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On Warped Product Gradient Ricci-Harmonic Soliton

Elismar BATISTA[1] Levi ADRIANO[2] Willian TOKURA[3]

Abstract In this paper, the authors study gradient Ricci-Harmonic solitons on warped product manifolds. First, they prove triviality results for the potential and warping functions that reach a maximum or a minimum. In order to provide nontrivial examples, they consider the base and the fiber conformal to a semi-Euclidean space, which is invariant under the action of a translation group of co-dimension one. This approach allows them to produce infinitely many examples of geodesically complete semi-Riemannian Ricci-Harmonic solitons not present in the literature.

Keywords Warped product, Gradient Ricci-Harmonic solitons, Semi-Riemannian metric, Group action
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1 Introduction

Let $(M^n, g(t))$ be an n-dimensional manifold with a family of semi-Riemannian metrics $g(t)$, let (N^m, g') be an m-dimensional manifold with a fixed semi-Riemannian metric and let $u(t): M \to N$ be a smooth map. The Ricci-Harmonic flow, introduced by Müller in [18], couples with Hamilton's Ricci flow (see [11]) and Eells and Sampson's harmonic map heat flow of u (see [6]), namely

$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)} + 2\theta(t)\nabla u(t) \otimes \nabla u(t), \\ \dfrac{\partial u}{\partial t} = \tau_{g(t)} u(t), \end{cases} \tag{1.1}$$

where $\theta(t) \geq 0$ is a time-dependent constant, $\nabla u(t) \otimes \nabla u(t) = u(t)^* g'$ is the pull-back of the metric g' via $u(t)$ and $\tau_{g(t)} u(t)$ is the tension field of $u(t)$ with respect to the metric $g(t)$. Many concepts and results in the Ricci flow can be carried over to the Ricci-Harmonic flow and some relevant works can be referred to [4, 9, 16–17, 25–26] and the references therein.

In this paper, we focus on gradient Ricci-Harmonic solitons, which correspond to self-similar solutions for system (1.1).

Definition 1.1 *Let (M, g) be a semi-Riemannian manifold. A metric g on M^n is a gradient Ricci-Harmonic* (*with respect to g'*) *soliton* (*GRHS for short*) *metric if for some map*

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[1]Department of Mathematics, IFBA-Barreiras, Barreiras-BA 47811-056, Brazil.
E-mail: elismardb@gmail.com
[2]Department of Mathematics, UFG, Goiania-GO, Brazil. E-mail: levi@ufg.br
[3]Department of Mathematics, UFGD, Dourados-MS, Brazil. E-mail: williamisaotokura@hotmail.com

$u:(M,g)\to(N,g')$, some potential smooth function $h:M\to\mathbb{R}$ and some constant λ, g satisfies the following coupled system:

$$\begin{cases} \mathrm{Ric}_g + \mathrm{Hess}\, h - \theta\nabla u\otimes\nabla u = \lambda g, \\ \tau_g u - g(\nabla u,\nabla h) = 0, \end{cases} \tag{1.2}$$

where $\tau_g u = \mathrm{trace}\nabla \mathrm{d}u$ denotes the tension field of u and θ is a time-dependent coupling constant. As usual, the quintuple (M^n,g,h,u,λ) is classified into three types according to the sign of λ: Shrinking if $\lambda>0$, steady if $\lambda=0$ and expanding if $\lambda<0$. Moreover, if h is a constant, the soliton is called trivial.

The function u in the previous definition is known as a harmonic map, and for $u=$ constant, (1.2) defines a gradient Ricci soliton, which was very important in Thurston's Geometrization conjecture studied by Perelman [21]. For more details on gradient Ricci soliton, see [7, 12, 22–23]. In the particular case, in which $(N,g')=(\mathbb{R},\mathrm{d}r^2)$ and u and h are constants, (1.2) defines an Einstein manifold, and for h constant and u harmonic, (1.2) defines a harmonic-Einstein manifold, which is a natural generalization of an Einstein manifold. After the introduction of the GRHS, many efforts have been devoted to understanding its geometry. For instance, in [27], the authors proved that any shrinking or steady GRHS is a gradient Ricci soliton if the sectional curvature of N is bounded from above by a constant. On the other hand, in [10], the authors provided triviality results for the potential function and harmonic map using the compactness of M.

In recent works on solitons, the notion of warped product introduced in [2] has become an important tool in the construction of manifolds with certain geometric properties (see [3, 5, 13]).

Definition 1.2 *Let (B^n,g_B), (F^m,g_F) be two semi-Riemannian manifolds and $f>0$ on B. The product manifold $M=B\times F$ furnished with the metric tensor*

$$g=\pi^*g_B+(f\circ\pi)^2\sigma^*g_F \tag{1.3}$$

is called a warped product. We denote it by $B^n\times_f F^m$, where $*$ is the pullback of canonical projections $\pi: B^n\times F^m\to B^n$, $\sigma: B^n\times F^m\to F^m$. The function f is called a warping function, B is called the base and F is called the fiber.

The authors in [15] studied gradient Ricci solitons on the warped product and proved that either the warping function is constant or the potential function satisfies

$$h=h_B\circ\pi,\quad h_B\in C^\infty(B). \tag{1.4}$$

We will consider the harmonic map as a smooth function $u:M\to\mathbb{R}$ and show that the decomposition (1.4) applied to a Ricci-Harmonic soliton is equivalent to a decomposition of the harmonic map u. More precisely, we obtain the following result.

Proposition 1.1 *Let $(B^n\times_f F^m,g,h,u,\lambda)$ be a GRHS with non-constant harmonic map u. Then in a neighborhood $\mathcal{V}$ of a point $(p,q)\in B^n\times F^m$, the harmonic map u can be represented as $u=u_B\circ\pi$ or $u=u_F\circ\sigma$ if and only if $h=h_B\circ\pi$.*

Motivated by the above proposition, we consider the study of the harmonic map in two cases, $u=u_B\circ\pi$ or $u=u_F\circ\sigma$. In this direction, we obtain a necessary and sufficient condition for the existence of GRHS on the warped product.

Theorem 1.1 *Let $(B^n \times_f F^m, g)$ be a warped product manifold. Then we have the following conditions:*

(a) *If $u = u_B \circ \pi$, then $(B^n \times_f F^m, g, h, u, \lambda)$ is a GRHS if and only if*

$$\begin{cases} \mathrm{Ric}_{g_B} - \dfrac{m}{f}\mathrm{Hess}_{g_B} f + \mathrm{Hess}_{g_B} h_B - \theta \nabla_{g_B} u_B \otimes \nabla_{g_B} u_B = \lambda g_B, \\ \Delta_\omega u_B = 0 \ \text{in } B, \end{cases} \tag{1.5}$$

F is Einstein with $\mathrm{Ric}_{g_F} = \mu g_F$,

$$f\Delta_{g_B} f + (m-1)\nabla_{g_B} f^2 + \lambda f^2 - f\nabla_{g_B} f(h_B) = \mu. \tag{1.6}$$

(b) *If $u = u_F \circ \sigma$, then $(B^n \times_f F^m, g, h, u, \lambda)$ is a GRHS if and only if*

$$\mathrm{Ric}_{g_B} - \frac{m}{f}\mathrm{Hess}_{g_B} f + \mathrm{Hess}_{g_B} h_B = \lambda g_B, \tag{1.7}$$

F is harmonic Einstein with

$$\begin{cases} \mathrm{Ric}_{g_F} - \theta \nabla_{g_F} u_F \otimes \nabla_{g_F} u_F = \mu g_F, \\ \Delta_{g_F} u_F = 0, \end{cases}$$

$$f\Delta_{g_B} f + (m-1)\nabla_{g_B} f^2 + \lambda f^2 - f\nabla_{g_B} f(h_B) = \mu, \tag{1.8}$$

where Ric_{g_B} *is the Ricci tensor of B,* Ric_{g_F} *is the Ricci tensor of F and $\Delta_\omega = \Delta - \langle \nabla, \nabla\omega \rangle$ with $\omega = h - m\nabla \log(f)$.*

Remark 1.1 Note that Theorem 1.1 is a generalization of [14, Corollary 3].

Proceeding, we obtain some triviality results for the warping function, potential function and harmonic map by means of the strong maximum principle.

Theorem 1.2 *Let $(B^n \times_f F^m, g, h, u, \lambda)$ be a GRHS with Riemannian base.*

(a) *If $u_B \in C^\infty(B)$ reaches an extremal value on B, then $u = u_B \circ \pi$ is a constant function, i.e., $(B^n \times_f F^m, g, h, u, \lambda)$ is a gradient Ricci soliton.*

(b) *If $\lambda \geq 0$, h_B reaches an extremal value on B and $\frac{m\Delta_{g_B} f}{f} \geq \mathrm{scal}_{g_B}$, then $h = h_B \circ \pi$ is a constant function, i.e., $(B^n \times_f F^m, g, h, u, \lambda)$ is a harmonic Einstein manifold, where* scal_{g_B} *is a scalar curvature of B.*

(c) *If f_B reaches the maximum and $\lambda \leq \frac{\mu}{f^2}$ in B, then $f = f_B \circ \pi$ is a constant function, i.e., $(B^n \times_f F^m, g, h, u, \lambda)$ is a product manifold. The same occurs if f_B reaches the minimum and $\lambda \geq \frac{\mu}{f^2}$.*

Corollary 1.1 *Let $(B^n \times_f F^m, g, h, u, \lambda)$ be a GRHS with compact Riemannian base.*

(a) *If $u_B \in C^\infty(B)$, then $u = u_B \circ \pi$ is a constant function, i.e., $(B^n \times_f F^m, g, h, u, \lambda)$ is a gradient Ricci soliton.*

(b) *If $\lambda \geq 0$ and $\frac{m\Delta_{g_B} f}{f} \geq \mathrm{scal}_{g_B}$, then $h = h_B \circ \pi$ is a constant function, i.e., $(B^n \times_f F^m, g, h, u, \lambda)$ is a harmonic Einstein manifold.*

(c) *If $\lambda \leq \frac{\mu}{f^2}$ in B, then $f = f_B \circ \pi$ is a constant function, i.e., $(B^n \times_f F^m, g, h, u, \lambda)$ is a product manifold. The same occurs if $\lambda \geq \frac{\mu}{f^2}$.*

The warped product has proved its efficiency in the construction of new examples of manifolds with certain geometric characteristics. For instance, in the scope of invariant solutions, Tokura et al. [24] provided explicit examples of geodesically complete gradient Yamabe solitons. On the other hand, in the same invariant solution context, Lemes de Sousa et al. [15] provided examples of non conformally flat steady gradient Ricci solitons.

In what follows, we focus our attention on the warped product $(B \times_f F, g)$, where the base or the fiber is conformally flat. In the case where $u = u_B \circ \pi$, we consider the warped product GRHS with the base conformal to an n-dimensional semi-Euclidean space, invariant under the action of an $(n-1)$-dimensional translation group. More precisely, consider the semi-Euclidean metric $(g_0)_{ij} = \varepsilon_i \delta_{ij}$ in local coordinates, $x_B = (x_1, \cdots, x_n)$ of $\mathbb{R}^n$, where $\varepsilon_i = \pm 1$. For an arbitrary choice of non-zero vector $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, we define the function $\xi : \mathbb{R}^n \to \mathbb{R}$ by

$$\xi(x_1, \cdots, x_n) = \sum_{i=1}^{n} \alpha_i x_i,$$

and we search for smooth functions $h, f, \varphi : (a,b) \subset \mathbb{R} \to \mathbb{R}$, $f, \varphi > 0$ such that composite functions $h = h \circ \xi$, $f = f \circ \xi$, $\varphi = \varphi \circ \xi : \xi^{-1}(a,b) \to \mathbb{R}$ satisfy (1.2) with $M = \xi^{-1}(a,b) \times F^m$ and metric tensor

$$g = \frac{1}{\varphi^2} g_0 + f^2 g_F. \tag{1.9}$$

The next result gives us a necessary and sufficient condition for the existence of GRHS warped product with base conformal to a semi-Euclidean space invariant by the action of an $(n-1)$-dimensional translation group when $u = u_B \circ \pi$.

Theorem 1.3 $(\mathbb{R}^n \times_f F^m, g, h, u = u_B \circ \pi, \lambda)$ *furnished with metric tensor* (1.9) *is a GRHS with non-constant harmonic map u and Einstein fiber constant μ if and only if functions* $f = f \circ \xi,\ h = h \circ \xi,\ \varphi = \varphi \circ \xi,\ u = u \circ \xi$ *verify the system below*:

$$(n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + h'' + 2\frac{\varphi'}{\varphi}h' - \theta(u')^2 = 0, \tag{1.10}$$

$$\left[\frac{\varphi''}{\varphi} - (n-1)\left(\frac{\varphi'}{\varphi}\right)^2 + m\frac{\varphi'}{\varphi}\frac{f'}{f} - \frac{\varphi'}{\varphi}h'\right]\|\alpha\|^2 = \frac{\lambda}{\varphi^2}, \tag{1.11}$$

$$\left[\frac{f''}{f} - (n-2)\frac{\varphi'}{\varphi}\frac{f'}{f} + (m-1)\left(\frac{f'}{f}\right)^2 - \frac{f'}{f}h'\right]\|\alpha\|^2 = \frac{\mu}{f^2\varphi^2} - \frac{\lambda}{\varphi^2}, \tag{1.12}$$

$$\left[u'' - (n-2)\frac{\varphi'}{\varphi}u' + mu'\frac{f'}{f} - u'h'\right]\|\alpha\|^2 = 0. \tag{1.13}$$

Corollary 1.2 *Under the hypotheses of Theorem* 1.3*, if* $\|\alpha\|^2 = 0$*, then* $(\mathbb{R}^n \times_f \mathbb{R}^m, g, h, u, \lambda)$ *is a steady GRHS and F^m is Ricci flat, that is,* $\lambda = 0$ *and* $\mu = 0$.

The next examples provide solutions of (1.10)–(1.13) in which $\|\alpha\|^2 = 0$ and $\|\alpha\|^2 \neq 0$, respectively.

Example 1.1 In Theorem 1.3, consider $\|\alpha\|^2 = 0$ and the Lorentzian space $(\mathbb{R}^n, \overline{g})$ with coordinates $(x_1, \cdots, x_n)$, signature $\varepsilon_1 = -1$, $\varepsilon_i = 1$, $\forall i \geq 2$, and fiber $(\mathbb{R}^m, \delta)$ where δ is the Euclidean metric. Let $\xi = x_1 + x_2$ and choose $k \in \mathbb{R} \setminus \{0\}$. Then

$$f(\xi) = \mathrm{e}^{k\xi}, \quad \varphi(\xi) = \mathrm{e}^{k\xi}, \quad u(\xi) = k\xi,$$

$$h(\xi) = \frac{k}{2}(2 - n + 3m + \theta)\xi - \frac{e^{-2k\xi}k_1}{2k} + k_2$$

define a family of geodesically complete steady GRHS on $(\mathbb{R}^n, \varphi^{-2}\overline{g}) \times_f (\mathbb{R}^m, \delta)$ with potential function h, harmonic map u and warping function f (see Section 2).

Example 1.2 In Theorem 1.3, consider $\|\alpha\|^2 = 1$, $m, n = 1$ and choose $k \in \mathbb{R}$ with $k^2 > 1$, then the functions

$$h(\xi) = (m - k(n-2))\ln(f(\xi)), \quad \varphi(\xi) = f(\xi)^k, \quad u(\xi) = k\xi,$$
$$f(\xi) = e^{\sqrt{\frac{\theta k^2}{(-k^2(n-2)-m)}}\xi}$$

provide a steady GRHS on $(\mathbb{R}^n, \varphi^{-2}\delta) \times_f (\mathbb{R}^m, \delta)$ defined for all $\xi \in \mathbb{R}$, where δ is the standard Euclidean metric.

In the case $u = u_F \circ \sigma$, we consider the warped product GRHS $(\mathbb{R}^n, \varphi^{-2}g_0) \times_f (\mathbb{R}^m, \tau^{-2}g_0)$ with the base and the fiber conformal to an n-dimensional and m-dimensional semi-Euclidean spaces, invariants under the action of an $(n-1)$-dimensional and $(m-1)$-dimensional translation group, respectively. In this way, similar to the function ξ, we define $\zeta : \mathbb{R}^m \to \mathbb{R}$ by

$$\zeta(x_{n+1}, \cdots, x_{n+m}) = \sum_{i=n+1}^{n+m} \beta_i x_i,$$

where $\beta = (\beta_{n+1}, \cdots, \beta_{n+m})$ is a non-zero vector and $x_F = (x_{n+1}, \cdots, x_{n+m}) \in \mathbb{R}^m$.

In the next result, we obtain conditions for functions $h \circ \xi$, $f \circ \xi$, $u \circ \zeta$, $\varphi \circ \xi$, $\tau \circ \zeta$ to satisfy (1.2) with $M = \xi^{-1}(a, b) \times \zeta^{-1}(c, d)$ and metric tensor

$$g = \frac{1}{\varphi^2}g_0 + f^2\frac{1}{\tau^2}g_0. \tag{1.14}$$

Theorem 1.4 $(\mathbb{R}^n \times_f \mathbb{R}^m, g, h, u = u_F \circ \sigma, \lambda)$ *furnished with the metric tensor* (1.14) *is a GRHS with non-constant harmonic map u and fiber harmonic Einstein constant* μ *if and only if functions* $f = f \circ \xi$, $h = h \circ \xi$, $\varphi = \varphi \circ \xi$, $u = u \circ \zeta$, $\tau = \tau \circ \zeta$ *verify the system below*:

$$(n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + h'' + 2\frac{\varphi'}{\varphi}h' = 0, \tag{1.15}$$

$$\Big[\frac{\varphi''}{\varphi} - (n-1)\Big(\frac{\varphi'}{\varphi}\Big)^2 + m\frac{\varphi'}{\varphi}\frac{f'}{f} - \frac{\varphi'}{\varphi}h'\Big]\|\alpha\|^2 = \frac{\lambda}{\varphi^2}, \tag{1.16}$$

$$\begin{aligned}&[f''\varphi^2 f - (n-2)\varphi'\varphi f f' + (m-1)(f')^2\varphi^2 - f' f\varphi^2 h']\|\alpha\|^2 + \lambda f^2\\ &\quad = [\tau''\tau - (m-1)(\tau')^2]\|\beta\|^2 = \mu,\end{aligned} \tag{1.17}$$

$$(m-2)\frac{\tau''}{\tau} - \theta(u')^2 = 0, \tag{1.18}$$

$$[\tau^2 u'' - (m-2)\tau\tau' u']\|\beta\|^2 = 0. \tag{1.19}$$

Corollary 1.3 *Under the hypotheses of Theorem* 1.4, *if we consider the condition* $\|\alpha\|^2 = 0$, *then* $(\mathbb{R}^n \times_f \mathbb{R}^m, g, h, u = u_F \circ \sigma, \lambda)$ *is a steady GRHS, that is,* $\lambda = 0$ *and* $\mu = 0$.

In the next results we describe all the solutions of Theorem 1.4 when $\lambda = 0$ and $\mu = 0$.

Theorem 1.5 *Consider $\lambda = 0$, $\mu = 0$ and $m \geq 0$ in Theorem 1.4. Then $(\mathbb{R}^n \times_f \mathbb{R}^m, g)$ is a steady GRHS if and only if functions $\varphi(\xi), f(\xi), h(\xi)$, $\tau(\zeta), u(\zeta)$ verify:*

(a) *For $\|\alpha\|^2 = 0$, $\|\beta\|^2 = 0$*

$$h(\xi) = \int \Big(\varphi^{-2}\Big[\int \Big(m\frac{f''}{f}\varphi^2 + 2m\varphi\varphi'\frac{f'}{f} - (n-2)\varphi\varphi''\Big)\mathrm{d}\xi + c_1\Big]\Big)\mathrm{d}\xi + c_2, \tag{1.20}$$

$$u(\zeta) = \int \pm\sqrt{\frac{(m-2)}{\theta}\frac{\tau''}{\tau}(\zeta)}\mathrm{d}\zeta + c_3, \tag{1.21}$$

where $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$.

(b) *For $\|\alpha\|^2 = 0$, $\|\beta\|^2 = 1$, the potential function $h(\xi)$ is defined by (1.20), and*

$$\tau(\zeta) = c_2(c_1 + (m-2)\zeta)^{\frac{1}{2-m}}, \tag{1.22}$$

$$u(\zeta) = c_2 \pm \frac{\sqrt{2 - 3m + m^2}((-2+m)\zeta + c_1)\log((-2+m)\zeta + c_1)}{(-2+m)\sqrt{\theta((-2+m)\zeta + c_1)^2}}. \tag{1.23}$$

These solutions are defined on the half space $\zeta > -\frac{c_1}{(m-2)}$.

(c) *For $||\alpha||^2 = 1$, $\|\beta\|^2 = 0$, either*

$$\begin{cases} \varphi_\pm(\xi) = \dfrac{c_4}{(N_\pm\xi + b)^{\frac{k}{N_\pm}}}, \\ f_\pm(\xi) = \dfrac{c_5}{(N_\pm\xi + b)^{\frac{1}{N_\pm}}}, \\ h_\pm(\xi) = -\dfrac{m - (n-2)k + N_\pm}{N_\pm}\ln(N_\pm\xi + b), \\ u(\zeta) = \displaystyle\int \pm\sqrt{\frac{(m-2)}{\theta}\frac{\tau''}{\tau}(\zeta)}\mathrm{d}\zeta + c_3, \end{cases} \tag{1.24}$$

where $c_4, c_5 > 0$, $k > 0$, b and $N_\pm$ are constants with $N_\pm = -k \pm \sqrt{m + k^2(n-1)}$, or

$$\begin{cases} \dfrac{\varphi'}{\varphi}(\xi) = k\dfrac{f'}{f}(\xi), \\ \dfrac{f'}{f}(\xi) = \psi(\xi), \\ h'(\xi) = [z(\xi) + m - k(n-2)]\psi(\xi), \\ u(\zeta) = \displaystyle\int \pm\sqrt{\frac{(m-2)}{\theta}\frac{\tau''}{\tau}(\zeta)}\mathrm{d}\zeta + c_3, \end{cases} \tag{1.25}$$

where $\psi(\xi)$ and $z(\xi)$ are solutions of

$$\begin{cases} \psi(\xi) = c_6(z(\xi) + k - \sqrt{m + k^2(n-1)})^{\frac{a-1}{2}}(z(\xi) + k + \sqrt{m + k^2(n-1)})^{\frac{-a+1}{2}}, \\ z'(\xi) = -c_6(z(\xi) + k - \sqrt{m + k^2(n-1)})^{\frac{a+1}{2}}(z(\xi) + k + \sqrt{m + k^2(n-1)})^{\frac{-a-1}{2}} \end{cases} \tag{1.26}$$

with non-constant function $z(\xi)$ and real constants k, $c_6 \geq 0$ and $a = \frac{k}{\sqrt{m+k^2(n-1)}}$.

(d) *For $\|\alpha\|^2 = 1$, $\|\beta\|^2 = 1$, functions $\varphi(\xi), f(\xi), h(\xi)$ are defined by either system (1.24) or (1.25), while functions $u(\zeta)$ and $\tau(\zeta)$ are determined by (1.22) and (1.23).*

Example 1.3 In Theorem 1.5, item (a) considers the Lorentzian spaces $(\mathbb{R}^n, g_1)$ and $(\mathbb{R}^m, g_2)$ with coordinates $(x_1, \cdots, x_n)$ and $(x_{n+1}, \cdots, x_{n+m})$, where $g_1 = -\mathrm{d}x_1^2 + \sum\limits_{i=2}^{n} \mathrm{d}x_i^2$ and $g_2 = -\mathrm{d}x_{n+1}^2 + \sum\limits_{j=n+2}^{n+m} \mathrm{d}x_j^2$. Let $\xi = x_1 + x_2$ and $\zeta = x_{n+1} + x_{n+2}$. Then

$$\begin{aligned}
&f(\xi) = k\mathrm{e}^{A\xi}, \quad \varphi(\xi) = k\mathrm{e}^{A\xi}, \quad \tau(\zeta) = \zeta^2 + 1, \quad A \neq 0, \ k > 0\\
&h(\xi) = (2 - n + 3m + \theta)\frac{A}{2}\xi - \frac{\mathrm{e}^{-2A\xi}c_7}{2Ak^2} + c_8, \quad c_7, c_8 \in \mathbb{R},\\
&u(\zeta) = -\frac{\sqrt{2}\sqrt{-2+m}\sqrt{1+\zeta^2}\sinh^{-1}(\zeta)}{\sqrt{\left(1+\zeta^2\right)\theta}} + c_9, \quad c_9 \in \mathbb{R}
\end{aligned}$$

define a family of geodesically complete steady GRHS on $(\mathbb{R}^n, \varphi^{-2}g_1) \times_f (\mathbb{R}^m, \tau^{-2}g_2)$ with potential function h, harmonic map u and warping function f (see Section 2).

Example 1.4 In Theorem 1.5, item (c) considers $k = 1$, $z(\xi)$ a non-zero constant, $(\mathbb{R}^2, g_0)$ the base and $(\mathbb{R}^3, g)$ the Lorentzian fiber with coordinates (y_1, y_2, y_3) and $\zeta = y_1 + y_2$. Then, we obtain

$$\begin{cases} \varphi(\xi) = \dfrac{c_5}{\xi + b}, \\ f(\xi) = \dfrac{c_4}{\xi + b}, \\ h(\xi) = -4\ln(\xi + b), \end{cases} \qquad \begin{cases} \tau(\zeta) = \mathrm{e}^{c_4\zeta}, \\ u(\zeta) = \dfrac{c_4}{\sqrt{\theta}}\zeta, \end{cases}$$

which describes a family of steady GRHS defined in the half-space $\xi > -b$.

2 Proof of the Main Results

Proof of Proposition 1.1 Let $\mathcal{L}(B)$, $\mathcal{L}(F)$ be the lifts spaces of the vector fields on B and F to $B \times F$, respectively. Consider $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$, so we have the following well-known formula for the Ricci tensor on warped products (see [20]),

$$\begin{aligned}
&\mathrm{Ric}(X, Y) = \mathrm{Ric}_{g_B}(X, Y) - \frac{m}{f}\mathrm{Hess}_{g_B} f(X, Y),\\
&\mathrm{Ric}(X, V) = 0, \\
&\mathrm{Ric}(V, W) = \mathrm{Ric}_{g_F}(V, W) - (f\Delta_B f + (m-1)g_B(\nabla f, \nabla f))g_F(V, W)
\end{aligned} \tag{2.1}$$

for all $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$.

Suppose that u can be represented as $u = u_B \circ \pi$ or $u = u_F \circ \sigma$, then (1.2) becomes

$$\mathrm{Hess}\, h(X, V) = X(V(h)) - \nabla_X V(h) = \theta \nabla u(X) \nabla u(V) = 0, \quad \forall X \in \mathcal{L}(B), \ \forall V \in \mathcal{L}(F).$$

It follows from [15] that $h = h_B \circ \pi$.

Now, assuming (1.4), then (1.2) applied to X, V is equivalent to

$$0 = \lambda g(X, V) = \theta \nabla u(X) \nabla u(V). \tag{2.2}$$

Note that the metric tensor (1.3) applied to X, V is null.

Since the harmonic map u is not constant, there exists a field $L = X + V \in \mathcal{L}(B^n \times F^m)$, $(p, q) \in B^n \times F^m$ and a neighborhood $\mathcal{V}$ such that $\nabla u(X+V)\nabla u(X+V) \neq 0$ in $\mathcal{V}$, which implies

$$\nabla u(X)^2 + 2\nabla u(X)\nabla u(V) + \nabla u(V)^2 \neq 0. \tag{2.3}$$

From (2.2), we derive

$$\nabla u(X)^2 + \nabla u(V)^2 \neq 0. \tag{2.4}$$

Therefore, either $\nabla u(X) \neq 0$ or $\nabla u(V) \neq 0$ in $\mathcal{V}$.

Proof of Theorem 1.1 First, consider $u = u_B \circ \pi$. Combining the first equation in (1.2) with (2.1), we arrive at

$$\mathrm{Ric}_{g_B} - \frac{m}{f}\mathrm{Hess}_{g_B} f(X,Y) + \mathrm{Hess}_{g_B} h(X,Y) - \theta \nabla_{g_B} u \otimes \nabla_{g_B} u(X,Y) = \lambda g_B(X,Y)$$

for all $X, Y \in \mathcal{L}(B)$. Now, combining [20, Proposition 35] and the second equation of (1.2), we have

$$\Delta u = \left[\Delta_{g_B} u_B + \frac{m}{f} g_B(\nabla_{g_B} u_B, \nabla_{g_B} f)\right] \circ \pi = g_B(\nabla u_B, \nabla h_B) \circ \pi. \tag{2.5}$$

Denoting $\Delta_\omega := \Delta - g(\nabla \omega, \nabla \cdot)$, (2.5) becomes

$$\Delta_\omega u_B = 0,$$

which proves (1.5).

For $V, W \in \mathcal{L}(F)$, we have from (1.2) and (2.1) that

$$\begin{aligned} \lambda f^2 g_F(V,W) = {} & \mathrm{Ric}_{g_F}(V,W) + \mathrm{Hess}\, h(V,W) \\ & - (f\Delta_B f + (m-1) g_B(\nabla f, \nabla f)) g_F(V,W), \end{aligned} \tag{2.6}$$

and from the connection expression of warped product metrics, we obtain the following expression for the Hessian $\mathrm{Hess}\, h(V,W)$,

$$\begin{aligned} \mathrm{Hess}\, h(V,W) &= V(W(h)) - (\nabla_V W)(h) \\ &= -({}^F\nabla_V W - f g_F(V,W)\nabla_{g_B} f)(h) \\ &= f g_F(V,W)\nabla_{g_B} f(h). \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) it follows that $\mathrm{Ric}_{g_F} = \mu g_F$, where

$$\mu = f\Delta_{g_B} f + (m-1)\nabla_{g_B} f^2 + \lambda f^2 - f\nabla_{g_B} f(h). \tag{2.8}$$

Therefore, F is an Einstein manifold.

In the case $u = u_F \circ \sigma$ just combine (1.2) and (2.1) to obtain

$$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_{g_B} - \frac{m}{f}\mathrm{Hess}_{g_B} f(X,Y) = \lambda g(X,Y) - \mathrm{Hess}\, h(X,Y) + \theta \nabla u \otimes \nabla u(X,Y)$$

for all $X, Y \in \mathcal{L}(B)$. It is equivalent to

$$\mathrm{Ric}_{g_B} - \frac{m}{f}\mathrm{Hess}_{g_B} f(X,Y) + \mathrm{Hess}_{g_B} h(X,Y) = \lambda g_B(X,Y)$$

for all $X, Y \in \mathcal{L}(B)$.

On the other hand,

$$\begin{aligned}\mathrm{Ric}_g(V,W) &= \lambda g(V,W) - \mathrm{Hess}\, h(V,W) + \theta \nabla u \otimes \nabla u(V,W) \\ &= \mathrm{Ric}_{g_F}(V,W) - (f\Delta_B f + (m-1) g_B(\nabla f, \nabla f)) g_F(V,W)\end{aligned}$$

for all $V, W \in \mathcal{L}(F)$. From the warped product metric we arrive at

$$\begin{aligned}\mathrm{Ric}_{g_F}(V,W) - \theta \nabla_{g_F} u \otimes \nabla_{g_F} u(V,W) &= \lambda f^2 g_F(V,W) - f g_F(V,W) \nabla_{g_B} f(h) \\ &\quad + (f\Delta_B f + (m-1) g_B(\nabla f, \nabla f)) g_F(V,W) \\ &= (f\Delta_{g_B} f + (m-1)\nabla_{g_B} f^2 + \lambda f^2 \\ &\quad - f\nabla_{g_B} f(h)) g_F.\end{aligned}$$

Consider the second equation of (1.2) with $u = u_F \circ \sigma$. By [20, Proposition 35], we derive

$$\Delta u = \frac{\Delta_{g_F} u_F}{f^2} \circ \sigma = g(\nabla u, \nabla h) = 0.$$

Note that $\mu = f\Delta_{g_B} f + (m-1)\nabla_{g_B} f^2 + \lambda f^2 - f\nabla_{g_B} f(h)$ is constant in relation to fiber, so (F^m, g_F) is a harmonic Einstein manifold and this completes the proof.

Proof of Theorem 1.2 (a) Suppose that $u = u_B \circ \pi$. Proceeding in the same way as in the proof of Theorem 1.1, we arrive at

$$\Delta u = \Big[\Delta_{g_B} u_B + \frac{m}{f} g_B(\nabla_{g_B} u_B, \nabla_{g_B} f)\Big] \circ \pi = g_B(\nabla u_B, \nabla h_B) \circ \pi \tag{2.9}$$

and

$$\Delta_\omega u_B = 0,$$

where $\omega = h_B - m\log(f)$. Hence, from the strong maximum principle (see [8]), u_B is a constant, which implies that $u = u_B \circ \pi$ is a constant. Therefore, $(B^n \times_f F^m, g, h, u, \lambda)$ is a gradient Ricci soliton.

(b) Combining (1.2) and (2.1), we produce

$$\begin{aligned}&\lambda g_B(X,Y) - \mathrm{Hess}_{g_B} h(X,Y) + \theta \nabla_{g_B} u(X) \nabla_{g_B} u(Y) \\ &= \mathrm{Ric}_{g_B}(X,Y) - \frac{m}{f}\mathrm{Hess}_{g_B} f(X,Y)\end{aligned} \tag{2.10}$$

for all $X, Y \in \mathcal{L}(B)$. Now, taking the trace in both sides of (2.10), we get

$$\Delta_{g_B} h_B = n\lambda + \theta \|\mathrm{d}\pi(\nabla u)\|^2 - \mathrm{scal}_{g_B} + \frac{m\Delta_{g_B} f}{f}.$$

From the hypothesis, we deduce that $\Delta_{g_B} h_B \geq 0$. Therefore, applying the strong maximum principle we conclude that h_B is a constant, i.e., the potential function $h = h_B \circ \pi$ is a constant.

(c) Consider the elliptic operator of second order given by

$$\Xi(\cdot) = \Delta(\cdot) - \nabla h(\cdot) + \frac{m-1}{f}\nabla f(\cdot).$$

It follows from (1.6) that

$$\Xi(f) = \frac{\mu - \lambda f^2}{f} \geq 0.$$

So, as a direct consequence of the strong maximum principle, one has that f is a constant function and this concludes the proof.

Proof of Theorem 1.3 Since Theorem 1.1 provides a necessary and sufficient condition to the warped product $(B^n \times_f F^m, g)$ to be a GRHS, we will utilize its equivalence condition in combination with the invariant solution approach to produce equations (1.10)–(1.13).

First, for an arbitrary choice of a non-zero vector $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, we define the function $\xi : \mathbb{R}^n \to \mathbb{R}$ by $\xi(x_1, \cdots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Since we assume that $\varphi = \varphi(\xi)$, $h = h(\xi)$, $u = u(\xi)$ and $f = f(\xi)$, then we have

$$\begin{array}{llll} \varphi_{,x_i} = \varphi'\alpha_i, & f_{,x_i} = f'\alpha_i, & h_{,x_i} = h'\alpha_i, & u_{,x_i} = u'\alpha_i, \\ \varphi_{,x_i x_j} = \varphi''\alpha_i\alpha_j, & f_{,x_i x_j} = f''\alpha_i\alpha_j, & h_{,x_i x_j} = h''\alpha_i\alpha_j, & u_{,x_i x_j} = u''\alpha_i\alpha_j. \end{array}$$

Now, let (B, g_B) be the Euclidean space endowed with the conformal metric $g_B = \varphi^{-2} g_0$. It is well known that for the conformal metric g_B, the Ricci tensor is given by [1]:

$$\mathrm{Ric}_{g_B} = \frac{1}{\varphi^2}\{(n-2)\varphi \mathrm{Hess}_{g_0}(\varphi) + [\varphi\Delta_{g_0}\varphi - (n-1)\nabla_{g_0}\varphi^2]g_0\}.$$

Using the tensorial notation $\mathrm{Hess}_{g_0}(\varphi)(\partial_i, \partial_j) = (\mathrm{Hess}_{g_0}(\varphi))_{i,j}$, where $\{\partial_i\}_{i=1}^n$ is a Euclidean orthonormal basis, we obtain

$$\begin{aligned} &(\mathrm{Hess}_{g_0}(\varphi))_{i,j} = \varphi''\alpha_i\alpha_j, \\ &\Delta_{g_0}\varphi = \sum_k \varepsilon_k (\mathrm{Hess}_{g_0}(\varphi))_{kk} = \varphi''\|\alpha\|^2, \\ &\nabla_{g_0}\varphi^2 = \sum_k \varepsilon_k \varphi^2_{,x_k} = \varphi'\alpha^2. \end{aligned}$$

Hence, the Ricci tensor takes the following form

$$(\mathrm{Ric}_{g_B})_{i,j} = \frac{1}{\varphi}\{(n-2)\varphi''\alpha_i\alpha_j\}, \quad i \neq j, \tag{2.11}$$

$$(\mathrm{Ric}_{g_B})_{i,i} = \frac{1}{\varphi^2}\{(n-2)\varphi\varphi''(\alpha_i)^2 + [\varphi\varphi''\|\alpha\|^2 - (n-1)(\varphi')^2\|\alpha\|^2]\varepsilon_i\} \tag{2.12}$$

for $i = 1, 2, \cdots, n$. Computing $\mathrm{Hess}(h)$ relatively to g_B, we have

$$(\mathrm{Hess}_{g_B}(h))_{ij} = h_{,x_i x_j} - \sum_{k=1}^n \Gamma_{ij}^k h_{,x_k},$$

where the Christoffel symbols Γ_{ij}^k for distinct i, j, k are given by

$$\Gamma_{ij}^k = 0, \quad \Gamma_{ij}^i = -\frac{\varphi_{,x_j}}{\varphi}, \quad \Gamma_{ii}^k = \varepsilon_i\varepsilon_k\frac{\varphi_{,x_k}}{\varphi}, \quad \Gamma_{ii}^i = -\frac{\varphi_{,x_i}}{\varphi}.$$

Therefore,

$$
\begin{aligned}
(\mathrm{Hess}_{g_B}(h))_{ij} &= h_{,x_ix_j} + \varphi^{-1}(\varphi_{,x_i}h_{,x_j} + \varphi_{,x_j}h_{,x_i}) - \delta_{ij}\varepsilon_i \sum_k \varepsilon_k \varphi^{-1}\varphi_{,x_k}h_{,x_k} \\
&= \alpha_i\alpha_j h'' + (2\alpha_i\alpha_j - \delta_{ij}\varepsilon_i\|\alpha\|^2)\varphi^{-1}\varphi' h'
\end{aligned}
\tag{2.13}
$$

and

$$
\Delta_{g_B} f = \sum_k \varphi^2\varepsilon_k(\mathrm{Hess}_{g_B}(f))_{kk} = \|\alpha\|^2\varphi^2(f'' - (n-2)\varphi^{-1}\varphi' f'). \tag{2.14}
$$

On the other hand, the expressions of $\nabla f(h)$, ∇f^2 and $(\nabla u \otimes \nabla u)_{ij}$ on the conformal metric g_B are given by

$$
\begin{aligned}
&\nabla_{g_B} f(h) = \langle \nabla_{g_B} f, \nabla_{g_B} h\rangle = \varphi^2 \sum_k \varepsilon_k f_{,x_k}h_{,x_k} = \|\alpha\|^2\varphi^2 f'h', \\
&\nabla_{g_B} f^2 = \varphi^2 \sum_k \varepsilon_k f_{,x_k}^2 = \|\alpha\|^2\varphi^2(f')^2, \\
&(\nabla_{g_B} u \otimes \nabla_{g_B} u)_{ij} = u_{,x_i}u_{,x_j} = \alpha_i\alpha_j(u')^2.
\end{aligned}
\tag{2.15}
$$

Then substituting (2.12)–(2.13) and (2.15), for $i = j$, into (1.5) we obtain (1.11).

Now, from (2.11) and (2.13), for $i \neq j$, we obtain

$$
\alpha_i\alpha_j\Big((n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + h'' + 2\frac{\varphi'}{\varphi}h' - \theta(u')^2\Big) = 0.
$$

If there exist i, j with $i \neq j$ such that $\alpha_i\alpha_j \neq 0$, then we get

$$
(n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + h'' + 2\frac{\varphi'}{\varphi}h' - \theta(u')^2 = 0,
$$

which is (1.10).

Now, we need to consider the case $\alpha_i\alpha_j = 0$, $\forall i \neq j$. For this, consider k_0 fixed and $\alpha_{k_0} = 1$, $\alpha_k = 0$ for $k \neq k_0$. In this case, substituting (2.12)–(2.13) and (2.15) into (1.5), we obtain (1.11) for $i \neq k_0$, that is, $\alpha_i = 0$, and when $i = k_0$, we obtain (1.10), that is, $\alpha_{k_0} = 1$.

Substituting (2.14)–(2.15) into (1.6), we obtain (1.12). Finally, substituting (2.14)–(2.15) into (2.9), we obtain (1.13) and this completes the proof.

Proof of Theorem 1.4 By a similar way as in the proof of Theorem 1.3, substituting equations (2.11) and (2.15) into (1.7) with $i \neq j$, we obtain (1.15), and to produce (1.16) it is sufficient to use (2.12), (2.15) in (1.7) with $i = j$.

We know by Theorem 1.1 that F is a harmonic Einstein manifold, that is

$$
\mathrm{Ric}_{g_F} - \theta\nabla_{g_F}u_{g_F} \otimes \nabla_{g_F}u_{g_F} = \mu g_F, \tag{2.16}
$$

where $\theta > 0$ and

$$
\Delta_{g_B} f + (m-1)\nabla_{g_B} f^2 + \lambda f^2 - f\nabla_{g_B} f(h) = \mu. \tag{2.17}
$$

For an arbitrary choice of a non-zero vector $\beta = (\beta_1, \cdots, \beta_m)$, consider $\tau : \mathbb{R}^m \to (0, \infty)$ and $\zeta : \mathbb{R}^m \to \mathbb{R}$ the conformal factor of the fiber and the invariant by translation function, respectively. Since we are assuming $u = u(\zeta)$, we have

$$
(\nabla_{g_F}u_{g_F} \otimes \nabla_{g_F}u_{g_F})_{ij} = u_{,x_{n+i}}u_{,x_{n+j}} = (u')^2\beta_i\beta_j \quad i, j = 1, \cdots, m. \tag{2.18}
$$

Substituting (2.15) into (2.17), we have

$$[f''\varphi^2 f-(n-2)\varphi'\varphi ff'+(m-1)(f')^2\varphi^2-f'f\varphi^2h']\|\alpha\|^2+\lambda f^2=\mu. \tag{2.19}$$

Substituting (2.11)–(2.12) and (2.18)–(2.19) into (2.16), we obtain

$$\begin{aligned}&[f''\varphi^2 f-(n-2)\varphi'\varphi ff'+(m-1)(f')^2\varphi^2-f'f\varphi^2h']\|\alpha\|^2+\lambda f^2\\ &=[\tau''\tau-(m-1)(\tau')^2]\|\beta\|^2=\mu\end{aligned} \tag{2.20}$$

for $i=j$, and

$$(m-2)\frac{\tau''}{\tau}-\theta(u')^2=0 \tag{2.21}$$

for $i\neq j$. Note that (2.20) and (2.21) are precisely (1.17) and (1.18), respectively. Finally, by Theorem 1.1(b), u is a harmonic function, then

$$\Delta_{g_F}u_F=\sum_k\tau^2\varepsilon_k(\mathrm{Hess}_{g_F}(u))_{kk}=\|\beta\|^2\tau^2(u''-(m-2)\tau^{-1}\tau'u')=0,$$

which corresponds to equation (1.19) and this concludes the proof.

Proof of Theorem 1.5 (a) Taking parameters $\lambda=0$ and $\|\alpha\|^2=||\beta||^2=0$ in Theorem 1.4, we have

$$(n-2)\frac{\varphi''}{\varphi}-m\frac{f''}{f}-2m\frac{\varphi'}{\varphi}\frac{f'}{f}+h''+2\frac{\varphi'}{\varphi}h'=0, \tag{2.22}$$

$$(m-2)\frac{\tau''}{\tau}-\theta(u')^2=0. \tag{2.23}$$

(1.20) and (1.21) follow by integration of above ODEs (2.22) and (2.23).

(b) Combining parameters $\|\beta\|^2=1$, $\|\alpha\|^2=0$, $\lambda=0$ with Theorem 1.4, we have

$$\frac{\tau''}{\tau}-(m-1)\Big(\frac{\tau'}{\tau}\Big)^2=0, \tag{2.24}$$

$$(m-2)\frac{\tau''}{\tau}-\theta(u')^2=0, \tag{2.25}$$

$$\tau^2u''-(m-2)\tau\tau'u'=0. \tag{2.26}$$

Integrating (2.24) and substituting it into (2.25), we produce (1.22), while integrating (2.24) and substituting it into (2.26), we obtain (1.23).

(c) Consider $\|\alpha\|^2=1$, $\|\beta\|^2=0$ and $\lambda=0$. Then from equations (1.15)–(1.18) of Theorem 1.4, we deduce

$$\begin{cases}(n-2)\dfrac{\varphi''}{\varphi}-m\dfrac{f''}{f}-2m\dfrac{\varphi'}{\varphi}\dfrac{f'}{f}+h''+2\dfrac{\varphi'}{\varphi}h'=0,\\ \dfrac{\varphi''}{\varphi}-(n-1)\Big(\dfrac{\varphi'}{\varphi}\Big)^2+m\dfrac{\varphi'}{\varphi}\dfrac{f'}{f}-\dfrac{\varphi'}{\varphi}h'=0,\\ f''\varphi^2f-(n-2)\varphi'\varphi ff'+(m-1)(f')^2\varphi^2-f'f\varphi^2h'=0,\\ (m-2)\dfrac{\tau''}{\tau}-\theta(u')^2=0.\end{cases} \tag{2.27}$$

The first three equations of the above system involving φ, f and h were solved in [15]. On the other hand, integrating the last equation of system (2.27), we obtain (1.24).

(d) It follows immediately from the previous cases.

In order to prove the completeness of Examples 1.1 and 1.3, we recall the following definition.

Definition 2.1 *A semi-Riemannian manifold in which every geodesic is defined on the entire real line is said to be geodesically complete, or just complete.*

Given a curve γ in $M \times_f F$, we can write $\gamma(s) = (\gamma_B(s), \gamma_F(s))$, where $\gamma_B = \pi \circ \gamma$ and $\gamma_F = \sigma \circ \gamma$. The following proposition guarantees a necessary and sufficient condition for a curve γ to be geodesic.

Proposition 2.1 (see [20]) *A curve $\gamma = (\gamma_B, \gamma_F)$ in $B \times_f F$ is geodesic if and only if*
(1) $\gamma_B'' = g_F(\gamma_F', \gamma_F') f \circ \gamma_B \nabla f$ *in B,*
(2) $\gamma_F'' = \dfrac{-2}{f \circ \gamma_B} \dfrac{\mathrm{d}(f \circ \gamma_B)}{\mathrm{d}s} \gamma_F'$ *in F.*

Proof of Completeness of Example 1.1 Proceeding similarly as in [19, 24], we will now prove that the GRHS defined in Example 1.1 is complete by showing that any geodesic $\gamma(s) = (\gamma_B(s), \gamma_F(s))$ is defined for all $t \in \mathbb{R}$.

Consider $(\mathbb{R}^n, \overline{g})$ to be the Lorentzian space with coordinates $(x_1, \cdots, x_n)$ with signature $\varepsilon_1 = -1$, $\varepsilon_i = 1$, $\forall i \geq 2$. Take $k \in \mathbb{R} \setminus \{0\}$ and functions

$$\begin{aligned} &f(\xi) = \mathrm{e}^{k\xi}, \quad \varphi(\xi) = \mathrm{e}^{k\xi}, \quad u(\xi) = k\xi, \\ &h(\xi) = \frac{k}{2}(2 - n + 3m + \theta)\xi - \frac{\mathrm{e}^{-2k\xi} k_1}{2k} + k_2. \end{aligned}$$

Define $g_B := \varphi^{-2}\overline{g} = \mathrm{e}^{-2k\xi}\overline{g}$, then the gradient $\nabla_{g_B} f$ becomes

$$\nabla_{g_B} f = (-k\mathrm{e}^{3k\xi}, k\mathrm{e}^{3k\xi}, 0, \cdots, 0).$$

Let $\gamma(s) = (\gamma_B(s), \gamma_F(s))$ be a geodesic curve so that $\gamma_B(s) = (y_1(s), \cdots, y_n(s))$ and $\gamma_F(s) = (y_{n+1}(s), \cdots, y_{n+m}(s))$. It follows from Proposition 2.1 that the curve $\gamma(s)$ is a geodesic curve of $(\mathbb{R}^n, g_B) \times_f (\mathbb{R}^m, \delta)$ if and only if the system of equations below is satisfied

$$\begin{cases} y_1''(s) = -k\delta(\gamma_F', \gamma_F')\mathrm{e}^{4k(y_1(s)+y_2(s))}, & \text{(I)} \\ y_2''(s) = k\delta(\gamma_F', \gamma_F')\mathrm{e}^{4k(y_1(s)+y_2(s))}, & \text{(II)} \\ y_r''(s) = 0, \quad \text{for } r \in \{3, \cdots, n\}, & \text{(III)} \\ y_{n+l}''(s) = -2k[y_1'(s) + y_2'(s)]y_{n+l}'(s) \quad \text{for } l \in \{1, \cdots, m\}. & \text{(IV)} \end{cases}$$

Integrating the sum of differential equations (I)–(II) and substituting them into (IV), we obtain the second order linear ordinary differential equation

$$y_{n+l}''(s) + 2kc_1 y_{n+l}'(s) = 0 \quad \text{for each } l \in \{1, \cdots, m\},\ c_1 \in \mathbb{R}, \tag{2.28}$$

whose general solutions $y_{n+l}(s)$ are defined on the entire real line $\mathbb{R}$. Integrating (III) we arrive at $y_r(s) = c_{5,r} + c_{6,r}s$, for $c_{5,r}, c_{6,r} \in \mathbb{R}$, whose domain is also the real line, then it is only necessary to prove that the solutions of (I) and (II) are also defined in $\mathbb{R}$.

Solving differential equation (2.28) and substituting its result into (I), we obtain that $y_1''(s) =$ constant, therefore, $y_1(s)$ is defined on the entire real line $\mathbb{R}$. Except for the sign, the same occurs for $y_2(s)$. Thus, all the geodesics $\gamma(s)$ are defined for the entire real line.

Proof of Completeness of Example 1.3 Let $(\mathbb{R}^n, g_1)$ and $(\mathbb{R}^m, g_2)$ be the standard semi-Euclidean spaces as in Example 1.3. Take $A \in \mathbb{R} \setminus \{0\}$, $k > 0$ and consider functions

$$\varphi(\xi) = k\mathrm{e}^{A\xi}, \quad f(\xi) = k\mathrm{e}^{A\xi}, \quad h(\xi) = (2 - n + 3m + \theta)\frac{A}{2}\xi - \frac{\mathrm{e}^{-2A\xi}c_7}{2Ak^2} + c_8, \quad c_7, c_8 \in \mathbb{R}$$

and

$$\tau(\zeta) = \zeta^2 + 1, \quad u(\zeta) = -\frac{\sqrt{2}\sqrt{-2+m}\sqrt{1+\zeta^2}\sinh^{-1}(\zeta)}{\sqrt{(1+\zeta^2)\theta}} + c_9, \quad c_9 \in \mathbb{R},$$

where f, φ, h are defined in $(\mathbb{R}^n, g_1)$ and τ, u are defined in $(\mathbb{R}^m, g_2)$.

In the conformal metric $g_B := \varphi^{-2}g_1 = k^{-2}\mathrm{e}^{-2A\xi}g_1$, gradient $\nabla_{g_B} f$ is given by

$$\nabla_{g_B} f = \sum_{r,s=1}^{n} g_B^{rs} f_{,x_s}\partial_r = \sum_{r,s=1}^{n} \varphi^2\varepsilon_r\delta_{rs}f'\alpha_s\partial_s = \sum_{s=1}^{n} k^3 A\varepsilon_s\alpha_s\mathrm{e}^{3A\xi}\partial_s.$$

Since $\alpha_1 = \alpha_2 = 1$, $\alpha_i = 0$, for $i \geq 3$, and $\varepsilon_1 = -1$, $\varepsilon_i = 1$, for $i \geq 2$, we obtain

$$\nabla_{g_B} f = (-k^3 A\mathrm{e}^{3A\xi}, k^3 A\mathrm{e}^{3A\xi}, 0, \cdots, 0).$$

Then, considering $\gamma_B(s) = (y_1(s), \cdots, y_n(s))$ and $\gamma_F(s) = (y_{n+1}(s), \cdots, y_{n+m}(s))$ in Proposition 2.1, we have

$$\begin{cases} y_1''(s) = -k^4 A g_F(\gamma_F', \gamma_F')\mathrm{e}^{4A(y_1(s)+y_2(s))}, & \text{(I}'\text{)} \\ y_2''(s) = k^4 A g_F(\gamma_F', \gamma_F')\mathrm{e}^{4A(y_1(s)+y_2(s))}, & \text{(II}'\text{)} \\ y_r''(s) = 0 \quad \text{for } r \in \{3, \cdots, n\}, & \text{(III}'\text{)} \\ y_{n+l}''(s) = -2A[y_1'(s) + y_2'(s)]y_{n+l}'(s) \quad \text{for } l \in \{1, \cdots, m\}. & \text{(IV}'\text{)} \end{cases}$$

Summing the differential equations (I′) and (II′) we have $y_1''(s) + y_2''(s) = 0$, then by integration

$$y_1'(s) + y_2'(s) = c_1, \quad y_1(s) + y_2(s) = c_1 s + c_2, \quad c_1, c_2 \in \mathbb{R}. \tag{2.29}$$

Substituting (2.29) into (IV′), we obtain the second order linear ordinary differential equation

$$y_{n+l}''(s) + 2Ac_1 y_{n+l}'(s) = 0 \quad \text{for each } l \in \{1, \cdots, m\},$$

whose general solutions are

$$y_{n+l}(s) = \begin{cases} c_{3,l} + c_{4,l}s, & \text{if } c_1 = 0, \\ c_{3,l} + c_{4,l}e^{-2Ac_1 s}, & \text{if } c_1 \neq 0, \end{cases} \tag{2.30}$$

where $c_{3,l}, c_{4,l} \in \mathbb{R}$. Therefore, for each $l \in \{1, \cdots, m\}$, functions $y_{n+l}(s)$ are defined on the entire real line $\mathbb{R}$. Notice that the solutions of (III′) are given by $y_r(s) = c_{5,r} + c_{6,r}s$, for $c_{5,r}, c_{6,r} \in \mathbb{R}$, whose domain is also the real line. Finally, to conclude it is only necessary to prove that the solutions of (I′) and (II′) are also defined in $\mathbb{R}$, and for this we will express $g_F(\gamma'_F, \gamma'_F)$ in function of the parameter s.

Since $g_F = \tau^{-2} g_2$, $\beta_{n+1} = \beta_{n+2} = 1$, $\beta_{n+j} = 0$, for $j \geq 3$, and $\varepsilon_{n+1} = -1$, $\varepsilon_{n+j} = 1$, for $j \geq 2$, we obtain

$$\begin{aligned} g_F(\gamma'_F, \gamma'_F) &= \tau(\zeta \circ \gamma_F)^{-2} g_2(\gamma'_F, \gamma'_F) \\ &= \tau(\zeta \circ \gamma_F)^{-2}[-y'_{n+1}(s)^2 + y'_{n+2}(s)^2 + \cdots + y'_{n+m}(s)^2] \\ &= \frac{1}{((y_{n+1}(s) + y_{n+2}(s))^2 + 1)^2}[-y'_{n+1}(s)^2 + y'_{n+2}(s)^2 + \cdots + y'_{n+m}(s)^2], \end{aligned}$$

and substituting (2.30) into (I′), we have

$$\begin{aligned} y''_1(s) = &-k^4 A[(c_{7,1} + c_{7,2} e^{-2Ac_1 s})^2 + 1]^{-2} \\ &\times [-c_{8,1}^2 + c_{8,2}^2 + \cdots + c_{8,r}^2] e^{-4Ac_1 s} e^{4A(y_1(s)+y_2(s))}, \end{aligned}$$

where $c_{7,1}, c_{7,2}, c_{8,1}, c_{8,2}, \cdots, c_{8,r} \in \mathbb{R}$.

Now, from (2.29) we arrive at

$$\begin{aligned} y''_1(s) &= c_{9,1}[(c_{7,1} + c_{7,2}\mathrm{e}^{-2Ac_1 s})^2 + 1]^{-2} \mathrm{e}^{-4Ac_1 s} \mathrm{e}^{4A(c_1 s + c_2)} \\ &= c_{9,1}[(c_{7,1} + c_{7,2}\mathrm{e}^{-2Ac_1 s})^2 + 1]^{-2} \mathrm{e}^{4Ac_2} \\ &= c_{10,1}[(c_{7,1} + c_{7,2}\mathrm{e}^{-2Ac_1 s})^2 + 1]^{-2}, \quad c_{9,1}, c_{10,1} \in \mathbb{R}. \end{aligned}$$

Note that $y'_1(s)$ is smooth at every point and has a bounded derivative. Indeed,

$$y''_1(s) = \|c_{10,1}[(c_{7,1} + c_{7,2}\mathrm{e}^{-2Ac_1 s})^2 + 1]^{-2}\| < \|c_{10,1}\|.$$

Thus, the system defined by

$$\begin{cases} y'_1(s) = z_1(s), \\ z'_1(s) = [(c_{7,1} + c_{7,2}\mathrm{e}^{-2Ac_1 s})^2 + 1]^{-2} \end{cases}$$

has solutions whose domain is $\mathbb{R}$. Except for the sign, the same occurs for $y_2(s)$. Thus, all the geodesics $\gamma = (\gamma_B, \gamma_F)$ are defined on the entire real line, which means that

$$(\mathbb{R}^n, \varphi^{-2} g_1) \times_f (\mathbb{R}^m, \tau^{-2} g_2)$$

is geodesically complete.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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