Boundary Behavior of Large Solutions for Equations of Monge-Ampère Type*

Zhijun ZHANG[1]

Abstract This paper is concerned with the existence and optimal boundary behavior of large solutions to the Monge-Ampère type equations $\det D^2u(x) = \lambda u^n(x) + b(x)g(|\nabla u(x)|)$, $x \in \Omega$, where Ω is a uniformly convex, bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$, $b \in C^\infty(\Omega)$ is positive in Ω, $g \in C[0,\infty) \cap C^1(0,\infty)$, $g(0) = 0$ and g is increasing on $[0,\infty)$. The author finds new structure conditions on g which play a crucial role in boundary behavior of such solutions.

Keywords Monge-Ampère type equations, A nonlinear gradient term, Large convex solutions, Boundary behavior
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1 Introduction

In this paper, we analyze the existence and boundary behavior of convex solutions to the following boundary blow-up problem of Monge-Ampère type

$$\det D^2u(x) = \lambda u^n(x) + b(x)g(|\nabla u(x)|), \quad u(x) > 0, \ x \in \Omega, \ u|_{\partial\Omega} = \infty, \tag{1.1}$$

where the boundary condition means that $u(x) \to +\infty$ as $d(x) = \text{dist}(x, \partial\Omega) \to 0$, and the solution is called "a large solution" or "an explosive solution", Ω is a uniformly convex, bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$, $D^2u(x) = \left(\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right)$ denotes the Hessian of $u(x)$, $\det D^2u$ is the so-called Monge-Ampère operator, $\lambda > 0$, $b \in C^\infty(\Omega)$ is positive in Ω, and g satisfies

($\mathbf{g_1}$) $g \in C[0,\infty) \cap C^1(0,\infty)$, $g(0) = 0$ and g is increasing on $[0,\infty)$;
($\mathbf{g_2}$) $\int_1^\infty \frac{s^{n-1}}{g(s)}\mathrm{d}s < \infty$.
Firstly, let us consider the following more general problem

$$\det D^2u(x) = h(x, u(x), \nabla u(x)), \quad x \in \Omega, \ u|_{\partial\Omega} = +\infty. \tag{1.2}$$

Problem (1.2) arises in Riemannian geometry. When h does not depend on ∇u, there are many papers which have been dedicated to resolving the issues of existence, uniqueness and asymptotic behavior of solutions to the problem, see, for instance, [7–8, 10–11, 16–17, 20–21, 26, 31, 33, 36–37, 39, 43] and the references therein. When h strongly depends on ∇u, only partial results are known for the problem, see, for instance, [15, 27, 35, 38, 40] and the references therein.

Some basic results are listed as follows.

In [35], Urbas established the following result (see [35, Theorem 4.12, Remark]). Let Ω be a $C^{1,1}$ uniformly bounded convex domain in $\mathbb{R}^n$. Suppose that $u \in C^2(\Omega)$ is a convex solution of equation in (1.2), and $h \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function satisfying

$$h(x, s, \nu) \leq c_0 (d(x))^\sigma (1 + |\nu|^2)^{\frac{q}{2}}, \quad \forall (x, s, \nu) \in D \times \mathbb{R} \times \mathbb{R}^n,$$

where $D \in \Omega$ is a neighbourhood of $\partial\Omega$, $c_0 > 0$, $q > n$ and $\sigma > q - n - 1$, then $\lim\limits_{d(x) \to 0} u(x) = +\infty$.

In [15], Guan and Jian established the existence, nonexistence, uniqueness and global asymptotic behavior of (strictly) convex solutions to problem (1.2) for several classes of positive functions h.

When $h(x, s, \nu) = p(x)a(s) + |\nu|^q$, $q \in [0, n]$, the author (see [40]) considered the existence and boundary behavior of (strictly) convex solutions to problem (1.2) under some additional conditions on a and p.

Secondly, when the Monge-Ampère operator $\det D^2 u$ is replaced by the classical Laplace operator Δu, many papers have been dedicated to resolving the issues of existence, uniqueness and asymptotic behavior for solutions of the corresponding problem, see, for instance, [1–3, 6, 9, 12, 18–19, 22, 24, 28–29, 42] and the references therein.

Some basic results are listed as follows.

Lasry and Lions [19] firstly established the following model

$$\triangle u(x) = \lambda u(x) + |\nabla u(x)|^q, \quad u(x) > 0, \ x \in \Omega, \ u|_{\partial\Omega} = \infty, \tag{1.3}$$

which arises from the description of the basic stochastic control problem with state constraints, and showed the following results. Model (1.3) has a unique solution $u \in C^2(\Omega)$ if and only if $q \in (1, 2]$. Moreover, u satisfies

$$\begin{cases} \lim\limits_{d(x) \to 0} u(x)(d(x))^{\frac{2-q}{q-1}} = \dfrac{(q-1)^{-\frac{2-q}{q-1}}}{2-q}, & \text{when } 1 < q < 2, \\ \lim\limits_{d(x) \to 0} \dfrac{u(x)}{-\ln(d(x))} = 1, & \text{when } q = 2. \end{cases}$$

Alarcón, García-Melián and Quaas [1] analyzed the existence, nonexistence, boundary behavior and uniqueness of solutions in $C^{1,\varsigma}(\Omega)$ for $\varsigma \in (0, 1)$ to the following problem

$$\triangle u(x) = f(u(x)) + g(|\nabla u(x)|), \quad u(x) \geq 0, \ x \in \Omega, \ u|_{\partial\Omega} = \infty, \tag{1.4}$$

where $f, g \in C[0, \infty)$ satisfy $f(0) = g(0) = 0$, f, g are increasing on $[0, \infty)$, $\int_1^\infty \frac{s}{g(s)} \mathrm{d}s = \infty$, $\int_1^\infty \frac{\mathrm{d}s}{g(s)} < \infty$ and the further appropriate conditions.

For convenience, we introduce several kinds of functions.

Firstly, the regularly varying function is presented as follows.

Definition 1.1 (see [32, Definition 1.1]) *A positive continuous function ϕ defined on $[a, \infty)$ for some $a > 0$, is called regularly varying at infinity with index ρ, denoted by $\phi \in \mathrm{RV}_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,*

$$\lim_{s \to \infty} \frac{\phi(\xi s)}{\phi(s)} = \xi^\rho. \tag{1.5}$$

In particular, when $\rho = 0$, ϕ is called slowly varying at infinity.

Clearly, if $\phi \in RV_\rho$, then $L(s) := \frac{\phi(s)}{s^\rho}$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are

($\mathbf{i_1}$) every continuous function on $[a, \infty)$ which has a positive limit at infinity;

($\mathbf{i_2}$) $(\ln s)^\mu$ $(s > 1)$ and $(\ln(\ln s))^\mu$ $(s > e)$, $\mu \in \mathbb{R}$;

($\mathbf{i_3}$) $\exp((\ln s)^\mu)$ $(s > 0)$, $0 < \mu < 1$.

We also say that a positive continuous function ϕ_0 defined on $(0, s_0)$ for some $s_0 > 0$, is regularly varying at zero with index ρ (and denoted by $\phi_0 \in RVZ_\rho$) if $s \to \phi_0(\frac{1}{s})$ belongs to $RV_{-\rho}$.

Proposition 1.1 (Uniform convergence theorem) (see [30, Proposition 0.5]) *If $\phi \in RV_\rho$, then (1.5) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 1.2 (The Karamata representation theorem) (see [32, Theorem 1.2, 30, Corollary, p. 17]) *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s) = z(s) \exp \Big(\int_{S_0}^s \frac{y(\tau)}{\tau} \mathrm{d}\tau \Big), \quad s \geq S_0 \; for \; some \; S_0 \geq a, \tag{1.6}$$

where the functions z and y are continuous and for $s \to \infty$, $y(s) \to 0$ and $z(s) \to c_0$, with $c_0 > 0$.

Definition 1.2 (see [25, Definition 0.3])

$$\widehat{L}(s) = c_0 \exp \Big(\int_{S_0}^s \frac{y(\tau)}{\tau} \mathrm{d}\tau \Big), \quad s \geq S_0 \tag{1.7}$$

is normalized slowly varying at infinity and

$$\phi(s) = s^\rho \widehat{L}(s), \quad s \geq S_0 \tag{1.8}$$

is normalized regularly varying at infinity with index ρ (and denoted by $\phi \in NRV_\rho$).

Equivalently, a function $\phi \in C^1[S_0, \infty)$ for some $S_0 > 0$ and $\phi \in NRV_\rho$ if and only if

$$\lim_{s \to \infty} \frac{s\phi'(s)}{\phi(s)} = \rho. \tag{1.9}$$

Similarly, ϕ_0 is called normalized regularly varying at zero with index ρ, denoted by $\phi_0 \in NRVZ_\rho$ if $s \to \phi_0(\frac{1}{s})$ belongs to $NRV_{-\rho}$.

Secondly, let Λ denote the set of all positive nondecreasing functions θ in $C^1(0, \delta_0)$ $(\delta_0 > 0)$ which satisfy

$$\lim_{t \to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{\Theta(t)}{\theta(t)} \Big) := D_\theta \in [0, \infty), \quad \Theta(t) := \int_0^t \theta(s)\mathrm{d}s. \tag{1.10}$$

The set Λ was first introduced by Cîrstea and Rădulescu [9].

Three basic examples of $\theta \in \Lambda$ are

($\mathbf{i_1}$) $\theta(t) = t^\sigma$ with $\sigma \geq 0$, $\Theta(t) = (1 + \sigma)^{-1} t^{1+\sigma}$, $D_\theta = (1 + \sigma)^{-1}$;

($\mathbf{i_2}$) $\theta(t) = (-\ln t)^{-\sigma}$, $t \in (0, \frac{1}{2})$, $\sigma > 0$, $D_\theta = 1$;

($\mathbf{i_3}$) $\theta(t) = \exp(-t^{-\sigma})$ with $\sigma > 0$, $D_\theta = 0$.

Next, we denote by $\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})$ the principal curvatures of $\partial\Omega$ at $\overline{x}$ and

$$M_0 := \max_{\overline{x} \in \partial\Omega} \prod_{i=1}^{n-1} \kappa_i(\overline{x}), \quad m_0 := \min_{\overline{x} \in \partial\Omega} \prod_{i=1}^{n-1} \kappa_i(\overline{x}). \tag{1.11}$$

Then we can see that $m_0 > 0$ provided Ω is a uniformly convex, bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$. In particular, when $\Omega = B_R$ which is a ball of radius R centered at the origin, $M_0 = m_0 = R^{1-n}$.

For convenience, when g satisfies $(\mathbf{g_1})$ and $(\mathbf{g_2})$, let

$$H(t) = \int_t^\infty \frac{s^{n-1}}{g(s)} \mathrm{d}s, \quad t > 0,$$

and we denote by ψ the inverse of H, i.e., ψ satisfies

$$\int_{\psi(t)}^\infty \frac{v^{n-1}}{g(v)} \mathrm{d}v = t, \quad \forall t > 0. \tag{1.12}$$

We also denote

$$\Psi(t) = \int_t^1 \psi(\tau) \mathrm{d}\tau, \quad t \in (0,1]. \tag{1.13}$$

In this paper, when $b \equiv 1$ in Ω, we show boundary behavior of solutions to problem (1.1) under the following new structure conditions on g:

$(\mathbf{g_3})$ either $g \in \mathrm{RV}_q$ with index $q \in (n, n+1)$, or $g \in \mathrm{RV}_{n+1}$ and $\int_0^1 \psi(\tau)\mathrm{d}\tau = \infty$.

Three basic examples of g which satisfy $(\mathbf{g_3})$ are

$(\mathbf{i_1})$ $g(s) = s^q$, $s \in [0,\infty)$, $q \in (n, n+1]$;

$(\mathbf{i_2})$ $g(s) = s^q(\ln s)^\mu$ on $[S_0, \infty)$ for sufficiently large $S_0 > 0$, $q \in (n, n+1)$ and $\mu \in \mathbb{R}$;

$(\mathbf{i_3})$ $g(s) = s^{n+1}(\ln s)^{-\mu}$ on $[S_0, \infty)$ for sufficiently large $S_0 > 0$, $\mu > 0$.

In particular, when $g(s) = s^q$, $q > n$,

$$\psi(t) = \left(\frac{1}{q-n}\right)^{\frac{1}{q-n}} t^{-\frac{1}{q-n}}, \quad t > 0. \tag{1.14}$$

Moreover, when $q \leq n+1$, for $t \in (0,1]$,

$$\Psi(t) = \begin{cases} \dfrac{q-n}{n+1-q}\left(\dfrac{1}{q-n}\right)^{\frac{1}{q-n}}(t^{-\frac{n+1-q}{q-n}} - 1), & q < n+1, \\ -\ln t, & q = n+1. \end{cases} \tag{1.15}$$

The complete characterizations of Θ, ψ and Ψ are provided in Lemmas 2.1 and 2.3, respectively.

Moreover, when b vanishes on the boundary properly, and $g(s) = s^q$, $q \in (n, n+1]$, we also show boundary behavior of such solutions.

Specifically, our main results are summarized as follows.

Theorem 1.1 *Let* $b \equiv 1$ *in* Ω, $q \in (n, n+1]$, *and* g *satisfy* $(\mathbf{g_1})$–$(\mathbf{g_3})$. *Then, for any convex solution* u *to problem* (1.1), *there holds*

$$m_0^{\frac{1}{q-n}} \leq \lim_{d(x)\to 0} \inf \frac{u(x)}{\Psi(d(x))} \leq \lim_{d(x)\to 0} \sup \frac{u(x)}{\Psi(d(x))} \leq M_0^{\frac{1}{q-n}}. \tag{1.16}$$

In particular, when $\Omega = B_R$, u satisfies

$$\lim_{d(x) \to 0} \frac{u(x)}{\Psi(d(x))} = R^{\frac{n-1}{q-n}}. \tag{1.17}$$

Theorem 1.2 *Let $g(s) = s^q$ with $q \in (n, n+1]$, and b satisfy the additional condition that*
(b_1) *there exist $\theta \in \Lambda$ and positive constants b_i $(i = 1, 2)$ such that*

$$b_1 := \lim_{d(x) \to 0} \inf \frac{b(x)}{(\theta(d(x)))^{n+1-q}} \le b_2 := \lim_{d(x) \to 0} \sup \frac{b(x)}{(\theta(d(x)))^{n+1-q}}.$$

(I_1) *If $q < n+1$, then, for any convex solution u to problem (1.1), there holds*

$$\frac{q-n}{n+1-q} \Big(\frac{m_0}{b_2} \Big(\frac{n+1-q}{q-n} + D_\theta \Big) \Big)^{\frac{1}{q-n}}$$

$$\le \lim_{d(x) \to 0} \inf u(x) (\Theta(d(x)))^{\frac{n+1-q}{q-n}}$$

$$\le \lim_{d(x) \to 0} \sup u(x) (\Theta(d(x)))^{\frac{n+1-q}{q-n}}$$

$$\le \frac{q-n}{n+1-q} \Big(\frac{M_0}{b_1} \Big(\frac{n+1-q}{q-n} + D_\theta \Big) \Big)^{\frac{1}{q-n}}. \tag{1.18}$$

(I_2) *If $q = n+1$ and $D_\theta > 0$, then, for any convex solution u to problem (1.1), there holds*

$$\frac{m_0 D_\theta}{b_2} \le \lim_{d(x) \to 0} \inf \frac{u(x)}{-\ln(\Theta(d(x)))} \le \lim_{d(x) \to 0} \sup \frac{u(x)}{-\ln(\Theta(d(x)))} \le \frac{M_0 D_\theta}{b_1}.$$

Finally, to consider the existence of convex solutions to problem (1.1), we need the following function φ. As we know, for a uniformly convex, bounded smooth domain Ω in $\mathbb{R}^n$ $(n \ge 2)$, there is a function $\varphi \in C^2(\overline{\Omega})$ with the properties (see [20, 34]): $\varphi(x) < 0$, $\forall x \in \Omega$, $\varphi|_{\partial\Omega} = 0$, $\nabla\varphi(x) \ne 0$, $\forall x \in \partial\Omega$, and $D^2\varphi(x)$ is positive definite on $\overline{\Omega}$. It follows that there exist $\delta_0 > 0$ and positive constants c_i $(i = 1, 2)$ such that

$$|\nabla\varphi(x)| > 0, \quad \nabla\varphi(x)(D^2\varphi(x))^{-1}(\nabla\varphi(x))^{\mathrm{T}} > 0, \quad \forall x \in \overline{\Omega}_{\delta_0} \tag{1.19}$$

and

$$c_1 d(x) \le -\varphi(x) \le c_2 d(x), \quad \forall x \in \Omega, \tag{1.20}$$

where $\nabla\varphi(x) = \big(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \cdots, \frac{\partial\varphi}{\partial x_n} \big)$ is the gradient of $\varphi(x)$, $(\nabla\varphi(x))^{\mathrm{T}}$ denotes the transpose of $\nabla\varphi(x)$, $(D^2\varphi(x))^{-1}$ denotes the inverse matrix of $D^2\varphi(x)$, and $\Omega_{\delta_0} = \{x \in \Omega : d(x) < \delta_0\}$.

For convenience, let

$$v = -\varphi \quad \text{and assume} \max_{x \in \overline{\Omega}} |\varphi(x)| < 1. \tag{1.21}$$

We have the following results.

Theorem 1.3 *Let $g(s) = s^q$ for $s \ge 0$, $q \in (n, n+1]$, and $b \in C^\infty(\Omega)$.*
(I_1) *If b satisfies the condition that*
(b_2) *there exist $\sigma > -(n+1-q)$ and positive constants b_i $(i = 1, 2)$ such that*

$$b_1(v(x))^\sigma \le b(x) \le b_2(v(x))^\sigma, \quad x \in \Omega;$$

then problem (1.1) *has a convex solution in the order interval* $[m_1(v(x))^{-\beta}, M_1(v(x))^{-\beta}]$, *where* $\beta = \frac{n+1+\sigma-q}{q-n}$, $m_1 \in (0,1)$ *is a sufficiently small constant satisfying*

$$\min_{x \in \overline{\Omega}}(\beta^n(-1)^n \det D^2 v(x)(v(x) + (1+\beta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}}))$$

$$\geq \max_{x \in \overline{\Omega}}(\lambda(v(x))^{n+1} + b_2 m_1^{q-n}\beta^q|\nabla v(x)|^q), \tag{1.22}$$

and $M_1 \geq 1$ *is a sufficiently large constant satisfying*

$$\max_{x \in \overline{\Omega}}(\beta^n(-1)^n \det D^2 v(x)(v(x) + (1+\beta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}}))$$

$$\leq \min_{x \in \overline{\Omega}}(\lambda(v(x))^{n+1} + b_1 M_1^{q-n}\beta^q|\nabla v(x)|^q). \tag{1.23}$$

($\mathbf{I_2}$) *If* b *satisfies the condition that*

($\mathbf{b_3}$) *there exist positive constants* b_i $(i = 1,2)$ *such that*

$$b_1(v(x))^\sigma \leq b(x) \leq b_2(v(x))^\sigma, \quad x \in \Omega, \ \sigma = -(n+1-q),$$

then problem (1.1) *has a convex solution in the order interval* $[-m_2 \ln(v(x)), -M_2 \ln(v(x))]$, *where* $m_2 \in (0,1)$ *is a sufficiently small constant satisfying*

$$\min_{x \in \overline{\Omega}}((-1)^n \det D^2 v(x)(v(x) + \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}}))$$

$$\geq \max_{x \in \overline{\Omega}}(\lambda(-\ln(v(x)))^n(v(x))^{n+1} + b_2 m_2^{q-n}|\nabla v(x)|^q), \tag{1.24}$$

and $M_2 \geq 1$ *is a sufficiently large constant satisfying*

$$\max_{x \in \overline{\Omega}}((-1)^n \det D^2 v(x)(v(x) + \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}}))$$

$$\leq \min_{x \in \overline{\Omega}}(\lambda(-\ln(v(x)))^n(v(x))^{n+1} + b_1 M_2^{q-n}|\nabla v(x)|^q). \tag{1.25}$$

The outline of this paper is as follows. Some preliminary considerations are presented in Section 2. In Section 3, we give the proofs of Theorems 1.1–1.2. The proof of Theorem 1.3 is provided in Section 4.

2 Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic processes (see [4, 30, 32] and the references therein) and in the existence and asymptotic formulas of regular solutions for both second-order linear ordinary differential equations and some nonlinear ordinary differential equations (see [25] and the references therein). Moreover, Cîrstea and Rădulescu [9] firstly introduced the theory to study boundary behavior of large solutions of semilinear elliptic equations.

In this section, we present some further basic facts from the theory in order to show the complete characterizations of the exact behavior near zero of Θ, ψ and Ψ in (1.10) and (1.12)–(1.13).

Proposition 2.1 (see [4, Proposition 1.3.6]) *If functions L, L_1 are slowly varying at infinity, then*

($\mathbf{i_1}$) L^ρ *(for every $\rho \in \mathbb{R}$), $c_1 L + c_2 L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \cdot L_1$ and $L \circ L_1$ (if $L_1(s) \to \infty$ as $s \to \infty$) are also slowly varying at infinity.*

($\mathbf{i_2}$) *For every $\varepsilon > 0$ and $s \to \infty$, $s^\varepsilon L(s) \to \infty$, $s^{-\varepsilon} L(s) \to 0$.*

Proposition 2.2 (see [4, Proposition 1.5.7]) *If $\phi \in \mathrm{RV}_\rho$, $\phi_1 \in \mathrm{RV}_{\rho_1}$ with $\lim\limits_{s \to \infty} \phi_1(s) = \infty$, then $\phi^\varsigma \in \mathrm{RV}_{\varsigma\rho}$ (for every $\varsigma \in \mathbb{R}$) and $\phi \circ \phi_1 \in \mathrm{RV}_{\rho\rho_1}$.*

Proposition 2.3 (see [4, Karamata's Theorem]) *If a function $\phi \in \mathrm{RV}_{-\rho}$ with $\rho > 1$, then $\int_1^\infty \phi(\tau)\mathrm{d}\tau < \infty$, $\int_s^\infty \phi(\tau)\mathrm{d}\tau$ ($s \geq 1$) belongs to $\mathrm{RV}_{1-\rho}$ and*

$$\lim_{s \to \infty} \frac{s\phi(s)}{\displaystyle\int_s^\infty \phi(\tau)\mathrm{d}\tau} = \rho - 1.$$

Lemma 2.1 (see [39, Lemma 2.1]) *Let $\theta \in \Lambda$. We have*

($\mathbf{i_1}$) $\lim\limits_{t \to 0^+} \frac{\Theta(t)}{\theta(t)} = 0$, $\lim\limits_{t \to 0^+} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = 1 - D_\theta$, *and $D_\theta \in [0, 1]$;*

($\mathbf{i_2}$) *when $D_\theta > 0$, $\theta \in \mathrm{NRV}Z_{(1-D_\theta)/D_\theta}$ and $\Theta \in \mathrm{NRV}Z_{D_\theta^{-1}}$;*

($\mathbf{i_3}$) *when $D_\theta = 0$, θ and Θ grow faster than any t^σ ($\sigma > 1$) near zero.*

Lemma 2.2 *If $D_\theta > 0$, then*

$$\lim_{t \to 0} \left(\frac{\Theta(t)}{\theta(t)}\right)^{n+1} (-\ln(\Theta(t)))^n = 0.$$

Proof From $D_\theta > 0$, the definition of Λ and the l'Hospital's rule, we have

$$\lim_{t \to 0} \frac{\Theta(t)}{t\theta(t)} = \lim_{t \to 0} \frac{\frac{\Theta(t)}{\theta(t)}}{t} = \lim_{t \to 0} \left(\frac{\Theta(t)}{\theta(t)}\right)' = D_\theta > 0,$$

and $-\ln(\Theta(t))$ is normalized regularly varying at zero. Moreover, Proposition 2.1 implies

$$\lim_{t \to 0} \left(\frac{\Theta(t)}{\theta(t)}\right)^{n+1} (-\ln(\Theta(t)))^n = \lim_{t \to 0} \left(\frac{\Theta(t)}{t\theta(t)}\right)^{n+1} \lim_{t \to 0} t^{n+1}(-\ln(\Theta(t)))^n = 0.$$

Recall that

$$\int_{\psi(t)}^\infty \frac{v^{n-1}}{g(v)}\mathrm{d}v = t, \quad \forall t > 0, \qquad \Psi(t) = \int_t^1 \psi(\tau)\mathrm{d}\tau, \quad t \in (0, 1).$$

Lemma 2.3 *Let $q \in (n, n+1]$ and g satisfy ($\mathbf{g_1}$)–($\mathbf{g_3}$). We have*

($\mathbf{i_1}$) $-(\psi(t))^{n-1}\psi'(t) = g(\psi(t))$, $t > 0$; $\psi(t) > 0$ *for $t > 0$, $\psi \in \mathrm{NRV}Z_{-\frac{1}{q-n}}$, and $\psi(0) := \lim\limits_{t \to 0^+} \psi(t) = \infty$;*

($\mathbf{i_2}$) *when $g \in \mathrm{RV}_q$ with index $q \in (n, n+1)$, $\int_0^1 \psi(\tau)\mathrm{d}\tau = \infty$;*

($\mathbf{i_3}$) $\Psi'(t) = -\psi(t)$, $\Psi''(t) = -\psi'(t)$, $t \in (0, 1)$, $\Psi \in \mathrm{NRV}Z_{-\frac{n+1-q}{q-n}}$, $\Psi(0) := \lim\limits_{t \to 0^+} \Psi(t) = \infty$, *and $\lim\limits_{t \to 0^+} \frac{(\Psi(t))^n}{g(\psi(t))} = 0$.*

Proof $(\mathbf{i_1})$–$(\mathbf{i_2})$ The first part of the results is easy to be obtained. For the last part, let $s = \psi(t)$, we see from Proposition 2.3 ($\phi(s) = \frac{s^{n-1}}{g(s)}$, $\phi \in \mathrm{RV}_{-(q+1-n)}$ with $q + 1 - n > 1$) that

$$\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = -\lim_{t\to 0^+} \frac{tg(\psi(t))}{(\psi(t))^n} = -\lim_{t\to 0^+} \frac{g(\psi(t)) \displaystyle\int_{\psi(t)}^{\infty} \frac{v^{n-1}}{g(v)}\mathrm{d}v}{(\psi(t))^n}$$

$$= -\lim_{s\to\infty} \frac{\displaystyle\int_s^{\infty} \frac{v^{n-1}}{g(v)}\mathrm{d}v}{s\frac{s^{n-1}}{g(s)}} = -\frac{1}{q-n},$$

i.e., $\psi \in \mathrm{NRV}Z_{-\frac{1}{q-n}}$.

So, there is a sufficiently small $t_0 \in (0,1)$ and $\widehat{L}$ which is normalized slowly varying at zero such that $\psi(t) = t^{-\frac{1}{q-n}}\widehat{L}(t)$, $t \in (0, t_0]$. Similar to the proof of Proposition 2.1($\mathbf{i_2}$), we obtain that $\lim\limits_{t\to 0^+} \psi(t) = \infty$ and $\lim\limits_{t\to 0^+} t^{-(\frac{1}{q-n}-\gamma)}\widehat{L}(t) = \infty$ for $\gamma \in (1, \frac{1}{q-n})$ provided $q \in (n, n+1)$. It follows that there is a sufficiently small $t_1 \in (0, t_0)$ such that

$$\psi(t) = t^{-\gamma}t^{-(\frac{1}{q-n}-\gamma)}\widehat{L}(t) > t^{-\gamma}, \quad t \in (0, t_1],$$

and thus $\int_0^1 \psi(\tau)\mathrm{d}\tau = \infty$ provided $q \in (n, n+1)$.

$(\mathbf{i_3})$ By a direct calculation, we can show the first part of the results. Moreover, $\int_0^1 \psi(\tau)\mathrm{d}\tau = \infty$ implies $\Psi(0) = \infty$. Then, we see from the l'Hospital's rule that

$$\lim_{t\to 0^+} \frac{t\Psi'(t)}{\Psi(t)} = -\lim_{t\to 0^+} \frac{t\psi(t)}{\Psi(t)} = -\lim_{t\to 0^+} \frac{t\psi'(t) + \psi(t)}{\Psi'(t)}$$

$$= 1 + \lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = -\frac{n+1-q}{q-n},$$

i.e., $\Psi \in \mathrm{NRV}Z_{-\frac{n+1-q}{q-n}}$. It follows from Proposition 2.1 ($\mathbf{i_2}$) and Proposition 2.2 that $g(\psi) \in \mathrm{RV}Z_{-\frac{q}{q-n}}$, $\frac{\Psi^n}{g(\psi)} \in \mathrm{RV}Z_{n+1}$ and $\lim\limits_{t\to 0^+} \frac{(\Psi(t))^n}{g(\psi(t))} = 0$.

Lemma 2.4 (The comparison principle) (see [15, Lemma 2.1]) *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with $n \geq 2$, $I = [0, \infty)$ or $I = \mathbb{R}$, and let $h \in C^2(\Omega \times I \times \mathbb{R}^n)$ be increasing in s for all $(x, \nu) \in \Omega \times \mathbb{R}^n$. If $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are convex functions satisfying*

$(\mathbf{i_1})$ $\det D^2 u(x) \geq h(x, u(x), \nabla u(x))$, $x \in \Omega$;

$(\mathbf{i_2})$ $\det D^2 v(x) \leq h(x, v(x), \nabla v(x))$, $x \in \Omega$;

$(\mathbf{i_3})$ $u(x) \leq v(x)$, $x \in \partial\Omega$,

then we have $u \leq v$ in Ω.

The following result is an improvement of Lemma 2.4.

Lemma 2.5 (The comparison principle) *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with $n \geq 2$, $I = (0, \infty)$ or $I = \mathbb{R}$, and let $h \in C^2(\Omega \times I \times \mathbb{R}^n)$ be nondecreasing in s for all $(x, \nu) \in \Omega \times \mathbb{R}^n$. If $u, v \in C^2(\Omega)$ are convex functions satisfying*

$(\mathbf{i_1})$ $\det D^2 u(x) > h(x, u(x), \nabla u(x))$, $x \in \Omega$; (or $\det D^2 u(x) \geq h(x, u(x), \nabla u(x))$, $x \in \Omega$);

$(\mathbf{i_2})$ $\det D^2 v(x) \leq h(x, v(x), \nabla v(x))$, $x \in \Omega$; (or $\det D^2 u(x) < h(x, u(x), \nabla u(x))$, $x \in \Omega$);

$(\mathbf{i_3})$ $\lim\limits_{d(x)\to 0} \sup(u(x) - v(x)) \leq 0$,

then we have $u \leq v$ in Ω.

Proof Assuming the contrary, there exists $x_0 \in \Omega$ such that

$$u(x_0) - v(x_0) = \max_{x \in \overline{\Omega}}(u(x) - v(x)) > 0.$$

Therefore, the Hessian $D^2(v - u)$ is positive semidefinite at x_0, $\nabla u(x_0) = \nabla v(x_0)$, and $\det D^2 u(x_0) \le \det D^2 v(x_0)$.

On the other hand, since ϕ is nondecreasing in s for all $(x, \nu) \in \Omega \times \mathbb{R}^N$, we have

$$\det D^2 v(x_0) \le h(x_0, v(x_0), \nabla v(x_0)) \le h(x_0, u(x_0), \nabla v(x_0))$$
$$= h(x_0, u(x_0), \nabla u(x_0)) < \det D^2 u(x_0).$$

This contradiction shows that $u \le v$ in Ω.

Lemma 2.6 (see [20, Proof of Proposition 2.4, 41, Lemma 2.5]) *Let E be an open subset of $\mathbb{R}^n$ with $n \ge 2$, $\psi_0 \in C^2(\mathbb{R})$, $v_0 \in C^2(E)$ and $D^2 v_0(x)$ be invertible. We have*
($\mathbf{i_1}$) *it holds for $x \in E$*

$$\det D^2 \psi_0(v_0(x))$$
$$= (\psi_0'(v_0(x)))^{n-1} \det D^2 v_0(x)(\psi_0'(v_0(x)) + \psi_0''(v_0(x)) \nabla v_0(x)(D^2 v_0(x))^{-1}(\nabla v_0(x))^{\mathrm{T}});$$

($\mathbf{i_2}$) *if $D^2 v_0(x)$ is positive definite on E, $\psi_0'(s) > 0$ and $\psi_0''(s) \ge 0$ on $\mathbb{R}$, then $D^2 \psi_0(v_0(x))$ is positive definite on E.*

3 Boundary Behavior of Solutions

In this section, we prove Theorems 1.1–1.2.

For any $\delta > 0$, let

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

When Ω is C^m-smooth for $m \ge 2$, choose $\delta_1 \in (0, \delta_0)$ (δ_0 is given as in the definition of Λ) such that (see [14, Lemmas 14.16–14.17])

$$d \in C^m(\Omega_{\delta_1}), \quad |\nabla d(x)| = 1, \quad \forall x \in \Omega_{\delta_1}. \tag{3.1}$$

Moreover, let $\overline{x}$ be the projection of the point $x \in \Omega_{\delta_1}$ to $\partial\Omega$, and $\kappa_i(\overline{x})$ ($i = 1, \cdots, n-1$) be the principal curvatures of $\partial\Omega$ at $\overline{x}$, we have

$$D^2(d(x)) = \mathrm{diag}\Big[\frac{-\kappa_1(\overline{x})}{1 - d(x)\kappa_1(\overline{x})}, \cdots, \frac{-\kappa_{n-1}(\overline{x})}{1 - d(x)\kappa_{n-1}(\overline{x})}, 0\Big], \quad x \in \Omega_{\delta_1}. \tag{3.2}$$

Lemma 3.1 (see [10, Corollary 2.3]) *Let ψ_0 be a C^2-function on $(0, \delta_1)$. Then*

$$\det D^2 \psi_0(d(x)) = (-\psi_0'(d(x)))^{n-1} \psi_0''(d(x)) \Big(\prod_{i=1}^{n-1} \frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})}\Big), \quad x \in \Omega_{\delta_1}.$$

Proof of Theorem 1.1 For an arbitrary $\varepsilon \in \left(0, \frac{1}{4}\right)$, let $\xi_{+\varepsilon}$ be the unique positive root of

$$\xi^q - M_0(1 - \varepsilon)^{-1}\xi^n - 2\varepsilon = 0$$

and $\xi_{-\varepsilon}$ be the unique positive root of

$$\xi^q - m_0(1+\varepsilon)^{-1}\xi^n + 2\varepsilon = 0.$$

It follows that

$$m_0^{\frac{1}{q-n}} < \xi_{-\varepsilon} < \xi_{+\varepsilon} < (2M_0)^{\frac{1}{q-n}}.$$

By using $(\mathbf{g_3})$ and Lemma 2.3, we see that

$$\lim_{d(x)\to 0}(1 - d(x)\kappa_i(\overline{x})) = 1 \tag{3.3}$$

and

$$\lim_{d(x)\to 0}\left(\lambda\xi_{+\varepsilon}^n \frac{\Psi^n(d(x))}{g(\psi(d(x)))} + \frac{g(\xi_{+\varepsilon}\psi(d(x)))}{g(\psi(d(x)))}\right) = \xi_{+\varepsilon}^q \tag{3.4}$$

uniformly for $i = 1, 2, \cdots, n-1$ and $\xi_{+\varepsilon} \in \left[m_0^{\frac{1}{q-n}}, (2M_0)^{\frac{1}{q-n}}\right]$, respectively.

In addition, we obtain from (1.11) and (3.3) that there is a sufficiently small $\delta_\varepsilon \in \left(0, \min\left\{1, \frac{\delta_1}{2}\right\}\right)$ (δ_1 has been given in (3.1) corresponding to ε), such that

$$\frac{m_0}{1+\varepsilon} < \prod_{i=1}^{n-1}\frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})} < \frac{M_0}{1-\varepsilon}, \quad x \in \Omega_{2\delta_\varepsilon}. \tag{3.5}$$

For $\eta \in (0, \delta_\varepsilon)$, let

$$\overline{u}_\varepsilon(x) = \xi_{+\varepsilon}\Psi(d_1(x)), \quad x \in D_\eta^- = \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\eta, \quad d_1(x) := d(x) - \eta.$$

By Lemma 3.1 and a direct calculation, we see that for $x \in D_\eta^-$

$$\det D^2\overline{u}_\varepsilon(x) = \xi_{+\varepsilon}^n(\psi(d_1(x)))^{n-1}\psi'(d_1(x))\left(\prod_{i=1}^{n-1}\frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})}\right)$$

$$= \xi_{+\varepsilon}^n g(\psi(d_1(x)))\left(\prod_{i=1}^{n-1}\frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})}\right)$$

$$< g(\psi(d_1(x)))(\xi_{+\varepsilon}^n M_0(1-\varepsilon)^{-1} + \varepsilon)$$

and

$$\lambda\overline{u}_\varepsilon^n(x) + g(|\nabla\overline{u}_\varepsilon(x)|) = \lambda\xi_{+\varepsilon}^n(\Psi(d_1(x)))^n + g(\xi_{+\varepsilon}\psi(d_1(x)))$$

$$= g(\psi(d_1(x)))\left(\lambda\xi_{-\varepsilon}^n\frac{(\Psi(d_1(x)))^n}{g(\psi(d_1(x)))} + \frac{g(\xi_{+\varepsilon}\psi(d_1(x)))}{g(\psi(d_1(x)))}\right)$$

$$> g(\psi(d_1(x)))(\xi_{+\varepsilon}^n M_0(1-\varepsilon)^{-1} + \varepsilon).$$

Thus, $\det D^2\overline{u}_\varepsilon(x) \leq \lambda\overline{u}_\varepsilon^n(x) + g(|\nabla\overline{u}_\varepsilon(x)|)$ for $x \in D_\eta^-$, i.e., $\overline{u}_\varepsilon$ is a supersolution to (1.1) in D_η^-.

In a similar way, we can show that

$$\underline{u}_\varepsilon(x) = \xi_{-\varepsilon}\Psi(d_2(x)), \quad x \in D_\eta^+ = \Omega_{2\delta_\varepsilon - \eta}, \quad d_2(x) := d(x) + \eta$$

is a subsolution to (1.1) in D_η^+.

Now let u be an arbitrary solution to problem (1.1). We assert that there exists a positive constant M such that

$$u \leq M + \overline{u}_\varepsilon, \quad x \in D_\eta^-, \tag{3.6}$$

$$\underline{u}_\varepsilon \leq u + M, \quad x \in D_\eta^+. \tag{3.7}$$

In fact, we can choose a large M such that

$$u \leq M + \overline{u}_\varepsilon \quad \text{on } \Gamma_{2\delta_\varepsilon} := \{x \in \Omega : d(x) = 2\delta_\varepsilon\}.$$

Obviously, $\overline{u}_\varepsilon + M$ is also a supersolution of equation (1.1) in D_η^-. Since $u < \overline{u}_\varepsilon$ on $\Gamma_\sigma := \{x \in \Omega : d(x) = \eta\}$, (3.6) follows by Lemma 2.5.

In a similar way, we can show (3.7).

Hence, when $x \in D_\eta^- \cap D_\eta^+$, by letting $\eta \to 0$, we have

$$\xi_{-\varepsilon} - \frac{M}{\Psi(d(x))} \leq \frac{u(x)}{\Psi(d(x))} \leq \xi_{+\varepsilon} + \frac{M}{\Psi(d(x))}.$$

Consequently,

$$\xi_{-\varepsilon} \leq \lim_{d(x)\to 0} \inf \frac{u(x)}{\Psi(d(x))} \leq \lim_{d(x)\to 0} \sup \frac{u(x)}{\Psi(d(x))} \leq \xi_{+\varepsilon}. \tag{3.8}$$

Thus letting $\varepsilon \to 0$ in (3.8), we obtain (1.16).

Proof of Theorem 1.2

Case I $q < n + 1$.

For an arbitrary $\varepsilon \in \left(0, \min\left\{\frac{1}{4}, \frac{b_1}{4}\right\}\right)$, let $\xi_{+\varepsilon}$ be the unique positive root of

$$(b_1 - \varepsilon)\left(\frac{n+1-q}{q-n}\right)^q \xi^q - M_0(1-\varepsilon)^{-1}\left(\frac{n+1-q}{q-n}\right)^n \left(\frac{n+1-q}{q-n} + D_\theta\right)\xi^n - 2\varepsilon = 0,$$

and $\xi_{-\varepsilon}$ be the unique positive root of

$$(b_2 + \varepsilon)\left(\frac{n+1-q}{q-n}\right)^q \xi^q - m_0(1+\varepsilon)^{-1}\left(\frac{n+1-q}{q-n}\right)^n \left(\frac{n+1-q}{q-n} + D_\theta\right)\xi^n + 2\varepsilon = 0.$$

It follows that

$$\frac{q-n}{n+1-q}\left(\frac{m_0}{2b_2}\left(\frac{n}{q-n} + D_\theta\right)\right)^{\frac{1}{q-n}} < \xi_{-\varepsilon} < \xi_{+\varepsilon} < \frac{q-n}{n+1-q}\left(\frac{8M_0}{b_1}\left(\frac{n}{q-n} + D_\theta\right)\right)^{\frac{1}{q-n}}.$$

From Lemmas 2.1–2.2, we see that

$$\lim_{d(x)\to 0} \left(\lambda \xi_{+\varepsilon}^n \left(\frac{\Theta(d(x))}{\theta(d(x))}\right)^{n+1} + (b_1 - \varepsilon)\left(\frac{n+1-q}{q-n}\right)^q \xi_{+\varepsilon}^q - \varepsilon\right.$$

$$\left. - M_0(1-\varepsilon)^{-1}\left(\frac{n+1-q}{q-n}\right)^n \left(\frac{1}{q-n} - \frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}\right)\xi_{+\varepsilon}^n\right)$$

$$= (b_1 - \varepsilon)\left(\frac{n+1-q}{q-n}\right)^q \xi_{+\varepsilon}^q - \varepsilon$$

$$- M_0(1-\varepsilon)^{-1}\left(\frac{n+1-q}{q-n}\right)^n \left(\frac{n+1-q}{q-n} + D_\theta\right)\xi_{+\varepsilon}^n,$$

uniformly for $\xi_{+\varepsilon} \in [\underline{\xi}, \ \overline{\xi}]$.

Moreover, we obtain by $(\mathbf{b_1})$ and Lemma 2.2 $(\mathbf{i_2})$ that there is a sufficiently small $\delta_\varepsilon \in \left(0, \min\left\{1, \frac{\delta_1}{2}\right\}\right)$ corresponding to ε, such that for $\eta \in (0, \delta_\varepsilon)$,

$$(b_1 - \varepsilon)\theta^{n+1-q}(d(x) - \eta) < (b_1 - \varepsilon)\theta^{n+1-q}(d(x)) < b(x), \quad x \in D_\eta^- = \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\eta$$

and

$$b(x) < (b_2 + \varepsilon)\theta^{n+1-q}(d(x)) < (b_2 + \varepsilon)\theta^{n+1-q}(d(x) + \eta), \quad x \in D_\eta^+ = \Omega_{2\delta_\varepsilon - \eta}.$$

Let

$$\overline{u}_\varepsilon(x) = \xi_{+\varepsilon}(\Theta(d_1(x)))^{-\frac{n+1-q}{q-n}}, \quad x \in D_\eta^-, \ \ d_1(x) := d(x) - \eta.$$

By Lemma 3.1 and a direct calculation, we see that for $x \in D_\eta^-$

$$\det D^2 \overline{u}_\varepsilon(x) = \xi_{-\varepsilon}^n \left(\frac{n+1-q}{q-n}\right)^n \theta^{n+1}(d_1(x))(\Theta(d_1(x)))^{-\frac{q}{q-n}}$$

$$\left(\frac{1}{q-n} - \frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\right)\left(\prod_{i=1}^{n-1}\frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})}\right)$$

$$\leq \theta^{n+1}(d_1(x))(\Theta(d_1(x)))^{-\frac{q}{q-n}}\left(\xi_{-\varepsilon}^n \frac{M_0}{1-\varepsilon}\left(\frac{n+1-q}{q-n}\right)^n\right.$$

$$\left.\left(\frac{1}{q-n} - \frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\right) + \varepsilon\right)$$

and

$$\lambda \overline{u}_\varepsilon^n(x) + b(x)|\nabla \overline{u}_\varepsilon(x)|^q$$

$$= \theta^{n+1}(d_1(x))(\Theta(d_1(x)))^{-\frac{q}{q-n}}\left(\lambda\xi_{+\varepsilon}^n\left(\frac{\Theta(d_1(x))}{\theta(d_1(x))}\right)^{n+1}\right.$$

$$\left.+ \xi_{+\varepsilon}^q\left(\frac{n+1-q}{q-n}\right)^q\frac{b(x)}{\theta^{n+1-q}(d_1(x))}\right.$$

$$\geq \theta^{n+1}(d_1(x))(\Theta(d_1(x)))^{-\frac{q}{q+1-p}}\left(\lambda\xi_{+\varepsilon}^n\left(\frac{(\Theta(d_1(x)))}{\theta(d_1(x))}\right)^{n+1}\right.$$

$$\left.+ \xi_{+\varepsilon}^q(b_1 - \varepsilon)\left(\frac{n+1-q}{q-n}\right)^q\right)$$

$$\geq \theta^{n+1}(d_1(x))(\Theta(d_1(x)))^{-\frac{q}{q-n}}\left(\xi_{-\varepsilon}^n\frac{M_0}{1-\varepsilon}\left(\frac{n+1-q}{q-n}\right)^n\right.$$

$$\left.\left(\frac{1}{q-n} - \frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\right) + \varepsilon\right).$$

Thus, $\det D^2 \overline{u}_\varepsilon(x) \leq \lambda \overline{u}_\varepsilon^n(x) + b(x)|\nabla \overline{u}_\varepsilon(x)|^q$ for $x \in D_\eta^-$, i.e., $\overline{u}_\varepsilon$ is a supersolution to equation (1.1) in D_η^-.

In a similar way, we can show that

$$\underline{u}_\varepsilon(x) = \xi_{-\varepsilon}(\Theta(d_2(x)))^{-\frac{n+1-q}{q-n}}, \quad x \in D_\eta^+ = \Omega_{2\delta_\varepsilon - \eta}, \quad d_2(x) := d(x) + \eta$$

is a subsolution to equation (1.1) in D_η^+.

Case II $D_\theta > 0$ and $q = n + 1$.

For an arbitrary $\varepsilon \in \left(0, \min\left\{\frac{1}{4}, \frac{b_1}{4}\right\}\right)$, let $\xi_{+\varepsilon}$ be the unique positive root of

$$(b_1 - \varepsilon)\xi^{n+1} - M_0(1-\varepsilon)^{-1}D_\theta\xi^n - 2\varepsilon = 0,$$

and $\xi_{-\varepsilon}$ be the unique positive root of

$$(b_2 + \varepsilon)\xi^{n+1} - m_0(1-\varepsilon)^{-1}D_\theta\xi^n + 2\varepsilon = 0.$$

It follows that

$$\frac{m_0 D_\theta}{2b_2} < \xi_{-\varepsilon} < \xi_{+\varepsilon} < \frac{8M_0 D_\theta}{b_1}.$$

From Lemmas 2.1–2.2, we see that

$$\lim_{d(x)\to 0} \Big(\lambda\xi_{+\varepsilon}^n \Big(\frac{\Theta(d(x))}{\theta(d(x))}\Big)^{n+1}(-\ln(\Theta(d(x))))^n + (b_1 - \varepsilon)\xi_{+\varepsilon}^{n+1} - \varepsilon$$

$$- M_0(1-\varepsilon)^{-1}\xi_{+\varepsilon}^n \Big(1 - \frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}\Big)\Big)$$

$$= (b_1 - \varepsilon)\xi_{+\varepsilon}^{n+1} - \frac{M_0}{1-\varepsilon}D_\theta\xi_{+\varepsilon}^n - \varepsilon,$$

uniformly for $\xi_{+\varepsilon} \in \big[\frac{m_0 D_\theta}{2b_2}, \frac{8M_0 D_\theta}{b_1}\big]$.

In addition, we know from ($\mathbf{b_1}$) and Lemma 2.2 ($\mathbf{i_1}$) that there is a sufficiently small $\delta_\varepsilon \in (0, \min\{1, \frac{\delta}{2}\})$ corresponding to ε, such that

$$b_1 - \varepsilon < b(x) < b_2 + \varepsilon, \quad x \in \Omega_{2\delta_\varepsilon}.$$

Let $\overline{u}_\varepsilon(x) = \xi_{+\varepsilon}(-\ln(\Theta(d_1(x))))$, $x \in D_\eta^-$, $d_1(x) := d(x) - \eta$. By Lemma 3.1 and a direct calculation, we see that for $x \in D_\eta^-$

$$\det D^2\overline{u}_\varepsilon(x) = \xi_{+\varepsilon}^n \Big(\frac{\theta(d_1(x))}{\Theta(d_1(x))}\Big)^{n+1}\Big(1 - \frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\Big)\Big(\prod_{i=1}^{n-1}\frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})}\Big)$$

$$\leq \Big(\frac{\theta(d_1(x))}{\Theta(d_1(x))}\Big)^{n+1}\Big(\xi_{+\varepsilon}^n\frac{M_0}{1-\varepsilon}\Big(1 - \frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\Big) + \varepsilon\Big)$$

and

$$\lambda\overline{u}_\varepsilon^n(x) + b(x)|\nabla\overline{u}_\varepsilon(x)|^{n+1}$$

$$= \Big(\frac{\theta(d_1(x))}{\Theta(d_1(x))}\Big)^{n+1}\Big(\lambda\xi_{+\varepsilon}^n\Big(\frac{\Theta(d_1(x))}{\theta(d_1(x))}\Big)^{n+1}(-\ln(\Theta(d_1(x))))^n + \xi_{+\varepsilon}^{n+1}b(x)\Big)$$

$$\geq \Big(\frac{\theta(d_1(x))}{\Theta(d_1(x))}\Big)^{n+1}\Big(\lambda\xi_{+\varepsilon}^n\Big(\frac{\Theta(d_1(x))}{\theta(d_1(x))}\Big)^{n+1}(-\ln(\Theta(d_1(x))))^n + (b_1 - \varepsilon)\xi_{+\varepsilon}^{n+1}\Big)$$

$$\geq \Big(\frac{\theta(d_1(x))}{\Theta(d_1(x))}\Big)^{n+1}\Big(\xi_{+\varepsilon}^n\frac{M_0}{1-\varepsilon}\Big(1 - \frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\Big) + \varepsilon\Big).$$

Thus, $\det D^2\overline{u}_\varepsilon(x) \leq \lambda\overline{u}_\varepsilon^n(x) + b(x)g(|\nabla\overline{u}_\varepsilon(x)|)$ for $x \in D_\eta^-$, i.e., $\overline{u}_\varepsilon$ is a supersolution to equation (1.1) in D_η^-.

In a similar way, we can show that

$$\underline{u}_\varepsilon(x) = \xi_{-\varepsilon}(-\ln(\Theta(d_2(x)))), \quad x \in D_\eta^+ = \Omega_{2\delta_\varepsilon - \eta}, \ d_2(x) := d(x) + \eta,$$

is a subsolution to (1.1) in D_η^+.

The rest of the proof is similar to that of Theorem 1.1 and thus is omitted here.

4 Global Asymptotic Behavior

In this section, we consider the existence of convex solutions to problem (1.1) and prove Theorem 1.3. For this purpose, we establish a sub-supersolution method with the boundary restriction to problem (1.1) with $g(s) = s^q$ and $s \in [0, \infty)$.

Definition 4.1 *A strictly convex function $\underline{u} \in C^\infty(\Omega)$ is called a subsolution to problem (1.1) if*

$$\det D^2\underline{u}(x) \geq \lambda \underline{u}^n(x) + b(x)|\nabla \underline{u}(x)|^q, \quad \underline{u}(x) > 0, \ x \in \Omega, \ \underline{u}|_{\partial\Omega} = \infty. \tag{4.1}$$

Definition 4.2 *A strictly convex function $\overline{u} \in C^\infty(\Omega)$ is called a supersolution to problem (1.1) if*

$$\det D^2\overline{u}(x) \leq \lambda \overline{u}^n(x) + b(x)|\nabla \overline{u}(x)|^q, \quad \overline{u}(x) > 0, \ x \in \Omega, \ \overline{u}|_{\partial\Omega} = \infty. \tag{4.2}$$

The basis of our subsequent discussions is the following lemma which is formulated in terms of supersolution and subsolution.

Lemma 4.1 *Let $g(|\nu|) = |\nu|^q$, $q \in (n, n+1]$ and $b \in C^\infty(\Omega)$ be positive in Ω. Suppose that problem (1.1) has a supersolution $\overline{u}$ and a subsolution $\underline{u}$ such that $\underline{u} \leq \overline{u}$ on Ω, then problem (1.1) has at least one convex solution $u \in C^\infty(\Omega)$ in the order interval $[\underline{u}, \overline{u}]$.*

Proof Since b may be singular or equal to 0 on $\partial\Omega$, we cannot directly obtain the existence of convex solutions to problem (1.1) in the order interval $[\underline{u}, \overline{u}]$. To overcome this problem, let $\Omega_j \subset\subset \Omega$ be a C^∞-smooth and uniformly convex domain $(j = 1, 2, \cdots)$ with

$$\overline{\Omega}_j \subset \Omega_{j+1} \subset \Omega, \quad \bigcup_{j=1}^{\infty} \Omega_j = \Omega,$$

we consider the following perturbation problem

$$\det D^2u(x) = h(x, u(x), \nabla u(x)), \quad x \in \Omega_j, \ u|_{\partial\Omega_j} = \underline{u}, \tag{4.3}$$

where $h(x, s, \nu) = \lambda s^n + b(x)|\nu|^q$.

Obviously, $\overline{u}$ and $\underline{u}$ are also a supersolution and a subsolution of problem (4.3), respectively. It follows by [5, Theorem 7.1] that problem (4.3) has a strictly convex solution $u_j \in C^\infty(\overline{\Omega}_j)$ in order interval $[\underline{u}, \overline{u}]$.

Moreover, it follows by Lemma 2.5 that

$$\underline{u}(x) \leq u_j(x) \leq u_{j+1}(x) \leq \cdots \leq \overline{u}(x), \quad x \in \overline{\Omega}_j.$$

Note that for an arbitrary $x_0 \in \Omega$ there exists a natural number j_0 such that $x_0 \in \Omega_{j_0}$, we obtain that

$$\underline{u}(x_0) \leq u_{j_0}(x_0) \leq u_{j_0+1}(x_0) \leq \cdots \leq u_j(x_0) \leq u_{j+1}(x_0) \leq \cdots \leq \overline{u}(x_0). \tag{4.4}$$

So $\{u_j\}$ converges pointwise on $\overline{\Omega}$. We denote by $u(x)$ the limit of it. In the following we show that $u \in C^\infty(\Omega)$ is a solution to problem (1.1).

Now we need to estimate $\{u_j\}$. The method mainly comes from the proof of [15, Proposition 4.1].

For $l > 0$ and $j = 1, 2, \cdots$, we denote

$$A_l = \{x \in \Omega_j : \overline{u}(x) < l\},$$

$$U_{j,l} = \{x \in \Omega_j : u_j(x) < l\},$$

$$B_l = \{x \in \Omega_j : \underline{u}(x) < l\}.$$

From (4.4) we see that

$$A_l \subset U_{j,l} \subset B_l \subset\subset \Omega. \tag{4.5}$$

For an arbitrary C^∞-smooth uniformly convex domain K with $\overline{K} \subset \Omega$, we choose j_0 and $l > 0$ large enough such that $\overline{K} \subset A_{l/2}$. Let $C_0 = \max\limits_{x \in \overline{B}_{4l}} \overline{u}(x)$. We obtain that

$$\max\limits_{x \in \overline{U}_{j,2l}} u_j(x) \leq C_0 \quad \text{for } j = j_0, \ j_0 + 1, \cdots. \tag{4.6}$$

Moreover, by using the strict convexity of u_j,

$$\max\limits_{U_{j,2l}} |\nabla u_j| = \max\limits_{\partial U_{j,2l}} |\nabla u_j| \leq \max\limits_{x \in \partial B_{4l}} \frac{u_j(x) - 2l}{\text{dist}(\overline{U}_{j,2l}, \partial B_{4l})}$$

$$\leq \max\limits_{x \in \partial B_{4l}} \frac{\overline{u}(x) - 2l}{\text{dist}(\overline{B}_{2l}, \partial B_{4l})} \equiv C_1 \quad \text{for } j = j_0, \ j_0 + 1, \cdots, \tag{4.7}$$

where the last inequality follows from (4.4) and $U_{j,2l} \subset B_{2l}$.

Since $h \in C^\infty(\Omega \times [0,\infty) \times \mathbb{R}^n)$ is a positive function, it is clear that there exists a positive constant C_2 such that for all $(x, s, \nu) \in \overline{B}_{4l} \times [0, C_0] \times B_{C_1}(0)$, there hold

$$|\nabla \mathscr{F}(x, s, \nu)| \leq C_2, \quad |D^2 \mathscr{F}(x, s, \nu)| \leq C_2,$$

where $\mathscr{F} := \ln(h(x, s, \nu))$, $B_{C_1}(0)$ is a ball of radius C_1 centered at the origin.

Next, applying Pogorelov's interior estimates (see [14, 17.72, Theorem 17.19]), we obtain

$$|D^2 u_j(x)| \leq \frac{C_3}{\text{dist}(x, \partial U_{j,2l})}, \quad \forall x \in A_l \ \text{ for } j = j_0, \ j_0 + 1, \ \cdots, \tag{4.8}$$

where C_3 depends on C_0, C_1 and C_2, but is independent of j.

Since $A_l \subset A_{2l} \subset U_{j,2l}$, one can see that

$$\text{dist}(A_l, \partial U_{j,2l}) \geq \text{dist}(A_l, \partial A_{2l}).$$

It follows from (4.8) that

$$\|D^2 u_j\|_{C(\overline{A}_l)} \leq \frac{C_3}{\text{dist}(A_l, \partial A_{2l})}. \tag{4.9}$$

Finally, by using the Evans-Krylov theorem (see [13, Theorem A.42, Proposition A.43]), we obtain

$$\|u_j\|_{C^{2+\varsigma}(\overline{A}_{l/2})} \leq C_4, \quad j \geq j_0, \tag{4.10}$$

where C_5 is independent of $j \geq j_0$.

Thus

$$\|u_j\|_{C^{2,\varsigma}(\overline{K})} \leq C_4. \tag{4.11}$$

Thus $\{u_j\}_{j=j_0}^{\infty}$ has a subsequence that converges uniformly in the $C^{2+\varsigma}(\overline{K})$ norm to function $u \in C^2(\overline{K})$ and u satisfies the equation of (1.1) on K. From the fact that $\underline{u} \leq u \leq \overline{u}$ on K and K is arbitrary, we get $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ for any $x \in \Omega$, which implies that $\lim\limits_{d(x) \to 0} u(x) = +\infty$. Thus, we get a solution $u \in C^2(\Omega)$ to problem (1.1). Applying [5, Theorem 7.1] or [23, Theorem 1.1], we see that $u \in C^\infty(\Omega)$.

Proof of Theorem 1.3

Case I $q \in (n, n+1]$, $\sigma > -(n+1-q)$. By using $\beta = \frac{n+1+\sigma-q}{q-n} > 0$, we see that for $x \in \overline{\Omega}$,

$$\min_{x \in \overline{\Omega}} (v(x) + (1+\beta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}}) > 0.$$

Let $\underline{u}_\sigma(x) = m_1 v^{-\beta}(x)$, $x \in \Omega$, where m_1 is given as in (1.22). We have from $(\mathbf{b_2})$, Lemma 2.6 and a direct computation that $\underline{u}_\sigma$ is strictly convex on Ω and

$$
\begin{aligned}
\det D^2 \underline{u}_\sigma(x) &= (v(x))^{-\beta n - (n+1)} (m_1^n \beta^n (-1)^n \det D^2 v(x)(v(x) \\
&\quad + (1+\beta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}})) \\
&\geq (v(x))^{-\beta n - (n+1)} \min_{x \in \overline{\Omega}} (m_1^n \beta^n (-1)^n \det D^2 v(x)(v(x) \\
&\quad + (1+\beta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}})) \\
&\geq (v(x))^{\sigma - q(1+\beta)} \max_{x \in \overline{\Omega}} (\lambda m_1^n (v(x))^{n+1} + b_2 m_1^q \beta^q |\nabla v(x)|^q) \\
&\geq (v(x))^{\sigma - q(1+\beta)} (\lambda m_1^n (v(x))^{n+1} + b_2 m_1^q \beta^q |\nabla v(x)|^q) \\
&\geq \lambda \underline{u}_\sigma^n(x) + b(x) |\nabla \underline{u}_\sigma(x)|^q, \quad x \in \Omega,
\end{aligned}
$$

i.e., $\underline{u}_\sigma$ is a subsolution to problem (1.1) in Ω.

In a similar way, we can show that $\overline{u}_\sigma(x) = M_1 v^{-\beta}(x)$ is a supersolution to problem (1.1) in Ω, where M_1 is given as in (1.23). Obviously, $\underline{u}_\sigma \leq \overline{u}_\sigma$ on $\overline{\Omega}$. Thus, we see from Lemma 4.1 that problem (1.1) has a convex solution u_σ in the ordered interval $[\underline{u}_\sigma, \overline{u}_\sigma]$.

Case II $q = n+1+\sigma$. Let $\underline{u}_\sigma(x) = -m_2 \ln(v(x))$, $x \in \Omega$, where m_2 is given as in (1.24). We have from $(\mathbf{b_3})$, Lemma 2.6 and a direct computation that $\underline{u}_\sigma$ is strictly convex and

$$
\begin{aligned}
\det D^2 \underline{u}_\sigma(x) &= (v(x))^{-(n+1)} (m_2^n (-1)^n \det D^2 v(x)(v(x) + \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}})) \\
&\geq (v(x))^{-(n+1)} \min_{x \in \overline{\Omega}} (m_2^n (-1)^n \det D^2 v(x)(v(x) + \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^{\mathrm{T}})) \\
&\geq (v(x))^{\sigma - q} \max_{x \in \overline{\Omega}} (\lambda m_2^n (-\ln(v(x)))^n (v(x))^{n+1} + b_2 m_2^q |\nabla v(x)|^q) \\
&\geq \lambda \underline{u}_\sigma^n(x) + b(x) |\nabla \underline{u}_\sigma(x)|^q, \quad x \in \Omega,
\end{aligned}
$$

i.e., $\underline{u}_\sigma$ is a subsolution to problem (1.1) in Ω.

In a similar way, we can show that $\overline{u}_\sigma(x) = -M_2 \ln(v(x))$ is a supersolution to problem (1.1) in Ω, where M_2 is given as in (1.25).

Obviously, $\underline{u}_\sigma \le \overline{u}_\sigma$ on $\overline{\Omega}$. Hence the desired conclusion follows from Lemma 4.1. The proof is finished.

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Declarations

Conflicts of interest   The authors declare no conflicts of interest.

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