Chin. Ann. Math. Ser. B
47(2), 2026, 359–374
DOI: 10.1007/s11401-025-0046-8

Chinese Annals of Mathematics, Series B

A Big Picard Type Theorem Concerning Derivative and Its Application*

Shuxian LI[1] Xiaojun LIU[1]

Abstract In this paper, the authors prove a big Picard type theorem concerning derivative: Let $f(z)$ be meromorphic in $D = \{z : 0 < |z - z_0| < \delta\}$ for each $\delta > 0$, if z_0 is an essential singularity of $f(z)$, then either $f(z)$ assumes every finite value infinitely often or $f'(z)$ assumes every finite value except possibly zero infinitely often. As an application of this result, they extend Nevo, Pang and Zalcman's quasinormal criterion: Let $\{f_n(z)\}$ be a sequence of meromorphic functions on the plane domain D, all of whose zeros are multiple such that $f_n'(z) - 1$ has zeros of multiplicity at least n for all n on D, then $\{f_n(z)\}$ is quasinormal of order 1 on D. Then they obtain a corresponding result in value distribution theory: Let $f(z)$ be a meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple such that

$$\overline{\lim_{r\to+\infty}} \frac{T(r,f)}{(\log r)^2} = +\infty,$$

then there exist a positive integer M and $R_0 > 0$ such that for each $r > R_0$, there exists $z_0 \in \mathbb{C}$ satisfying $|z_0| > r$ such that z_0 is a zero of $f'(z) - 1$ with multiplicity at most M.

Keywords Essential singularity, Quasinormal families, Value distribution theory
2020 MR Subject Classification 30D30, 30D35, 30D45

1 Introduction and Main Results

Essential singularity is an important isolated singularity of complex functions. Studying the behavior of a complex function which is holomorphic or meromorphic in a punctured domain of its isolated essential singularity holds profound significance.

In 1879, Picard [9] proved the following famous big Picard theorem.

Theorem 1.1 (see [9]) *Let $f(z)$ be meromorphic in $D = \{z : 0 < |z - z_0| < R\}$, if z_0 is an isolated essential singularity, then $f(z)$ must take every complex number infinitely often times on D, with at most two exceptions. Let $f(z)$ be holomorphic in $D = \{z : 0 < |z - z_0| < R\}$, if z_0 is an isolated essential singularity, then $f(z)$ must take every complex number infinitely often times on D with at most one exception.*

This result promoted the development of Nevanlinna's value distribution theory. With this result, we can immediately infer whether an isolated singularity is an essential singularity of a function.

Manuscript received November 6, 2022. Revised October 18, 2023.
[1]Department of Mathematics, University of Shanghai for Science and Technology, Shanghai 200093, China. E-mail: shu-xian.li@outlook.com xiaojunliu2007@hotmail.com
*This work was supported by the National Natural Science Foundation of China (No. 11871216).

Let $r > 0$ and let $f(z)$ be holomorphic in $D = \{z : 0 < |z - z_0| < r\}$, if $f(z)$ is a constant on D or $f(z)$ omits two distinct values for $z \in D$, then z_0 is a pole or a removable singularity of $f(z)$.

We write $\langle f, \Omega \rangle \in P$ to indicate that the holomorphic function $f(z)$ is defined on the plane domain Ω with the property P. In order to determine whether an isolated singularity is an essential singularity of a function, Minda [5] obtained the following more general conclusion in 1985, which also revealed the connection between the big Picard theorem and the little Picard theorem.

Theorem 1.2 (see [5, Theorem 3]) *If P is a property of holomorphic functions satisfying*:

(1) *If $\langle f, \Omega \rangle \in P$ and $\Omega' \subset \Omega$, then $\langle f, \Omega' \rangle \in P$.*

(2) *If $\langle f, \Omega \rangle \in P$ and $\Phi(z) = \alpha z + \beta$, $\alpha \neq 0$, then $\langle f \circ \Phi, \Phi^{-1}(\Omega) \rangle \in P$.*

(3) *Let $\langle f_n, \Omega_n \rangle \in P$, where $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\mathbb{C} = \cup \Omega_n$. If $f_n(z) \to f$ spherically uniformly on compact subsets of $\mathbb{C}$, then $\langle f, \mathbb{C} \rangle \in P$.*

(4) *If $\langle f, \mathbb{C} \rangle \in P$, then f is constant.*

And $\langle f, \Delta'(z_0, r) \rangle \in P$, then f has a pole or a removable singularity at z_0.

However, the preceding results about the behavior of a complex function which is holomorphic or meromorphic in a punctured domain of its isolated essential singularity do not involve its derivative. In 1959, Hayman [1] established the following renowned result.

Theorem 1.3 (see [2, Corollary of Theorem 3.5]) *Suppose that $f(z)$ is meromorphic and transcendental in $\mathbb{C}$, then either $f(z)$ assumes every finite value infinitely often or $f'(z)$ assumes every finite value except possibly zero infinitely often.*

In Theorem 1.3, ∞ is an essential singularity of $f(z)$. Naturally, we want to know whether $f(z)$ has the same properties near a finite essential singular point. Based on the question, we obtain the following result.

Theorem 1.4 *Let $f(z)$ be meromorphic in $D = \{z : 0 < |z - z_0| < \delta\}$ for each $\delta > 0$, if z_0 is an essential singularity of $f(z)$, then either $f(z)$ assumes every finite value infinitely often or $f'(z)$ assumes every finite value except possibly zero infinitely often.*

To prove Theorem 1.4, we only need to verify the following result.

Theorem 1.5 *Let $\delta > 0$ and let $f(z)$ be meromorphic in $D = \{z : 0 < |z - z_0| < \delta\}$ such that $f(z) \neq a$ and $f'(z) \neq b$ for some constants $a, b \in \mathbb{C}$, $b \neq 0$ and all $z \in D$, then z_0 is not an essential singularity of $f(z)$.*

According to Theorem 1.5, we obtain a new method for determining if a point is an essential singularity of a function. Furthermore, we apply it to the quasinormal criterion and we get a corresponding result in value distribution theory.

Recall that a family $\mathcal{F}$ of functions meromorphic on a plane domain $D \subset \mathbb{C}$ is said to be quasinormal on D (see [6]) if from each sequence $\{f_n(z)\} \subset \mathcal{F}$ one can extract a subsequence $\{f_{n_k}(z)\}$ which converges locally uniformly with respect to the spherical metric on $D \backslash E$, where the set E (which may depend on $\{f_{n_k}(z)\}$) has no accumulation point in D. If E can always be chosen to satisfy $|E| \leq \nu$, $\mathcal{F}$ is said to quasinormal of order ν on D. Thus a family is quasinormal of order 0 on D if and only if it is normal on D. The family $\mathcal{F}$ is said to be (quasi) normal at $z_0 \in D$ if it is (quasi)normal on some neighborhood of z_0; thus $\mathcal{F}$ is quasinormal on D if and only if it is quasinormal at each point $z \in D$. On the other hand, $\mathcal{F}$ fails to be quasinormal of order ν on D precisely when there exist points $z_1, z_2, \cdots, z_{\nu+1}$ in D and a sequence $\{f_n(z)\} \subset \mathcal{F}$ such that no subsequence of $\{f_n(z)\}$ is normal at z_j, $j = 1, 2, \cdots, \nu+1$.

In 2007, Nevo, Pang and Zalcman [6] obtained the following results.

Theorem 1.6 (see [6, Theorem 1]) *Let $\{f_n(z)\}$ be a sequence of meromorphic functions on the plane domain D, all of whose zeros are multiple such that $f_n'(z) \neq 1$ for all n and all $z \in D$, then $\{f_n(z)\}$ is quasinormal of order 1 on D.*

Theorem 1.7 (see [6, Theorem 2]) *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple, then $f'(z) - 1$ has infinitely many zeros.*

As the application of Theorem 1.5, we generalize Theorem 1.6 as follows.

Theorem 1.8 *Let $\{f_n(z)\}$ be a sequence of meromorphic functions on the plane domain D, all of whose zeros are multiple such that $f_n'(z) - 1$ has zeros with multiplicity at least n for all n on D, then $\{f_n(z)\}$ is quasinormal of order 1 on D.*

Then we can use Theorem 1.8 to prove the following result in value distribution theory.

Theorem 1.9 *Let $f(z)$ be a meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple such that*

$$\overline{\lim_{r\to+\infty}} \frac{T(r,f)}{(\log r)^2} = +\infty,$$

then there exist a positive integer M and $R_0 > 0$ such that for each $r > R_0$, there exists $z_0 \in \mathbb{C}$ satisfying $|z_0| > r$ such that z_0 is a zero of $f'(z) - 1$ with multiplicity at most M.

Remark 1.1 Actually, if we have a meromorphic function $f(z)$ on $\mathbb{C}$ whose zeros are multiple such that $\overline{\lim\limits_{r\to+\infty}} \frac{T(r,f)}{(\log r)^2} = +\infty$, then $f'(z) - 1$ not only has infinitely many zeros but also has a sequence of these zeros that tends to infinity and multiplicity can be controlled.

2 Notations and Preliminaries

In this section, we introduce some notations and preliminary definitions.

Throughout, k is a positive integer. We write $\{f_n(z)\} \subset \mathcal{M}(D)$ to indicate that $\{f_n(z)\}$ is a sequence of functions meromorphic on the plane domain D. We denote by Δ the open unit disc in $\mathbb{C}$. For $z_0 \in \mathbb{C}$, $r > 0$ and $\rho > 0$, $\Delta(z_0, r) := \{z : |z - z_0| < r\}$, $\overline{\Delta}(z_0, r) := \{z : |z - z_0| \leq r\}$, $\Delta'(z_0, r) := \{z : 0 < |z - z_0| < r\}$ and $D_{\rho,r} := \{z : \rho < |z| < r\}$.

We write $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ on D to indicate that the sequence $\{f_n(z)\}$ converges to $f(z)$ in the spherical metric uniformly on compact subsets of D and $f_n(z) \Rightarrow f(z)$ on D if the convergence is in the Euclidean metric.

For $f(z)$ meromorphic on the plane domain D,

$$S(D, f) := \frac{1}{\pi} \iint_D [f^{\#}(z)]^2 \mathrm{d}x\mathrm{d}y,$$

where

$$f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2}$$

denotes the spherical derivative. And we set $S(r,f)=S(\Delta(0,r),f)$ and $S(\rho,r,f)=S(D_{\rho,r},f)$.

Recall that the order of a meromorphic function $f(z)$ on $\mathbb{C}$ can be defined as

$$\rho=\overline{\lim_{r\to\infty}}\frac{\log^{+}T(r,f)}{\log r},$$

where $T(r,f)$ is the Ahlfors-Shimizu form of the Nevanlinna characteristic function, given by

$$T(r,f)=\int_0^r\frac{S(t,f)}{t}\mathrm{d}t.$$

Clearly, if $f^{\#}(z)$ is bounded on $\mathbb{C}$, $f(z)$ has order at most 2.

Definition 2.1 (see [6]) *Let z_1, $z_2\in\mathbb{C}$ and put $\widetilde{z}=\frac{z_1+z_2}{2}$. We say that (z_1,z_2) is a nontrivial pair of zeros of $f(z)$ if*

(i) $f(z_1)=f(z_2)=0$;

(ii) *there exists z_3 such that $|z_3-\widetilde{z}|<|z_1-z_2|$ and $|f'(z_3)|>1$.*

Remark 2.1 Clearly, if all zeros of $f(z)$ are multiply, each pair of zeros satisfies (i). Note also that (ii) is equivalent to

(ii′) there exists z^* such that $|z^*|<1$ and $|h'(z^*)|>1$, where

$$h(z)=\frac{f(\widetilde{z}+(z_1-z_2)z)}{z_1-z_2}.$$

Since $|h'(z)|\geq h^{\#}(z)$, it suffices to have $h^{\#}(z^*)>1$ in (ii′).

3 Proof of Theorem 1.5

To prove Theorem 1.5, we require some preliminary results.

The spherical derivative can be a natural measure for the growth of a function that is meromorphic near the isolated singularity. Lehto and Virtanen [3–4] obtained the following important result.

Theorem 3.1 (see [3]) *If $f(z)$ is meromorphic in the neighborhood of the singularity $z=z_0$, an absolute constant $K>0$ exists such that*

$$\overline{\lim_{z\to z_0}}|z-z_0|f^{\#}(z)\geq K.$$

Furthermore, for meromorphic functions $f(z)$ omitting at least one value in a neighborhood of the singularity $z=z_0$, we always have

$$\overline{\lim_{z\to z_0}}|z-z_0|f^{\#}(z)=+\infty.$$

Lemma 3.1 (see [8, Lemma 2]) *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$;*

(b) *points z_n, $|z_n| < r$;*

(c) *functions $f_n \in \mathcal{F}$;*

(d) *positive numbers $\rho_n \to 0^+$*

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n+\rho_n\xi) \overset{\mathcal{X}}{\Rightarrow} g(\xi)$ on $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$ such that for every $\xi \in \mathbb{C}$, $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$.

Proof of Theorem 1.5 Otherwise, suppose that z_0 is an essential singularity of $f(z)$. We can assume that $z_0 = 0$, $a = 0$ and $b = 1$.

Following from Theorem 3.1, we have

$$\overline{\lim_{z\to 0}}\, |z| f^{\#}(z) = +\infty.$$

Then there exists a sequence $\{b_n\}$ in $\mathbb{C}$ such that $b_n \to 0$ and

$$\lim_{n\to\infty} |b_n| f^{\#}(b_n) = +\infty.$$

We define $f_n(z) = f(b_n z)$. Since $b_n \to 0$, $f_n(z)$ is defined for any $z \in \mathbb{C}\backslash\{0\}$ if n is sufficiently large. Then

$$f_n^{\#}(1) = |b_n| f^{\#}(b_n) \to \infty,$$

thus $\{f_n(z)\}$ is not normal at $\widetilde{z} = 1$.

Using Lemma 3.1 for $\alpha = 0$, we can extract a subsequence (which, renumbering, we continue to call $\{f_n(z)\}$), points $z_n \in \mathbb{C}$, $z_n \to 1$, $\rho_n > 0$, $\rho_n \to 0^+$ such that

$$g_n(\xi) = f_n(z_n + \rho_n\xi) \overset{\mathcal{X}}{\Rightarrow} g(\xi),$$

where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$.

For each $\varepsilon > 0$, there exists $\xi_0 \in \mathbb{C}$ such that $|g(\xi_0)| < \varepsilon$ and $g'(\xi_0) \neq 0$.

With $a_n = b_n(z_n + \rho_n\xi_0)$, we have $a_n \to 0$,

$$f(a_n) = f_n(z_n + \rho_n\xi_0) \to g(\xi_0)$$

and

$$b_n\rho_n f'(a_n) = \rho_n f_n'(z_n + \rho_n\xi_0) \to g'(\xi_0)$$

for n sufficiently large. Thus

$$a_n f'(a_n) = b_n(z_n + \rho_n\xi_0) f'(a_n) = b_n\rho_n f'(a_n)\Big(\xi_0 + \frac{z_n}{\rho_n}\Big) \to \infty$$

for n sufficiently large.

Since $\lim_{n\to\infty} |f(a_n)| = |g(\xi_0)| < \varepsilon$, there exists $N \in \mathbb{N}$, for each $n > N$, $|f(a_n)| < \varepsilon$.

Let $\varepsilon = \frac{1}{k}$, then there exist points $\xi_k \in \mathbb{C}$ and $n_k \in \mathbb{N}$ such that $a_{n_k,k} = b_{n_k,k}(z_{n_k,k}+\rho_{n_k,k}\xi_k)$ and $|f(a_{n_k,k})| < \frac{1}{k}$.

Altogether, we find a sequence $\{a_{n_k,k}\} \in \mathbb{C}$ such that $a_{n_k,k} \to 0$, $f(a_{n_k,k}) \to 0$ and $a_{n_k,k}f'(a_{n_k,k}) \to \infty$ for k sufficiently large.

Let $G(z) = \frac{f(z)}{z}$, then $z_0 = 0$ is also an essential singularity of $G(z)$. Thus we can also find a sequence $\{c_n\} \in \mathbb{C}$ such that $c_n \to 0$,

$$G(c_n) = \frac{f(c_n)}{c_n} \to 0$$

and

$$c_nG'(c_n) = -\frac{f(c_n)}{c_n} + f'(c_n) \to \infty$$

for n sufficiently large. Clearly, $\lim\limits_{n\to\infty} f'(c_n) = \infty$.

Let $h_n(z) = \frac{f(c_nz)}{c_n}$, $z \in \Delta(1, r)$, where $r > 0$. Then

$$h_n^{\#}(1) = \frac{|f'(c_n)|}{1 + \left|\dfrac{f(c_n)}{c_n}\right|^2} \to \infty$$

for n sufficiently large, so that $\{h_n(z)\}$ is not normal at $\widetilde{z} = 1$. But for each n, $h_n(z) \neq 0$ and $h_n'(z) \neq 1$ for every $z \in \Delta(1, r)$, thus $\{h_n(z)\}$ is normal at $\widetilde{z} = 1$. This contradiction shows that z_0 is not an essential singularity of $f(z)$.

4 Proof of Theorem 1.8

Before we give the proof of Theorem 1.8, we require the following results.

Lemma 4.1 (see [10]) *Let $f(z)$ be a nonconstant meromorphic function of finite order on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$. If $f^{(k)}(z) \neq 1$ on $\mathbb{C}$, then*

$$f(z) = \frac{1}{k!}\frac{(z-a)^{k+1}}{(z-b)}$$

for some a and $b\,(\neq a)$ in $\mathbb{C}$.

Lemma 4.2 *Let $\{f_n(z)\} \subset \mathcal{M}(D)$. If for each n, $f_n(z) \neq 0$ and $f_n'(z) - 1$ has zeros with multiplicity at least n on D, then $\{f_n(z)\}$ is normal on D.*

Proof Suppose not, $f_n(z)$ is not normal at some point $z_0 \in D$. Then by Lemma 3.1, we can extract a subsequence (which renumbering, we continue to call $\{f_n(z)\}$), points $z_n \in \mathbb{C}$, $z_n \to z_0$, $\rho_n > 0$, $\rho_n \to 0^+$, such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n\xi)}{\rho_n} \overset{\chi}{\Rightarrow} g(\xi),$$

where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$.

Clearly, $g_n(\xi) \neq 0$ and $g_n'(\xi) - 1 = f_n'(z_n + \rho_n\xi) - 1$ has zeros with multiplicity at least n.

Since $g(\xi)$ is nonconstant, it follows from Hurwitz' Theorem that $g(\xi) \neq 0$ on $\mathbb{C}$.

Since $g_n'(\xi)-1$ has zeros with multiplicity at least n and $g_n'(\xi)\Rightarrow g'(\xi)$ on the complement of the poles of $g(\xi)$, we claim that $g'(\xi)\neq 1$ for every $\xi\in\mathbb{C}$.

Indeed, $g'(\xi)\not\equiv 1$, otherwise $g'(\xi)\equiv 1$, it would contradict the fact that $g(\xi)\neq 0$ for every $\xi\in\mathbb{C}$. Suppose there exists $\xi_0\in\mathbb{C}$ such that $g'(\xi_0)=1$, we can assume ξ_0 as the zero of $g'(\xi)-1$ with multiplicity of $\widetilde{m}$ exactly. Since $g'(\xi)\not\equiv 1$, by Hurwitz' Theorem, there exist $N\in\mathbb{N}$ and points ξ_n in a neighborhood of ξ_0 such that $g_n'(\xi_n)=1$ and ξ_n as the zero of $g_n'(\xi)-1$ with multiplicity at most $\widetilde{m}$ for every $n>N$. On the other hand, for each n, $g_n'(\xi)-1$ has zeros with multiplicity at least n. Contradiction occurs when $n>\max(N,\widetilde{m})$, thus $g'(\xi)\neq 1$ for every $\xi\in\mathbb{C}$, as claimed.

Then by the well-known Hayman's alternative (see [1–2]), $g(\xi)$ is constant, which is a contradiction. Thus $\{f_n(z)\}$ is normal on D.

Lemma 4.3 *Let $\{a_k\}$ be a sequence in D which has no accumulation points in D. Let $\{f_n(z)\}\subset\mathcal{M}(D)$, all of whose zeros are multiple, such that $f_n'(z)-1$ has zeros with multiplicity at least n for all n on D. Suppose that*

(a) *no subsequence of $\{f_n(z)\}$ is normal at a_1;*

(b) *there exists $\delta>0$ such that each $f_n(z)$ has a single (multiple) zero on $\Delta(a_1,\delta)$;*

(c) *$f_n(z)\overset{\chi}{\Rightarrow} f(z)$ on $D\backslash\{a_j\}_{j=1}^{\infty}$.*

Then

(d) *there exists $\eta_0>0$ such that for each $0<\eta<\eta_0$, $f_n(z)$ has a single simple pole on $\Delta(a_1,\eta)$ for all sufficiently large n;*

(e) *$f(z)=z-a_1$ for $z\in D\backslash\{a_j\}_{j=2}^{\infty}$.*

Proof It suffices to prove that each subsequence of $\{f_n(z)\}$ has a subsequence which satisfies (d) and that (e) holds. So suppose that we have a subsequence of $\{f_n(z)\}$, which (to avoid complication in notation) we again call $\{f_n(z)\}$.

Since $\{f_n(z)\}$ is not normal at a_1, it follows from Lemma 3.1 that we can extract a subsequence (which renumbering, we continue to call $\{f_n(z)\}$), points $z_n\in\mathbb{C}$, $z_n\to a_1$, $\rho_n>0$, $\rho_n\to 0^+$ such that

$$g_n(\xi)=\frac{f_n(z_n+\rho_n\xi)}{\rho_n}\overset{\chi}{\Rightarrow} g(\xi),$$

where $g(\xi)$ is a nonconstant meromorphic function of finite order on $\mathbb{C}$, all of whose zeros are multiple.

Since $g_n'(\xi)-1=f_n'(z_n+\rho_n\xi)-1$ has zeros with multiplicity at least n and $g_n'(\xi)\Rightarrow g'(\xi)$ on the complement of the poles of $g(\xi)$, we claim that $g'(\xi)\neq 1$ for every $\xi\in\mathbb{C}$.

Indeed, $g'(\xi)\not\equiv 1$, otherwise $g'(\xi)\equiv 1$, it would contradict the fact that all zeros of $g(\xi)$ are multiple. Suppose that there exists $\xi_0\in\mathbb{C}$ such that $g'(\xi_0)=1$, we can assume ξ_0 as the zero of $g'(\xi)-1$ with multiplicity of $\widetilde{m}$ exactly. Since $g'(\xi)\not\equiv 1$, by Hurwitz' Theorem, there exist $N\in\mathbb{N}$ and points ξ_n in a neighborhood of ξ_0 satisfying $g_n'(\xi_n)=1$ and ξ_n as the zero of $g_n'(\xi)-1$ with multiplicity at most $\widetilde{m}$ for every $n>N$. On the other hand, for each n, the zeros of $g_n'(\xi)-1$ with multiplicity at least n. Contradiction occurs when $n>\max(N,\widetilde{m})$, thus $g'(\xi)\neq 1$ for every $\xi\in\mathbb{C}$, as claimed.

So by Lemma 4.1,

$$g(\xi)=\frac{(\xi-a)^2}{\xi-b}$$

for distinct complex numbers a and b.

It now follows from the argument principle that there exist sequences $\xi_n\to a$ and $\eta_n\to b$ such that $g_n(\xi_n)=0$ and $g_n(\eta_n)=\infty$ for n sufficiently large.

Thus, writing $z_{n,0}=z_n+\rho_n\xi_n$, $z_{n,1}=z_n+\rho_n\eta_n$, we have $z_{n,j}\to a_1$ $(j=0,1)$, $f_n(z_{n,0})=0$, $f_n(z_{n,1})=\infty$ and the multiplicity of $z_{n,0}$ as a zero of $f_n(z)$ is exactly 2 for n sufficiently large.

Let us now assume that (d) has been shown to hold. Since $f_n(z)\overset{\chi}{\Rightarrow} f(z)$ on $D\backslash\{a_j\}_{j=1}^{\infty}$, $f(z)$ is either meromorphic on $D\backslash\{a_j\}_{j=1}^{\infty}$ or identically infinite there.

Suppose first that $f(z)$ is meromorphic on $D\backslash\{a_j\}_{j=1}^{\infty}$.

By (b) and (d), there exists $0<\delta_0<\eta_0$ such that $f_n(z)$ only has a single (multiple) zero $z_{n,0}$ and a single simple pole $z_{n,1}$ on $\Delta'(a_1,\delta_0)$, thus $f(z)\neq 0$ and ∞ for every z in $\Delta'(a_1,\delta_0)$. Then $f_n^{(k)}(z)\Rightarrow f^{(k)}(z)$ on $\Delta'(a_1,\delta_0)$, $k=1,2$.

We claim that $f'(z)\equiv 1$ on $\Delta'(a_1,\delta_0)$. Otherwise, by Hurwitz' Theorem, $f'(z)\neq 1$ on $\Delta'(a_1,\delta_0)$, thus by Theorem 1.2, $z=a_1$ is not the essential singularity of $f(z)$. Hence $f(z)$ is meromorphic on $\Delta(a_1,\delta_0)$. For large enough n, we have

$$\frac{1}{2\pi \mathrm{i}}\int_{|z-a_1|=\frac{\delta_0}{2}}\frac{f_n''(z)}{{f_n}'(z)-1}\mathrm{d}z=\frac{1}{2\pi \mathrm{i}}\int_{|z-a_1|=\frac{\delta_0}{2}}\frac{f''(z)}{f'(z)-1}\mathrm{d}z.$$

But the integral on the left (for large n) tends to infinity, integral on the right is a finite value. This contradiction proves our claim.

Thus $f_n'(z)\Rightarrow f'(z)\equiv 1$ on $\Delta'(a_1,\delta_0)$. Also, $f(z)$ has a removable singularity at $z=a_1$, since $f'(z)$ dose. For large enough n, we also have

$$\frac{1}{2\pi \mathrm{i}}\int_{|z-a_1|=\frac{\delta_0}{2}}\frac{f_n'(z)}{f_n(z)}\mathrm{d}z=\frac{1}{2\pi \mathrm{i}}\int_{|z-a_1|=\frac{\delta_0}{2}}\frac{f'(z)}{f(z)}\mathrm{d}z.$$

The integral on the left is (for large n) the multiplicity of the zero of $f_n(z)$ at $z_{n,0}$ minus the multiplicity of the pole of $f_n(z)$ at $z_{n,1}$, which is $2-1=1$; thus the right hand side is also 1. But the integral on the right is the multiplicity of the zero of the analytic function $f(z)$ at a_1. Thus a_1 is the zero of $f(z)$ with multiplicity 1, so we have $f(z)=z-a_1$ for $z\in D\backslash\{a_j\}_{j=2}^{\infty}$.

Suppose now that $f\equiv\infty$ on $D\backslash\{a_j\}_{j=1}^{\infty}$. Let

$$F_n(z)=f_n(z)\frac{z-z_{n,1}}{(z-z_{n,0})^2}.$$

By (b), $F_n(z)\neq 0$ on $\Delta(a_1,\delta)$ for large n. Applying the maximum principle to the sequence $\left\{\frac{1}{F_n(z)}\right\}$ of holomorphic functions, we see that $F_n(z)\Rightarrow\infty$ on $\Delta(a_1,\delta)$. We have

$$\begin{aligned}g_n(\xi)&=\frac{f_n(z_n+\rho_n\xi)}{\rho_n}\\&=\frac{F_n(z_n+\rho_n\xi)}{\rho_n}\cdot\frac{(z_n+\rho_n\xi-(z_n+\rho_n\xi_n))^2}{z_n+\rho_n\xi-(z_n+\rho_n\eta_n)}\end{aligned}$$

$$= F_n(z_n+\rho_n\xi)\frac{(\xi-\xi_n)^2}{\xi-\eta_n}.$$

Thus for $\xi\neq a,b$, $F_n(z_n+\rho_n\xi)\rightarrow 1$ for n sufficiently large, which contradicts with $F_n(z)\Rightarrow\infty$ near a_1. Then $f(z)\not\equiv\infty$.

We have shown that when (d) obtains, (e) does as well. Now let us show that (d) must hold.

Suppose not, taking a subsequence and renumbering, we may assume that on any neighborhood of a_1, $f_n(z)$ has at least two poles for sufficiently large n.

Let $z_{n,2}\neq z_{n,1}$ such that $f_n(z_{n,2})=\infty$ and $f_n(z)$ has no poles in $\Delta'(z_{n,1},|z_{n,1}-z_{n,2}|)$. Write $z_{n,2}=z_n+\rho_n\eta_n^*$, then $z_{n,2}\rightarrow a_1$ and $\eta_n^*\rightarrow\infty$. Set

$$G_n(\zeta)=\frac{f_n(z_{n,1}+(z_{n,2}-z_{n,1})\zeta)}{z_{n,2}-z_{n,1}}.$$

Since $z_{n,2}-z_{n,1}\rightarrow 0$, $G_n(\zeta)$ is defined for any $\zeta\in\mathbb{C}$ if n is sufficiently large. And $G_n'(\zeta)-1=f_n'(z_{n,1}+(z_{n,2}-z_{n,1})\zeta)-1$ has zeros with multiplicity at least n. Note that

$$G_n(1)=\infty,\quad G_n(0)=\infty,\quad G_n\Big(\frac{z_{n,0}-z_{n,1}}{z_{n,2}-z_{n,1}}\Big)=0$$

and

$$\frac{z_{n,0}-z_{n,1}}{z_{n,2}-z_{n,1}}=\frac{\xi_n-\eta_n}{\eta_n^*-\eta_n}\rightarrow 0,$$

so $\{G_n(\zeta)\}$ is not normal at 0.

On the other hand, by (b), for n sufficiently large, $G_n(\zeta)$ has only a single zero (which tends to 0 as $n\rightarrow\infty$) on any compact subset of $\mathbb{C}$. Since $G_n'(\zeta)-1=f_n'(z_{n,1}+(z_{n,2}-z_{n,1})\zeta)-1$ has zeros with multiplicity at least n, it follows from Lemma 4.2 that $\{G_n(\zeta)\}$ is normal on $\mathbb{C}\setminus\{0\}$. Taking a subsequence and renumbering, we may assume that $G_n(\zeta)\overset{\mathcal{X}}{\Rightarrow}G(\xi)$ on $\mathbb{C}\setminus\{0\}$. Since $G_n(\zeta)$ has only a single pole on Δ, conditions (a)–(d) hold for the sequence $\{G_n(\zeta)\}$ (defined, say, on $\Delta(0,2)$ with $a_1=0$ and $\delta=1$). Thus, by the first part of the proof, $G(\zeta)$ is analytic on $\Delta(0,2)$. But this contradicts $G(1)=\infty$. This completes the proof of Lemma 4.3.

Lemma 4.4 *Let $\{f_n(z)\}\subset\mathcal{M}(\Delta)$, all of whose zeros are multiple, such that $f_n'(z)-1$ has zeros with multiplicity at least n for all n on Δ. Suppose that*

(a) *no subsequence of $\{f_n(z)\}$ is normal at z_0;*

(b) *for each $\delta>0$, $f_n(z)$ has at least two distinct zeros on $\Delta(z_0,\delta)$ for sufficiently large n. Then for each $\delta>0$, $f_n(z)$ has a nontrivial pair (a_n,c_n) of zeros on $\Delta(z_0,\delta)$ for sufficiently large n and the sequence*

$$\Big\{\frac{f_n(d_n+(a_n-c_n)\xi)}{a_n-c_n}\Big\}$$

is not normal on Δ. Here $d_n=\frac{a_n+c_n}{2}$.

Proof As in the proof of the previous lemma, it follows from Lemmas 3.1 and 4.1 that for each subsequence of $\{f_n(z)\}$, there exists a subsequence (which, renumbering, we continue to denote by $\{f_n(z)\}$), points $z_n\in\mathbb{C}$, $z_n\rightarrow z_0$, $\rho_n>0$, $\rho_n\rightarrow 0^+$ and distinct $a,b\in\mathbb{C}$ such that

$$g_n(\xi)=\frac{f_n(z_n+\rho_n\xi)}{\rho_n}\overset{\mathcal{X}}{\Rightarrow}g(\xi)=\frac{(\xi-a)^2}{\xi-b}$$

on $\mathbb{C}$. Thus there exist $\xi_n \to a, \eta_n \to b$ so that $a_n = z_n + \rho_n \xi_n \to z_0, b_n = z_n + \rho_n \eta_n \to z_0$ and $g_n(\xi_n) = f_n(a_n) = 0, g_n(\eta_n) = f_n(b_n) = \infty$ for sufficiently large n.

By (b), there also exists $c_n \neq a_n$, $c_n \to z_0$ such that $f_n(c_n) = 0$. Thus $c_n = z_n + \rho_n \xi_n^*$ and $\xi_n^* \to \infty$. Setting $d_n = \frac{a_n + c_n}{2}$, we define the function

$$h_n(\xi) = \frac{f_n(d_n + (a_n - c_n)\xi)}{a_n - c_n},$$

which is defined for any $\xi \in \mathbb{C}$ if n sufficiently large.

Since

$$h_n\Big(\frac{1}{2}\Big) = f_n(a_n) = 0, \quad h_n\Big(\frac{b_n - d_n}{a_n - c_n}\Big) = f_n(b_n) = \infty$$

and

$$\frac{b_n - d_n}{a_n - c_n} \to \frac{1}{2},$$

$\{h_n(z)\}$ is not normal at $\xi = \frac{1}{2}$. It follows from Marty's Theorem that

$$\lim_{n\to\infty} \sup_{|\xi - \frac{1}{2}| \leq \frac{1}{4}} [h_n{}^{\#}(\xi)] = \infty.$$

Then there exists ξ_n^*, $\left|\xi_n^* - \frac{1}{2}\right| \leq \frac{1}{4}$ such that $h_n^{\#}(\xi_n^*) > 1$, so (a_n, c_n) is a nontrivial pair of zeros of $f_n(z)$ for sufficiently large n.

Lemma 4.5 (see [6, Lemma 10]) *Let $f(z) \subset \mathcal{M}(\mathbb{C})$, all of whose zeros are multiple, such that $f'(z) \neq 1$, $z \in \mathbb{C}$. Then either $f(z)$ is rational, or there exist nontrivial pairs (a_n, c_n) of zeros of $f(z)$ such that $|a_n - c_n| \to 0$ and points $\{z_n^*\}$ such that $\left|z_n^* - \frac{a_n + c_n}{2}\right| < |a_n - c_n|$ and $f^{\#}(z_n^*) \to \infty$.*

Proof of Theorem 1.8 Let A be the set of points in D at which $\{f_n(z)\}$ fails to be normal. We consider two cases.

Case I For each $a \in A$, there exists $\delta_a > 0$ such that $f_n(z)$ has at most a single zero in $\Delta(a, \delta_a)$ for sufficiently large n. It then follows from Lemma 4.2 that $\{f_n(z)\}$ is quasinormal of order 1 in a neighborhood of each $a \in A$ and hence quasinormal in D. Thus, taking a subsequence and renumbering, we may assume that A is a (countable) discrete set in D.

Suppose that $\{f_n(z)\}$ has no subsequence having only a single point of nonnormality. Then there exist distinct points $a_1, a_2 \in A$ and a subsequence of $\{f_n(z)\}$ (which we continue to call $\{f_n(z)\}$) such that $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ on $D \backslash A$, but no subsequence of $\{f_n(z)\}$ is normal at a_1 or a_2. By Lemma 4.3, we have $f(z) = z - a_1 = z - a_2$, so that $a_1 = a_2$. This leads to a contradiction. Thus, in this case, $\{f_n(z)\}$ is quasinormal of order 1 on D.

Case II We now show that the assumption that Case I does not hold leads to a contradiction. Suppose then that the condition of Case I does not hold. Then there exists $a \in \Delta$, which we can (and do) henceforth assume to be 0, such that for each $\delta > 0$, the collection of indices for which $f_n(z)$ has at least two zeros in $\Delta(0, \delta)$ is infinite.

Taking an appropriate subsequence and renumbering, we may assume that for each $0 < \delta < 1$, there exists $N(\delta)$ such that $f_n(z)$ has at least two distinct zeros in $\Delta(0, \delta)$ if $n \geq N(\delta)$. Now fix $\delta > 0$. By Lemma 4.4, $f_n(z)$ has a nontrivial pair of zeros in $\Delta(0, \delta)$ for n sufficiently large.

Let (a_n, c_n) be such a pair for which

$$\left|\frac{a_n - c_n}{\delta - |d_n|}\right| \text{ is minimal, where } d_n = \frac{a_n + c_n}{2}. \tag{4.1}$$

We have

$$a_n - c_n \to 0 \text{ and thus } \frac{a_n - c_n}{\delta - |d_n|} \to 0. \tag{4.2}$$

Taking a subsequence if necessary, we may assume that $d_n \to d$. Set

$$h_n(\xi) = \frac{f_n(d_n + (a_n - c_n)\xi)}{a_n - c_n}.$$

Then for each $R > 0$, $h_n(\xi)$ is defined on $\overline{\Delta}(0, R)$ for sufficiently large n. Clearly all zeros of $h_n(\xi)$ are multiple and $h_n'(\xi) - 1$ has zeros with multiplicity at least n on $\mathbb{C}$.

We claim that no subsequence of $\{h_n(\xi)\}$ is normal on $\mathbb{C}$. Otherwise, taking a subsequence and renumbering we would have $h_n(\xi) \overset{\chi}{\rightrightarrows} h(\xi)$ on $\mathbb{C}$.

Since (a_n, c_n) is a nontrivial pair of zeros of $f_n(z)$, $h_n\big(\pm\frac{1}{2}\big) = h_n'\big(\pm\frac{1}{2}\big) = 0$ and $\sup\limits_{\Delta} |h_n'(\xi)| > 1$.

Clearly, $h(\xi) \not\equiv \infty$, otherwise $h(\xi) \equiv \infty$, but $\lim\limits_{n\to\infty} h_n\big(\pm\frac{1}{2}\big) = h\big(\pm\frac{1}{2}\big) = 0$, this contradiction shows that $h(\xi) \not\equiv \infty$ on $\mathbb{C}$. Then we have $h(\xi)$ is nonconstant, otherwise $h(\xi)$ is a constant, it would contradict $\sup\limits_{\Delta} |h_n'(\xi)| > 1$. It now follows that $h'(\xi) \neq 1$ on $\mathbb{C}$, since otherwise $h'(\xi) \equiv 1$, which would contradict $h'\big(\pm\frac{1}{2}\big) = 0$.

Now if $h(\xi)$ is of finite order, by Lemma 4.1,

$$h(\xi) = \frac{(\xi - a)^2}{\xi - b}$$

for distinct complex numbers a and b. But it contradicts $h\big(\pm\frac{1}{2}\big) = 0$. Thus $h(\xi)$ is a transcendental meromorphic function of infinite order.

It follows from Lemma 4.5 that there exists infinitely many nontrivial pairs (ξ_ℓ, η_ℓ) of zeros of $h(\xi)$ such that $\xi_\ell \to \infty$ and $\xi_\ell - \eta_\ell \to 0$, and points z_ℓ^* with $\big|z_\ell^* - \frac{\xi_\ell + \eta_\ell}{2}\big| < |\xi_\ell - \eta_\ell|$ such that $h^{\#}(z_\ell^*) \to \infty$.

Fix ℓ such that $h^{\#}(z_\ell^*) \geq 2$, $|\xi_\ell - \eta_\ell| < 1$. Then there exist $\xi_{n,\ell} \to \xi_\ell$ and $\eta_{n,\ell} \to \eta_\ell$ such that for n sufficiently large,

$$h_n(\xi_{n,\ell}) = h_n(\eta_{n,\ell}) = 0 \quad \text{and} \quad \left|z_\ell^* - \frac{\xi_{n,\ell} + \eta_{n,\ell}}{2}\right| < |\xi_{n,\ell} - \eta_{n,\ell}|.$$

Put

$$\begin{aligned} \xi_{n,\ell}^* &= d_n + (a_n - c_n)\xi_{n,\ell}, \\ \eta_{n,\ell}^* &= d_n + (a_n - c_n)\eta_{n,\ell}, \\ z_{n,\ell}^* &= d_n + (a_n - c_n)z_\ell^*. \end{aligned}$$

Then

$$\left|z_{n,\ell}^* - \frac{\xi_{n,\ell}^* + \eta_{n,\ell}^*}{2}\right| = |a_n - c_n|\left|z_\ell^* - \frac{\xi_{n,\ell} + \eta_{n,\ell}}{2}\right| < |a_n - c_n||\xi_{n,\ell} - \eta_{n,\ell}| = |\xi_{n,\ell}^* - \eta_{n,\ell}^*|$$

and $\xi^*_{n,\ell}, \eta^*_{n,\ell} \in \Delta(0,\delta)$ for sufficiently large n, and also

$$|f_n'(z^*_{n,\ell})| = |h_n'(z^*_\ell)| \geq h_n^{\#}(z^*_\ell) > 1.$$

Hence $(\xi^*_{n,\ell}, \eta^*_{n,\ell})$ is a nontrivial pair of zeros of $f_n(z)$ in $\Delta(0,\delta)$ (for large enough n), but

$$\begin{aligned}\frac{|\xi^*_{n,\ell}-\eta^*_{n,\ell}|}{\delta - \left|\dfrac{\xi^*_{n,\ell}+\eta^*_{n,t}}{2}\right|} &= \frac{|(a_n-c_n)(\xi_{n,\ell}-\eta_{n,\ell})|}{\delta - \left|d_n + (a_n-c_n)\left(\dfrac{\xi_{n,\ell}-\eta_{n,\ell}}{2}\right)\right|} \\ &= \frac{|a_n-c_n|}{\delta-|d_n|}\frac{|\xi_{n,\ell}-\eta_{n,\ell}|(\delta-|d_n|)}{\delta - \left|d_n + (a_n-c_n)\left(\dfrac{\xi_{n,\ell}+\eta_{n,\ell}}{2}\right)\right|}.\end{aligned}$$

Given $\varepsilon > 0$, by (4.2) we have, for large enough n,

$$\left|d_n + (a_n-c_n)\left(\frac{\xi_{n,\ell}+\eta_{n,\ell}}{2}\right)\right| < |d_n| + \varepsilon(\delta-|d_n|)\frac{|\xi_{n,\ell}+\eta_{n,\ell}|}{2}.$$

Choose $\varepsilon_0 < 1$ such that $|\xi_\ell - \eta_\ell| < \varepsilon_0$. Since $\xi_{n,\ell} - \eta_{n,\ell} \to \xi_\ell - \eta_\ell$, $|\xi_{n,\ell} - \eta_{n,\ell}| < \varepsilon_0$ for n sufficiently large. Then we have for ε sufficiently small and n sufficiently large,

$$\frac{|\xi^*_{n,\ell}-\eta^*_{n,\ell}|}{\delta - \left|\dfrac{\xi^*_{n,\ell}+\eta^*_{n,t}}{2}\right|} < \frac{|a_n-c_n|}{\delta-|d_n|}\cdot\varepsilon_0\frac{1}{1-\varepsilon\dfrac{|\xi_{n,\ell}+\eta_{n,\ell}|}{2}} < \frac{|a_n-c_n|}{\delta-|d_n|}.$$

This is a contradiction to (4.1). Thus no subsequence of $\{h_n(\xi)\}$ is normal on $\mathbb{C}$.

Let E be the set on which $\{h_n(\xi)\}$ is not normal. Suppose that for each $\xi \in E$, there is a neighborhood on which $h_n(\xi)$ has only a single (multiple) zero for sufficiently large n. Then by Lemma 4.2, $\{h_n(\xi)\}$ is quasinormal at each point of E and hence on all of $\mathbb{C}$.

Let $\xi_0 \in E$. Taking a subsequence, we may assume that no subsequence of $\{h_n(\xi)\}$ is normal at ξ_0 and that $\{h_n(\xi)\}$ converges locally spherically uniformly on $\mathbb{C}\backslash E_0$, where $E_0 \subset E$ is a discrete set containing ξ_0. By Lemma 4.3, $h_n(\xi) \overset{\chi}{\Rightarrow} \xi - \xi_0$ on $\mathbb{C}\backslash E_0$. Taking additional subsequences and diagonalizing, we may assume that no subsequence of $\{h_n(\xi)\}$ is normal at any point of E_0.

We claim that $E_0 = \{\xi_0\}$. Indeed, if $\xi_1 \in E_0$, by Lemma 4.3, we have that $h_n(\xi) \overset{\chi}{\Rightarrow} \xi - \xi_1$ on $\mathbb{C}\backslash E_0$, so that $\xi_0 = \xi_1$, $E_0 = \{\xi_0\}$ and $h_n(\xi) \overset{\chi}{\Rightarrow} \xi - \xi_0$ on $\mathbb{C}\backslash\{\xi_0\}$. But this contradicts $h_n\left(\pm\frac{1}{2}\right) = 0$.

Hence there exists $\xi_0 \in E$, such that for each $\delta > 0$, there is a subsequence of $\{h_n(\xi)\}$ (which we continue to call $\{h_n(\xi)\}$) such that each $h_n(\xi)$ has at least two distinct zeros in $\Delta(\xi_0,\delta)$ for sufficiently large n. Now it follows from Lemmas 3.1 and 4.1 that to each subsequence of $\{h_n(\xi)\}$, there corresponds a subsequence (which we continue to write as $\{h_n(\xi)\}$), points $z_n \in \Delta(\xi_0,\delta)$ such that $z_n \to \xi_0$, $\rho_n > 0$, $\rho_n \to 0^+$, and distinct complex numbers a and b such that

$$\widetilde{h}_n(\xi) = \frac{h_n(z_n+\rho_n\xi)}{\rho_n} \overset{\chi}{\Rightarrow} \frac{(\xi-a)^2}{\xi-b} \quad \text{on } \mathbb{C}.$$

Thus there exist sequences $\{\xi_{n,0}\}$ and $\{\xi_{n,1}\}$ in $\Delta(\xi_0,\delta)$ such that $\xi_{n,0} \to b$, $\xi_{n,1} \to a$, so that $z_{n,j} = z_n + \rho_n\xi_{n,j} \to \xi_0$ $(j = 0,1)$ and $\widetilde{h}_n(\xi_{n,0}) = h_n(z_{n,0}) = 0$, $\widetilde{h}_n(\xi_{n,1}) = h_n(z_{n,1}) = \infty$.

Since any neighborhood of ξ_0 contains at least two distinct zeros of $h_n(\xi)$ for large enough n, there exists $z_{n,2} \to \xi_0$, $z_{n,2} \neq z_{n,0}$, such that $h_n(z_{n,2}) = 0$. Setting $z_{n,2} = z_n + \rho_n \xi_{n,2}$, we have $\xi_{n,2} \to \infty$.

Now put $z^*_{n,j} = d_n + (a_n - c_n)z_n + \rho(a_n - c_n)\xi_{n,j}$, $j = 0, 1, 2$. Clearly $z_{n,j} \to d$. Define

$$G_n(\xi) = \frac{f_n\Big(\dfrac{z^*_{n,0} + z^*_{n,2}}{2} + (z^*_{n,0} - z^*_{n,2})\xi\Big)}{z^*_{n,0} - z^*_{n,2}}.$$

Then $\{G_n(\xi)\}$ is not normal at $\xi = \frac{1}{2}$. Indeed,

$$G_n\Big(\frac{1}{2}\Big) = 0, \quad G_n\Big(\frac{2\xi_{n,1} - \xi_{n,0} - \xi_{n,2}}{2(\xi_{n,0} - \xi_{n,2})}\Big) = \infty.$$

Since

$$\frac{2\xi_{n,1} - \xi_{n,0} - \xi_{n,2}}{2(\xi_{n,0} - \xi_{n,2})} \to \frac{1}{2},$$

$\{G_n(\xi)\}$ is not equicontinuous at $\xi = \frac{1}{2}$. It follows from Marty's Theorem that

$$\lim_{n\to\infty} \sup_{|\xi - \frac{1}{2}| \le \frac{1}{4}} [G^{\#}_n(\xi)] = \infty,$$

so that $(z^*_{n,0}, z^*_{n,2})$ is a nontrivial pair of zeros of $f_n(z)$ and $z^*_{n,0}, z^*_{n,2} \in \Delta(0, \delta)$ (for large enough n) and similarly,

$$\frac{|z^*_{n,0} - z^*_{n,2}|}{\delta - \Big|\dfrac{z^*_{n,0} + z^*_{n,2}}{2}\Big|} = \frac{|a_n - c_n| \cdot |z_{n,0} - z_{n,2}|}{\delta - \Big|d_n + (a_n - c_n) \cdot \dfrac{(z_{n,0} + z_{n,2})}{2}\Big|} < \frac{|a_n - c_n|}{\delta - |d_n|}$$

for n sufficiently large. This contradiction to (4.1) shows that Case II cannot obtain and completes the proof of Theorem 1.8.

5 Proof of Theorem 1.9

Before we give the proof of Theorem 1.9, we require the following results.

Lemma 5.1 (see [7, Lemma 2]) *Let $f(z)$ be a meromorphic function on $\mathbb{C}$. If*

$$\overline{\lim_{r\to+\infty}} \frac{T(r, f)}{(\log r)^2} = +\infty,$$

then there exists $r_n \to +\infty$ such that

$$\lim_{n\to\infty} S\Big(\frac{r_n}{2}, 2r_n, f\Big) = +\infty$$

and a sequence $\{a_n\}$ in $\mathbb{C}$ satisfying $\frac{r_n}{2} < |a_n| < 2r_n$ such that $f(a_n) \to 0$ and $a_n f'(a_n) \to \infty$.

Lemma 5.2 *Let $\{f_n(z)\} \subset \mathcal{M}(\Delta)$, all of whose zeros are multiple, such that $f'_n(z) - 1$ has zeros with multiplicity at least n for each n on Δ. Suppose that*

(a) *$\{f_n(z)\}$ is normal on $\Delta'(0, 1)$, but no subsequence of $\{f_n(z)\}$ is normal at* 0;

(b) *there exists* $\delta > 0$ *such that* $f_n(z)$ *has a single (multiple) zero on* $\Delta(0,\delta)$ *for all sufficiently large* n.

Then there exists a subsequence of $\{f_n(z)\}$ *(which we continue to call* $\{f_n(z)\}$*), such that for any* $a \in \mathbb{C}$, $f_n(z) - a$ *has at most 2 zeros (counting multiplicity) on* $\Delta\big(0, \frac{1}{2}\big)$.

Proof Taking a subsequence and renumbering, we may assume that $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ on $\Delta'(0,1)$. By Lemma 4.3, $f(z) = z$. Suppose that $|a| \le \frac{2}{3}$. Taking Γ to be the circle $\big\{|z| = \frac{3}{4}\big\}$ traversed once in the positive direction, we have

$$\frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{f_n'(z)}{f_n(z)-a}\mathrm{d}z \to \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{1}{z-a}\mathrm{d}z = 1.$$

However, the left hand side is the number of a-points of $f_n(z)$ minus the number of poles of $f_n(z)$ inside Γ, counting multiplicities. By Lemma 4.3, there exists $0 < \eta < \frac{3}{4}$ such that $f_n(z)$ has a single simple pole on $\Delta(0,\eta)$ for n sufficiently large. Since $f_n(z)$ converges uniformly to z on $\big\{z : \eta \le |z| \le \frac{3}{4}\big\}$, there exists N_1 such that if $n \ge N_1$, $f_n(z)$ has a single simple pole in $\Delta\big(0, \frac{3}{4}\big)$. Hence for $n \ge N_1$, $f_n(z)$ takes on the value a (counting multiplicities) exactly 2 times on $\Delta\big(0, \frac{3}{4}\big)$.

Suppose now that $|a| > \frac{2}{3}$. Let Γ' be the circle $\{|z| = \frac{5}{9}\}$ traversed in the positive direction. Then

$$\frac{1}{2\pi \mathrm{i}}\int_{\Gamma'} \frac{f_n'(z)}{f_n(z)-a}\mathrm{d}z \to \frac{1}{2\pi \mathrm{i}}\int_{\Gamma'} \frac{1}{z-a}\mathrm{d}z = 0,$$

so the number of a-points of $f_n(z)$ minus the number of poles of $f_n(z)$ (counting multiplicity) inside Γ' is 0 for large n. It follows as before that there exists N_2 such that $f_n(z)$ takes on the value exactly once (counting multiplicities) on $\Delta\big(0, \frac{5}{9}\big)$ if $n \ge N_2$.

Dropping the elements $f_n(z)$ with $n <\max(N_1, N_2)$ and renumbering, we obtain the desired sequence.

Proof of Theorem 1.9 We assume that for each positive integer m and for each $\widetilde{r}_0 > 0$, there exists $\widetilde{r} > \widetilde{r}_0$ such that $f'(z) - 1$ has zeros with multiplicity at least m as $|z| > \widetilde{r}$ and derive a contradiction.

Set $g(z) = \frac{f(z)}{z}$. Since $T(r,g) = T(r,f) + \boldsymbol{O}(\log r)$ as $r \to +\infty$, we have

$$\overline{\lim_{r\to+\infty}} \frac{T(r,g)}{(\log r)^2} = +\infty.$$

By Lemma 5.1, there exist $r_n \to +\infty$ and complex numbers a_n satisfying $\frac{r_n}{2} < |a_n| < 2r_n$ such that

$$S\Big(\frac{r_n}{2}, 2r_n, g\Big) \to \infty$$

and

$$g(a_n) = \frac{f(a_n)}{a_n} \to 0, \quad a_n g'(a_n) = -\frac{f(a_n)}{a_n} + f'(a_n) \to \infty \text{ as } n \to \infty.$$

Hence $\lim\limits_{n\to\infty} f'(a_n) = \infty$.

Since $r_n \to +\infty$, let $m = k$, $\widetilde{r}_0 = k$, then there exists $n_k \in \mathbb{N}$ and $\widetilde{r}_k > \widetilde{r}_0$ such that $\frac{r_{n_k}}{16} > \widetilde{r}_k > \widetilde{r}_0 = k$ and $f'(z) - 1$ has zeros with multiplicity at least $m = k$ as $16r_{n_k} > |z| > \frac{r_{n_k}}{16}$.

Altogether, we find a sequence $\{r_{n_k}\} \subset \{r_n\}$ satisfying $r_{n_k} \to \infty$ and a sequence $\{a_{n_k}\} \subset \{a_n\}$ satisfying $\frac{r_{n_k}}{2} < |a_{n_k}| < 2r_{n_k}$ such that

$$S\Big(\frac{r_{n_k}}{2}, 2r_{n_k}, g\Big) \to \infty, \quad \frac{f(a_{n_k})}{a_{n_k}} \to 0, \quad f'(a_{n_k}) \to \infty \quad \text{as } k \to \infty$$

and $f'(z) - 1$ has zeros with multiplicity at least k as $16r_{n_k} > |z| > \frac{r_{n_k}}{16}$.

Set $f_k(z) = \frac{f(a_{n_k} z)}{a_{n_k}}$ and put $D = \big\{z : \frac{1}{8} < |z| < 8\big\}$. Clearly,

$$f_k^{\#}(1) = \frac{|f'(a_{n_k})|}{1 + \Big|\dfrac{f(a_{n_k})}{a_{n_k}}\Big|^2} \to \infty$$

as $k \to \infty$, thus $\{f_k(z)\}$ is not normal at $z_1 = 1$.

Since $\frac{r_{n_k}}{16} < |a_{n_k} z| < 16r_{n_k}$, $f_k'(z) - 1$ has zeros with multiplicity at least k on D. Since the zeros of $f_k(z)$ are multiple, by Theorem 1.8, $\{f_k(z)\}$ is quasinormal of order 1 on D.

Taking subsequences, we may assume that no subsequence is normal at the point $z_1 = 1$ and that $\{f_k(z)\}$ is normal on $D \backslash \{z_1\}$.

By Lemma 5.2, there exists $\delta > 0$ such that $S(\Delta(z_1, \delta), f_k) \le 2$.

Now let $D' = \big\{\frac{1}{7} < |z| < 7\big\}$ and $K = \Delta(z_1, \delta)$. Since $\{f_k(z)\}$ is normal on $D \backslash \{z_1\}$, by Marty's Theorem there exists $M_0 > 0$ such that $f_k^{\#}(z) \le M_0$ for $z \in D' \backslash K$. It follows that

$$S\Big(\frac{1}{7}, 7, f_k\Big) \le S(D' \backslash K, f_k) + S(K, f_k) \le 49M_0^2 + 2 := M_1.$$

Let $g_k(z) = g(a_{n_k} z) = \frac{f_k(z)}{z}$, then

$$g_k^{\#}(z) = \frac{|zf_k'(z) - f_k(z)|}{|z|^2 + |f_k(z)|^2} \le \frac{|zf_k'(z)|}{|z|^2 + |f_k(z)|^2} + \frac{|f_k(z)|}{|z|^2 + |f_k(z)|^2},$$

so

$$[g_k^{\#}(z)]^2 \le \frac{2|zf_k'(z)|^2}{(|z|^2 + |f_k(z)|^2)^2} + \frac{2|f_k(z)|^2}{(|z|^2 + |f_k(z)|^2)^2} \le 2\max\Big(|z|^2, \frac{1}{|z|^2}\Big)[f_k^{\#}(z)]^2 + \frac{1}{2|z|^2}$$

and

$$[g_k^{\#}(z)]^2 \le 2 \cdot 7^2 [f_k^{\#}(z)]^2 + \frac{1}{2} \cdot 7^2.$$

Thus

$$S\Big(\frac{1}{7}, 7, g_k\Big) \le 2 \cdot 7^2 \cdot S\Big(\frac{1}{7}, 7, f_k\Big) + \frac{1}{2} \cdot 7^2 \cdot \pi \cdot 7^2 \le 2 \cdot 7^2 M_1 + \frac{1}{2} \cdot 7^4 \cdot \pi := M_2.$$

Since $\frac{r_{n_k}}{2} < |a_{n_k}| < 2r_{n_k}$, we have

$$\frac{|a_{nk}|}{7} < \frac{r_{n_k}}{2} < 2r_{n_k} < 7|a_{n_k}|,$$

so that

$$S\Big(\frac{r_{n_k}}{2}, 2r_{n_k}, g\Big) \le S\Big(\frac{|a_{n_k}|}{7}, 7|a_{n_k}|, g\Big) = S\Big(\frac{1}{7}, 7, g_k\Big) \le M_2,$$

which contradicts $S\left(\frac{r_{n_k}}{2}, 2r_{n_k}, g\right) \to \infty$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Hayman, W. K., Picard values of meromorphic functions and their derivatives, *Ann. Math.*, **70**(1), 1959, 9–42.

[2] Hayman, W. K., Meromorphic Functions, Oxford Univ. Press, Oxford, 1964.

[3] Lehto, O., The spherical derivative of a meromorphic function in the neighborhood of an isolated singularity, *Comment. Math. Helv.*, **33**, 1959, 169–205.

[4] Lehto, O. and Virtanen, K. I., On the behaviour of meromorphic functions in the neighbourhood of an isolated singularity, *Ann. Acad. Sci. Fenn.*, **240**, 1957, 1–9.

[5] Minda, D., A heuristic principle for a nonessential isolated singularity, *Proc. Am. Math. Soc.*, **93**(3), 1985, 443–447.

[6] Nevo, S., Pang, X. C. and Zalcman, L., Quasinormality and meromorphic functions with multiple zeros, *J. Anal. Math.*, **101**, 2007, 1–23.

[7] Pang, X. C., Nevo, S. and Zalcman, L., Derivatives of meromorphic functions with multiple zeros and rational functions, *Comput. Methods Funct. Theory*, **8**(2), 2008, 483–491.

[8] Pang, X. C. and Zalcman, L., Normal families and shared values, *Bull. Lond. Math. Soc.*, **32**(3), 2000, 325–331.

[9] Picard, E., Sur les fonctions analytiques uniformes dans le voisinage d'un point singular essentiel, *C. R. Acad. Sci.*, **89**, 1879, 745–747.

[10] Wang, Y. F. and Fang, M. L., Picard values and normal families of meromorphic functions with multiple zeros, *Acta Math. Sinica* (*N. S.*), **14**(1), 1998, 17–26.