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On Lie All-Derivable Points of $B(H)$*

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Abstract Let H be a Hilbert space of dimension greater than 2 and $B(H)$ be the algebra of all bounded linear operators on H. In this paper, the authors show that $G \in B(H)$ is a Lie all-derivable point of $B(H)$ if the range of G is not dense in H.

Keywords Derivation, Lie derivation, All-Derivable point
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1 Introduction

Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$. Recall that a linear map ϕ from $\mathcal{A}$ into itself is a derivation if $\phi(ST) = \phi(S)T + S\phi(T)$ for all $S, T \in \mathcal{A}$ (see [4, 11]). In recent years, there have been a number of papers on the study of conditions under which derivations of algebras can be completely determined by the action on some sets of elements. We say that a linear map $\phi : \mathcal{A} \to \mathcal{A}$ is derivable at $G \in \mathcal{A}$ if $\phi(ST) = \phi(S)T + S\phi(T)$ for all $S, T \in \mathcal{A}$ with $ST = G$ and an element G in $\mathcal{A}$ is an all-derivable point if every derivable map at G is a derivation of $\mathcal{A}$. For some algebras, the cases that G is zero, the unit, nontrivial idempotents, invertible elements, and so on were discussed by several authors (for example, see [1–3, 8, 10, 12] and the references therein).

More generally, a linear map $\phi : \mathcal{A} \to \mathcal{A}$ is said to be a Lie derivation if $\phi([S, T]) = [\phi(S), T] + [S, \phi(T)]$ for all $S, T \in \mathcal{A}$ and Lie derivable at $G \in \mathcal{A}$ if $\phi([S, T]) = [\phi(S), T] + [S, \phi(T)]$ for all $S, T \in \mathcal{A}$ with $ST = G$, where $[S, T] = ST - TS$ is the usual Lie product. It is obvious that the condition of maps Lie derivable at some points is much weaker than the condition of being a Lie derivation. In [9], Lu and Jing gave a characterization for Lie derivable maps. Let X be a Banach space of dimension greater than 2 and $B(X)$ be the algebra of all bounded linear operators acting on X. It is proved in [9] that if $\delta : B(X) \to B(X)$ is Lie derivable at zero (resp., P, where P is a fixed nontrivial idempotent of $B(X)$), then δ is standard, that is, $\delta = d + \tau$, where d is a derivation of $B(X)$ and $\tau : B(X) \to \mathbb{C}I$ is a linear map vanishing at commutators $[A, B]$ with $AB = 0$ (resp., $AB = P$).

In this paper, we say that an element $G \in \mathcal{A}$ is a Lie all-derivable point of $\mathcal{A}$ if every linear map from $\mathcal{A}$ into itself which is Lie derivable at G can be decomposed as the standard form. So zero and nontrivial idempotent P are Lie all-derivable points of $B(X)$. In [6], Ji and Qi proved

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the same is true for triangular algebras. Du and Wang [5] showed that zero is a Lie all-derivable point of generalized matrix algebras. It is natural and interesting to ask the question whether or not there exists a Lie all-derivable point that is neither zero nor nontrivial idempotent. The purpose of the present paper is to give some new Lie all-derivable points of $B(H)$.

Throughout this paper, denote by H and $B(H)$ the Hilbert space of dimension greater than 2 and the algebra of all bounded linear operators on H, respectively. We use the symbol $\operatorname{ran} G$ for the range of $G \in B(H)$. The closure of $\operatorname{ran} G$ is denoted by $\overline{\operatorname{ran} G}$.

2 Result and Proof

In this section, we show that every operator with non-dense range in $B(H)$ is a Lie all-derivable point of $B(H)$. We begin with the following lemma which is frequently used in the rest of this section.

Lemma 2.1 *Let ϕ be a linear map from $B(H)$ into itself. Then the following statements hold*:

(i) *If $\phi(X) = 0$ for any invertible operator $X \in B(H)$, then $\phi \equiv 0$.*

(ii) *If $\phi(X)Y = Y\phi(X)$ for any invertible operators $X, Y \in B(H)$, then $\phi(X)Y = Y\phi(X)$ for all $X, Y \in B(H)$.*

Proof For each operator $X \in B(H)$, there exists a real number $\alpha > \|X\|$ such that $\alpha I - X$ is invertible in $B(H)$. Then $\phi(\alpha I - X) = 0$ for all $X \in B(H)$. It follows from the linearity of ϕ that $\phi \equiv 0$.

Similarly, for each operator $X \in B(H)$ and $Y \in B(H)$, there exist two real numbers $\alpha > \|X\|$ and $\beta > \|Y\|$ such that $\alpha I - X$ and $\beta I - Y$ are invertible operators in $B(H)$. Then $\phi(\alpha I - X)(\beta I - Y) = (\beta I - Y)\phi(\alpha I - X)$. It follows from the linearity of ϕ that $\phi(X)Y = Y\phi(X)$ for all $X, Y \in B(H)$.

The following is the main result of the paper.

Theorem 2.1 *Let $G \in B(H)$ be an operator with $\overline{\operatorname{ran} G} \neq H$. Then G is a Lie all-derivable point of $B(H)$.*

Proof For the case that $G = 0$, it was proved in [9] that G is a Lie all-derivable point of $B(H)$.

For the case that $G \neq 0$, suppose that linear map $\phi : B(H) \to B(H)$ is Lie derivable at G. If $\dim(\operatorname{ran} G) > 1$, then we take $H_1 = \overline{\operatorname{ran} G}$ and $H_2 = (\operatorname{ran} G)^{\perp}$. In the case that $\dim(\operatorname{ran} G) = 1$, we take $H_1 = \overline{\operatorname{ran} G} \oplus \operatorname{span}\{e\}$ and $H_2 = (\operatorname{ran} G)^{\perp} \ominus \operatorname{span}\{e\}$, where $0 \neq e \in (\operatorname{ran} G)^{\perp}$. Thus we always assume that $\dim(H_1) > 1$. Since the dimension of H is greater than 2 and $\overline{\operatorname{ran} G} \neq H$, we see that G can be represented as a 2×2 operator matrix relative to the orthogonal decomposition $H = H_1 \oplus H_2$ as follows:

$$G = \begin{pmatrix} a_0 & m_0 \\ 0 & 0 \end{pmatrix},$$

where $a_0 \in B(H_1)$, $m_0 \in B(H_2, H_1)$. For each operator $S \in B(H)$, it can also be expressed as the following operator matrix in the orthogonal decomposition of $H = H_1 \oplus H_2$ as follows

$$S = \begin{pmatrix} a & m \\ n & b \end{pmatrix},$$

where $a \in B(H_1)$, $m \in B(H_2, H_1)$, $n \in B(H_1, H_2)$ and $b \in B(H_2)$. Since ϕ is linear, we may write

$$\begin{cases} \phi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f_{11}(a) & f_{12}(a) \\ f_{21}(a) & f_{22}(a) \end{pmatrix}, \\ \phi\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g_{11}(m) & g_{12}(m) \\ g_{21}(m) & g_{22}(m) \end{pmatrix}, \\ \phi\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} l_{11}(n) & l_{12}(n) \\ l_{21}(n) & l_{22}(n) \end{pmatrix}, \\ \phi\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} h_{11}(b) & h_{12}(b) \\ h_{21}(b) & h_{22}(b) \end{pmatrix}, \end{cases}$$

where f_{ij}, g_{ij}, l_{ij} and h_{ij} are linear maps from $B(H_1)$, $B(H_2, H_1)$, $B(H_1, H_2)$ and $B(H_2)$ into $B(H_j, H_i)$ $(i, j = 1, 2)$, respectively. For any S, $T \in B(H)$ with $ST = G$, we write

$$S = \begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix}, \quad T = \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix},$$

where $a_1a_2 + m_1n_2 = a_0$, $a_1m_2 + m_1b_2 = m_0$, $n_1a_2 + b_1n_2 = 0$ and $n_1m_2 + b_1b_2 = 0$. Since ϕ is Lie derivable at G, we have

$$\begin{aligned} &\begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{pmatrix} \\ &= \phi([S,T]) = [\phi(S), T] + [S, \phi(T)] \\ &= \begin{pmatrix} f_{11}(a_1) + g_{11}(m_1) + l_{11}(n_1) + h_{11}(b_1) & f_{12}(a_1) + g_{12}(m_1) + l_{12}(n_1) + h_{12}(b_1) \\ f_{21}(a_1) + g_{21}(m_1) + l_{21}(n_1) + h_{21}(b_1) & f_{22}(a_1) + g_{22}(m_1) + l_{22}(n_1) + h_{22}(b_1) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix} \\ &\quad - \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} f_{11}(a_1) + g_{11}(m_1) + l_{11}(n_1) + h_{11}(b_1) & f_{12}(a_1) + g_{12}(m_1) + l_{12}(n_1) + h_{12}(b_1) \\ f_{21}(a_1) + g_{21}(m_1) + l_{21}(n_1) + h_{21}(b_1) & f_{22}(a_1) + g_{22}(m_1) + l_{22}(n_1) + h_{22}(b_1) \end{pmatrix} \\ &\quad + \begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} f_{11}(a_2) + g_{11}(m_2) + l_{11}(n_2) + h_{11}(b_2) & f_{12}(a_2) + g_{12}(m_2) + l_{12}(n_2) + h_{12}(b_2) \\ f_{21}(a_2) + g_{21}(m_2) + l_{21}(n_2) + h_{21}(b_2) & f_{22}(a_2) + g_{22}(m_2) + l_{22}(n_2) + h_{22}(b_2) \end{pmatrix} \\ &\quad - \begin{pmatrix} f_{11}(a_2) + g_{11}(m_2) + l_{11}(n_2) + h_{11}(b_2) & f_{12}(a_2) + g_{12}(m_2) + l_{12}(n_2) + h_{12}(b_2) \\ f_{21}(a_2) + g_{21}(m_2) + l_{21}(n_2) + h_{21}(b_2) & f_{22}(a_2) + g_{22}(m_2) + l_{22}(n_2) + h_{22}(b_2) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix}, \end{aligned}$$

where

$$\Delta_1 = f_{11}(a_0 - a_2a_1 - m_2n_1) + g_{11}(m_0 - a_2m_1 - m_2b_1)$$

$$
\begin{aligned}
&+ l_{11}(-n_2a_1 - b_2n_1) + h_{11}(-n_2m_1 - b_2b_1),\\
\Delta_2 &= f_{12}(a_0 - a_2a_1 - m_2n_1) + g_{12}(m_0 - a_2m_1 - m_2b_1)\\
&\quad + l_{12}(-n_2a_1 - b_2n_1) + h_{12}(-n_2m_1 - b_2b_1),\\
\Delta_3 &= f_{21}(a_0 - a_2a_1 - m_2n_1) + g_{21}(m_0 - a_2m_1 - m_2b_1)\\
&\quad + l_{21}(-n_2a_1 - b_2n_1) + h_{21}(-n_2m_1 - b_2b_1),\\
\Delta_4 &= f_{22}(a_0 - a_2a_1 - m_2n_1) + g_{22}(m_0 - a_2m_1 - m_2b_1)\\
&\quad + l_{22}(-n_2a_1 - b_2n_1) + h_{22}(-n_2m_1 - b_2b_1).
\end{aligned}
$$

Then the above matrix equation implies the following:

$$
\begin{aligned}
&f_{11}(a_0 - a_2a_1 - m_2n_1) + g_{11}(m_0 - a_2m_1 - m_2b_1) + l_{11}(-n_2a_1 - b_2n_1)\\
&+ h_{11}(-n_2m_1 - b_2b_1)\\
&= f_{11}(a_1)a_2 + g_{11}(m_1)a_2 + l_{11}(n_1)a_2 + h_{11}(b_1)a_2 + f_{12}(a_1)n_2 + g_{12}(m_1)n_2\\
&+ l_{12}(n_1)n_2 + h_{12}(b_1)n_2 - a_2f_{11}(a_1) - a_2g_{11}(m_1) - a_2l_{11}(n_1) - a_2h_{11}(b_1)\\
&- m_2f_{21}(a_1) - m_2g_{21}(m_1) - m_2l_{21}(n_1) - m_2h_{21}(b_1) + a_1f_{11}(a_2) + a_1g_{11}(m_2)\\
&+ a_1l_{11}(n_2) + a_1h_{11}(b_2) + m_1f_{21}(a_2) + m_1g_{21}(m_2) + m_1l_{21}(n_2) + m_1h_{21}(b_2)\\
&- f_{11}(a_2)a_1 - g_{11}(m_2)a_1 - l_{11}(n_2)a_1 - h_{11}(b_2)a_1 - f_{12}(a_2)n_1 - g_{12}(m_2)n_1\\
&- l_{12}(n_2)n_1 - h_{12}(b_2)n_1, \qquad (2.1)\\
&f_{12}(a_0 - a_2a_1 - m_2n_1) + g_{12}(m_0 - a_2m_1 - m_2b_1) + l_{12}(-n_2a_1 - b_2n_1)\\
&+ h_{12}(-n_2m_1 - b_2b_1)\\
&= f_{11}(a_1)m_2 + g_{11}(m_1)m_2 + l_{11}(n_1)m_2 + h_{11}(b_1)m_2 + f_{12}(a_1)b_2 + g_{12}(m_1)b_2\\
&+ l_{12}(n_1)b_2 + h_{12}(b_1)b_2 - a_2f_{12}(a_1) - a_2g_{12}(m_1) - a_2l_{12}(n_1) - a_2h_{12}(b_1)\\
&- m_2f_{22}(a_1) - m_2g_{22}(m_1) - m_2l_{22}(n_1) - m_2h_{22}(b_1) + a_1f_{12}(a_2) + a_1g_{12}(m_2)\\
&+ a_1l_{12}(n_2) + a_1h_{12}(b_2) + m_1f_{22}(a_2) + m_1g_{22}(m_2) + m_1l_{22}(n_2) + m_1h_{22}(b_2)\\
&- f_{11}(a_2)m_1 - g_{11}(m_2)m_1 - l_{11}(n_2)m_1 - h_{11}(b_2)m_1 - f_{12}(a_2)b_1 - g_{12}(m_2)b_1\\
&- l_{12}(n_2)b_1 - h_{12}(b_2)b_1, \qquad (2.2)\\
&f_{21}(a_0 - a_2a_1 - m_2n_1) + g_{21}(m_0 - a_2m_1 - m_2b_1) + l_{21}(-n_2a_1 - b_2n_1)\\
&+ h_{21}(-n_2m_1 - b_2b_1)\\
&= f_{21}(a_1)a_2 + g_{21}(m_1)a_2 + l_{21}(n_1)a_2 + h_{21}(b_1)a_2 + f_{22}(a_1)n_2 + g_{22}(m_1)n_2\\
&+ l_{22}(n_1)n_2 + h_{22}(b_1)n_2 - n_2f_{11}(a_1) - n_2g_{11}(m_1) - n_2l_{11}(n_1) - n_2h_{11}(b_1)\\
&- b_2f_{21}(a_1) - b_2g_{21}(m_1) - b_2l_{21}(n_1) - b_2h_{21}(b_1) + n_1f_{11}(a_2) + n_1g_{11}(m_2)\\
&+ n_1l_{11}(n_2) + n_1h_{11}(b_2) + b_1f_{21}(a_2) + b_1g_{21}(m_2) + b_1l_{21}(n_2) + b_1h_{21}(b_2)\\
&- f_{21}(a_2)a_1 - g_{21}(m_2)a_1 - l_{21}(n_2)a_1 - h_{21}(b_2)a_1 - f_{22}(a_2)n_1 - g_{22}(m_2)n_1\\
&- l_{22}(n_2)n_1 - h_{22}(b_2)n_1, \qquad (2.3)\\
&f_{22}(a_0 - a_2a_1 - m_2n_1) + g_{22}(m_0 - a_2m_1 - m_2b_1) + l_{22}(-n_2a_1 - b_2n_1)\\
&+ h_{22}(-n_2m_1 - b_2b_1)\\
&= f_{21}(a_1)m_2 + g_{21}(m_1)m_2 + l_{21}(n_1)m_2 + h_{21}(b_1)m_2 + f_{22}(a_1)b_2 + g_{22}(m_1)b_2
\end{aligned}
$$

$$+ l_{22}(n_1)b_2 + h_{22}(b_1)b_2 - n_2 f_{12}(a_1) - n_2 g_{12}(m_1) - n_2 l_{12}(n_1) - n_2 h_{12}(b_1)$$
$$- b_2 f_{22}(a_1) - b_2 g_{22}(m_1) - b_2 l_{22}(n_1) - b_2 h_{22}(b_1) + n_1 f_{12}(a_2) + n_1 g_{12}(m_2)$$
$$+ n_1 l_{12}(n_2) + n_1 h_{12}(b_2) + b_1 f_{22}(a_2) + b_1 g_{22}(m_2) + b_1 l_{22}(n_2) + b_1 h_{22}(b_2)$$
$$- f_{21}(a_2)m_1 - g_{21}(m_2)m_1 - l_{21}(n_2)m_1 - h_{21}(b_2)m_1 - f_{22}(a_2)b_1 - g_{22}(m_2)b_1$$
$$- l_{22}(n_2)b_1 - h_{22}(b_2)b_1. \tag{2.4}$$

Now we organize the proof in a series of claims.

Claim 1 For any $a \in B(H_1)$ and $b \in B(H_2)$, the following is true:
(i) $f_{12}(a) = af_{12}(I_1)$;
(ii) $f_{21}(a) = f_{21}(I_1)a$;
(iii) $h_{12}(b) = -f_{12}(I_1)b$;
(iv) $h_{21}(b) = -bf_{21}(I_1)$.

For any invertible operator $a \in B(H_1)$ and any real number $\lambda > 0$, taking $a_1 = \lambda a$, $m_1 = m_0$, $n_1 = 0$, $b_1 = 0$, $a_2 = \lambda^{-1}a^{-1}a_0$, $m_2 = 0$, $n_2 = 0$ and $b_2 = I_2$ into (2.2)–(2.3), respectively, we get

$$\begin{aligned} &f_{12}(a_0 - a^{-1}a_0 a) + g_{12}(m_0 - \lambda^{-1}a^{-1}a_0 m_0) \\ &= \lambda(f_{12}(a) + ah_{12}(I_2)) + \lambda^{-1}(-a^{-1}a_0 g_{12}(m_0) - f_{11}(a^{-1}a_0)m_0 + m_0 f_{22}(a^{-1}a_0)) \\ &\quad + g_{12}(m_0) - a^{-1}a_0 f_{12}(a) + af_{12}(a^{-1}a_0) + m_0 h_{22}(I_2) - h_{11}(I_2)m_0 \end{aligned}$$

and

$$\begin{aligned} &f_{21}(a_0 - a^{-1}a_0 a) + g_{21}(m_0 - \lambda^{-1}a^{-1}a_0 m_0) \\ &= \lambda(-f_{21}(a) - h_{21}(I_2)a) + \lambda^{-1}g_{21}(m_0)a^{-1}a_0 - g_{21}(m_0) - f_{21}(a^{-1}a_0)a \\ &\quad + f_{21}(a)a^{-1}a_0. \end{aligned}$$

It follows that $f_{12}(a) + ah_{12}(I_2) = 0$ and $f_{21}(a) + h_{21}(I_2)a = 0$ for any invertible $a \in B(H_1)$. By Lemma 2.1, we obtain that

$$f_{12}(a) + ah_{12}(I_2) = 0 \tag{2.5}$$

and

$$f_{21}(a) + h_{21}(I_2)a = 0 \tag{2.6}$$

for all $a \in B(H_1)$.

Similarly, for any invertible operator $b \in B(H_2)$ and any real number $\lambda > 0$, putting $a_1 = I_1$, $m_1 = 0$, $n_1 = 0$, $b_1 = 0$, $a_2 = a_0$, $m_2 = m_0$, $n_2 = 0$, $b_2 = \lambda b$ into (2.2)–(2.3), respectively, we have

$$0 = \lambda(f_{12}(I_1)b + h_{12}(b)) - a_0 f_{12}(I_1) + f_{11}(I_1)m_0 - m_0 f_{22}(I_1) + f_{12}(a_0)$$

and

$$g_{21}(m_0) = -\lambda(bf_{21}(I_1) + h_{21}(b)) + f_{21}(I_1)a_0 - f_{21}(a_0) - g_{21}(m_0).$$

Then the above two equations imply that $f_{12}(I_1)b + h_{12}(b) = 0$ and $bf_{21}(I_1) + h_{21}(b) = 0$ for any invertible $b \in B(H_2)$. By Lemma 2.1, we obtain

$$f_{12}(I_1)b + h_{12}(b) = 0 \tag{2.7}$$

and

$$bf_{21}(I_1) + h_{21}(b) = 0 \tag{2.8}$$

for all $b \in B(H_2)$. Combining (2.5) and (2.7), we have

$$f_{12}(a) = af_{12}(I_1)$$

and

$$h_{12}(b) = -f_{12}(I_1)b.$$

Combining (2.6) and (2.8), we get

$$f_{21}(a) = f_{21}(I_1)a$$

and

$$h_{21}(b) = -bf_{21}(I_1)$$

for all $a \in B(H_1)$ and $b \in B(H_2)$.

Claim 2 The following statements hold:
(i) $h_{11}(b) \in \mathbb{C}I_1$ for all $b \in B(H_2)$;
(ii) $f_{22}(a) \in \mathbb{C}I_2$ for all $a \in B(H_1)$.

For any invertible $a \in B(H_1)$, $b \in B(H_2)$ and any real number $\lambda > 0$, putting $a_1 = \lambda a$, $m_1 = 0$, $n_1 = 0$, $b_1 = 0$, $a_2 = \lambda^{-1}a^{-1}a_0$, $m_2 = \lambda^{-1}a^{-1}m_0$, $n_2 = 0$ and $b_2 = \lambda b$ into (2.1) and (2.4), respectively, we arrive at

$$\begin{aligned} &f_{11}(a_0 - a^{-1}a_0a) + g_{11}(m_0) \\ &= \lambda^2(ah_{11}(b) - h_{11}(b)a) + f_{11}(a)a^{-1}a_0 - a^{-1}a_0f_{11}(a) - a^{-1}m_0f_{21}(a) \\ &\quad + af_{11}(a^{-1}a_0) + ag_{11}(a^{-1}m_0) - f_{11}(a^{-1}a_0)a - g_{11}(a^{-1}m_0)a \end{aligned}$$

and

$$f_{22}(a_0 - a^{-1}a_0a) + g_{22}(m_0) = f_{21}(a)a^{-1}m_0 + \lambda^2(f_{22}(a)b - bf_{22}(a)).$$

Then the above two equations imply that $ah_{11}(b) - h_{11}(b)a = 0$ and $f_{22}(a)b - bf_{22}(a) = 0$ for all invertible $a \in B(H_1)$ and $b \in B(H_2)$. By Lemma 2.1, we obtain that

$$ah_{11}(b) - h_{11}(b)a = 0$$

and

$$f_{22}(a)b - bf_{22}(a) = 0$$

for all $a \in B(H_1)$ and $b \in B(H_2)$, which yield that $h_{11}(b) \in \mathbb{C}I_1$ and $f_{22}(a) \in \mathbb{C}I_2$.

Claim 3 For any $a \in B(H_1)$, $m \in B(H_2, H_1)$ and $n \in B(H_1, H_2)$, there hold

(i) $l_{11}(n) = -f_{12}(I_1)n$;
(ii) $l_{12}(n) = 0$;
(iii) $(g_{11}(m) + mf_{21}(I_1)) \oplus (g_{22}(m) - f_{21}(I_1)m) \in \mathbb{C}I$;
(iv) $l_{22}(n) = nf_{12}(I_1)$;
(v) $h_{22}(nm) = mg_{12}(m) + l_{21}(n)m + f_{22}(mn)$;
(vi) $l_{21}(na) = l_{21}(n)a + nf_{11}(a) - f_{22}(a)n$;
(vii) $f_{11}(mn) = g_{12}(m)n + ml_{21}(n) + h_{11}(nm)$.

For any $n \in B(H_1, H_2)$, arbitrary nonzero real numbers λ and μ, putting $a_1 = I_1$, $m_1 = \mu m$, $n_1 = 0$, $b_1 = 0$, $a_2 = a_0 - \lambda\mu mn$, $m_2 = m_0$, $n_2 = \lambda n$ and $b_2 = 0$ into (2.1)–(2.4), respectively, we obtain that

$$\begin{aligned} &\lambda\mu f_{11}(mn) + g_{11}(m_0) - \mu g_{11}(a_0 m) + \lambda\mu^2 g_{11}(mnm) - \lambda l_{11}(n) - \lambda\mu h_{11}(nm) \\ &= \mu(g_{11}(m)a_0 - a_0 g_{11}(m) - m_0 g_{21}(m) + mf_{21}(a_0) + mg_{21}(m_0)) + f_{11}(I_1)a_0 \\ &\quad + \lambda\mu(-f_{11}(I_1)mn + g_{12}(m)n + mnf_{11}(I_1) + ml_{21}(n)) + \lambda f_{12}(I_1)n \\ &\quad - a_0 f_{11}(I_1) - m_0 f_{21}(I_1) + \lambda\mu^2(-g_{11}(m)(mn) + mng_{11}(m) - mf_{21}(mn)), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\lambda\mu f_{12}(mn) + g_{12}(m_0) - \mu g_{12}(a_0 m) + \lambda\mu^2 g_{12}(mnm) - \lambda l_{12}(n) - \lambda\mu h_{12}(nm) \\ &= f_{11}(I_1)m_0 - a_0 f_{12}(I_1) - m_0 f_{22}(I_1) + f_{12}(a_0) + g_{12}(m_0) + \lambda\mu mnf_{12}(I_1) \\ &\quad - \lambda\mu f_{12}(mn) + \lambda\mu ml_{22}(n) - \lambda\mu l_{11}(n)m + \mu(-g_{11}(m_0)m + g_{11}(m)m_0 \\ &\quad - a_0 g_{12}(m) - m_0 g_{22}(m)) + \lambda\mu^2 mng_{12}(m) - \lambda\mu^2 mf_{22}(mn) + \lambda\mu^2 f_{11}(mn)m \\ &\quad + \mu(mf_{22}(a_0) + mg_{22}(m_0) - f_{11}(a_0)m) + \lambda l_{12}(n), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\lambda\mu f_{21}(mn) + g_{21}(m_0) - \mu g_{21}(a_0 m) + \lambda\mu^2 g_{21}(mn)m - \lambda\mu h_{21}(nm) \\ &= \lambda\mu g_{22}(m)n - \lambda\mu ng_{11}(m) + f_{21}(I_1)a_0 - f_{21}(a_0) - g_{21}(m_0) + \mu g_{21}(m)a_0 \\ &\quad - \lambda\mu^2 g_{21}(m)mn + \lambda f_{22}(I_1)n - \lambda nf_{11}(I_1) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\lambda\mu f_{22}(mn) + g_{22}(m_0 - \mu a_0 m + \lambda\mu^2 mnm) - \lambda l_{22}(n) - \lambda\mu h_{22}(nm) \\ &= \lambda\mu^2 f_{21}(mn)m - \lambda\mu(ng_{12}(m) + l_{21}(n)m) - \lambda nf_{12}(I_1) + \mu(g_{21}(m)m_0 \\ &\quad - g_{21}(m_0)m - f_{21}(a_0)m) + f_{21}(I_1)m_0. \end{aligned} \tag{2.12}$$

It follows from (2.9) that

$$l_{11}(n) = -f_{12}(I_1)n$$

and

$$f_{11}(mn) - h_{11}(nm) = g_{12}(m)n + ml_{21}(n) + mnf_{11}(I_1) - f_{11}(I_1)mn \tag{2.13}$$

for all $m \in B(H_2, H_1)$ and $n \in B(H_1, H_2)$.

By (2.10), we get

$$l_{12}(n) = 0$$

for all $n \in B(H_1, H_2)$.

(2.11) reduces to

$$f_{21}(mn) - h_{21}(nm) = g_{22}(m)n - ng_{11}(m).$$

The above equation and Claim 1(ii), (iv) together imply that

$$n(g_{11}(m)+mf_{21}(I_1))=(g_{22}(m)-f_{21}(I_1)m)n$$

for all $m\in B(H_2,H_1)$ and $n\in B(H_1,H_2)$. This leads to

$$(g_{11}(m)+mf_{21}(I_1))\oplus(g_{22}(m)-f_{21}(I_1)m)\in\mathbb{C}I$$

for all $m\in B(H_2,H_1)$.

By (2.12), we have

$$l_{22}(n)=nf_{12}(I_1)$$

and

$$h_{22}(nm)=ng_{12}(m)+l_{21}(n)m+f_{22}(mn)$$

for all $m\in B(H_2,H_1)$ and $n\in B(H_1,H_2)$.

Furthermore, for any invertible $a\in B(H_1)$, any $n\in B(H_1,H_2)$ and any real number $\lambda>0$, putting $a_1=a$, $m_1=0$, $n_1=0$, $b_1=0$, $a_2=a^{-1}a_0$, $m_2=a^{-1}m_0$, $n_2=\lambda n$ and $b_2=0$ into (2.3), we have that

$$\begin{aligned}&f_{21}(a_0-a^{-1}a_0a)-\lambda l_{21}(na)\\&=\lambda(f_{22}(a)n-nf_{11}(a)-l_{21}(n)a)+f_{21}(a)a^{-1}a_0-f_{21}(a^{-1}a_0)a-g_{21}(a^{-1}m_0)a,\end{aligned}$$

which implies that $l_{21}(na)=l_{21}(n)a+nf_{11}(a)-f_{22}(a)n$ for all invertible $a\in B(H_1)$ and all $n\in B(H_1,H_2)$. By Lemma 2.1, we obtain that

$$l_{21}(na)=l_{21}(n)a+nf_{11}(a)-f_{22}(a)n$$

for all $a\in B(H_1)$ and $n\in B(H_1,H_2)$. Replacing a by I_1 in the above equation, we get that

$$nf_{11}(I_1)=f_{22}(I_1)n$$

for all $n\in B(H_1,H_2)$, which implies that there exists a complex number λ such that $f_{11}(I_1)=\lambda I_1$ and $f_{22}(I_1)=\lambda I_2$. Combining this and (2.13), we obtain that

$$f_{11}(mn)=g_{12}(m)n+ml_{21}(n)+h_{11}(nm)$$

for all $n\in B(H_1,H_2)$ and $m\in B(H_2,H_1)$, completing the proof of Claim 3.

Claim 4 Let $a\in B(H_1)$, $b\in B(H_2)$ and $m\in B(H_2,H_1)$. Then
(i) $g_{12}(mb)=g_{12}(m)b+mh_{22}(b)-h_{11}(b)m$;
(ii) $g_{12}(am)=ag_{12}(m)+f_{11}(a)m-mf_{22}(a)$;
(iii) $g_{21}(m)=0$.

For any $m\in B(H_2,H_1)$, any invertible $b\in B(H_2)$ and any nonzero real numbers λ and μ, taking $a_1=I_1$, $m_1=\mu m$, $n_1=0$, $b_1=0$, $a_2=a_0$, $m_2=m_0-\lambda\mu mb$, $n_2=0$ and $b_2=\lambda b$ into (2.2), one can easily check that

$$g_{12}(m_0-\mu a_0m)$$

$$\begin{aligned}&= -\lambda\mu^2 g_{11}(m)mb + \lambda\mu^2 mbg_{22}(m) - \lambda\mu^2 mg_{22}(mb) + \lambda\mu^2 g_{11}(mb)m \\ &\quad + \mu(g_{11}(m)m_0 + mg_{22}(m_0) - g_{11}(m_0)m - m_0 g_{22}(m)) \\ &\quad + \lambda\mu(g_{12}(m)b - g_{12}(mb) + mh_{22}(b) - h_{11}(b)m) \\ &\quad + \mu(mf_{22}(a_0) - f_{11}(a_0)m - a_0 g_{12}(m)) + g_{12}(m_0).\end{aligned}$$

It follows that $g_{12}(mb) = g_{12}(m)b + mh_{22}(b) - h_{11}(b)m$ for all $m \in B(H_2, H_1)$ and all invertible $b \in B(H_2)$. By Lemma 2.1, we obtain

$$g_{12}(mb) = g_{12}(m)b + mh_{22}(b) - h_{11}(b)m$$

for all $m \in B(H_2, H_1)$ and $b \in B(H_2)$.

Moreover, for any invertible $a \in B(H_1)$, any $n \in B(H_1, H_2)$ and any nonzero real numbers λ and μ, putting $a_1 = \lambda a$, $m_1 = m_0 - \lambda\mu am$, $n_1 = 0$, $b_1 = 0$, $a_2 = \lambda^{-1}a^{-1}a_0$, $m_2 = \mu m$, $n_2 = 0$ and $b_2 = I_2$ into (2.2)–(2.4), respectively, we have

$$\begin{aligned}&f_{12}(a_0 - a^{-1}a_0 a) + g_{12}(m_0) - g_{12}(\lambda^{-1}a^{-1}a_0 m_0) + g_{12}(a^{-1}a_0 a\mu m) \\ &= \lambda\mu(f_{11}(a)m - g_{12}(am) - mf_{22}(a) + ag_{12}(m) - amh_{22}(I_2) + h_{11}(I_2)am) \\ &\quad + \mu(g_{11}(m_0)m + a^{-1}a_0 g_{12}(am) - mg_{22}(m_0) - amf_{22}(a^{-1}a_0) + m_0 g_{22}(m) \\ &\quad + f_{11}(a^{-1}a_0)am - g_{11}(m)m_0) + g_{12}(m_0) - a^{-1}a_0 f_{12}(a) + af_{12}(a^{-1}a_0) \\ &\quad + m_0 h_{22}(I_2) - h_{11}(I_2)m_0 + \lambda\mu^2(-g_{11}(am)m + mg_{22}(am) - amg_{22}(m) \\ &\quad + g_{11}(m)am) + \lambda^{-1}(-a^{-1}a_0 g_{12}(m_0) + m_0 f_{22}(a^{-1}a_0) - f_{11}(a^{-1}a_0)m_0), \\ &f_{21}(a_0 - a^{-1}a_0 a) - \lambda^{-1}g_{21}(a^{-1}a_0 m_0) + \mu g_{21}(a^{-1}a_0 am) \\ &= f_{21}(a)(a^{-1}a_0) - g_{21}(am)a^{-1}a_0 - f_{21}(a^{-1}a_0)a + \lambda\mu(g_{21}(am) - g_{21}(m)a)\end{aligned}$$

and

$$\begin{aligned}&f_{22}(a_0 - a^{-1}a_0 a) + g_{22}(m_0) - \lambda^{-1}g_{22}(a^{-1}a_0 m_0) + \mu g_{22}(a^{-1}a_0 am) \\ &= -\lambda^{-1}f_{21}(a^{-1}a_0)m_0 + \mu(f_{21}(a^{-1}a_0)am - g_{21}(m)m_0) - h_{21}(I_2)m_0 \\ &\quad + \lambda\mu^2(g_{21}(m)am - g_{21}(am)m) + \lambda\mu(f_{21}(a)m + h_{21}(I_2)am).\end{aligned}$$

The above three equations and Claim 1(ii) imply that

$$g_{12}(am) = ag_{12}(m) + f_{11}(a)m - mf_{22}(a) + h_{11}(I_2)am - amh_{22}(I_2), \tag{2.14}$$

$$g_{21}(am)a^{-1}a_0 = 0, \tag{2.15}$$

$$g_{21}(am) = g_{21}(m)a \tag{2.16}$$

and

$$-g_{21}(m)m_0 = g_{22}(a^{-1}a_0 am) - f_{21}(a^{-1}a_0)am \tag{2.17}$$

for all invertible $a \in B(H_1)$ and all $m \in B(H_2, H_1)$, respectively. Replacing a by I_1 in (2.14)–(2.15), respectively, we obtain that

$$f_{11}(I_1)m - mf_{22}(I_1) + h_{11}(I_2)m - mh_{22}(I_2) = 0 \tag{2.18}$$

and

$$g_{21}(m)a_0 = 0 \tag{2.19}$$

for all $m \in B(H_2, H_1)$. Note that we have proved $f_{11}(I_1) = \lambda I_1$ and $f_{22}(I_1) = \lambda I_2$. Combining these and (2.18), we get

$$h_{11}(I_2)m - mh_{22}(I_2) = 0 \tag{2.20}$$

for all $m \in B(H_2, H_1)$. Comparing (2.14) and (2.20), we see that $g_{12}(am) = ag_{12}(m)+f_{11}(a)m-mf_{22}(a)$ for all invertible $a \in B(H_1)$ and all $m \in B(H_2, H_1)$. Using Lemma 2.1 again, we arrive at

$$g_{12}(am) = ag_{12}(m) + f_{11}(a)m - mf_{22}(a)$$

for all $a \in B(H_1)$ and $m \in B(H_2, H_1)$.

Next, we will show that $g_{21}(m) = 0$. Indeed, replacing a by I_1 in (2.17), we have

$$-g_{21}(m)m_0 = g_{22}(a_0m) - f_{21}(a_0)m \tag{2.21}$$

for all $m \in B(H_2, H_1)$. Moreover, for any $m \in B(H_2, H_1)$, any real number $\lambda > 0$, putting $a_1 = I_1$, $m_1 = m_0 + \lambda a_0m$, $n_1 = 0$, $b_1 = 0$, $a_2 = a_0$, $m_2 = -\lambda a_0m$, $n_2 = 0$ and $b_2 = I_2$ into (2.1), we have

$$-g_{11}(a_0m) = g_{11}(m)a_0 - a_0g_{11}(m) - m_0g_{21}(m) + mf_{21}(a_0) \tag{2.22}$$

for all $m \in B(H_2, H_1)$. It follows from Claim 1(ii) and Claim 3(iii) that

$$g_{11}(m)a_0 + mf_{21}(a_0) = a_0g_{11}(m) + a_0mf_{21}(I_1) \tag{2.23}$$

for all $m \in B(H_2, H_1)$. Applying (2.22)–(2.23) and Claim 3(iii), we get

$$m_0g_{21}(m) = g_{11}(a_0m) + a_0mf_{21}(I_1) \in \mathbb{C}I_1 \tag{2.24}$$

for all $m \in B(H_2, H_1)$. Replacing m by am in (2.24), we obtain

$$m_0g_{21}(am) = g_{11}(a_0am) + a_0amf_{21}(I_1) \in \mathbb{C}I_1.$$

Combining this and (2.16) reduces to

$$m_0g_{21}(m)a = g_{11}(a_0am) + a_0amf_{21}(I_1) \in \mathbb{C}I_1$$

for all $a \in B(H_1)$ and $m \in B(H_2, H_1)$. Note that $\dim(H_1) > 1$. It follows that

$$m_0g_{21}(m) = 0$$

for all $m \in B(H_2, H_1)$. Then it follows from (2.24) that

$$g_{11}(a_0m) + a_0mf_{21}(I_1) = 0$$

for all $m \in B(H_2, H_1)$. Combining this and Claim 3(iii) reduces to

$$g_{22}(a_0m) - f_{21}(I_1)a_0m = 0$$

for all $m \in B(H_2, H_1)$. Thus, by the above equation and (2.21), we have

$$g_{21}(m)m_0 = 0 \tag{2.25}$$

for all $m \in B(H_2, H_1)$. Therefore, (2.19) and (2.25) yield that

$$g_{21}(m) = 0$$

for all $m \in B(H_2, H_1)$, as desired.

By Claim 2(ii), we see that $f_{22}(a) \in \mathbb{C}I_2$ for all $a \in B(H_1)$. So there exists a linear functional τ_1 on $B(H_1)$ such that $f_{22}(a) = \tau_1(a)I_2$ for all $a \in B(H_1)$.

Claim 5 $\delta_1 = f_{11} - \tau_1 I_1$ is a derivation of $B(H_1)$.

For any $a_1, a_2 \in B(H_1)$ and $m \in B(H_2, H_1)$, on the one hand, by Claim 4(ii), we have

$$\begin{aligned} g_{12}(a_1a_2m) &= a_1 g_{12}(a_2m) + \delta_1(a_1)a_2m \\ &= a_1a_2g_{12}(m) + a_1\delta_1(a_2)m + \delta_1(a_1)a_2m. \end{aligned}$$

On the other hand,

$$g_{12}(a_1a_2m) = a_1a_2g_{12}(m) + \delta_1(a_1a_2)m.$$

Comparing these two equalities, we have that

$$a_1\delta_1(a_2)m + \delta_1(a_1)a_2m = \delta_1(a_1a_2)m,$$

which is equivalent to

$$(\delta_1(a_1a_2) - a_1\delta_1(a_2) - \delta_1(a_1)a_2)I_1B(H)I_2 = 0.$$

Since $B(H)$ is prime, we get

$$\delta_1(a_1a_2) = a_1\delta_1(a_2) + \delta_1(a_1)a_2$$

for all $a_1, a_2 \in B(H_1)$. Hence δ_1 is a derivation of $B(H_1)$.

Similarly, by Claim 2(i), we see that $h_{11}(b) \in \mathbb{C}I_1$ for all $b \in B(H_2)$. So there exists a linear functional τ_2 on $B(H_2)$ such that $h_{11}(b) = \tau_2(b)I_1$ for all $b \in B(H_2)$. With the similar argument in Claim 5, using Claim 4(i), one can get the following claims.

Claim 6 $\delta_2 = h_{22} - \tau_2 I_2$ is a derivation of $B(H_2)$.

Claim 7 For any $m \in B(H_2, H_1)$, $n \in B(H_1, H_2)$ and $b \in B(H_2)$, we claim that
(i) $h_{11}(nm) \oplus f_{22}(mn) \in \mathbb{C}I$;
(ii) $l_{21}(bn) = bl_{21}(n) + h_{22}(b)n - nh_{11}(b)$.

For any $a \in B(H_1)$, $m \in B(H_2, H_1)$ and $n \in B(H_1, H_2)$, by Claim 3(vii), on the one hand, we have

$$f_{11}(mna) = g_{12}(m)na + ml_{21}(na) + h_{11}(nam),$$

on the other hand,

$$f_{11}(mn)a = g_{12}(m)na + ml_{21}(n)a + h_{11}(nm)a.$$

Comparing the above two equalities, we see that

$$f_{11}(mn)a = f_{11}(mna) - ml_{21}(na) - h_{11}(nam) + ml_{21}(n)a + h_{11}(nm)a,$$

which implies

$$f_{11}(mn)a = f_{11}(mna) - h_{11}(nam) + h_{11}(nm)a - mnf_{11}(a) + mn\tau_1(a)I_1 \tag{2.26}$$

for all $a \in B(H_1)$, $m \in B(H_2, H_1)$ and $n \in B(H_1, H_2)$ by Claim 3(vi). Applying Claim 5, we get

$$\begin{aligned} f_{11}(mna) - \tau_1(mna)I_1 &= \delta_1(mna) \\ &= \delta_1(mn)a + mn\delta_1(a) \\ &= f_{11}(mn)a - \tau_1(mn)a + mnf_{11}(a) - mn\tau_1(a)I_1. \end{aligned}$$

This and (2.26) yield that

$$h_{11}(nm)a - \tau_1(mn)a = h_{11}(nam) - \tau_1(mna)I_1$$

for all $a \in B(H_1)$, $m \in B(H_2, H_1)$ and $n \in B(H_1, H_2)$. Note that $\dim(H_1) > 1$. This leads to

$$h_{11}(nm) - \tau_1(mn)I_1 = 0,$$

and then

$$h_{11}(nm) \oplus f_{22}(mn) \in \mathbb{C}I$$

for all $m \in B(H_2, H_1)$, $n \in B(H_1, H_2)$.

Furthermore, for any $n \in B(H_1, H_2)$, $m \in B(H_2, H_1)$, $b \in B(H_2)$, by Claim 7(i), we have $h_{11}(bnm) \oplus f_{22}(mbn) \in \mathbb{C}I$ and $h_{11}(nmb) \oplus f_{22}(mbn) \in \mathbb{C}I$. It follows that

$$h_{11}(bnm) = h_{11}(nmb). \tag{2.27}$$

For any $n \in B(H_1, H_2)$, $m \in B(H_2, H_1)$ and $b \in B(H_2)$, by Claim 3(vii), on the one hand, we have

$$g_{12}(mb)n = f_{11}(mbn) - mbl_{21}(n) - h_{11}(nmb).$$

On the other hand, by Claim 4(i), we have

$$g_{12}(mb)n = g_{12}(m)bn + mh_{22}(b)n - h_{11}(b)mn.$$

Comparing these two equalities, we see that

$$f_{11}(mbn) - mbl_{21}(n) - h_{11}(nmb) = g_{12}(m)bn + mh_{22}(b)n - h_{11}(b)mn. \tag{2.28}$$

Moreover, combining Claim 3(vii) and (2.27), we have

$$ml_{21}(bn) = f_{11}(mbn) - h_{11}(nmb) - g_{12}(m)bn. \tag{2.29}$$

Thus (2.28)–(2.29) imply that

$$ml_{21}(bn) = mbl_{21}(n) + mh_{22}(b)n - mnh_{11}(b),$$

which is equivalent to

$$I_1 B(H) I_2 (l_{21}(bn) - bl_{21}(n) - h_{22}(b)n + nh_{11}(b)) = 0.$$

Since $B(H)$ is prime, we get

$$l_{21}(bn) = bl_{21}(n) + h_{22}(b)n - nh_{11}(b)$$

for all $n \in B(H_1, H_2)$ and $b \in B(H_2)$.

Now, by Claim 1, Claim 3(i)–(ii), (iv) and Claim 4(ii), we obtain that

$$\begin{aligned}
\phi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} f_{11}(a) & af_{12}(I_1) \\ f_{21}(I_1)a & f_{22}(a) \end{pmatrix}, \\
\phi\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} g_{11}(m) & g_{12}(m) \\ 0 & g_{22}(m) \end{pmatrix}, \\
\phi\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} &= \begin{pmatrix} -f_{12}(I_1)n & 0 \\ l_{21}(n) & nf_{12}(I_1) \end{pmatrix}, \\
\phi\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} &= \begin{pmatrix} h_{11}(b) & -f_{12}(I_1)b \\ -bf_{21}(I_1) & h_{22}(b) \end{pmatrix}.
\end{aligned}$$

Define two linear maps $\psi : B(H) \to B(H)$ and $\gamma : B(H) \to \mathbb{C}I$ by

$$\psi\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \delta_1(a) & g_{12}(m) \\ l_{21}(n) & \delta_2(b) \end{pmatrix} + \begin{pmatrix} -mf_{21}(I_1) - f_{12}(I_1)n & af_{12}(I_1) - f_{12}(I_1)b \\ f_{21}(I_1)a - bf_{21}(I_1) & nf_{12}(I_1) + f_{21}(I_1)m \end{pmatrix}$$

and

$$\begin{aligned}
\gamma\begin{pmatrix} a & m \\ n & b \end{pmatrix} &= \begin{pmatrix} \tau_1(a)I_1 + h_{11}(b) & 0 \\ 0 & \tau_2(b)I_2 + f_{22}(a) \end{pmatrix} \\
&\quad + \begin{pmatrix} g_{11}(m) + mf_{21}(I_1) & 0 \\ 0 & g_{22}(m) - f_{21}(I_1)m \end{pmatrix}
\end{aligned}$$

for all $\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in B(H)$.

It is clear that $\phi = \psi + \gamma$. Moreover, by Claim 3(v)–(vii) and Claims 4–7, the following statements hold:

(i) $\delta_1(a)$ is a derivation of $B(H_1)$ with $\delta_1(mn) = g_{12}(m)n + ml_{21}(n)$;

(ii) $\delta_2(b)$ is a derivation of $B(H_2)$ with $\delta_2(nm) = ng_{12}(m) + l_{21}(n)m$;

(iii) $g_{12}(am) = ag_{12}(m) + \delta_1(a)m$ and $g_{12}(mb) = g_{12}(m)b + m\delta_2(b)$;

(iv) $l_{21}(na) = l_{21}(n)a + n\delta_1(a)$ and $l_{21}(bn) = bl_{21}(n) + \delta_2(b)n$.

Thus, using [7, Proposition 4.2], one can easily check that ψ is a derivation of $B(H)$.

Furthermore, for any $S, T \in B(H)$ with $ST = G$, we have

$$\begin{aligned}
\gamma([S,T]) &= \phi([S,T]) - \psi([S,T]) \\
&= [\phi(S), T] + [S, \phi(T)] - \psi(ST - TS) \\
&= [\psi(S) + \gamma(S), T] + [S, \psi(T) + \gamma(T)] - \psi(ST - TS) \\
&= [\psi(S), T] + [S, \psi(T)] - \psi(ST - TS) \\
&= 0.
\end{aligned}$$

The proof is completed.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] An, R. and Cai, Y., Derivations and derivable maps on von Neumann algebras, *Linear Multilinear Algebra*, **69**, 2021, 2806–2812.

[2] An, R. and Hou, J., Characterizations of derivations on triangular rings: Additive maps derivable at idempotents, *Linear Algebra Appl.*, **431**, 2009, 1070–1080.

[3] Chebotar, M. A., Ke, W. F. and Lee, P. H., Maps characterized by action on zero products, *Pacific J. Math.*, **216**, 2004, 217–228.

[4] Christensen, E., Derivations of nest algebras, *Math. Ann.*, **229**, 1977, 155–161.

[5] Du, Y. and Wang, Y., Lie derivations of generalized matrix algebras, *Linear Algebra Appl.*, **437**, 2012, 2719–2726.

[6] Ji, P. and Qi, W., Characterizations of Lie derivations of triangular algebras, *Linear Algebra Appl.*, **435**, 2011, 1137–1146.

[7] Li, Y. and Wei, F., Semi-centralizing maps of generalized matrix algebras, *Linear Algebra Appl.*, **436**, 2012, 1122–1153.

[8] Lu, F., Characterizations of derivations and Jordan derivations on Banach algebras, *Linear Algebra Appl.*, **430**, 2009, 2233–2239.

[9] Lu, F. and Jing, W., Characterizations of Lie derivations of $B(X)$, *Linear Algebra Appl.*, **432**, 2010, 89–99.

[10] Pan, Z., Derivable maps and derivational points, *Linear Algebra Appl.*, **436**, 2012, 4251–4260.

[11] Sakai, S., Derivations of W^*-algebras, *Ann. of Math.*, **83**, 1966, 273–279.

[12] Zhu, J. and Xiong, C., Derivable mappings at unit operator on nest algebras, *Linear Algebra Appl.*, **422**, 2007, 721–735.