_n}$, $n \in \mathbb{Z}$, we will denote it by $\omega \sim \delta$.

Next, we consider the standard model of multivariable. Let $\delta = \{\delta_0, \delta_1, \cdots, \delta_n, \cdots\}$ be another sequence of positive numbers and $\sup\limits_n \frac{\delta_{n+1}}{\delta_n} < +\infty$. Then we can construct $H^2(\omega, \delta) := H^2(\omega) \otimes H^2(\delta)$ as the Hilbert space consisting of analytic functions

$$f(z, w) = \sum_{n,m=0}^{+\infty} a_{n,m} z^n w^m$$

such that

$$\|f\|^2 = \|f\|^2_{\omega,\delta} := \sum_{n,m=0}^{+\infty} \omega_n \delta_m |a_{n,m}|^2 < \infty.$$

We can lift M_z (resp. M_w) on $H^2(\omega)$ (resp. $H^2(\delta)$) to $M_z \otimes I$ (resp. $I \otimes M_w$) on $H^2(\omega, \delta)$. In this case, we still write M_z (resp. M_w) in place of $M_z \otimes I$ (resp. $I \otimes M_w$). In other words, we have $M_z z^n w^m = z^{n+1} w^m$ and $M_w z^n w^m = z^n w^{m+1}$.

For M_z on $H^2(\omega, \delta)$, we can take $H_n = \overline{\mathrm{span}}\{z^n w^m : m \in \mathbb{Z}_+\}$, $n \in \mathbb{Z}_+$, then $\{H_n\}_{n\in\mathbb{Z}_+}$ is an $\mathcal{S}_{M_z}$-grading of $H^2(\omega, \delta)$. Furthermore, this is even double stable and surely not minimal. Of course, it is natural to consider it as the tensor product of the standard model of single variable $(M_z, H^2(\omega))$ with $(I, H^2(\delta))$.

By the multiplication operator M_{z+w} with the symbol $z + w$ on $H^2(\omega, \delta)$, we mean the bounded operator $M_z + M_w$ on $H^2(\omega, \delta)$. In this case, we can put $H_n = \overline{\mathrm{span}}\{z^s w^t : s+t = n\}$, then we can check that $\{H_n\}_{n\in\mathbb{Z}_+}$ is an $\mathcal{S}_{M_{z+w}}$-grading of $H^2(\omega, \delta)$. This will be called the type I of standard model with two variables $(M_{z+w}, H^2(\omega, \delta))$.

Proposition 5.2 *For the type I of standard model with two variables $(M_{z+w}, H^2(\omega, \delta))$, we have*

(i) $\{H_n\}_{n\in\mathbb{Z}_+}$ *is bounded below, reverse stable.*

(ii) $(M_{z+w}, H^2(\omega, \delta))$ *is irreducible if and only if* $\{H_n\}_{n\in\mathbb{Z}_+}$ *is stable.*

(iii) $\ker_n \mathcal{S}^{-1}_{M_{z+w}} \subseteq \mathbb{C}\{\chi_n\}$, $n \in \mathbb{Z}_+$, *where* $\chi_n = \sum\limits_{i=0}^{n} \frac{(-1)^i z^i w^{n-i}}{\omega_i \delta_{n-i}}$.

(iv) *Denote by* $I := \{n \in \mathbb{Z}_+ : \ker_n \mathcal{S}^{-1}_{M_{z+w}} = \mathbb{C}\{\chi_n\}\}$, *then* $i \in I$ *if and only if* $\mathcal{S}^0_{M_{z+w}}\chi_i = \mathbb{C}\{\chi_i\}$.

(v) $H^2(\omega, \delta) = \bigoplus\limits_{i\in I} [\chi_i]_{M_{z+w}}$ *and* $[\chi_i]_{M_{z+w}}$ *is a minimal reducing subspace of* T *for all* $i \in I$.

(vi) *For any* $i, j \in I$, $[\chi_i]_{M_{z+w}}$ *is* $\mathcal{S}$*-graded isomorphic to* $[\chi_j]_{M_{z+w}}$ *if and only if* $\mathcal{S}^0_{M_{z+w}}\{\chi_i + \chi_j\} = \mathbb{C}\{\chi_i + \chi_j\}$.

(vii) *If I is finite and for all $i,j\in I$, $i\neq j$, $\mathcal{S}^0_{M_{z+w}}\{\chi_i+\chi_j\}\neq\mathbb{C}\{\chi_i+\chi_j\}$, then $\mathcal{V}^*(M_{z+w})\cong\bigoplus_{|I|}\mathbb{C}$.*

Proof Since M_{z+w} is injective, (i) is easy.

(ii) If $(M_{z+w},H^2(\omega,\delta))$ is irreducible, then $\mathcal{S}_{M_{z+w}}$-graded module $H^2(\omega,\delta)$ is minimal and $\{H_n\}_{n\in\mathbb{Z}_+}$ is double stable. If $\{H_n\}_{n\in\mathbb{Z}_+}$ is stable, then $H^2(\omega,\delta)\ominus\mathcal{S}^1_{M_{z+w}}H^2(\omega,\delta)=H_0$, thus $H^2(\omega,\delta)$ has $\mathcal{S}_{M_{z+w}}$-codimension one property. By Proposition 2.2, $(M_{z+w},H^2(\omega,\delta))$ is irreducible.

(iii) $\ker_n\mathcal{S}^{-1}_{M_{z+w}}\subseteq\ker M^*_{z+w}\cap H_n$ and $\ker M^*_{z+w}\cap H_n=\mathbb{C}\{\chi_n\}$.

(iv) If $\ker_i\mathcal{S}^{-1}_{M_{z+w}}=\mathbb{C}\{\chi_i\}$, then $\mathcal{S}^{-1}_{M_{z+w}}\mathcal{S}^0_{M_{z+w}}\chi_i=\mathcal{S}^{-1}_{M_{z+w}}\chi_i=0$, thus $\mathcal{S}^0_{M_{z+w}}\chi_i\subseteq\ker_i\mathcal{S}^{-1}_{M_{z+w}}=\mathbb{C}\{\chi_i\}$. Since $\chi_i\in\mathcal{S}^0_{M_{z+w}}\chi_i$, equality holds. Conversely, we have $M_{z+w}(\mathcal{S}^{-1}_{M_{z+w}}\chi_i)\subseteq\mathcal{S}^0_{M_{z+w}}\chi_i$ and $M_{z+w}(\mathcal{S}^{-1}_{M_{z+w}}\chi_i)\perp\chi_i$ since $\chi_i\in\ker M^*_{z+w}$. Using $\mathcal{S}^0_{M_{z+w}}\chi_i=\mathbb{C}\{\chi_i\}$ it follows that $M_{z+w}(\mathcal{S}^{-1}_{M_{z+w}}\chi_i)$ must be zero. Thus $\mathcal{S}^{-1}_{M_{z+w}}\chi_i=0$ by injective of M_{z+w}.

(v) Since $[\ker_i S^{-1}_{M_{z+w}}]_{M_{z+w}}=[\chi_i]_{M_{z+w}}$, use Theorem 4.1. Note that for any $i\in I$, $[\chi_i]_{M_{z+w}}$ has $\mathcal{S}_{M_{z+w}}$-codimension one property.

(vi) If $[\chi_i]_{M_{z+w}}$ is $\mathcal{S}$-graded isomorphic to $[\chi_j]_{M_{z+w}}$, then there are two unitaries $U_0\colon\mathbb{C}\{\chi_i\}\to\mathbb{C}\{\chi_j\}$ and $U_1\colon\mathcal{S}^1_{M_{z+w}}\chi_i\to\mathcal{S}^1_{M_{z+w}}\chi_j$ such that $U_1M_{z+w}=M_{z+w}U_0$. Assume $U_0\chi_i=c\chi_j$, and then for all $A\in\mathcal{S}^0_{M_{z+w}}$, $A\chi_i=\lambda_A\chi_i$ by $\mathcal{S}^0_{M_{z+w}}\chi_i=\mathbb{C}\{\chi_i\}$, and we have

$$A(\chi_i+\chi_j)=A\Big(\chi_i+\frac{1}{c}U_0\chi_i\Big)=\lambda_A\Big(\chi_i+\frac{1}{c}U_0\chi_i\Big)=\lambda_A(\chi_i+\chi_j).$$

Thus $\mathcal{S}^0_{M_{z+w}}\{\chi_i+\chi_j\}=\mathbb{C}\{\chi_i+\chi_j\}$.

If $\mathcal{S}^0_{M_{z+w}}\{\chi_i+\chi_j\}=\mathbb{C}\{\chi_i+\chi_j\}$, then for $A\in\mathcal{S}^0_{M_{z+w}}$, assume $A\chi_i=\lambda_{A,i}\chi_i$, $A\chi_j=\lambda_{A,j}\chi_j$. By $A(\chi_i+\chi_j)=\lambda_{A,i}\chi_i+\lambda_{A,j}\chi_j$, we find that $\lambda_{A,i}=\lambda_{A,j}$ for all $A\in\mathcal{S}^0_{M_{z+w}}$. Define $U_0\colon\mathbb{C}\{\chi_i\}\to\mathbb{C}\{\chi_j\}$ by $U_0\chi_i=c\chi_j$. Take c such that U_0 is unitary. Since $[\chi_i]_{M_{z+w}}=\bigoplus_{n\geq0}S^n_{M_{z+w}}\chi_i$ and $[\chi_j]_{M_{z+w}}=\bigoplus_{n\geq0}S^n_{M_{z+w}}\chi_j$, if we define $U\colon[\chi_i]_{M_{z+w}}\to[\chi_j]_{M_{z+w}}$ directly by $UB\chi_i=cB\chi_j$, where $B\in\mathcal{S}^n_{M_{z+w}}$ for all $n\geq0$, and extend it linearly. Then for any $B_1,B_2\in\mathcal{S}^n_{M_{z+w}}$,

$$\begin{aligned}\langle cB_1\chi_j,cB_2\chi_j\rangle&=\langle cB_2^*B_1\chi_j,c\chi_j\rangle=\langle c\lambda_{B_2^*B_1,j}\chi_j,c\chi_j\rangle\\&=\lambda_{B_2^*B_1,j}\langle U_0\chi_j,U_0\chi_j\rangle=\lambda_{B_2^*B_1,j}\langle\chi_i,\chi_i\rangle\\&=\langle\lambda_{B_2^*B_1,j}\chi_i,\chi_i\rangle=\langle\lambda_{B_2^*B_1,i}\chi_i,\chi_i\rangle\\&=\langle B_2^*B_1\chi_i,\chi_i\rangle=\langle B_1\chi_i,B_2\chi_i\rangle.\end{aligned}$$

If we have $B_1\chi_i=B_2\chi_i$, then $(B_1-B_2)\chi_i=0$. By the above result we get that $(B_1-B_2)\chi_j=0$ and $B_1\chi_j=B_2\chi_j$. Thus U is a well defined $\mathcal{S}$-isomorphism between $[\chi_i]_{M_{z+w}}$ and $[\chi_j]_{M_{z+w}}$. Hence $[\chi_i]_{M_{z+w}}$ is $\mathcal{S}$-graded isomorphic to $[\chi_j]_{M_{z+w}}$ by Theorem 3.3.

(vii) If I is finite, then $\dim\ker\mathcal{S}^{-1}_{M_{z+w}}=|I|$ and $\mathcal{V}^*$ is $*$-isomorphic into a closed $*$-subalgebra of $\mathcal{B}(\mathbb{C}^{|I|})$. If for all $i,j\in I$, $i\neq j$, $\mathcal{S}^0_{M_{z+w}}\{\chi_i+\chi_j\}\neq\mathbb{C}\{\chi_i+\chi_j\}$, then $[\chi_i]_{M_{z+w}}$ is not $\mathcal{S}_{M_{z+w}}$-isomorphic to $[\chi_j]_{M_{z+w}}$. By the theory of von Neumann algebra, $\mathcal{V}^*\cong\bigoplus_{|I|}\mathbb{C}$.

Remark 5.2 If we directly use the $\ker\mathcal{S}^{-1}_{M_{z+w}}$ to generate the $\mathcal{S}_{M_{z+w}}$-grading, then $H^2(\omega,\delta)$ is already double stable $\mathcal{S}_{M_{z+w}}$-graded module. It also provides another interesting example that double stability is irrelevant to minimality if we do not pose any restriction on the concrete graded structure.

Inspired by this proof, we want invite the reader to verify the following general propositions for $\mathcal{S}$-graded isomorphism and the kernel method.

Proposition 5.3 *Suppose that M is an $\mathcal{S}_A$-graded module with $\mathcal{S}_A$-grading $\{M_n\}_{n\in\mathbb{Z}}$, N is an $\mathcal{S}_B$-graded module with $\mathcal{S}_B$-grading $\{N_n\}_{n\in\mathbb{Z}}$, and they satisfy the following properties:*

(a) *$\{M_n\}_{n\in\mathbb{Z}}$ and $\{N_n\}_{n\in\mathbb{Z}}$ are bounded below and stable.*

(b) *Take $n_1 = n_l(M)$, $n_2 = n_l(N)$, then $\dim M_{n_1} = \dim N_{n_2} = 1$.*

Let $M_{n_1} = \mathbb{C}\{\chi_1\}$ and $N_{n_2} = \mathbb{C}\{\chi_2\}$. Then M is $\mathcal{S}$-graded isomorphic to N if and only if for all $T_1 \in \mathcal{S}_A^0$ and $T_2 \in \mathcal{S}_B^0$ which is get by replacing A with B in T_1, then χ_1 is an eigenvector of T_1 , χ_2 is an eigenvector ofT_2 and they have the same eigenvalue.

Proposition 5.4 *Suppose that an $\mathcal{S}_T$-graded module H has a bounded below $\mathcal{S}_T$-grading $\{H_n\}_{n\in\mathbb{Z}}$. Let $I = \{n : \dim \ker_n \mathcal{S}_T^{-1} = 1\}$ and assume that for all $i \in I$, $\ker_i \mathcal{S}_T^{-1} = \mathbb{C}\{\chi_i\}$.*

(i) *For $i, j \in I$, $[\ker_i \mathcal{S}_T^{-1}]_T$ is $\mathcal{S}$-graded isomorphic to $[\ker_j \mathcal{S}_T^{-1}]_T$ if and only if $\mathcal{S}_T^0\{\chi_i + \chi_j\} = \mathbb{C}\{\chi_i + \chi_j\}$.*

(ii) *If $I = \{n : \ker_n \mathcal{S}_T^{-1} \neq 0\}$, I is finite and for all $i, j \in I$, $i \neq j$, $\mathcal{S}_T^0\{\chi_i+\chi_j\} \neq \mathbb{C}\{\chi_i+\chi_j\}$, then $\mathcal{V}^*(T) \cong \bigoplus_{|I|} \mathbb{C}$.*

Of course they all have dual versions.

The question of stability of $(M_{z+w}, H^2(\omega, \delta))$ is really difficult. For common results we only know that when $\omega \sim \delta$, $(M_{z+w}, H^2(\omega, \delta))$ is not stable. For general case, we conjecture that $I \subseteq \{0, 1\}$.

By the Toeplitz operator $T_{z+\overline{w}}$ with the symbol $z + \overline{w}$ on $H^2(\omega, \delta)$, we mean the bounded operator $M_z + M_w^*$ on $H^2(\omega, \delta)$.

For $T_{z+\overline{w}}$, we can set $H_n = \overline{\text{span}}\{z^s w^t : s - t = n\}$, then we can check that $\{H_n\}_{n\in\mathbb{Z}_+}$ is an unbounded $\mathcal{S}_{T_{z+\overline{w}}}$-grading of $H^2(\omega, \delta)$. This will be called the type II standard model with two variables $(T_{z+\overline{w}}, H^2(\omega, \delta))$. Since $T_{z+\overline{w}}$ is injective and $(T_{z+\overline{w}}, H^2(\omega, \delta))$'s adjoint is unitarily equivalent to $(T_{z+\overline{w}}, H^2(\delta, \omega))$, we get the following result.

Proposition 5.5 *The type II of standard model with two variables $(T_{z+\overline{w}}, H^2(\omega, \delta))$ is double stable.*

Note that there are a few results showing that $(T_{z+\overline{w}}, H^2(\omega, \delta))$ is irreducible and the rest is still unknown. The usefulness of standard models is indicated as follows: All of the tuples $(M_{z^n}, H^2(\omega))$, $(M_{az^n+bw^m}, H^2(\omega, \delta))$, $(T_{az^n+b\overline{w}^m}, H^2(\omega, \delta))$ can be represented as a direct sum of finitely many standard models.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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