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Homogenization of Semilinear Parabolic PDEs with the Third Boundary Conditions*

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Abstract In this paper, the authors study the homogenization of the third boundary value problem for semilinear parabolic PDEs with rapidly oscillating periodic coefficients in the weak sense. Their method is entirely probabilistic, and builds upon the work of Tanaka (2020) and Buckdahn (1999). Backward stochastic differential equations with singular coefficients play an important role in this approach.

Keywords Homogenization, Weak solution, Third boundary value problem, Backward stochastic differential equations
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1 Introduction

In the study of porous media, composite materials and other systems of physics and mechanics, they frequently involve the boundary value problems with periodic structures. The process of establishing the macroscopic rigorous characterizations of such microscopic systems is called homogenization. It has been a highly active research area in mathematics for a long time, and a vast literature exists on this topic. See [3–4, 11, 15–16, 21] and references therein.

Homogenization theory has motivated the development of various notions of weak convergence in analysis. Such convergence can be better understood from the direction of probabilistic interpretation of the equation. Generally speaking, the probabilistic method begins with representing the quantities of interest by means of stochastic processes, and then attempts to prove convergence in laws of these processes. Hence it is also known as the averaging principle.

The goal of this paper is to use a probabilistic approach to study the homogenization of the following third boundary value problem for semilinear parabolic PDEs with rapidly oscillating periodic coefficients:

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$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) + L^\varepsilon u^\varepsilon(t,x) + f(x, u^\varepsilon(t,x), \nabla u^\varepsilon(t,x)) = 0, & (t,x) \in [0,T) \times \mathcal{O}, \\ \dfrac{1}{2}\dfrac{\partial u^\varepsilon}{\partial \upsilon^\varepsilon}(t,x) + c\Big(\dfrac{x}{\varepsilon}\Big) u^\varepsilon(t,x) = 0, & (t,x) \in [0,T) \times \partial\mathcal{O}, \\ u^\varepsilon(T,x) = g(x), & x \in \overline{\mathcal{O}}, \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is a bounded convex domain in $\mathbb{R}^d$ ($d \geq 2$) and the boundary $\partial\mathcal{O}$ is said to be of class C_b^1, that is, there exists a function $\Psi \in C_b^1(\mathbb{R}^d, \mathbb{R})$ such that

$$\mathcal{O} = \{x \in \mathbb{R}^d : \Psi(x) > 0\}, \quad \inf_{x \in \partial\mathcal{O}} |\nabla \Psi(x)| > 0. \tag{1.2}$$

Here $\frac{\partial u^\varepsilon}{\partial \upsilon^\varepsilon} = \vec{n}_i(x) a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u^\varepsilon}{\partial x_j}$ denotes the conormal derivative associated with L^ε, and $\vec{n}$ is the inward unit normal to $\partial\mathcal{O}$. The family of divergence-form operators L^ε is given by

$$\begin{aligned} L^\varepsilon &:= \frac{1}{2}\nabla \cdot \Big(A\Big(\frac{\cdot}{\varepsilon}\Big)\nabla\Big) + \frac{1}{\varepsilon} b\Big(\frac{\cdot}{\varepsilon}\Big) \cdot \nabla \\ &= \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\Big(\frac{\cdot}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\Big) + \frac{1}{\varepsilon}\sum_{i=1}^d b_i\Big(\frac{\cdot}{\varepsilon}\Big)\frac{\partial}{\partial x_i}, \end{aligned} \tag{1.3}$$

where the coefficients a and b are periodic.

$$A(x) = (a_{ij}(x))_{1 \leq i,j \leq d} : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$$

is a smooth, symmetric matrix-valued function satisfying the uniformly elliptic condition

$$\lambda^{-1} I_{d \times d} \leq A(\cdot) \leq \lambda I_{d \times d} \tag{1.4}$$

for some constant $\lambda \geq 1$. The function $c : \mathbb{R}^d \to [-\alpha, 0]$ is non-positive for some constant $\alpha \in [0, \infty)$ and periodic with respect to the orthonormal basis $\{(e_1, \cdots, e_d)\}$ of $\mathbb{R}^d$, i.e.,

$$c \in C_b(\mathbb{R}^d) \quad \text{and} \quad c(x + e_i) = c(x) \leq 0$$

for $i = 1, \cdots, d$. Assumptions 1–3 are made for the coefficients and will be listed in Section 2. The main result of this paper is the following theorem, which we will prove at the end of Section 5.

Theorem 1.1 *Let Assumptions* 1–3 *hold, then the semilinear PDE* (1.1) *has a unique weak solution* u^ε *for each* $\varepsilon > 0$. *Moreover, under* (3.6) *and* (5.7), *we have for each* $t \in [0,T], x \in \overline{\mathcal{O}}$,

$$u^\varepsilon(t,x) \to u^0(t,x) \qquad \text{as } \varepsilon \to 0,$$

where u^0 *satisfies the limit semilinear PDE*

$$\begin{cases} \dfrac{\partial u^0}{\partial t}(t,x) + \overline{L} u^0(t,x) + \overline{f}(x, u^0(t,x), \nabla u^0(t,x)) = 0, & (t,x) \in [0,T) \times \mathcal{O}, \\ \dfrac{1}{2}\dfrac{\partial u^0}{\partial \upsilon^0}(t,x) + \overline{C} u^0(t,x) = 0, & (t,x) \in [0,T) \times \partial\mathcal{O}, \\ u^0(T,x) = g(x), & x \in \overline{\mathcal{O}}. \end{cases} \tag{1.5}$$

Here

$$\overline{L} := \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(\overline{a}_{ij}\frac{\partial}{\partial x_j}\Big), \tag{1.6}$$

$\overline{f}$ *is given by* (5.2) *and* $\frac{\partial u^0}{\partial \overline{v}^0}$ *is the conormal derivative associated with* $\overline{L}$*. Moreover, the homogenized* (*or effective*) *coefficients* $\overline{A} = (\overline{a}_{ij})_{1\le i,j\le d}$ *and* $\overline{C}$ *are constant, which are given by* (5.1) *and* (5.3)*, respectively.*

For the linear case (i.e., $f = 0$ in (1.1)), the original probabilistic approach to the homogenization of the second order parabolic partial differential operators is presented in Chapter 3 of [2], which is based on the thoughts of Freĭdlin [9], that is finding harmonic functions (the so-called "auxiliary" problem). Hence the problem can be reduced to transforming the underlying Markov process into a martingale, whose quadratic variation has a deterministic limit by the ergodic theorem. By now, this method has been extended in various directions, such as in the case of periodic coefficients (see [15–16, 21, 28]) and the case when the coefficients are stationary random fields (see [4, 16, 20]). But for the nonlinear case (i.e., $f \neq 0$ in (1.1)), such PDEs are generally very hard to solve. Since the notion of nonlinear backward stochastic differential equations (BSDEs for short) was first introduced by Pardoux and Peng [22], BSDEs have been used effectively to solve certain PDEs with nonlinear terms (see [1, 19, 25] and references therein). In view of the connection between them, the probabilistic tool has also been widely used in order to prove homogenization results for certain classes of nonlinear PDEs. When rapidly oscillating nonlinear terms are of the type $f\big(t, \frac{x}{\varepsilon}, u^\varepsilon\big) + \nabla u^\varepsilon\, \widehat{f}\big(t, \frac{x}{\varepsilon}, u^\varepsilon\big)$ (see [16, 21]) or the type $f(x, u^\varepsilon) + \widehat{f}(x, u^\varepsilon)\, \|\nabla u^\varepsilon\|^2$ (see [11]), it successfully proves convergence in laws of the stochastic processes involved with the help of BSDEs. However, for more general nonlinearity in the gradient, we need another method in [3], which is to exploit the stability results of BSDEs. This strategy has also been employed in the homogenization of random PDEs (see [4]).

In this paper, we stress that the solutions we considered for PDEs are weak solutions, not viscosity solutions. In addition, we are interested in homogenization problems for which it is necessary to identify both the homogenized PDEs and the homogenized boundary conditions. There are few works dealing with homogenization of PDEs with boundary conditions by probabilistic methods (see [2, 16, 23] for Dirichlet boundary conditions and [5, 19] for Neumann boundary conditions). We mainly consider PDEs with the third boundary conditions, which will give rise to several difficulties both in analytic and probabilistic aspects. Firstly, due to the asymmetry of operator L in the auxiliary problems (3.1), we cannot expect an analogue of Meyers' result (see [18, Theorem 2]) to hold for every $p > 2$. We are able to prove, however, that under weak hypotheses (3.6) it continues to hold for some $p > 2$ by [12, Theorem 1]. Hence the L^p-estimates ($p > 2$) for gradients of solutions can be established in Theorem 3.1, which plays an important role in the proof of homogenization in Subsection 5.3. Secondly, for the property (5.8) of process X^ε, the method of [2, Lemma 3.9.2] is no longer applicable because of the appearance of the local time term and the highly oscillating term $\varepsilon^{-1}\widetilde{b}\big(\frac{X^\varepsilon}{\varepsilon}\big)\,\mathrm{d}s$ in (2.1). Here we are highly motivated by [28, Lemma 6.1] and flexibly use the property of the local time.

The organization of the paper is as follows. In Section 2, we define a $\mathbb{T}^d$-valued Markov process with generator L (defined by (2.2)) having periodic coefficients and a $\mathbb{T}^{d-1}$-valued

Markov process on the boundary with generator H_γ (defined by (2.4)). We then consider the invariant measures of these two processes respectively. We finally state the assumptions made throughout the paper. In Section 3, we deal with the periodic solutions of the auxiliary problems (3.1) and obtain the L^p-estimates (for some $p > 2$) of the gradient of the solution (Theorem 3.1). In Section 4, we mainly study the existence and uniqueness for a class of BSDEs with singular coefficients (Theorem 4.1). In Section 5, the homogenization result for the problem (1.1) is proved.

We use the following notation in this paper. For a matrix σ, its transpose and Hilbert-Schmidt norm are expressed by σ^* and $\|\sigma\| = (\sum\limits_{ij} \sigma_{ij}^2)^{\frac{1}{2}}$. Denote by $|x|$ the Euclidean norm of x in $\mathbb{R}^d$ and by $\langle x, y\rangle$ the inner product of $x, y \in \mathbb{R}^d$. The torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ will be used frequently and we shall always identify the periodic functions on $\mathbb{R}^d$ of period 1 with their restrictions on the torus $\mathbb{T}^d$. Denote by $L^2(\mathbb{T}^d)$ and $H^1(\mathbb{T}^d)$ the spaces of functions locally in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ which are $\mathbb{T}^d$-periodic. Thanks to the compactness of the torus $\mathbb{T}^d$, the injection from the space $H^1(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$ is compact.

2 General Assumptions and Preliminaries

This section is devoted to finding a $\mathbb{T}^d$-valued Markov process with generator L (defined by (2.2)) having periodic coefficients and a $\mathbb{T}^{d-1}$-valued Markov process on the boundary with generator H_γ (defined by (2.4)). We then consider the invariant measures of these two processes respectively, which will be used to define the homogenized coefficients in Section 5. In the end, we state the assumptions more precisely made on the system (1.1).

Given $\varepsilon > 0, x \in \mathcal{O}$, the differential operator L^ε inside $\mathcal{O}$ together with the Neumann boundary condition $\big\langle \gamma\big(\frac{x}{\varepsilon}\big), \nabla\cdot\big\rangle := \big\langle A\big(\frac{x}{\varepsilon}\big)\vec{n}, \nabla\cdot\big\rangle = 0$ on $\partial\mathcal{O}$ determines a unique reflecting diffusion process $(X^\varepsilon(s), \mathbb{P}^\varepsilon_{t,x}, t \in [0,T], x \in \mathcal{O})$ starting from x at time t. $\mathcal{F}^{X^\varepsilon}$ is the minimal admissible filtration generated by X^ε. Set $\widetilde{b} := (\widetilde{b}_1, \cdots, \widetilde{b}_d)^{\mathrm{T}}$, where $\widetilde{b}_i = \frac{1}{2}\sum\limits_{j=1}^{d} \frac{\partial a_{ij}}{\partial x_j} + b_i$. Then according to [17], it has the following decomposition

$$X^\varepsilon_s = x + M^\varepsilon_s + \frac{1}{\varepsilon}\int_t^s \widetilde{b}\Big(\frac{X^\varepsilon_r}{\varepsilon}\Big)\,\mathrm{d}r + \int_t^s \gamma\Big(\frac{X^\varepsilon_r}{\varepsilon}\Big)\,\mathrm{d}K^\varepsilon_r, \quad 0 \le t \le s \le T,\ \mathbb{P}^\varepsilon_{t,x} - \text{a.s.}, \tag{2.1}$$

where M^ε is a martingale additive functional of X^ε with quadratic cross-variation

$$d \ll M^{\varepsilon,i}, M^{\varepsilon,j} \gg_s = a_{ij}\Big(\frac{X^\varepsilon_s}{\varepsilon}\Big)\,\mathrm{d}s$$

and $K^\varepsilon_s = \int_t^s I_{\{X^\varepsilon_r \in \partial\mathcal{O}\}}\,\mathrm{d}K^\varepsilon_r$ is the boundary local time of X^ε.

Via the canonical quotient map $\pi : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$, we can define a $\mathbb{T}^d$-valued Markov process with generator L (defined by (2.2)) having periodic coefficients. Meanwhile, we also want to find a $\mathbb{T}^{d-1}$-valued Markov process on the boundary with generator H_γ (defined by (2.4)) by using the local time that serves as a time change function. Now consider the invariant measures of these two processes, respectively.

The invariant measure $m(x)\mathrm{d}x$ Let

$$L := \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big) + \sum_{i=1}^{d} b_i\frac{\partial}{\partial x_i}, \tag{2.2}$$

then it is well-known that the divergence-form operator L generates a Markov process on $\mathbb{R}^d$. By mapping all trajectories of this process on $\mathbb{R}^d$ to the torus $\mathbb{T}^d$, we can define a Markov process $\overline{X}$, which is $\mathbb{T}^d$-valued and generated by the operator L having periodic coefficients (see [2, Section 3.3.2] or [15, Section 3] for details). In view of the compactness of the torus $\mathbb{T}^d$, the process $\overline{X}$ is ergodic.

Moreover, from the maximum principle and the $\mathbb{T}^d$-periodicity of functions in $L^2(\mathbb{T}^d)$, it follows that any solution to $Lu = 0$ is constant. Hence by the Fredholm alternative theorem, there exists a unique solution to $L^*m = 0$ such that $\int_{\mathbb{T}^d} m(x)\,\mathrm{d}x = 1$. According to [2, 13], we can obtain that m is positive, continuous and in fact the density of the invariant measure of the process $\overline{X}$.

The invariant measure $\widetilde{m}(\mathrm{d}x)$ Given a function $\varphi \in C(\partial\mathcal{O})$ with bounded partial derivatives of order ≤ 2 and we consider the problem

$$\begin{cases} L\widetilde{u}(x) = 0 & \text{in } \mathcal{O}, \\ \widetilde{u}(x) = \varphi(x) & \text{on } \partial\mathcal{O}, \end{cases} \tag{2.3}$$

where $L = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla$. According to [6, Theorem 1.1], $\widetilde{u}(x) = E_x[\varphi(X_{\tau(\mathcal{O})})]$ is the unique continuous weak solution to (2.3), where X is the continuous diffusion process generated by the operator L and $\tau(\mathcal{O})$ is the first exit time from $\mathcal{O}$, that is $\tau(\mathcal{O}) := \inf\{t > 0 : X_t \notin \mathcal{O}\}$. Then define

$$H_\gamma\varphi(x) := \langle \gamma, \nabla\widetilde{u}\rangle(x), \quad x \in \partial\mathcal{O}. \tag{2.4}$$

Denote by (X^1, K^1) the solution of (2.1) with $\varepsilon = 1$. Since the local time K^1 increases if and only if X^1 hits the boundary $\partial\mathcal{O}$, we can obtain a Markov process on the boundary by putting $\widetilde{X}^1(s) := X^1(K^{-1}(s))$, where $K^{-1}(s)$ is the right continuous inverse $\sup\{t : K^1(t) \leq s\}$ of K^1. In [29, Theorem 4], it is shown that the operator H_γ is the generator of the process $\widetilde{X}^1$. By the periodicity of the coefficients, the Markov process $\widetilde{X}^1$ induces a Markov process $\widetilde{X}^1_{\mathbb{T}^{d-1}}$ on the torus $\mathbb{T}^{d-1}$. Combined with the compactness of $\mathbb{T}^{d-1}$ and Doeblin's theorem , we can deduce by a similar argument to that in [28, Lemma 4.3] that there exists a unique invariant measure $\widetilde{m}$ of the Markov process $\widetilde{X}^1_{\mathbb{T}^{d-1}}$.

We now list some general assumptions for the semilinear PDEs (1.1).

Assumption 1 *The functions a, b, c are all periodic of period 1 in each component. The coefficient b is bounded and c satisfies*

$$-\infty < -\alpha \leq c(x) \leq 0, \quad \forall x \in \mathbb{R}^d \tag{2.5}$$

for some positive constant α. Moreover, we assume that for every $1 \leq i \leq d$, $\sum_{j=1}^{d} \frac{\partial a_{ij}}{\partial x_j}(x) \in L^\infty(\mathbb{R}^d)$ and the following condition holds

$$-\frac{1}{2}\sum_{j=1}^{d}\int_{\mathbb{T}^d} a_{ij}(x)\,\frac{\partial m(x)}{\partial x_j}\,\mathrm{d}x + \int_{\mathbb{T}^d} b_i(x)m(x)\,\mathrm{d}x = 0. \tag{2.6}$$

Remark 2.1 The condition (2.6) is common and natural in the homogenization problem and the following comments will be helpful to understand it. Note that a_{ij} is smooth enough, then

(i) the operators L^ε can be rewritten as

$$L^\varepsilon = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\varepsilon}\sum_{i=1}^{d}\Big(\frac{1}{2}\sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial x_j}\Big(\frac{x}{\varepsilon}\Big) + b_i\Big(\frac{x}{\varepsilon}\Big)\Big)\frac{\partial}{\partial x_i}.$$

It is easy to see that (2.6) is the centering condition in the book of Bensoussan et al. [2] (see also [3, 16, 21, 28]).

(ii) The operator L can be written as

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\Big(\frac{1}{2}\sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial x_j}(x) + b_i(x)\Big)\frac{\partial}{\partial x_i}.$$

As in [28, Remark 4], (2.6) ensures that each component process $X_i(t)$ of the L-diffusion $X(t)$ is recurrent in the sense that $X_i(t)$ hits any state in $\mathbb{R}^d$ with probability 1.

(iii) In the Neumann boundary case, (2.6) corresponds to the condition (H.3) in [28], and in the Dirichlet boundary case, it corresponds to [16, (38)]. Moreover, under the condition (2.6), there will be a unique periodic solution to the auxiliary problem (3.1) in Section 3 for each $i = 1, \cdots, d$. Hence the solution of the problem (3.1) can be given by $\omega_i(x) = \int_0^\infty E_x[\widetilde{b}_i(\overline{X}_t)]\,\mathrm{d}t$ when $\overline{X}$ starts from $x \in \mathbb{T}^d$ at time 0.

Assumption 2 *The function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a bounded uniformly continuous function which satisfies*

(1) $(y_1 - y_2)(f(x, y_1, z) - f(x, y_2, z)) \le c_1(x)\,|y_1 - y_2|^2$, *where $c_1(x)$ is a Borel measurable function.*

(2) $|f(x, y, z_1) - f(x, y, z_2)| \le c_2\,|z_1 - z_2|$. *Here c_2 is a positive constant.*

Moreover, we assume

$$E_x^\varepsilon[\mathrm{e}^{2\int_0^T c_1^+(X^\varepsilon(s))\,\mathrm{d}s}] < \infty \tag{2.7}$$

for any fixed ε and also

$$E_x^0[\mathrm{e}^{2\int_0^T c_1^+(X^0(r))\mathrm{d}r}] < \infty \tag{2.8}$$

holds, where E_x^0 denotes the expectation under the law of the reflected Brownian motion X^0 with covariance matrix $\bar{a}$. Here (2.7)–(2.8) are imposed to ensure that the inequalities (5.23)–(5.24) in the proof of homogenization hold, respectively. In particular, they are clearly true when c_1 is a negative or bounded function.

Assumption 3 $g : \overline{\mathcal{O}} \to \mathbb{R}$ *is continuous and bounded. In this paper, we always extend the definition of the function g to $\mathbb{R}^d$ by setting its values to be zero off $\mathcal{O}$.*

3 The Auxiliary Periodic Problems

In this section, we study the periodic solutions of the auxiliary problems (3.1). This will make us to get rid of the highly oscillating terms in treating the reflecting diffusion process

X^ε in Section 5. The main result is Theorem 3.1, which gives the L^p-estimates ($p > 2$) of the gradient of the solution of the auxiliary problem. We are mainly inspired by the thoughts of [12, Theorem 1] and [18, Theorem 2].

Based on (1.4) and Assumption 1, we now consider the 1-periodic solutions in $H^1(\mathbb{T}^d)$ of the auxiliary problems

$$\begin{cases} L\omega_i = -\Big(\dfrac{1}{2}\displaystyle\sum_{j=1}^{d}\dfrac{\partial a_{ij}}{\partial x_j} + b_i\Big), \\ \displaystyle\int_{\mathbb{T}^d} \omega_i(x)\, m(x)\mathrm{d}x = 0 \end{cases} \tag{3.1}$$

in the weak sense for $i = 1, \cdots, d$. Combined with (2.6) and the Fredholm alternative theorem, it follows from [2, Theorem 3.3.5] that the solution to (3.1) exists and is unique. On the other hand, as stated in the proof of [15, Proposition 1] or [27, Theorem 7.2], each function ω_i is continuous and bounded. Define

$$\widetilde{\omega}_i(x) := x_i + \omega_i^\sharp(x), \quad \widetilde{\omega}_i^\varepsilon(x) := \varepsilon\, \widetilde{\omega}_i\Big(\frac{x}{\varepsilon}\Big)$$

for $i = 1, \cdots, d$, where $\omega_i^\sharp$ is the extension of ω_i by periodicity to the whole $\mathbb{R}^d$. Then in the weak sense, $\widetilde{\omega}_i$ satisfies

$$\frac{1}{2}\nabla \cdot (A\, \nabla\widetilde{\omega}_i) + b \cdot \nabla\widetilde{\omega}_i = 0.$$

Meanwhile,

$$\frac{1}{2}\nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big)\, \nabla\widetilde{\omega}_i^\varepsilon(x)\Big) + \frac{1}{\varepsilon}\, b\Big(\frac{x}{\varepsilon}\Big) \cdot \nabla\widetilde{\omega}_i^\varepsilon(x) = 0$$

holds for each function $\widetilde{\omega}_i^\varepsilon$. Hence $\widetilde{\omega}_i$ and $\widetilde{\omega}_i^\varepsilon$ are harmonic functions for operators L and L^ε, respectively. Since $\widetilde{\omega}_i^\varepsilon$ belongs to the domain of the quadratic form associated with the process X^ε, it follows from Fukushima's decomposition (see [10]) that

$$\begin{aligned} \mathrm{d}\widetilde{\omega}_i^\varepsilon(X_s^\varepsilon) &= \Big\langle \nabla\widetilde{\omega}_i\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big), \mathrm{d}M_s^\varepsilon\Big\rangle + \langle \nabla\widetilde{\omega}_i, \gamma\rangle\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\, \mathrm{d}K_s^\varepsilon \\ &=: \mathrm{d}\widetilde{M}_s^{\varepsilon,i} + \widetilde{\gamma}_i\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\, \mathrm{d}K_s^\varepsilon, \quad i = 1, \cdots, d,\ 0 \le t \le s \le T, \end{aligned} \tag{3.2}$$

starting from x at time t. Moreover, $\widetilde{M}_s^\varepsilon = (\widetilde{M}_s^{\varepsilon,1}, \cdots, \widetilde{M}_s^{\varepsilon,d})^{\mathrm{T}}$ is a local martingale with cross-variations

$$\begin{aligned} \ll \widetilde{M}^{\varepsilon,i}, \widetilde{M}^{\varepsilon,j} \gg_s &= \int_0^t \langle A\nabla\widetilde{\omega}_i, \nabla\widetilde{\omega}_j\rangle\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\, \mathrm{d}s \\ &=: \int_0^t \widehat{a}_{ij}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\, \mathrm{d}s. \end{aligned} \tag{3.3}$$

We take $\nabla\widetilde{\omega}_i(x)$ as the column vectors to form a matrix, and denote it by $\nabla\widetilde{\omega}(x)$. Let $\widetilde{\gamma}^\varepsilon(x) := (A\nabla\widetilde{\omega})^*\big(\frac{x}{\varepsilon}\big)\, \vec{n}(x)$ and $\widetilde{\omega}^\varepsilon(x) := (\widetilde{\omega}_1^\varepsilon(x), \cdots, \widetilde{\omega}_d^\varepsilon(x))^{\mathrm{T}}$, then it yields from (3.2) that

$$\mathrm{d}\widetilde{\omega}^\varepsilon(X_s^\varepsilon) = \mathrm{d}\widetilde{M}_s^\varepsilon + \widetilde{\gamma}^\varepsilon(X_s^\varepsilon)\, \mathrm{d}K_s^\varepsilon. \tag{3.4}$$

Remark 3.1 The operator $L = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla$ inside $\mathcal{O}$ equipped with the Neumann boundary condition $\langle A(x)\vec{n}, \nabla\cdot\rangle = 0$ on $\partial\mathcal{O}$ determines the reflecting diffusion process X^1.

Then the scaling relation shows that $\varepsilon X^1\big(\frac{\cdot}{\varepsilon^2}\big)$ is equivalent in law to $X^\varepsilon(\cdot)$. In addition, the measure $m(x)\mathrm{d}x$ is invariant for the reflected process X^1. That is, we have

$$\int_{\overline{\mathcal{O}}} E_x[f(X^1(t))]\, m(x)\mathrm{d}x = \int_{\overline{\mathcal{O}}} f(x)\, m(x)\mathrm{d}x \tag{3.5}$$

for any bounded and continuous function f over $\overline{\mathcal{O}}$.

It suffices to prove (3.5) holds for the function $f \in C_0^\infty(\mathcal{O})$. Indeed, it is well known that $v(t,x) := E_x[f(X^1(t))]$ gives the probabilistic representation to the problem

$$\begin{cases} \partial_t v(t,x) = Lv(t,x), & [0,t]\times\mathcal{O}, \\ \dfrac{\partial v}{\partial\gamma}(t,x) = 0, & [0,t]\times\partial\mathcal{O}, \\ v(0,x) = f(x), & x\in\mathcal{O}. \end{cases}$$

Then we obtain

$$\begin{aligned} \partial_t\Big(\int_{\overline{\mathcal{O}}} v(t,x)\, m(x)\mathrm{d}x\Big) &= \int_{\overline{\mathcal{O}}} Lv(t,x)\, m(x)\mathrm{d}x \\ &= \int_{\overline{\mathcal{O}}} v(t,x)\, L^* m(x)\mathrm{d}x \\ &= 0. \end{aligned}$$

This implies $\int_{\overline{\mathcal{O}}} v(t,x)\, m(x)\mathrm{d}x$ is constant with respect to t. In view of $v(0,x) = f(x)$, it yields that the equality (3.5) holds.

Based on the above displays, it should be pointed out that we can also use the ideas of [21, Proposition 2.4] to establish the following convergence result

$$\int_0^t \widehat{a}_{ij}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\mathrm{d}s \overset{\text{dist}}{=} \varepsilon^2 \int_0^{\frac{t}{\varepsilon^2}} \widehat{a}_{ij}(X_s^1)\mathrm{d}s \xrightarrow{\varepsilon\to 0} t\int_{\mathbb{T}^d} \langle A\nabla\widetilde{\omega}_i, \nabla\widetilde{\omega}_j\rangle(x)\, m(x)\mathrm{d}x$$

by applying the properties of m. More generally, we can prove the conclusion (5.10), which plays an important role in homogenization in Section 5.

Theorem 3.1 *Let $B(0,R)$ be the ball in $\mathbb{R}^d$ centered at 0 with radius $R>0$. Define the function $\mathcal{G} : B(0,2R)\times\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$,*

$$\begin{aligned} \mathcal{G}_j(x,\zeta) &:= \frac{1}{2}\sum_{i=1}^d a_{ij}(x)\zeta_i, \quad j = 1,\cdots,d, \\ \mathcal{G}_0(x,\zeta) &:= \sum_{i=1}^d b_i(x)\zeta_i \end{aligned}$$

for $\zeta = (\zeta_0,\cdots,\zeta_d)^{\mathrm{T}}\in\mathbb{R}^{d+1}$ and assume that the following conditions

$$\begin{aligned} &\langle\mathcal{G}(x,\zeta)-\mathcal{G}(x,\vartheta), \zeta-\vartheta\rangle \ge q_1|\zeta-\vartheta|^2, \quad q_1 > \frac{1}{2\lambda}, \\ &|\mathcal{G}(x,\zeta)-\mathcal{G}(x,\vartheta)| \le q_2|\zeta-\vartheta|, \quad q_2<\infty, \end{aligned} \tag{3.6}$$

hold for any $x\in B(0,2R)$ and $\zeta,\vartheta\in\mathbb{R}^{d+1}$. Then there exists a constant $Q(\lambda,d,q_1,q_2)>2$ such that for all $p\in[2,Q(\lambda,d,q_1,q_2))$, $\nabla\omega_i\in L^p(\mathbb{T}^d)$, $i=1,\cdots,d$.

Proof Setting $\widetilde{\omega}_{i,R}(x) := \widetilde{\omega}_i(x) - \int_{B(0,2R)} \widetilde{\omega}_i(x)\mathrm{d}x$, then it is easy to see that $\int_{B(0,2R)} \widetilde{\omega}_{i,R}(x)\mathrm{d}x = 0$ and each function $\widetilde{\omega}_{i,R} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies

$$\frac{1}{2}\nabla \cdot (A\,\nabla\widetilde{\omega}_{i,R}) + b \cdot \nabla\widetilde{\omega}_{i,R} = 0, \quad i = 1, \cdots, d$$

in the weak sense. In view of (1.4) and (3.6), it yields that the function $\mathcal{G}$ satisfies the conditions (4.1) in [12]. For $u \in H^1(B(0,2R))$, define the operator $\Lambda \in \mathcal{L}(H^1; L^2(B(0,2R); \mathbb{R}^{d+1}))$ by $\Lambda u := (u, \nabla u)^{\mathrm{T}}$. Let the operator $J : W_0^{1,2}(B(0,2R)) \to W^{-1,2}(B(0,2R))$ be

$$\begin{aligned}\langle Ju, v\rangle &:= \int_{B(0,2R)} \mathcal{G}(\cdot, \Lambda u) \cdot \Lambda v \,\mathrm{d}x\\ &= \int_{B(0,2R)} \Big(\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i}v(x)\Big)\,\mathrm{d}x, \quad \forall v \in W_0^{1,2}. \end{aligned} \tag{3.7}$$

Hence by [12, Theorem 1] and a similar argument as that in [18, Theorem 2], we can deduce that the existence of constants $Q(\lambda, d, q_1, q_2) > 2$ and $C(\lambda, p, d) > 0$ such that for all $p \in [2, Q(\lambda, d, q_1, q_2))$,

$$\|\nabla\widetilde{\omega}_{i,R}\|_{p,B(0,R)} \le C(\lambda, p, d)\, R^{d(\frac{1}{p}-\frac{1}{2})-1}\,\|\widetilde{\omega}_{i,R}\|_{2,B(0,2R)}, \tag{3.8}$$

where $\|\cdot\|_{p,B(0,R)}$ denotes the norm in $L^p(B(0,R), \mathrm{d}x)$.

On the other hand, in view of $\int_{B(0,2R)} \widetilde{\omega}_{i,R}(x)\mathrm{d}x = 0$, it follows from Poincaré's inequality that

$$\|\widetilde{\omega}_{i,R}\|_{2,B(0,2R)} \le C(d)\, R\, \|\nabla\widetilde{\omega}_{i,R}\|_{2,B(0,2R)}.$$

Hence combined with (3.8), we have

$$R^{-\frac{d}{p}}\|\nabla\widetilde{\omega}_{i,R}\|_{p,B(0,R)} \le C(\lambda, p, d)\, R^{-\frac{d}{2}}\|\nabla\widetilde{\omega}_{i,R}\|_{2,B(0,2R)}.$$

Moreover, since

$$\begin{aligned}&\limsup_{R\to\infty} R^{-d}\,\|\nabla\widetilde{\omega}_{i,R}\|^2_{2,B(0,2R)}\\ &= \limsup_{R\to\infty} \frac{1}{R^d}\int_{B(0,2R)} \|\nabla\widetilde{\omega}_{i,R}(x)\|^2\,\mathrm{d}x\\ &= \limsup_{R\to\infty} \frac{1}{R^d}\int_{B(0,2R)} \|e_i + \nabla\omega_i^\sharp(x)\|^2\,\mathrm{d}x\\ &= |B(0,2)|\int_{\mathbb{T}^d} \|e_i + \nabla\omega_i(x)\|^2\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\limsup_{R\to\infty} R^{-d}\,\|\nabla\widetilde{\omega}_{i,R}\|^p_{p,B(0,R)}\\ &= \limsup_{R\to\infty} \frac{1}{R^d}\int_{B(0,R)} \|\nabla\widetilde{\omega}_{i,R}(x)\|^p\,\mathrm{d}x\\ &\ge \limsup_{R\to\infty} \frac{1}{R^d}\int_{B(0,R)} [\|e_i + \nabla\omega_i^\sharp(x)\| \wedge n]^p\,\mathrm{d}x\end{aligned}$$

$$= |B(0,1)| \int_{\mathbb{T}^d} [\|e_i + \nabla\omega_i(x)\| \wedge n]^p \,\mathrm{d}x,$$

the Monotone Convergence Theorem implies

$$\|e_i + \nabla\omega_i(x)\|_{L^p(\mathbb{T}^d)} \le C(\lambda, p, d)\|e_i + \nabla\omega_i(x)\|_{L^2(\mathbb{T}^d)} < \infty.$$

The proof is complete.

Remark 3.2 The condition (3.6) mean that the operator J defined by (3.7) is strongly monotone and Lipschitzian. More precisely, $(\zeta_0 - \vartheta_0)\big(\sum\limits_{i=1}^{d} b_i(x)(\zeta_i - \vartheta_i)\big) \ge \big(q_1 - \frac{1}{2\lambda}\big)\,|\zeta - \vartheta|^2$ and $\big(\sum\limits_{i=1}^{d} b_i(x)(\zeta_i - \vartheta_i)\big)^2 + \sum\limits_{j=1}^{d}\big(\sum\limits_{i=1}^{d} a_{ij}(x)(\zeta_i - \vartheta_i)\big)^2 \le 5q_2^2|\zeta - \vartheta|^2$. Clearly, (3.6) holds when the coefficients a_{ij} and b are bounded.

4 BSDEs with Singular Coefficients

This section is independent of other sections and devoted to studying the existence and uniqueness of a class of BSDEs with singular coefficients, which involves the integral with respect to the local time of a reflecting diffusion process. The main result is Theorem 4.1 and it implies in Section 5 that for any fixed ε, there exists a unique pair $(Y_s^\varepsilon, Z_s^\varepsilon)_{s\in[t,T]}$ of progressively measurable processes satisfying (5.5)–(5.6).

For any fixed ε, let $(\Omega, \mathbb{P}, \mathcal{F}_t)$ be the probability space carrying the reflecting diffusion process $X(t), t \ge 0$ described in Section 2 and $M(t), K(t)$ are the martingale part and local time of X, respectively. By the martingale representation theorem in [31, Theorem 2.1], we mainly study the existence and uniqueness of solutions for a class of BSDEs associated with the martingale part $M(t)$ and the local time $K(t)$.

Let $F(\omega, s, y, z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $h(\omega, s, y) : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be given progressively measurable functions. For simplicity, we omit the random parameter ω. Assume that they are continuous in y and satisfy the following conditions:

(A.1) $(y_1 - y_2)(F(s, y_1, z) - F(s, y_2, z)) \le d_1(s)|y_1 - y_2|^2$,

(A.2) $(y_1 - y_2)(h(s, y_1) - h(s, y_2)) \le \beta(s)|y_1 - y_2|^2$,

(A.3) $|F(s, y, z_1) - F(s, y, z_2)| \le d_2|z_1 - z_2|$,

(A.4) $|F(s, y, z)| \le |F(s, 0, z)| + d_3(s)(1 + |y|)$,

(A.5) $|h(s, y)| \le |h(s, 0)| + d_3(s)(1 + |y|)$,

where $d_1(s), d_3(s)$ are progressively measurable stochastic processes, d_2 is a positive constant and $\beta(s) < 0$ for all $s \in [0,T]$. Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$.

Lemma 4.1 *Denote*

$$\varphi(s) := \int_0^s d(u)\mathrm{d}u + \int_0^s \mu(u)\,\mathrm{d}K_u,$$

where $d(s) := 2d_1^+(s)$ *and* $\mu(s) := 2(\beta(s)+1)$. *Let* $E[\mathrm{e}^{\varphi(T)}|\xi|^2] < \infty$, $E[\int_0^T \mathrm{e}^{\varphi(s)}\,|h(s,0)|^2\mathrm{d}K_s] < \infty$ *and*

$$E\Big[\int_0^T \mathrm{e}^{\varphi(s)}\,(|F(s,0,0)|^2 + |d_3(s)|^2)\mathrm{d}s\Big] < \infty,$$

then there exists a unique solution (Y, Z) *to the following BSDE*:

$$Y(t) = \xi + \int_t^T F(s, Y(s), Z(s))\,\mathrm{d}s + \int_t^T h(s, Y(s))\,\mathrm{d}K_s - \int_t^T \langle Z(s), \mathrm{d}M(s)\rangle. \tag{4.1}$$

Furthermore,

$$E\Big[\sup_{t\in[0,T]} \mathrm{e}^{\varphi(t)}|Y(t)|^2\Big] < \infty, \quad E\Big[\int_0^T \mathrm{e}^{\varphi(s)}\|Z(s)\|^2\,\mathrm{d}s\Big] < \infty \tag{4.2}$$

and

$$E\Big[\int_0^T \mathrm{e}^{\varphi(s)}|Y(s)|^2\,\mathrm{d}K_s\Big] < \infty. \tag{4.3}$$

Proof Uniqueness. Suppose $(Y^1(t), Z^1(t)), (Y^2(t), Z^2(t))$ are two solutions to (4.1). By Itô's formula and (A.1)–(A.3), we have

$$\begin{aligned}
&\mathrm{e}^{\varphi(t)}|Y^1(t) - Y^2(t)|^2 + \int_t^T \mathrm{e}^{\varphi(s)}\langle a(X(s))(Z^1(s) - Z^2(s)), Z^1(s) - Z^2(s)\rangle\,\mathrm{d}s \\
= &-\int_t^T d(s)\,\mathrm{e}^{\varphi(s)}|Y^1(s) - Y^2(s)|^2\,\mathrm{d}s - \int_t^T \mu(s)\,\mathrm{e}^{\varphi(s)}|Y^1(s) - Y^2(s)|^2\,\mathrm{d}K_s \\
&+ 2\int_t^T \mathrm{e}^{\varphi(s)}(Y^1(s) - Y^2(s))\,(F(s, Y^1(s), Z^1(s)) - F(s, Y^2(s), Z^2(s)))\,\mathrm{d}s \\
&+ 2\int_t^T \mathrm{e}^{\varphi(s)}(Y^1(s) - Y^2(s))\,(h(s, Y^1(s)) - h(s, Y^2(s)))\,\mathrm{d}K_s \\
&- 2\int_t^T \mathrm{e}^{\varphi(s)}(Y^1(s) - Y^2(s))\,\langle Z^1(s) - Z^2(s), \mathrm{d}M(s)\rangle \\
\le &-\int_t^T d(s)\,\mathrm{e}^{\varphi(s)}|Y^1(s) - Y^2(s)|^2\,\mathrm{d}s + 2\int_t^T d_1(s)\,\mathrm{e}^{\varphi(s)}\,|Y^1(s) - Y^2(s)|^2\mathrm{d}s \\
&+ 2\int_t^T d_2\,\mathrm{e}^{\varphi(s)}\,|Y^1(s) - Y^2(s)||Z^1(s) - Z^2(s)|\mathrm{d}s - 2\int_t^T \mathrm{e}^{\varphi(s)}|Y^1(s) - Y^2(s)|^2\,\mathrm{d}K_s \\
&- 2\int_t^T \mathrm{e}^{\varphi(s)}(Y^1(s) - Y^2(s))\,\langle Z^1(s) - Z^2(s), \mathrm{d}M(s)\rangle \\
\le &\,\frac{1}{2}\int_t^T \frac{1}{\lambda}\mathrm{e}^{\varphi(s)}\,|Z^1(s) - Z^2(s)|^2\mathrm{d}s + 8d_2^2\lambda\int_t^T \mathrm{e}^{\varphi(s)}\,|Y^1(s) - Y^2(s)|^2\mathrm{d}s \\
&- 2\int_t^T \mathrm{e}^{\varphi(s)}(Y^1(s) - Y^2(s))\,\langle Z^1(s) - Z^2(s), \mathrm{d}M(s)\rangle - 2\int_t^T \mathrm{e}^{\varphi(s)}|Y^1(s) - Y^2(s)|^2\,\mathrm{d}K_s.
\end{aligned}$$

Thus combined with (1.4) and (A.2), we can obtain

$$\begin{aligned}
&\mathrm{e}^{\varphi(t)}|Y^1(t) - Y^2(t)|^2 + \frac{1}{2}\int_t^T \mathrm{e}^{\varphi(s)}\langle a(X(s))(Z^1(s) - Z^2(s)), Z^1(s) - Z^2(s)\rangle\,\mathrm{d}s \\
\le &-2\int_t^T \mathrm{e}^{\varphi(s)}(Y^1(s) - Y^2(s))\,\langle Z^1(s) - Z^2(s), \mathrm{d}M(s)\rangle \\
&+ 8d_2^2\lambda\int_t^T \mathrm{e}^{\varphi(s)}\,|Y^1(s) - Y^2(s)|^2\mathrm{d}s.
\end{aligned} \tag{4.4}$$

Taking expectation in the above inequality, it yields that

$$E[\mathrm{e}^{\varphi(t)}|Y^1(t)-Y^2(t)|^2] \le C_\lambda \int_t^T E[\mathrm{e}^{\varphi(s)}\,|Y^1(s)-Y^2(s)|^2]\,\mathrm{d}s.$$

By Gronwall's inequality, we conclude that $\forall t, Y^1(t) = Y^2(t)$, a.s. and hence $Z^1(t) = Z^2(t)$, a.s. by (4.4).

Existence. Define $F_n(t,y,z) := \int_{\mathbb{R}} F(t,x,z)\phi_n(y-x)\mathrm{d}x$ and $h_n(t,y) := \int_{\mathbb{R}} h(t,x)\phi_n(y-x)\mathrm{d}x$, where $\phi_n(x) := n\phi(nx)$ and $\phi \in C_0^\infty(\mathbb{R})$ is an even, nonnegative function with $\int_{\mathbb{R}} \phi(x)\mathrm{d}x = 1$. Hence, it is easy to see that for each $n \ge 1$,

$$\begin{aligned}&|F_n(t,y_1,z) - F_n(t,y_2,z)| \le C_n|y_1-y_2|,\\ &|h_n(t,y_1) - h_n(t,y_2)| \le C_n'|y_1-y_2|, \quad y_1,y_2 \in \mathbb{R}\end{aligned} \tag{4.5}$$

for some constants C_n and C_n'. Furthermore, since functions F and h are continuous in y, we have $F_n(t,y,z) \to F(t,y,z)$ and $h_n(t,y) \to h(t,y)$ as $n \to \infty$.

Consider the following BSDE:

$$Y_n(t) = \xi + \int_t^T F_n(s,Y_n(s),Z_n(s))\mathrm{d}s + \int_t^T h_n(s,Y_n(s))\mathrm{d}K_s - \int_t^T \langle Z_n(s), \mathrm{d}M(s)\rangle. \tag{4.6}$$

In view of (4.5) and the assumptions (A.3)–(A.5), we deduce from [24, Theorem 1.6] that (4.6) admits a unique solution (Y_n, Z_n). Now, our aim is to show that there exists a convergent subsequence (Y_{n_k}, Z_{n_k}). Indeed by Itô's formula, it yields that

$$\begin{aligned}&\mathrm{e}^{\varphi(t)}|Y_n(t)|^2 + \int_t^T \mathrm{e}^{\varphi(s)}\mu(s)|Y_n(s)|^2\,\mathrm{d}K_s + \int_t^T \mathrm{e}^{\varphi(s)}\langle a(X(s))Z_n(s), Z_n(s)\rangle\,\mathrm{d}s\\ &= \mathrm{e}^{\varphi(T)}|\xi|^2 - \int_t^T \mathrm{e}^{\varphi(s)}\,d(s)|Y_n(s)|^2\,\mathrm{d}s + 2\int_t^T \mathrm{e}^{\varphi(s)}\,Y_n(s)\,F_n(s,Y_n(s),Z_n(s))\,\mathrm{d}s\\ &\quad + 2\int_t^T \mathrm{e}^{\varphi(s)}\,Y_n(s)\,h_n(s,Y_n(s))\,\mathrm{d}K_s - 2\int_t^T \mathrm{e}^{\varphi(s)}\,Y_n(s)\,\langle Z_n(s), \mathrm{d}M(s)\rangle.\end{aligned}$$

By (A.1)–(A.4), (1.4) and Young's inequality, we have

$$\begin{aligned}&2\int_t^T \mathrm{e}^{\varphi(s)}\,Y_n(s)F_n(s,Y_n(s),Z_n(s))\,\mathrm{d}s\\ &\le 2\int_t^T \mathrm{e}^{\varphi(s)}\,d_1(s)|Y_n(s)|^2\,\mathrm{d}s + \frac{1}{2}\int_t^T \mathrm{e}^{\varphi(s)}\langle a(X(s))Z_n(s), Z_n(s)\rangle\,\mathrm{d}s\\ &\quad + (8d_2^2\lambda + 1)\int_t^T \mathrm{e}^{\varphi(s)}\,|Y_n(s)|^2\mathrm{d}s + \int_t^T \mathrm{e}^{\varphi(s)}\,|F(s,0,0)|^2\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&2\int_t^T \mathrm{e}^{\varphi(s)}\,Y_n(s)h_n(s,Y_n(s))\,\mathrm{d}K_s\\ &\le 2\int_t^T \mathrm{e}^{\varphi(s)}\,\beta(s)|Y_n(s)|^2\,\mathrm{d}K_s + 2\int_t^T \mathrm{e}^{\varphi(s)}\,|Y_n(s)||h(s,0)|^2\,\mathrm{d}K_s\\ &\le \int_t^T \mathrm{e}^{\varphi(s)}\,(2\beta(s)+1)|Y_n(s)|^2\,\mathrm{d}K_s + \int_t^T \mathrm{e}^{\varphi(s)}\,|h(s,0)|^2\,\mathrm{d}K_s.\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\mathrm{e}^{\varphi(t)}|Y_n(t)|^2+\int_t^T \mathrm{e}^{\varphi(s)}|Y_n(s)|^2\,\mathrm{d}K_s+\frac{1}{2}\int_t^T \mathrm{e}^{\varphi(s)}\langle a(X(s))Z_n(s),Z_n(s)\rangle\,\mathrm{d}s\\
&\le \mathrm{e}^{\varphi(T)}|\xi|^2+C_\lambda\int_t^T \mathrm{e}^{\varphi(s)}\,|Y_n(s)|^2\mathrm{d}s+\int_t^T \mathrm{e}^{\varphi(s)}\,|F(s,0,0)|^2\mathrm{d}s\\
&\quad+\int_t^T \mathrm{e}^{\varphi(s)}\,|h(s,0)|^2\,\mathrm{d}K_s-2\int_t^T \mathrm{e}^{\varphi(s)}\,Y_n(s)\langle Z_n(s),\mathrm{d}M(s)\rangle. \qquad (4.7)
\end{aligned}
$$

Taking expectation and by Gronwall's inequality, it yields that

$$
\begin{aligned}
&\sup_n \sup_{0\le t\le T} E[\mathrm{e}^{\varphi(t)}|Y_n(t)|^2]\\
&\le C\Big\{E[\mathrm{e}^{\varphi(T)}|\xi|^2]+E\Big[\int_t^T \mathrm{e}^{\varphi(s)}\,|F(s,0,0)|^2\mathrm{d}s\Big]+E\Big[\int_t^T \mathrm{e}^{\varphi(s)}\,|h(s,0)|^2\,\mathrm{d}K_s\Big]\Big\}\\
&<\infty. \qquad (4.8)
\end{aligned}
$$

Hence, we deduce that

$$
\sup_n E\Big[\int_t^T \mathrm{e}^{\varphi(s)}\langle a(X(s))Z_n(s),Z_n(s)\rangle\,\mathrm{d}s\Big]<\infty \qquad (4.9)
$$

and

$$
\sup_n E\Big[\int_t^T \mathrm{e}^{\varphi(s)}|Y_n(s)|^2\,\mathrm{d}K_s\Big]<\infty. \qquad (4.10)
$$

Furthermore, combined with the conditions of this lemma, we also obtain from (4.7)–(4.8) that there exists some constant C such that

$$
\begin{aligned}
&E\Big[\sup_{t\in[0,T]} \mathrm{e}^{\varphi(t)}|Y_n(t)|^2\Big]\\
&\le E[\mathrm{e}^{\varphi(T)}|\xi|^2]+CE\Big[\int_0^T \mathrm{e}^{\varphi(s)}\,|Y_n(s)|^2\mathrm{d}s\Big]\\
&\quad+E\Big[\int_0^T \mathrm{e}^{\varphi(s)}\,|F(s,0,0)|^2\mathrm{d}s\Big]+E\Big[\int_0^T \mathrm{e}^{\varphi(s)}\,|h(s,0)|^2\,\mathrm{d}K_s\Big]\\
&\quad+CE\Big[\sup_{t\in[0,T]}\int_0^t \mathrm{e}^{\varphi(s)}\,Y_n(s)\langle Z_n(s),\mathrm{d}M(s)\rangle\Big]\\
&\le C+CE\Big[\sup_{t\in[0,T]}\int_0^t \mathrm{e}^{\varphi(s)}\,Y_n(s)\langle Z_n(s),\mathrm{d}M(s)\rangle\Big].
\end{aligned}
$$

By both Burkhölder's and Young's inequalities, it implies

$$
\begin{aligned}
&CE\Big[\sup_{t\in[0,T]}\int_0^t \mathrm{e}^{\varphi(s)}\,Y_n(s)\langle Z_n(s),\mathrm{d}M(s)\rangle\Big]\\
&\le CE\Big[\Big(\int_0^T \mathrm{e}^{2\varphi(s)}\,Y_n^2(s)\langle a(X(s))Z_n(s),Z_n(s)\rangle\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\
&\le \frac{1}{2}E\Big[\sup_{s\in[0,T]} \mathrm{e}^{\varphi(s)}\,|Y_n(s)|^2\Big]+CE\Big[\int_0^T \mathrm{e}^{\varphi(s)}\,\langle a(X(s))Z_n(s),Z_n(s)\rangle\,\mathrm{d}s\Big].
\end{aligned}
$$

Then it follows from (4.9) that

$$\sup_n E\Big[\sup_{t\in[0,T]} \mathrm{e}^{\varphi(t)}|Y_n(t)|^2\Big] < \infty. \tag{4.11}$$

In view of (4.9)–(4.11), we can extract a subsequence n_k such that $Y_{n_k}(t)\mathrm{e}^{\frac{1}{2}\varphi(t)}$ converges to some $\widehat{Y}(t)$ in $L^2(\Omega, L^\infty[0,T])$ equipped with the weak star topology. In addition, $Z_{n_k}(t)\mathrm{e}^{\frac{1}{2}\varphi(t)}$ converges weakly to some $\widehat{Z}(t)$ in $L^2([0,T]\times\Omega;\mathbb{R})$. Since

$$\begin{aligned}\mathrm{e}^{\frac{1}{2}\varphi(t)}Y_{n_k}(t) &= \mathrm{e}^{\frac{1}{2}\varphi(T)}\xi + \int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)}\, F_{n_k}(s, Y_{n_k}(s), Z_{n_k}(s))\,\mathrm{d}s \\ &\quad + \int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)}\, h_{n_k}(s, Y_{n_k}(s))\,\mathrm{d}K_s - \frac{1}{2}\int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)}\, Y_{n_k}(s)d(s)\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)}\, Y_{n_k}(s)\mu(s)\,\mathrm{d}K_s - \int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)}\,\langle Z_{n_k}(s), \mathrm{d}M(s)\rangle,\end{aligned}$$

letting $k\to\infty$ and by the same arguments in the proof of [24, Proposition 1.8, p. 546–547], we conclude that the limit $(\widehat{Y}, \widehat{Z})$ satisfies

$$\begin{aligned}\widehat{Y}(t) &= \mathrm{e}^{\frac{1}{2}\varphi(T)}\xi + \int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)} \times F\big(s, \mathrm{e}^{-\frac{1}{2}\varphi(s)}\widehat{Y}(s), \mathrm{e}^{-\frac{1}{2}\varphi(s)}\widehat{Z}(s)\big)\,\mathrm{d}s \\ &\quad + \int_t^T \mathrm{e}^{\frac{1}{2}\varphi(s)}\, h(s, \mathrm{e}^{-\frac{1}{2}\varphi(s)}\widehat{Y}(s))\,\mathrm{d}K_s - \int_t^T \langle\widehat{Z}(s), \mathrm{d}M(s)\rangle \\ &\quad - \frac{1}{2}\int_t^T \widehat{Y}(s)d(s)\,\mathrm{d}s - \frac{1}{2}\int_t^T \widehat{Y}(s)\mu(s)\,\mathrm{d}K_s.\end{aligned}$$

Define $Y(t) := \mathrm{e}^{-\frac{1}{2}\varphi(t)}\,\widehat{Y}(t)$, and $Z(t) := \mathrm{e}^{-\frac{1}{2}\varphi(t)}\,\widehat{Z}(t)$, then Itô's formula yields that

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\mathrm{d}s + \int_t^T h(s, Y(s))\mathrm{d}K_s - \int_t^T \langle\widehat{Z}(s), \mathrm{d}M(s)\rangle,$$

which implies (Y, Z) is a solution to the backward equation (4.1). Applying Fatou's lemma, (4.2)–(4.3) follow from the above proof.

Now we apply Lemma 4.1 to a particular situation. Let $F(x,y,z) : \mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $h(x,y) : \mathbb{R}^d\times\mathbb{R}$ be Borel measurable functions. Assume that they are continuous in y and satisfy the following conditions:

(B.1) $(y_1-y_2)(F(x,y_1,z)-F(x,y_2,z)) \le d_1(x)|y_1-y_2|^2$,

(B.2) $(y_1-y_2)(h(x,y_1)-h(x,y_2)) \le \beta(x)|y_1-y_2|^2$,

(B.3) $|F(x,y,z_1)-F(x,y,z_2)| \le d_2|z_1-z_2|$,

(B.4) $|F(x,y,z)| \le |F(x,0,z)| + d_3(x)(1+|y|)$,

(B.5) $|h(x,y)| \le |h(x,0)| + d_3(x)(1+|y|)$,

where d_1 and d_3 are Borel measurable functions on $\mathbb{R}^d$, d_2 is a positive constant and β is a bounded negative measurable function on $\mathbb{R}^d$. Given $g\in C_b(\mathbb{R}^d)$ and consider the BSDE:

$$\begin{aligned}Y(t) &= g(X(T)) + \int_t^T F(X(s), Y(s), Z(s))\,\mathrm{d}s \\ &\quad + \int_t^T h(X(s), Y(s))\,\mathrm{d}K_s - \int_t^T \langle Z(s), \mathrm{d}M(s)\rangle,\end{aligned} \tag{4.12}$$

where $M(s)$ is the martingale part of $X(s)$. Set $d(x) := 2d_1^+(x)$ and $\mu(x) := 2(\beta(x)+1)$, the following result follows from Lemma 4.1.

Theorem 4.1 *Let* (B.1)–(B.5) *hold. Assume moreover* $E[\mathrm{e}^{\int_0^T d(X(s))\mathrm{d}s+\int_0^T \mu(X(s))\mathrm{d}K_s}] < \infty$,

$$E\Big[\int_0^T \mathrm{e}^{\int_0^s d(X(u))\mathrm{d}u+\int_0^s \mu(X(u))\mathrm{d}K_u}\,|h(X(s),0)|^2\mathrm{d}K_s\Big] < \infty$$

and

$$E\Big[\int_0^T \mathrm{e}^{\int_0^s d(X(u))\mathrm{d}u+\int_0^s \mu(X(u))\mathrm{d}K_u}\,(|F(X(s),0,0)|^2+|d_3(X(s))|^2)\mathrm{d}s\Big] < \infty,$$

then the BSDE (4.12) *admits a unique solution.*

5 Homogenization of Parabolic Systems

In this section, we are concerned with the homogenization of the parabolic systems (1.1). In Subsection 5.1, the homogenized coefficients are defined (see (5.1)–(5.3)) and we show that the homogenized boundary value problem (5.4) has a unique weak solution (see Theorem 5.1). In Subsection 5.2, inspired by the ideas of [28, Lemma 6.1, Lemma 6.3], two important lemmas are proved (see Lemmas 5.1–5.2). Based on them, the homogenization result can be obtained in the case where the coefficients f and g are smooth (see Lemma 5.3). At the end of Subsection 5.3, Theorem 1.1 is proved by a regularization procedure.

5.1 Homogenized PDEs with the third boundary conditions

Define

$$\overline{a}_{ij} := \int_{\mathbb{T}^d} \langle A\nabla\widetilde{\omega}_i, \nabla\widetilde{\omega}_j\rangle(\eta)\, m(\eta)\mathrm{d}\eta, \tag{5.1}$$

$$\begin{aligned}\overline{f}(x,y,z) &:= \int_{\mathbb{T}^d} f\big(x,y,\nabla\widetilde{\omega}(\eta)\,z\big)\, m(\eta)\mathrm{d}\eta\\ &= \int_{\mathbb{T}^d} f\big(x,y,(Id+\nabla\omega(\eta))\,z\big)\, m(\eta)\mathrm{d}\eta\end{aligned} \tag{5.2}$$

and

$$\overline{C} := \int_{\mathbb{T}^{d-1}} c(\eta)\,\widetilde{m}(\mathrm{d}\eta). \tag{5.3}$$

Theorem 5.1 *Let* (1.4) *and Assumptions* 1–3 *hold, then* $\overline{A}$ *defined by* (5.1) *is a strictly positive symmetric matrix, and* $\overline{f}$ *satisfies Assumption* 2 *with constant* $c_2\int_{\mathbb{T}^d}\|(Id+\nabla\omega(\eta)\|\,\mathrm{d}\eta$. *Moreover, the homogenized boundary value problem*:

$$\begin{cases}\dfrac{\partial u^0}{\partial t}(t,x)+\overline{L}u^0(t,x)+\overline{f}(x,u^0(t,x),\nabla u^0(t,x))=0, & (t,x)\in[0,T)\times\mathcal{O},\\[2mm] \dfrac{1}{2}\dfrac{\partial u^0}{\partial v^0}(t,x)+\overline{C}u^0(t,x)=0, & (t,x)\in[0,T)\times\partial\mathcal{O},\\[2mm] u^0(T,x)=g(x), & x\in\overline{\mathcal{O}}\end{cases} \tag{5.4}$$

has a unique weak solution in the space

$$\mathcal{W}_1^2(0,T,H^1(\mathcal{O}),L^2(\mathcal{O})) := \{u(t,x)\in L^2([0,T];H^1(\mathcal{O}))\ \textit{such that}\ \partial_t u(t,x)\in L^2([0,T],H^{-1}(\mathcal{O}))\}.$$

Proof In view of $\overline{a}_{ij}=\int_{\mathbb{T}^d}\langle A\nabla\widetilde{\omega}_i,\nabla\widetilde{\omega}_j\rangle(x)\,m(x)\mathrm{d}x$, $\overline{A}$ is clearly a non-negative symmetric matrix. Hence, it suffices to prove that $\overline{A}$ is non-degenerate. Assume that there exists $\xi\in\mathbb{R}^d$ such that $\langle\overline{A}\xi,\xi\rangle=0$. Using the fact that A is uniformly elliptic, we can have $\int_{\mathbb{T}^d}\langle\nabla\widetilde{\omega}_i(x),\xi\rangle\,\mathrm{d}x=0$ for every $i=1,\cdots,d$. That is

$$\xi_i+\sum_{j=1}^d\int_{\mathbb{T}^d}(\nabla\omega_i)_j(x)\,\xi_j\,\mathrm{d}x=0,\quad\forall i=1,\cdots,d.$$

In view of $\omega\in H^1(\mathbb{T}^d)$, then it yields $\int_{\mathbb{T}^d}\nabla\omega_i(x)\,\mathrm{d}x=0$. This implies that $\xi=0$.

The assertion concerning $\bar{f}$ is an easy consequence of Assumption 2 and $\nabla\omega_i\in L^2(\mathbb{T}^d)$ for $i=1,\cdots,d$. Since the coefficient $\bar{A}$ is a constant matrix, the existence and uniqueness of a weak solution to (5.4) can be deduced from [30, Theorem 1].

5.2 Two lemmas

Under Assumptions 1–3, one can deduce by a similar argument as that in [24, Proposition 3.2] together with Khas'minskii's lemma that for any fixed ε, $E_x^\varepsilon[\exp\{2(\alpha+1)\,K_T^\varepsilon\}]<C(\alpha,T)$. Hence by (2.7), Theorem 4.1 and the boundedness of function f, there exists a unique pair $(Y_s^\varepsilon,Z_s^\varepsilon)_{s\in[t,T]}$ of progressively measurable process satisfying

$$\begin{aligned}Y^\varepsilon(s)=g(X^\varepsilon(T))&+\int_s^T f(X^\varepsilon(r),Y^\varepsilon(r),Z^\varepsilon(r))\,\mathrm{d}r+\int_s^T c\Big(\frac{X^\varepsilon(r)}{\varepsilon}\Big)Y^\varepsilon(r)\,\mathrm{d}K_r^\varepsilon\\&-\int_s^T\langle Z^\varepsilon(r),\mathrm{d}M^\varepsilon(r)\rangle,\quad t\le s\le T,\quad\mathbb{P}_{t,x}^\varepsilon-\text{a.s.}\end{aligned}\tag{5.5}$$

and

$$E_{t,x}^\varepsilon\Big[\sup_{t\le s\le T}|Y^\varepsilon(s)|^2+\int_t^T\|Z^\varepsilon(s)\|^2\mathrm{d}s\Big]<\infty.\tag{5.6}$$

Moreover, for each fixed ε, we deduce from [30, Corollary 3] that $Y^\varepsilon(s)=u^\varepsilon(s,X^\varepsilon(s))$ where u^ε is a continuous version of the weak solution of system (1.1). Therefore $u^\varepsilon(t,x)=Y^\varepsilon(t)$. Let E_x^ε be the expectation under $\mathbb{P}_{t,x}^\varepsilon$. We are going to prove that for all $p\in\big(1,\frac{Q(\lambda,d,q_1,q_2)}{2}\big)$,

$$\lim_{\varepsilon\to0}E_x^\varepsilon[|Y^\varepsilon(t)-u^0(t,x)|^p]=0,$$

where $Q(\lambda,d,q_1,q_2)$ is the constant in Theorem 3.1. To avoid heavy notations, we will take in all the sequel $t=0$. To this end, we prove the following two lemmas, which will play an important role in the homogenization of Theorem 5.2.

Lemma 5.1 *Let $\psi(s,x,\eta):[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be a bounded, uniformly continuous function which is periodic in η. Assume that for any $x\in\partial\mathcal{O}$,*

$$\Big\langle\Big(\int_{\mathbb{T}^{d-1}}(A\nabla\widetilde{\omega})_{ij}(\eta)\,\widetilde{m}(\mathrm{d}\eta)\Big)\,\nabla\Psi(x),\nabla\Psi(x)\Big\rangle>0,\tag{5.7}$$

where the matrix $\big(\int_{\mathbb{T}^{d-1}} (A\nabla\widetilde{\omega})_{ij}(\eta)\,\widetilde{m}(\mathrm{d}\eta)\,\big)_{1\le i,j\le d}$ *is a constant and matrix* $\nabla\widetilde{\omega}$ *is composed of* $\nabla\widetilde{\omega}_i$ *as column vectors, that is,* $\nabla\widetilde{\omega}(x) = (\nabla\widetilde{\omega}_1(x), \nabla\widetilde{\omega}_2(x), \cdots, \nabla\widetilde{\omega}_d(x))$*. If* $\int_{\mathbb{T}^d} \psi(s,x,\eta)\, m(\eta)\mathrm{d}\eta = 0$ *holds for any* $s \in [0,T]$ *and* $x \in \mathbb{R}^d$*, then we have*

$$\lim_{\varepsilon\to 0} E^{\varepsilon}_{t,x}\Big[\Big|\int_0^s \psi\Big(r, X^{\varepsilon}_r, \frac{X^{\varepsilon}_r}{\varepsilon}\Big)\,\mathrm{d}r\Big|^2\Big] = 0. \tag{5.8}$$

Proof By a standard smooth approximation procedure, it is sufficient to prove (5.8) holds when the function ψ is C_b^{∞} and periodic in η. Indeed, we consider the equation

$$\begin{cases} \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial \eta_i}\Big(a_{ij}(\eta)\frac{\partial \varphi}{\partial \eta_j}\Big) + \sum_{i=1}^{d} b_i(\eta)\frac{\partial \varphi}{\partial \eta_i} = -\psi, \\ \displaystyle\int_{\mathbb{T}^d} \varphi(s,x,\eta)\, m(\eta)\mathrm{d}\eta = 0. \end{cases}$$

By the regularity on ψ and on the coefficients, then we can use Itô's formula to obtain

$$\begin{aligned} \varphi\Big(s, X^{\varepsilon}_s, \frac{X^{\varepsilon}_s}{\varepsilon}\Big) &= \varphi\Big(t, X^{\varepsilon}_t, \frac{X^{\varepsilon}_t}{\varepsilon}\Big) + \int_t^s \frac{\partial \varphi}{\partial \tau}\Big(r, X^{\varepsilon}_r, \frac{X^{\varepsilon}_r}{\varepsilon}\Big)\,\mathrm{d}r + \int_t^s \Big\langle \Big(\frac{\partial \varphi}{\partial x} + \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial \eta}\Big), \mathrm{d}M^{\varepsilon}_r \Big\rangle \\ &\quad + \int_t^s \Big(\frac{1}{\varepsilon}\Big\langle \frac{\partial \varphi}{\partial x}, \widetilde{b}\Big\rangle + \frac{1}{\varepsilon^2} L\varphi\Big)\,\mathrm{d}r + \int_t^s \Big\langle \Big(\frac{\partial \varphi}{\partial x} + \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial \eta}\Big), \gamma\Big(\frac{X^{\varepsilon}_r}{\varepsilon}\Big)\Big\rangle\,\mathrm{d}K^{\varepsilon}_r \\ &\quad + \frac{1}{2}\int_t^s tr\Big(a\Big(\frac{X^{\varepsilon}_r}{\varepsilon}\Big)\Big(\frac{\partial^2 \varphi}{\partial x^2} + \frac{2}{\varepsilon}\frac{\partial^2 \varphi}{\partial x \partial \eta}\Big)\Big)\,\mathrm{d}r. \end{aligned}$$

Hence

$$\begin{aligned} \int_t^s \psi\Big(r, X^{\varepsilon}_r, \frac{X^{\varepsilon}_r}{\varepsilon}\Big)\,\mathrm{d}r &= \varepsilon^2\Big[\varphi\Big(t, X^{\varepsilon}_t, \frac{X^{\varepsilon}_t}{\varepsilon}\Big) - \varphi\Big(s, X^{\varepsilon}_s, \frac{X^{\varepsilon}_s}{\varepsilon}\Big)\Big] \\ &\quad + \varepsilon^2 \int_t^s \frac{\partial \varphi}{\partial \tau}\Big(r, X^{\varepsilon}_r, \frac{X^{\varepsilon}_r}{\varepsilon}\Big)\,\mathrm{d}r + \int_t^s \Big\langle \Big(\varepsilon^2\frac{\partial \varphi}{\partial x} + \varepsilon\frac{\partial \varphi}{\partial \eta}\Big), \mathrm{d}M^{\varepsilon}_r \Big\rangle \\ &\quad + \varepsilon\int_t^s \Big\langle \frac{\partial \varphi}{\partial x}, \widetilde{b}\Big\rangle\,\mathrm{d}r + \int_t^s \Big\langle \Big(\varepsilon^2\frac{\partial \varphi}{\partial x} + \varepsilon\frac{\partial \varphi}{\partial \eta}\Big), \gamma\Big(\frac{X^{\varepsilon}_r}{\varepsilon}\Big)\Big\rangle\,\mathrm{d}K^{\varepsilon}_r \\ &\quad + \frac{\varepsilon^2}{2}\int_t^s tr\Big(a\Big(\frac{X^{\varepsilon}_r}{\varepsilon}\Big)\frac{\partial^2 \varphi}{\partial x^2}\Big)\,\mathrm{d}r + \varepsilon\int_t^s tr\Big(a\Big(\frac{X^{\varepsilon}_r}{\varepsilon}\Big)\frac{\partial^2 \varphi}{\partial x \partial \eta}\Big)\,\mathrm{d}r. \end{aligned}$$

If $E^{\varepsilon}_{t,x}[(K^{\varepsilon}(T))^2]$ is bounded in ε for any fixed $T > 0$, it is easy to see that

$$E^{\varepsilon}_{t,x}\Big[\Big|\int_0^s \psi\Big(r, X^{\varepsilon}_r, \frac{X^{\varepsilon}_r}{\varepsilon}\Big)\,\mathrm{d}r\Big|^2\Big] \le C\varepsilon,$$

which implies (5.8) follows. In fact, based on the ideas of [28, Lemma 5.2, Lemma 6.1], we can obtain that $E^{\varepsilon}_{t,x}[(K^{\varepsilon}(T))^p]$ is bounded in ε for each $p \ge 1$. For each function $\widetilde{\omega}_i$, we consider the solution ϕ_i of the following equation:

$$\begin{cases} \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial \phi}{\partial x_j}\Big) + \sum_{i=1}^{d} b_i(x)\frac{\partial \phi}{\partial x_i} = 0, & x \in \mathcal{O}, \\ \dfrac{\partial \phi}{\partial v}(x) = \Big\langle A(x)\nabla\widetilde{\omega}_i(x) - \Big(\displaystyle\int_{\mathbb{T}^{d-1}} (A\nabla\widetilde{\omega}_i)_j(\eta)\,\widetilde{m}(\mathrm{d}\eta)\Big), \vec{n}(x)\Big\rangle, & x \in \partial\mathcal{O}, \end{cases}$$

where the $(\int_{\mathbb{T}^{d-1}}(A\nabla\widetilde{\omega}_i)_j(\eta)\,\widetilde{m}(\mathrm{d}\eta))_{1\le j\le d}$ is a constant vector. As $\phi_i \in W^{1,2}(\mathcal{O})$, it yields

$$\begin{aligned}\mathrm{d}\Big(\varepsilon\phi_i\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\Big) &= \Big\langle \nabla\phi_i\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big), \mathrm{d}M_s^\varepsilon\Big\rangle + \Big\langle (A\nabla\widetilde{\omega}_i)\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big), \vec{n}(X_s^\varepsilon)\Big\rangle \\ &\quad - \Big\langle \Big(\int_{\mathbb{T}^{d-1}}(A\nabla\widetilde{\omega}_i)_j(\eta)\,\widetilde{m}(\mathrm{d}\eta)\Big), \vec{n}(X_s^\varepsilon)\Big\rangle.\end{aligned}$$

Let $\phi(x) := (\phi_1(x), \cdots, \phi_d(x))^{\mathrm{T}}$, then we have

$$\begin{aligned}\int_0^t (A\nabla\widetilde{\omega})^*\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\,\vec{n}(X_s^\varepsilon)\mathrm{d}K_s^\varepsilon &= \int_0^t \Big(\int_{\mathbb{T}^{d-1}}(A\nabla\widetilde{\omega})_{ij}(\eta)\,\widetilde{m}(\mathrm{d}\eta)\Big)^*\,\vec{n}(X_s^\varepsilon)\mathrm{d}K_s^\varepsilon \\ &\quad + \varepsilon\Big(\phi\Big(\frac{X_t^\varepsilon}{\varepsilon}\Big) - \phi\Big(\frac{x}{\varepsilon}\Big)\Big) - \int_0^t \Big\langle \nabla\phi\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big), \mathrm{d}M_s^\varepsilon\Big\rangle.\end{aligned}$$

Consequently, combined with the definition of $\widetilde{\omega}_i^\varepsilon$ and (3.4), it implies

$$X_t^\varepsilon = \widehat{X}_t^\varepsilon + \int_0^t \Big(\int_{\mathbb{T}^{d-1}}(A\nabla\widetilde{\omega})_{ij}(\eta)\,\widetilde{m}(\mathrm{d}\eta)\Big)^*\,\vec{n}(X_s^\varepsilon)\mathrm{d}K_s^\varepsilon, \tag{5.9}$$

where

$$\begin{aligned}\widehat{X}_t^\varepsilon &= x + \Big[\widetilde{M}_t^\varepsilon - \int_0^t \Big\langle \nabla\phi\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big), \mathrm{d}M_s^\varepsilon\Big\rangle\Big] + \varepsilon\Big[\phi^\varepsilon\Big(\frac{X_t^\varepsilon}{\varepsilon}\Big) - \phi^\varepsilon\Big(\frac{x}{\varepsilon}\Big)\Big] \\ &\quad - \varepsilon\Big[\widetilde{\omega}^\varepsilon\Big(\frac{X_t^\varepsilon}{\varepsilon}\Big) - \widetilde{\omega}^\varepsilon\Big(\frac{x}{\varepsilon}\Big)\Big].\end{aligned}$$

In view of (1.2) and (5.7), then we can obtain

$$\Big\langle \Big(\int_{\mathbb{T}^{d-1}}(A\nabla\widetilde{\omega})_{ij}(\eta)\,\widetilde{m}(\mathrm{d}\eta)\Big)\,\vec{n}(x), \vec{n}(x)\Big\rangle > 0, \quad \forall x \in \partial\mathcal{O},$$

which implies that (5.9) can be regarded as a Skorohod equation with respect to $\widehat{X}^\varepsilon(t)$ and $K^\varepsilon(t)$. From [8, Theorem 2.2], we can know that $K^\varepsilon(T)$ can be controlled by $\sup\limits_{0\le t_1\le t_2\le T}|\widehat{X}^\varepsilon(t_1) - \widehat{X}^\varepsilon(t_2)|$. Hence, $E_{t,x}^\varepsilon[(K^\varepsilon(T))^p]$ is bounded in ε for any $p \ge 1$. The proof of this lemma is completed.

Remark 5.1 It should be noted that the method of [2, Lemma 3.9.2] is no longer applicable because of the appearance of the local time term and the highly oscillating term $\varepsilon^{-1}\widetilde{b}\big(\frac{X^\varepsilon}{\varepsilon}\big)\,\mathrm{d}s$ in (2.1). Under condition (5.7) and inspired by [28, Lemma 6.1], process X^ε can be rewritten to another equation (5.9) of Skorohod type. Hence from the property of the local time, we can prove $E_{t,x}^\varepsilon[(K^\varepsilon(T))^p]$ is bounded in ε for any $p \ge 1$. The lemma is further proved to be true.

In addition, since $\widetilde{\omega}^\varepsilon(X_s^\varepsilon) = X_s^\varepsilon + \varepsilon\big(\omega\big(\frac{X_s^\varepsilon}{\varepsilon}\big) - \omega\big(\frac{x}{\varepsilon}\big)\big)$ and each function ω_i is bounded, we can also conclude that

$$E_{t,x}^\varepsilon\Big[\Big|\int_0^s \psi\Big(r, \widetilde{\omega}^\varepsilon(X_s^\varepsilon), \frac{X_r^\varepsilon}{\varepsilon}\Big)\,\mathrm{d}r\Big|^2\Big] \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.10}$$

Meanwhile, we have the following similar convergence result for integrals of local time K^ε.

Lemma 5.2 *Let $h : \mathbb{R}^d \to \mathbb{R}$ be bounded, continuous and periodic of period one in each direction. If the function h satisfies $\int_{\mathbb{T}^{d-1}} h(\eta)\,\widetilde{m}(\mathrm{d}\eta) = 0$, then*

$$\lim_{\varepsilon\to 0} E_{t,x}^\varepsilon\Big[\Big|\int_0^s h\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\,\mathrm{d}K_r^\varepsilon\Big|^2\Big] = 0. \tag{5.11}$$

Proof By a standard smooth approximation procedure, it suffices to prove (5.11) holds for the function $h \in C_b^\infty$. Let ϕ be the solution of $L\phi = 0$ in $\mathcal{O}$ and $\frac{\partial \phi}{\partial \gamma} = h$ on $\partial \mathcal{O}$.

Then by Itô's formula, we have

$$\begin{aligned}
\varepsilon \phi\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big) &= \varepsilon \phi\Big(\frac{x}{\varepsilon}\Big) + \frac{1}{\varepsilon} \int_0^s L\phi\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\,\mathrm{d}r + \int_0^s \Big\langle \nabla \phi\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big), \mathrm{d}M_r^\varepsilon \Big\rangle \\
&\quad + \int_0^s \Big\langle \nabla \phi\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big), \gamma\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\Big\rangle \,\mathrm{d}K_r^\varepsilon \\
&= \varepsilon \phi\Big(\frac{x}{\varepsilon}\Big) + \int_0^s \Big\langle \nabla \phi\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big), \mathrm{d}M_r^\varepsilon \Big\rangle + \int_0^s h\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\,\mathrm{d}K_r^\varepsilon \\
&=: \varepsilon \phi\Big(\frac{x}{\varepsilon}\Big) + I_{1,\varepsilon}(s) + I_{2,\varepsilon}(s),
\end{aligned} \tag{5.12}$$

which implies

$$E[|I_{1,\varepsilon}(s) + I_{2,\varepsilon}(s)|^2] = \varepsilon^2 E\Big[\Big|\phi\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big) - \phi\Big(\frac{x}{\varepsilon}\Big)\Big|^2\Big] \le 4\varepsilon^2 |\phi|_\infty < \infty.$$

Now let Q_ε be the probability measure induced by the process $(I_{1,\varepsilon}(s), I_{2,\varepsilon}(s))$, then as the arguments in [28, Lemma 6.3], we can choose a subsequence such that Q_ε converges weakly to some limit probability measure Q_0 as $\varepsilon \to 0$. Moreover, it follows from (5.12) that the limit $(I_1(s), I_2(s))$ satisfies $I_1(s) + I_2(s) = 0$ (Q_0 a.s.) and they are a Q_0-martingale and a bounded variation process, respectively. Hence $I_1(s) = I_2(s) = 0$ (Q_0 a.s.). We have proved the conclusion of the lemma.

5.3 Homogenization

Now, we are going to prove Theorem 1.1 in the case where the coefficients f and g are smooth. To this end, we introduce the following assumptions (C.1)–(C.2).

(C.1) f is bounded and $f(x,0,0) \in L^2(\mathbb{R}^d)$. Moreover, f is Lipschitz, that is,

$$|f(x,y,z) - f(x',y',z')| \le C(|x-x'| + |y-y'| + |z-z'|), \quad \forall x, x' \in \mathbb{R}^d,\ y, y' \in \mathbb{R},\ z, z' \in \mathbb{R}^d.$$

(C.2) $g : \mathbb{R}^d \to \mathbb{R}$ is a C^2 function.

Then we know from [14, Theorem 7.4] that the system (5.4) has a unique classical solution $u^0 \in C^{1,2}([0,T] \times \mathcal{O}; \mathbb{R})$.

Theorem 5.2 *Let Assumption* 1, (1.4) *and* (C.1)–(C.2) *hold. Define*

$$\begin{cases}
\widetilde{Y}^\varepsilon(s) := Y^\varepsilon(s) - u^0(s, \widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \\
\widetilde{Z}^\varepsilon(s) := Z^\varepsilon(s) - \nabla \widetilde{\omega}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big) \nabla u^0(s, \widetilde{\omega}^\varepsilon(X_s^\varepsilon)),
\end{cases}$$

then we have

$$\begin{aligned}
\widetilde{Y}^\varepsilon(s) &= g(X_T^\varepsilon) - g(\widetilde{\omega}^\varepsilon(X_T^\varepsilon)) + \int_s^T F^\varepsilon(r, \widetilde{Y}_r^\varepsilon, \widetilde{Z}_r^\varepsilon)\,\mathrm{d}r \\
&\quad + \int_s^T \widehat{h}^\varepsilon(r, \widetilde{Y}_r^\varepsilon)\,\mathrm{d}K_s^\varepsilon - \int_s^T \langle \widetilde{Z}_r^\varepsilon, \mathrm{d}M_s^\varepsilon \rangle, \quad \mathbb{P}_{t,x}^\varepsilon - a.s.,
\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}F^\varepsilon(s,y,z) &:= f\Big(X_s^\varepsilon, y+u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), z+\nabla\widetilde{\omega}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big)\\ &\quad -\overline{f}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon), u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)))\\ &\quad +\frac{1}{2}\Big(\Big(\widehat{a}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)-\overline{a}\Big)_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j}(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big),\\ \widehat{h}^\varepsilon(s,y) &:= 2c\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)(y+u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)))-2\overline{C}u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\\ &\quad +\Big\langle\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \widetilde{\gamma}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)-\overline{A}\vec{n}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big\rangle.\end{aligned}$$

Moreover for all $t\in[0,T]$*, we can deduce that as* ε *tends to* 0*,* $E_x^\varepsilon[|\int_0^t F^\varepsilon(s,0,0)\,\mathrm{d}s|^2]\to 0$ *and* $E_x^\varepsilon[|\int_0^t \widehat{h}^\varepsilon(s,0)\,\mathrm{d}K_s^\varepsilon|^2]\to 0$.

Proof By Itô's formula, we have

$$\begin{aligned}&\mathrm{d}(Y^\varepsilon(s)-u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)))\\ &= -\Big[f(X_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon)-\overline{f}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon), u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)))\\ &\quad +\frac{1}{2}\Big(\Big(\widehat{a}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)-\overline{a}\Big)_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j}\Big)\Big]\mathrm{d}s-\Big[2c\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)Y_s^\varepsilon-2\overline{C}u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\\ &\quad +\Big\langle\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)),\,\widetilde{\gamma}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)-\overline{A}\vec{n}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big\rangle\Big]\,\mathrm{d}K_s^\varepsilon\\ &\quad +\Big\langle Z_s^\varepsilon-\nabla\widetilde{\omega}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\,,\,\mathrm{d}M_s^\varepsilon\Big\rangle,\end{aligned}$$

which implies (5.13). Moreover,

$$\begin{aligned}F^\varepsilon(s,0,0) &= \psi\Big(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon),\frac{X_s^\varepsilon}{\varepsilon}\Big)+\Big[f\Big(X_s^\varepsilon,u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)),\nabla\widetilde{\omega}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big)\\ &\quad -f\Big(\widetilde{\omega}^\varepsilon(X_s^\varepsilon),u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)),\nabla\widetilde{\omega}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big)\Big]\\ &=: \psi\Big(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon),\frac{X_s^\varepsilon}{\varepsilon}\Big)+I_1^\varepsilon(s)\end{aligned}$$

with

$$\begin{aligned}\psi(s,x,\eta) &:= [f(x,u^0(s,x),\nabla\widetilde{\omega}(\eta)\nabla u^0(s,x))-\overline{f}(x,u^0(s,x),\nabla u^0(s,x))]\\ &\quad +\frac{1}{2}\Big((\widehat{a}(\eta)-\overline{a})_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j}(s,x)\Big)\end{aligned}$$

and

$$\begin{aligned}\widehat{h}^\varepsilon(s,0) &= \Big\langle\nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)),\widetilde{\gamma}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)-\overline{A}\vec{n}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big\rangle\\ &\quad +2\Big[c\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)-\overline{C}\Big]\,u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)).\end{aligned}$$

Since function f is Lipschitz with respect to x and each solution ω_i of the auxiliary problems is bounded, we obtain

$$E_x^\varepsilon\Big[\Big|\int_0^t I_1^\varepsilon(s)\,\mathrm{d}s\Big|^2\Big]\le CE_x^\varepsilon\Big[\int_0^t\|X_s^\varepsilon-\widetilde{\omega}^\varepsilon(X_s^\varepsilon)\|^2\,\mathrm{d}s\Big]$$

$$= C\varepsilon^2 \int_0^t E_x^\varepsilon\Big[\Big\|\omega^\sharp\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big)\Big\|^2\Big]\,\mathrm{d}s$$
$$\to 0$$

as $\varepsilon \to 0$. By the definition of the homogenized coefficients, we obtain $\int_{\mathbb{T}^d} \psi(s,x,\eta)\, m(\eta)\mathrm{d}\eta = 0$. Then from Lemma 5.1 and (5.10), it yields that

$$\lim_{\varepsilon\to 0} E_x^\varepsilon\Big[\Big|\int_0^t F^\varepsilon(s,0,0)\,\mathrm{d}s\Big|^2\Big] = 0. \tag{5.14}$$

On the other hand, by [7, Proposition 8.5], we know that $a\big(\frac{X_s^\varepsilon}{\varepsilon}\big)\nabla\widetilde{\omega}\big(\frac{X_s^\varepsilon}{\varepsilon}\big)$ converges weakly to the homogenized matrix $\overline{a}$ in $L^2(\mathcal{O})^d$. Hence letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\Big\langle \nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \widetilde{\gamma}\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big) - \overline{A}\vec{n}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon))\Big\rangle \\
&= \Big\langle \nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \Big((A\nabla\widetilde{\omega})\Big(\frac{X_s^\varepsilon}{\varepsilon}\Big) - \overline{A}\Big)^* \vec{n}(X_s^\varepsilon)\Big\rangle \\
&\quad + \langle \nabla u^0(s,\widetilde{\omega}^\varepsilon(X_s^\varepsilon)), \overline{A}(\vec{n}(X_s^\varepsilon) - \vec{n}(\widetilde{\omega}^\varepsilon(X_s^\varepsilon)))\rangle \\
&\to 0.
\end{aligned}$$

Combined with Lemma 5.2, it implies

$$\lim_{\varepsilon\to 0} E_x^\varepsilon\Big[\Big|\int_0^t \widehat{h}^\varepsilon(s,0)\,\mathrm{d}K_s^\varepsilon\Big|^2\Big] = 0. \tag{5.15}$$

The theorem is proved.

Lemma 5.3 *Let Assumptions* 1–3, (1.4) *and* (C.1)–(C.2) *hold, then*

$$\lim_{\varepsilon\to 0} u^\varepsilon(0,x) = u^0(0,x).$$

Proof Define

$$\begin{aligned}
\widehat{Y}_t^\varepsilon &:= \widetilde{Y}_t^\varepsilon + \int_0^t F^\varepsilon(r,0,0)\,\mathrm{d}r + \int_0^t \widehat{h}^\varepsilon(r,0)\,\mathrm{d}K_r^\varepsilon \\
&= [g(X_T^\varepsilon) - g(\widetilde{\omega}^\varepsilon(X_T^\varepsilon))] + \int_0^T F^\varepsilon(r,0,0)\,\mathrm{d}r + \int_t^T [F^\varepsilon(r,\widetilde{Y}_r^\varepsilon,\widetilde{Z}_r^\varepsilon) - F^\varepsilon(r,0,0)]\mathrm{d}r \\
&\quad - \int_t^T \langle \widetilde{Z}_r^\varepsilon, \mathrm{d}M_r^\varepsilon\rangle + \int_0^T \widehat{h}^\varepsilon(r,0)\,\mathrm{d}K_r^\varepsilon + \int_t^T [\widehat{h}^\varepsilon(r,\widetilde{Y}_r^\varepsilon) - \widehat{h}^\varepsilon(r,0)]\mathrm{d}K_r^\varepsilon.
\end{aligned}$$

Then it suffices to show that for all $p \in (1, \frac{Q(\lambda,d,q_1,q_2)}{2} \wedge 2)$, $|\widehat{Y}_0^\varepsilon|^p$ tends to 0 as $\varepsilon \to 0$.

Note that for all $p \in (1, \frac{Q(\lambda,d,q_1,q_2)}{2})$, Theorem 3.1 implies

$$E_x^\varepsilon\Big[\int_0^T |F^\varepsilon(r,0,0)|^p\,\mathrm{d}r\Big] \le C E_x^\varepsilon\Big[\int_0^T (1 + \|\nabla\omega^\sharp\|^{2p})\,\mathrm{d}r\Big] < \infty.$$

Moreover, by Hölder's inequality and in view of the boundedness of functions $u^0, \nabla u^0, c$, it yields that

$$E_x^\varepsilon\Big[\Big(\int_0^T \widehat{h}^\varepsilon(r,0)\,\mathrm{d}K_r^\varepsilon\Big)^p\Big] \le E_x^\varepsilon\Big[\Big(\int_0^T |\widehat{h}^\varepsilon(r,0)|^p\,\mathrm{d}K_r^\varepsilon\Big)\Big(\int_0^T 1\,\mathrm{d}K_r^\varepsilon\Big)^{p-1}\Big]$$

$$\leq C E_x^\varepsilon\Big[(K_T^\varepsilon)^{p-1}\cdot\Big(\int_0^T(1+\|\nabla\omega^\sharp\|^p)\,\mathrm{d}K_r^\varepsilon\Big)\Big]$$
$$\leq C E_x^\varepsilon[(K_T^\varepsilon)^2].$$

Hence combined with (5.6), then for all $p\in(1,\frac{Q(\lambda,d,q_1,q_2)}{2}\wedge 2)$,

$$E_x^\varepsilon\Big[\sup_{0\leq t\leq T}|\widehat{Y}_t^\varepsilon|^p\Big]<\infty. \tag{5.16}$$

Meanwhile, from the boundedness of ∇u^0 and (5.6), it also follows that

$$E_x^\varepsilon\Big[\int_0^T\|\widetilde{Z}_r^\varepsilon\|^2\,\mathrm{d}r\Big]<\infty, \tag{5.17}$$

where we have used the fact that $\omega\in L^2(\mathbb{T}^d)$.

Let τ_n be the stopping time

$$\tau_n:=\inf\Big\{t\geq 0:|\widehat{Y}_t^\varepsilon|\leq\frac{1}{n}\Big\}.$$

By Itô's formula, we have for all $p\in(1,\frac{Q(\lambda,d,q_1,q_2)}{2}\wedge 2)$,

$$\begin{aligned}
&|\widehat{Y}_{t\wedge\tau_n}^\varepsilon|^p+\frac{p}{2}\int_{t\wedge\tau_n}^{T\wedge\tau_n}|\widehat{Y}_r^\varepsilon|^{p-2}\Big\langle\widehat{a}\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\widetilde{Z}_r^\varepsilon,\widetilde{Z}_r^\varepsilon\Big\rangle\,\mathrm{d}r\\
&=|\widehat{Y}_{T\wedge\tau_n}^\varepsilon|^p+p\int_{t\wedge\tau_n}^{T\wedge\tau_n}|\widehat{Y}_r^\varepsilon|^{p-1}\,(F^\varepsilon(r,\widetilde{Y}_r^\varepsilon,\widetilde{Z}_r^\varepsilon)-F^\varepsilon(r,0,0))\,\mathrm{d}r\\
&\quad-p\int_{t\wedge\tau_n}^{T\wedge\tau_n}|\widehat{Y}_r^\varepsilon|^{p-1}\,\langle\widetilde{Z}_r^\varepsilon,\mathrm{d}M_r^\varepsilon\rangle+p\int_{t\wedge\tau_n}^{T\wedge\tau_n}|\widehat{Y}_r^\varepsilon|^{p-1}\,(\widehat{h}^\varepsilon(r,\widetilde{Y}_r^\varepsilon)-\widehat{h}^\varepsilon(r,0))\,\mathrm{d}K_r^\varepsilon.
\end{aligned} \tag{5.18}$$

In view of $\frac{p}{2}<1$ and applying Young's inequality, we obtain

$$\begin{aligned}
&E_x^\varepsilon\Big[\sup_{0\leq t\leq T}\Big|\int_0^t|\widehat{Y}_r^\varepsilon|^{p-1}\,\langle\widetilde{Z}_r^\varepsilon,\mathrm{d}M_r^\varepsilon\rangle\Big|\Big]\\
&\leq C E_x^\varepsilon\Big[\Big(\int_0^T|\widehat{Y}_r^\varepsilon|^{2(p-1)}\,\Big\langle\widehat{a}\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\widetilde{Z}_r^\varepsilon,\widetilde{Z}_r^\varepsilon\Big\rangle\,\mathrm{d}r\Big)^{\frac{1}{2}}\Big]\\
&\leq C E_x^\varepsilon\Big[\Big(\sup_{0\leq r\leq T}|\widehat{Y}_r^\varepsilon|^{p-1}\Big)\cdot\Big(\int_0^T\|\widetilde{Z}_r^\varepsilon\|^2\,\mathrm{d}r\Big)^{\frac{1}{2}}\Big]\\
&\leq C E_x^\varepsilon\Big[\sup_{0\leq r\leq T}|\widehat{Y}_r^\varepsilon|^p\Big]+C E_x^\varepsilon\Big[\Big(\int_0^T\|\widetilde{Z}_r^\varepsilon\|^2\,\mathrm{d}r\Big)^{\frac{p}{2}}\Big]\\
&\leq C E_x^\varepsilon\Big[\sup_{0\leq r\leq T}|\widehat{Y}_r^\varepsilon|^p\Big]+C E_x^\varepsilon\Big(\Big[\int_0^T\|\widetilde{Z}_r^\varepsilon\|^2\,\mathrm{d}r\Big]\Big)^{\frac{p}{2}},
\end{aligned}$$

which implies the local martingale $\int_0^t|\widehat{Y}_r^\varepsilon|^{p-1}\,\langle\widetilde{Z}_r^\varepsilon,\mathrm{d}M_r^\varepsilon\rangle$ is actually a martingale. Denote

$$H^\varepsilon(r):=\int_0^r F^\varepsilon(u,0,0)\,\mathrm{d}u+\int_0^r\widehat{h}^\varepsilon(u,0)\,\mathrm{d}K_u^\varepsilon,$$

and taking expectation in (5.18), then

$$E_x^\varepsilon[|\widehat{Y}_{t\wedge\tau_n}^\varepsilon|^p]+\frac{p}{2}E_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n}|\widehat{Y}_r^\varepsilon|^{p-2}\Big\langle\widehat{a}\Big(\frac{X_r^\varepsilon}{\varepsilon}\Big)\widetilde{Z}_r^\varepsilon,\widetilde{Z}_r^\varepsilon\Big\rangle\,\mathrm{d}r\Big]$$

$$
\begin{aligned}
&\le E_x^\varepsilon[|\widehat{Y}^\varepsilon_{T\wedge\tau_n}|^p] + pCE_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^{p-1}\,(|\widetilde{Y}^\varepsilon_r| + \|\widetilde{Z}^\varepsilon_r\|)\,\mathrm{d}r\Big]\\
&\quad + pCE_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^{p-1}\,|\widetilde{Y}^\varepsilon_r|\mathrm{d}K^\varepsilon_r\Big]\\
&\le E_x^\varepsilon[|\widehat{Y}^\varepsilon_{T\wedge\tau_n}|^p] + CE_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^{p-1}\,(|\widehat{Y}^\varepsilon_r| + |H^\varepsilon(r)| + \|\widetilde{Z}^\varepsilon_r\|)\,\mathrm{d}r\Big]\\
&\quad + CE_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^{p-1}\,(|\widehat{Y}^\varepsilon_r| + |H^\varepsilon(r)|)\,\mathrm{d}K^\varepsilon_r\Big],
\end{aligned}
$$

where we have used the fact that functions $F^\varepsilon(r,y,z)$ and $\widehat{h}^\varepsilon(r,y)$ are uniformly Lipschitz in (y,z) and y, respectively. By Young's inequality, we have

$$
\begin{aligned}
|\widehat{Y}^\varepsilon_r|^{p-1}\,|H^\varepsilon(r)| &\le \frac{1}{q}|\widehat{Y}^\varepsilon_r|^{(p-1)q} + \frac{1}{p}|H^\varepsilon(r)|^p\\
&= \frac{1}{q}|\widehat{Y}^\varepsilon_r|^{p} + \frac{1}{p}|H^\varepsilon(r)|^p
\end{aligned}
$$

and

$$
\begin{aligned}
|\widehat{Y}^\varepsilon_r|^{p-1}\,\|\widetilde{Z}^\varepsilon_r\| &= |\widehat{Y}^\varepsilon_r|^{\frac{p}{2}}\cdot(|\widehat{Y}^\varepsilon_r|^{\frac{p}{2}-1}\,\|\widetilde{Z}^\varepsilon_r\|)\\
&\le \frac{1}{\delta}|\widehat{Y}^\varepsilon_r|^p + \delta|\widehat{Y}^\varepsilon_r|^{p-2}\,\|\widetilde{Z}^\varepsilon_r\|^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&E_x^\varepsilon[|\widehat{Y}^\varepsilon_{t\wedge\tau_n}|^p] + C(1-\delta)E_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^{p-2}\,\|\widetilde{Z}^\varepsilon_r\|^2\,\mathrm{d}r\Big]\\
&\le E_x^\varepsilon[|\widehat{Y}^\varepsilon_{T\wedge\tau_n}|^p] + CE_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |H^\varepsilon(r)|^p\,\mathrm{d}r\Big] + CE_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |H^\varepsilon(r)|^p\,\mathrm{d}K^\varepsilon_r\Big]\\
&\quad + C\Big(1+\frac{1}{\delta}\Big)E_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^p\,\mathrm{d}r\Big] + C\Big(1+\frac{1}{\delta}\Big)E_x^\varepsilon\Big[\int_{t\wedge\tau_n}^{T\wedge\tau_n} |\widehat{Y}^\varepsilon_r|^p\,\mathrm{d}K^\varepsilon_r\Big].
\end{aligned}
$$

Choosing δ sufficiently small so that $1-\delta > 0$ and by the version of Gronwall's lemma in [26, Lemma 3], we deduce that

$$
E_x^\varepsilon[|\widehat{Y}^\varepsilon_{t\wedge\tau_n}|^p] \le CE_x^\varepsilon[|\widehat{Y}^\varepsilon_{T\wedge\tau_n}|^p] + C\Big(E_x^\varepsilon\Big[\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}r\Big] + E_x^\varepsilon\Big[\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K^\varepsilon_r\Big]\Big).
$$

Let $\tau_\infty := \lim_{n\to\infty} \tau_n$, that is, $\tau_\infty = \inf\{t \ge 0, \widehat{Y}_t = 0\}$. For all $t \in [0,T]$, $\widehat{Y}^\varepsilon_{t\wedge\tau_n}$ is dominated by $\sup_{0\le t\le T} |\widehat{Y}^\varepsilon_t|$. By (5.16) and letting n tends to ∞, dominated convergence theorem implies that

$$
E_x^\varepsilon[|\widehat{Y}^\varepsilon_{t\wedge\tau_\infty}|^p] \le CE_x^\varepsilon[|\widehat{Y}^\varepsilon_{T\wedge\tau_\infty}|^p] + C\Big(E_x^\varepsilon\Big[\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}r\Big] + E_x^\varepsilon\Big[\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K^\varepsilon_r\Big]\Big).
$$

In view of

$$
|\widehat{Y}^\varepsilon_{T\wedge\tau_\infty}| = |\widehat{Y}^\varepsilon_T|\,1_{\{T\le\tau_\infty\}} \le |g(X^\varepsilon_T) - g(\widetilde{\omega}^\varepsilon(X^\varepsilon_T)) + H^\varepsilon(T)|,
$$

then

$$
E_x^\varepsilon[|\widehat{Y}^\varepsilon_{t\wedge\tau_\infty}|^p] \le CE_x^\varepsilon[|g(X^\varepsilon_T) - g(\widetilde{\omega}^\varepsilon(X^\varepsilon_T)) + H^\varepsilon(T)|^p]
$$

$$
\begin{aligned}
&+ C\Big(E_x^\varepsilon\Big[\int_0^T |H^\varepsilon(r)|^p \,\mathrm{d}r\Big] + E_x^\varepsilon\Big[\int_0^T |H^\varepsilon(r)|^p \,\mathrm{d}K_r^\varepsilon\Big]\Big)\\
&=: I_{1,\varepsilon} + C(I_{2,\varepsilon} + I_{3,\varepsilon}).
\end{aligned}
\tag{5.19}
$$

Now, we are going to prove that $E_x^\varepsilon[|\widehat{Y}^\varepsilon_{t\wedge\tau_\infty}|^p]$ tends to zero as $\varepsilon \to 0$. Indeed, by Theorem 5.2, the first term in the right-hand side converges to zero, and for all $r \in [0,T]$, $E_x^\varepsilon[|H^\varepsilon(r)|^p] \overset{\varepsilon\to 0}{\longrightarrow} 0$. On the other hand, it follows from Hölder's inequality that

$$
\begin{aligned}
E_x^\varepsilon[|H^\varepsilon(r)|^p] &\le 2E_x^\varepsilon\Big[\Big|\int_0^r F^\varepsilon(u,0,0)\,\mathrm{d}u\Big|^p\Big] + 2E_x^\varepsilon\Big[\Big|\int_0^r \widehat{h}^\varepsilon(u,0)\,\mathrm{d}K_u^\varepsilon\Big|^p\Big]\\
&\le 2E_x^\varepsilon\Big[\Big(\int_0^r 1\,\mathrm{d}r\Big)^{p-1}\Big(\int_0^r |F^\varepsilon(u,0,0)|^p\,\mathrm{d}u\Big)\Big]\\
&\quad + 2E_x^\varepsilon\Big[\Big(\int_0^r 1\,\mathrm{d}K_u^\varepsilon\Big)^{p-1}\Big(\int_0^r |\widehat{h}^\varepsilon(u,0)|^p\,\mathrm{d}K_u^\varepsilon\Big)\Big]\\
&\le CT^{p-1}E_x^\varepsilon\Big[\int_0^T (1+\|\nabla\omega^\sharp\|^{2p})\,\mathrm{d}u\Big] + CE_x^\varepsilon\Big[(K_T^\varepsilon)^{p-1}\Big(\int_0^T (1+\|\nabla\omega^\sharp\|^p)\,\mathrm{d}K_u^\varepsilon\Big)\Big]\\
&\le CT^p(1+\|\nabla\omega^\sharp\|^{2p}) + C(1+\|\nabla\omega^\sharp\|^p)\,E_x^\varepsilon[(K_T^\varepsilon)^p].
\end{aligned}
$$

Hence by dominated convergence, the term $I_{2,\varepsilon}$ in (5.19) converges also to zero. Since for all $r \in [0,T]$, $E_x^\varepsilon[|H^\varepsilon(r)|^p] \overset{\varepsilon\to 0}{\longrightarrow} 0$, which implies for any $\alpha > 0$, $\mathbb{P}_x^\varepsilon(|H^\varepsilon(r)|^p > \alpha) \overset{\varepsilon\to 0}{\longrightarrow} 0$. Meanwhile for any fixed $\delta > 0$, we have

$$
\begin{aligned}
&\mathbb{P}_x^\varepsilon\Big[\Big(\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K_r^\varepsilon\Big) > \delta\Big]\\
&= \mathbb{P}_x^\varepsilon\Big[\Big(\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K_r^\varepsilon\Big) > \delta,\ |H^\varepsilon(r)|^p > \alpha\Big]\\
&\quad + \mathbb{P}_x^\varepsilon\Big[\Big(\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K_r^\varepsilon\Big) > \delta,\ |H^\varepsilon(r)|^p \le \alpha\Big]\\
&\le \mathbb{P}_x^\varepsilon[|H^\varepsilon(r)|^p > \alpha] + \mathbb{P}_x^\varepsilon\Big[K_T^\varepsilon > \frac{\delta}{\alpha},\ |H^\varepsilon(r)|^p \le \alpha\Big].
\end{aligned}
$$

From the definition of H^ε and the boundedness of $\widehat{h}^\varepsilon(u,0)$, it yields for any fixed δ, we can choose appropriate α such that the second term in the right-hand side equals to 0. Thus for any $\delta > 0$, it yields

$$
\mathbb{P}_x^\varepsilon\Big[\Big(\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K_r^\varepsilon\Big) > \delta\Big] \overset{\varepsilon\to 0}{\longrightarrow} 0.
$$

Since $\int_0^T |H^\varepsilon(r)|^p\,\mathrm{d}K_r^\varepsilon$ is nonnegative and integrable, the term $I_{3,\varepsilon}$ in (5.19) converges to zero as ε tends to 0. Consequently, we have proved that $E_x^\varepsilon[|\widehat{Y}^\varepsilon_{t\wedge\tau_\infty}|^p] \overset{\varepsilon\to 0}{\longrightarrow} 0$ for all $t \in [0,T]$. Taking $t = 0$, we have the desired conclusion.

Proof of Theorem 1.1 Let us assume now that f and g satisfy Assumptions 2–3. Let $\rho_1 : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a C_c^∞ function with $\int \rho_1(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 1$. Define $f_n := \rho_n * f$ with $\rho_n(x,y,z) = n^{2d+1}\rho(nx,ny,nz)$, then f_n is infinitely differentiable with bounded derivatives. Since g is bounded and continuous, we can also approximate g by a sequence g_n of functions in $C_b^\infty(\mathbb{R}^d)$.

Let $(Y_n^\varepsilon, Z_n^\varepsilon)$ be the solution of the BSDE:

$$Y_n^\varepsilon(s) = g_n(X^\varepsilon(T)) + \int_s^T f_n(X^\varepsilon(r), Y_n^\varepsilon(r), Z_n^\varepsilon(r))\,\mathrm{d}r + \int_s^T c\Big(\frac{X^\varepsilon(r)}{\varepsilon}\Big)Y_n^\varepsilon(r)\,\mathrm{d}K_r^\varepsilon - \int_s^T \langle Z_n^\varepsilon(r), \mathrm{d}M^\varepsilon(r)\rangle, \quad t \le s \le T,\ \mathbb{P}_{t,x}^\varepsilon - \text{a.s.}$$

satisfying $E_{t,x}^\varepsilon[\sup\limits_{t\le s\le T} |Y_n^\varepsilon(s)|^2 + \int_t^T \|Z_n^\varepsilon(s)\|^2\mathrm{d}s] < \infty$. By Itô's formula, we have

$$\begin{aligned}
&\mathrm{e}^{2\int_0^s c_1^+(X_r^\varepsilon)\mathrm{d}r}|Y_n^\varepsilon(s) - Y^\varepsilon(s)|^2 \\
&+ \int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\langle a(X(r))(Z_n^\varepsilon(r) - Z^\varepsilon(r)), Z_n^\varepsilon(r) - Z^\varepsilon(r)\rangle \mathrm{d}r \\
=\ & \mathrm{e}^{2\int_0^T c_1^+(X_r^\varepsilon)\mathrm{d}r}|g_n(X^\varepsilon(T)) - g(X^\varepsilon(T))|^2 - 2\int_s^T c_1^+(X_r^\varepsilon)\mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\,|Y_n^\varepsilon(r) - Y^\varepsilon(r)|^2\,\mathrm{d}r \\
&- 2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}(Y_n^\varepsilon(r) - Y^\varepsilon(r))\,\langle Z_n^\varepsilon(r) - Z^\varepsilon(r), \mathrm{d}M^\varepsilon(r)\rangle \\
&+ 2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}(Y_n^\varepsilon(r) - Y^\varepsilon(r))\,[f_n(X^\varepsilon(r), Y_n^\varepsilon(r), Z_n^\varepsilon(r)) \\
&- f(X^\varepsilon(r), Y^\varepsilon(r), Z^\varepsilon(r))]\,\mathrm{d}r \\
&+ 2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}c\Big(\frac{X^\varepsilon(r)}{\varepsilon}\Big)|Y_n^\varepsilon(r) - Y^\varepsilon(r)|^2\,\mathrm{d}K_r^\varepsilon.
\end{aligned}$$

Since the function c is nonpositive, $-(c_1^+(X_r^\varepsilon) - c_1(X_r^\varepsilon)) \le 0$ and

$$\begin{aligned}
&f_n(X^\varepsilon(r), Y_n^\varepsilon(r), Z_n^\varepsilon(r)) - f(X^\varepsilon(r), Y^\varepsilon(r), Z^\varepsilon(r)) \\
&= [f_n(X^\varepsilon(r), Y_n^\varepsilon(r), Z_n^\varepsilon(r)) - f_n(X^\varepsilon(r), Y^\varepsilon(r), Z_n^\varepsilon(r))] \\
&\quad + [f_n(X^\varepsilon(r), Y^\varepsilon(r), Z_n^\varepsilon(r)) - f_n(X^\varepsilon(r), Y^\varepsilon(r), Z^\varepsilon(r))] \\
&\quad + [f_n(X^\varepsilon(r), Y^\varepsilon(r), Z^\varepsilon(r)) - f(X^\varepsilon(r), Y^\varepsilon(r), Z^\varepsilon(r)],
\end{aligned}$$

we can obtain by usual computations as (4.7) that

$$\begin{aligned}
&\mathrm{e}^{2\int_0^s c_1^+(X_r^\varepsilon)\mathrm{d}r}|Y_n^\varepsilon(s) - Y^\varepsilon(s)|^2 \\
&+ \int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\langle a(X(r))(Z_n^\varepsilon(r) - Z^\varepsilon(r)), Z_n^\varepsilon(r) - Z^\varepsilon(r)\rangle \mathrm{d}r \\
\le\ & \mathrm{e}^{2\int_0^T c_1^+(X_r^\varepsilon)\mathrm{d}r}|g_n(X^\varepsilon(T)) - g(X^\varepsilon(T))|^2 \\
&- 2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}(Y_n^\varepsilon(r) - Y^\varepsilon(r))\,\langle Z_n^\varepsilon(r) - Z^\varepsilon(r), \mathrm{d}M^\varepsilon(r)\rangle \\
&+ 2c_2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}|Y_n^\varepsilon(r) - Y^\varepsilon(r)|\,|Z_n^\varepsilon(r) - Z^\varepsilon(r))|\,\mathrm{d}r \\
&+ 2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}|Y_n^\varepsilon(r) - Y^\varepsilon(r)|\,|(f_n - f)(X^\varepsilon(r), Y^\varepsilon(r), Z^\varepsilon(r))|\,\mathrm{d}r \\
\le\ & \mathrm{e}^{2\int_0^T c_1^+(X_r^\varepsilon)\mathrm{d}r}|g_n(X^\varepsilon(T)) - g(X^\varepsilon(T))|^2 + C_\lambda\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}|Y_n^\varepsilon(r) - Y^\varepsilon(r)|^2\,\mathrm{d}r \\
&+ \frac{1}{2}\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\langle a(X(r))(Z_n^\varepsilon(r) - Z^\varepsilon(r)), Z_n^\varepsilon(r) - Z^\varepsilon(r)\rangle \mathrm{d}r
\end{aligned}$$

$$+\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}|(f_n-f)(X^\varepsilon(r),Y^\varepsilon(r),Z^\varepsilon(r))|^2\,\mathrm{d}r$$
$$-2\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}(Y_n^\varepsilon(r)-Y^\varepsilon(r))\,\langle Z_n^\varepsilon(r)-Z^\varepsilon(r),\mathrm{d}M^\varepsilon(r)\rangle. \tag{5.20}$$

Taking expectation and by Gronwall's inequality, it yields that

$$\begin{aligned}
&E_x^\varepsilon[\mathrm{e}^{2\int_0^s c_1^+(X_r^\varepsilon)\mathrm{d}r}|Y_n^\varepsilon(s)-Y^\varepsilon(s)|^2]\\
&\quad+E_x^\varepsilon\Big[\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\langle a(X(r))(Z_n^\varepsilon(r)-Z^\varepsilon(r)),Z_n^\varepsilon(r)-Z^\varepsilon(r)\rangle\mathrm{d}r\Big]\\
&\le CE_x^\varepsilon\Big[\mathrm{e}^{2\int_0^T c_1^+(X_r^\varepsilon)\mathrm{d}r}|g_n(X_T^\varepsilon)-g(X_T^\varepsilon)|^2\\
&\quad+\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\,|f_n(X_r^\varepsilon,Y_r^\varepsilon,Z_r^\varepsilon)-f(X_r^\varepsilon,Y_r^\varepsilon,Z_r^\varepsilon)|^2\,\mathrm{d}r\Big].
\end{aligned}\tag{5.21}$$

By both Burkhölder's and Young's inequalities, it implies

$$\begin{aligned}
&2E_x^\varepsilon\Big[\sup_{s\in[0,T]}\Big|\int_0^s \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}(Y_n^\varepsilon(r)-Y^\varepsilon(r))\,\langle Z_n^\varepsilon(r)-Z^\varepsilon(r),\mathrm{d}M^\varepsilon(r)\rangle\Big|\Big]\\
&\le CE_x^\varepsilon\Big[\Big(\int_0^T \mathrm{e}^{4\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}|Y_n^\varepsilon(r)-Y^\varepsilon(r)|^2\,\langle a(X(r))(Z_n^\varepsilon(r)-Z^\varepsilon(r)),Z_n^\varepsilon(r)-Z^\varepsilon(r)\rangle\mathrm{d}r\Big)^{\frac12}\Big]\\
&\le \frac12 E_x^\varepsilon\Big[\sup_{s\in[0,T]}\mathrm{e}^{2\int_0^s c_1^+(X_r^\varepsilon)\mathrm{d}r}|Y_n^\varepsilon(s)-Y^\varepsilon(s)|^2\Big]\\
&\quad+CE_x^\varepsilon\Big[\int_s^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\langle a(X(r))(Z_n^\varepsilon(r)-Z^\varepsilon(r)),Z_n^\varepsilon(r)-Z^\varepsilon(r)\rangle\mathrm{d}r\Big].
\end{aligned}\tag{5.22}$$

Consequently, in view of (5.20)–(5.22), we conclude that

$$\begin{aligned}
&E_x^\varepsilon\Big[\sup_{s\in[0,T]}|Y_n^\varepsilon(s)-Y^\varepsilon(s)|^2\Big]\\
&\le E_x^\varepsilon\Big[\sup_{s\in[0,T]}\mathrm{e}^{2\int_0^s c_1^+(X_r^\varepsilon)\mathrm{d}r}|Y_n^\varepsilon(s)-Y^\varepsilon(s)|^2\Big]\\
&\le CE_x^\varepsilon\Big[\mathrm{e}^{2\int_0^T c_1^+(X_r^\varepsilon)\mathrm{d}r}|g_n(X_T^\varepsilon)-g(X_T^\varepsilon)|^2\\
&\quad+\int_0^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\,|f_n(X_r^\varepsilon,Y_r^\varepsilon,Z_r^\varepsilon)-f(X_r^\varepsilon,Y_r^\varepsilon,Z_r^\varepsilon)|^2\,\mathrm{d}r\Big]\\
&\le C\Big(\|g_n-g\|_{L^\infty}^2E_x^\varepsilon[\mathrm{e}^{2\int_0^T c_1^+(X_r^\varepsilon)\mathrm{d}r}]+\omega_n(f)^2\,E_x^\varepsilon\Big[\int_0^T \mathrm{e}^{2\int_0^r c_1^+(X_u^\varepsilon)\mathrm{d}u}\,\mathrm{d}r\Big]\Big)\\
&\le C(\|g_n-g\|_{L^\infty}^2+\omega_n(f)^2),
\end{aligned}\tag{5.23}$$

where the fact for all $(x,y,z)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$,

$$|f_n(x,y,z)-f(x,y,z)|\le \sup_{\|(x,y,z)-(x',y',z')\|\le\frac1n}|f(x,y,z)-f(x',y',z')|=:\omega_n(f)$$

has been used.

In the same way, define $\overline{f}_n(x,y,z):=\int_{\mathbb{T}^d}f_n(x,y,z\,\nabla\widetilde{\omega}(\eta))\,m(\eta)\mathrm{d}\eta$. Then $\overline{f}_n$ satisfies Assumption 2 with constant independent of n. Also for all (x,y,z), we have $|\overline{f}_n(x,y,z)-\overline{f}(x,y,z)|\le\omega_n(f)$. Let E_x^0 denote the expectation under the law of a reflected Brownian motion X^0 with covariance matrix $\bar{a}$, and M^0 and K^0 be the martingale part and local time of X^0, respectively. If $(\overline{Y}_n, \overline{Z}_n)$ is the solution of the BSDE:

$$\begin{aligned}\overline{Y}_n(t) &= g_n(X^0(T)) + \int_t^T \overline{f}_n(X^0_r, \overline{Y}_n(r), \overline{Z}_n(r))\,\mathrm{d}r \\ &\quad - \int_t^T \langle \overline{Z}_n(r), \mathrm{d}M^0(r)\rangle + \overline{C}\int_t^T \overline{Y}_n(r)\,\mathrm{d}K^0_r, \quad t\in[0,T],\ \mathbb{P}^0_x - \text{a.s.},\end{aligned}$$

and if $(\overline{Y}, \overline{Z})$ is the solution of the BSDE:

$$\begin{aligned}\overline{Y}(t) &= g(X^0(T)) + \int_t^T \overline{f}(X^0_r, \overline{Y}(r), \overline{Z}(r))\,\mathrm{d}r \\ &\quad - \int_t^T \langle \overline{Z}(r), \mathrm{d}M^0(r)\rangle + \overline{C}\int_t^T \overline{Y}(r)\,\mathrm{d}K^0_r, \quad t\in[0,T],\ \mathbb{P}^0_x - \text{a.s.},\end{aligned}$$

then we similarly have

$$\begin{aligned}&E^0_x\Big[\sup_{s\in[0,T]}|\overline{Y}_n(s)-\overline{Y}(s)|^2\Big] \\ &\le C\Big(\|g_n-g\|^2_{L^\infty}E^0_x[\mathrm{e}^{2\int_0^T c_1^+(X^0_r)\mathrm{d}r}] + \omega_n(f)^2\,E^0_x\Big[\int_0^T \mathrm{e}^{2\int_0^r c_1^+(X^0_u)\mathrm{d}u}\,\mathrm{d}r\Big]\Big) \\ &\le C(\|g_n-g\|^2_{L^\infty}+\omega_n(f)^2). \end{aligned} \tag{5.24}$$

From (5.23)–(5.24) and Lemma 5.3, we can conclude that $\lim_{\varepsilon\to 0}|Y^\varepsilon(0)-Y(0)|^p = 0$ holds for any $p\in(1, \frac{Q(\lambda,d,q_1,q_2)}{2}\wedge 2)$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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