Chin. Ann. Math. Ser. B
47(2), 2026, 413–420
DOI: 10.1007/s11401-026-0044-5

Chinese Annals of Mathematics, Series B

Residue Class Ring with Identical Representation Function*

Shiqiang CHEN[1]

Abstract For an integer $m \geq 2$, let $\mathbb{Z}/m\mathbb{Z}$ be the set of all residue classes $\bmod\, m$. For $S \subseteq \mathbb{Z}/m\mathbb{Z}$ and $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, $R_S(\overline{n})$ is defined as the number of solutions to the equation $\overline{n} = \overline{s} + \overline{s'}$ with an unordered pair $(\overline{s}, \overline{s'}) \in S^2$ and $\overline{s} \neq \overline{s'}$. In this paper, the author determines the structures of sets A and B such that $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $A \cap B = \overline{k}\mathbb{Z}$ and $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, where k is an integer.

Keywords Residue class, Representation function
2020 MR Subject Classification 11B13

1 Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a set $X \subseteq \mathbb{N}$ and an integer $n \in \mathbb{N}$, the representation function $R'_X(n)$ is defined as the number of solutions of the equation $n = x + x'$ with $x < x'$ and $x, x' \in X$. The topic of representation functions has been studied for many years. Sárközy ever asked if there exist two integer sets A and B with infinite symmetric difference and $R'_A(n) = R'_B(n)$ for all sufficiently large integers n. In 2002, Dombi [3] proved that the set of positive integers can be partitioned into two subsets C and D such that $R'_C(n) = R'_D(n)$ for every positive integer n.

The intersection of sets C and D constructed by Dombi is the empty set. In 2012, Yu and Tang [13] considered the case that the intersection of sets being infinite arithmetic sequences, and proposed the following conjecture.

Conjecture 1.1 (see [13, Conjecture 4]) Let $m \in \mathbb{N}$ and $R \subset \{0, 1, \cdots, m-1\}$. If $\mathbb{N} = A \cup B$ and $A \cap B = \{r + km : k \in \mathbb{N},\ r \in R\}$, then $R'_A(n) = R'_B(n)$ cannot hold for all sufficiently large integers n.

In 2016, Chen and Lev [2] disproved Conjecture 1.1.

Theorem A (see [2, Theorem 1]) *For any given positive integer ℓ, there exist two subsets A and B with $A \cup B = \mathbb{N}$ and $A \cap B = \{2^{2\ell} - 1 + (2^{2\ell+1} - 1)k : k \in \mathbb{N}\}$ such that $R'_A(n) = R'_B(n)$ for every positive integer n.*

In 2021, Chen and Chen [1] proved the following result.

Theorem B (See [1, Theorem 1.1]) *Let $m \geq 2$ and $r \geq 0$ be two integers and let A and B be two sets with $A \cup B = \mathbb{N}$ and $A \cap B = \{r + mk : k \in \mathbb{N}\}$ such that $R_A(n) = R_B(n)$ for every positive integer n. Then there exists an integer $\ell \geq 1$ such that $r = 2^{2\ell} - 1$ and $m = 2^{2\ell+1} - 1$.*

Manuscript received April 17, 2022. Revised October 12, 2023.
[1]School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, Anhui, China.
E-mail: csq20180327@163.com
*This work was supported by the National Natural Science Foundation of China (No. 12301003), the Anhui Provincial Natural Science Foundation (No. 2308085QA02) and the University Natural Science Research Project of Anhui Province (No. 2022AH050171).

The investigation of the partitions of $\mathbb{N}$ with the identical representation functions is a popular topic and the related results can be found in [4–10].

For an integer $m \geq 2$, let $\mathbb{Z}/m\mathbb{Z} = \{\overline{0}, \overline{1}, \cdots, \overline{m-1}\}$ be the set of all residue classes mod m, where $\overline{x}$ is denoted as the residue class $x \pmod{m}$. For two different residue classes $\overline{a}$ and $\overline{b}$, there exist integers $0 \leq a', b' \leq m-1$ such that $\overline{a'} = \overline{a}$ and $\overline{b'} = \overline{b}$, and we define the ordering $\overline{a} < \overline{b}$ if $a' < b'$. For a given set $S \subseteq \mathbb{Z}/m\mathbb{Z}$ and $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, the representation function $R_S(\overline{n})$ is defined as the number of solutions to the equation $\overline{n} = \overline{s} + \overline{s'}$ with $\overline{s} < \overline{s'}$ and $\overline{s}, \overline{s'} \in S$. For $\overline{a} \in \mathbb{Z}/m\mathbb{Z}$ and $X \subseteq \mathbb{Z}/m\mathbb{Z}$, define $\overline{a} + X = \{\overline{a} + \overline{x} : \overline{x} \in X\}$.

In 2012, Yang and Chen [11] studied the analogue of Sárközy's problem in $\mathbb{Z}/m\mathbb{Z}$. They determined all subsets $A \subseteq \mathbb{Z}/m\mathbb{Z}$ such that $R_A(\overline{n}) = R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$.

Theorem C (see [11, Theorem 1]) *The equality $R_A(\overline{n}) = R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n})$ holds for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$ if and only if m is even and $(\mathbb{Z}/m\mathbb{Z}) \setminus A = A + \overline{\frac{m}{2}}$.*

In this paper, we determine the structures of sets A and B such that $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $A \cap B = \overline{k}\mathbb{Z}$ and $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, where k is an integer.

If $m \mid k$, then $A \cap B = \{\overline{0}\}$ and m is odd from Lemma 2.1. By the proof of [12, Theorem 3.3], we can determine the structures of sets A and B such that $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $A \cap B = \{\overline{0}\}$ and $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$. Hence, it remains to consider the case of $m \nmid k$.

Theorem 1.1 *Let k be an integer and m be a positive integer with $m \nmid k$. Let A and B be two sets with $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $A \cap B = \overline{k}\mathbb{Z}$. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$ if and only if m is even, $2 \mid \frac{m}{(m,k)}$ and $B = A + \overline{\frac{m}{2}}$.*

Remark 1.1 Let $m \geq 2$ and k be two integers such that $m \nmid k$ and $2 \mid \frac{m}{(m,k)}$. Then sets A and B satisfy $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $A \cap B = \overline{k}\mathbb{Z}$ and $B = A + \overline{\frac{m}{2}}$ if and only if $A = C \cup \left(D + \overline{\frac{m}{2}}\right) \cup \overline{k}\mathbb{Z}$ and $B = D \cup \left(C + \overline{\frac{m}{2}}\right) \cup \overline{k}\mathbb{Z}$, where $C \cup D = \left[\overline{0}, \overline{\frac{m}{2}-1}\right] \setminus \overline{k}\mathbb{Z}$ and $C \cap D = \emptyset$.

Corollary 1.1 *Let k and r be two integers and m be a positive integer with $m \nmid k$. Let A and B be two sets with $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $A \cap B = \overline{r} + \overline{k}\mathbb{Z}$. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$ if and only if m is even, $2 \mid \frac{m}{(m,k)}$ and $B = A + \overline{\frac{m}{2}}$.*

Proof For sets $A, B \subseteq \mathbb{Z}/m\mathbb{Z}$, if $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, then for any integer r, we have $R_{A+\overline{r}}(\overline{n}) = R_{B+\overline{r}}(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$. Therefore, Theorem 1.1 implies Corollary 1.1.

Throughout this paper, the characteristic function of the set $A \subseteq \mathbb{Z}/m\mathbb{Z}$ is denoted by

$$\chi_A(t) = \begin{cases} 0, & \overline{t} \notin A, \\ 1, & \overline{t} \in A. \end{cases}$$

For any sets $A, B \subseteq \mathbb{Z}/m\mathbb{Z}$, $R_{A,B}(\overline{n})$ is defined as the number of solutions to the equation $\overline{a} + \overline{b} = \overline{n}$ with $\overline{a} \in A$ and $\overline{b} \in B$.

Our paper will be organized as follows: In Section 2, we give some auxiliary lemmas. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give a concrete example.

2 Lemmas

Lemma 2.1 *Let m, t be positive integers. Let A and B be two sets such that $A \cup B = \mathbb{Z}/m\mathbb{Z}$, $|A \cap B| = t$ and $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, then $2 \mid m + t$ and $|A| = |B| = \frac{m+t}{2}$.*

Proof Since $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, it follows that

$$\binom{|B|}{2} = \sum_{\overline{n} \in \mathbb{Z}/m\mathbb{Z}} R_B(\overline{n}) = \sum_{\overline{n} \in \mathbb{Z}/m\mathbb{Z}} R_A(\overline{n}) = \binom{|A|}{2}.$$

This implies that $|A| = |B|$. Since

$$|A| + |B| = |A \cup B| + |A \cap B| = m + t,$$

it follows from $|A| = |B|$ that

$$2 \mid m + t, \quad |A| = |B| = \frac{m+t}{2}.$$

This completes the proof of Lemma 2.1.

Lemma 2.2 *Let m be a positive integer and let A be a set with $A \subseteq \mathbb{Z}/m\mathbb{Z}$. Then*

$$R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) = \sum_{0 \le i < \frac{n}{2}} 1 + \sum_{n+1 \le i < \frac{n+m}{2}} 1 - |A| + \chi_A\Big(\frac{n}{2}\Big) + \chi_A\Big(\frac{n+m}{2}\Big) + R_A(\overline{n})$$

for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$.

Proof For each residue class $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$, without loss of generality, we may assume that $0 \le n \le m-1$. Then

$$\begin{aligned}
&R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) \\
&= |\ \{(a, a') : \overline{a}, \overline{a'} \in (\mathbb{Z}/m\mathbb{Z}) \setminus A,\ 0 \le a < a' \le m-1,\ a + a' = n \text{ or } a + a' = n + m\}\ | \\
&= \sum_{0 \le i < \frac{n}{2}} (1 - \chi_A(i))(1 - \chi_A(n-i)) + \sum_{n+1 \le i < \frac{n+m}{2}} (1 - \chi_A(i))(1 - \chi_A(n+m-i)) \\
&= \sum_{0 \le i < \frac{n}{2}} 1 - \sum_{0 \le i \le n} \chi_A(i) + \chi_A\Big(\frac{n}{2}\Big) + \sum_{0 \le i < \frac{n}{2}} \chi_A(i)\chi_A(n-i) + \sum_{n+1 \le i < \frac{n+m}{2}} 1 \\
&\quad - \sum_{n+1 \le i \le m-1} \chi_A(i) + \chi_A\Big(\frac{n+m}{2}\Big) + \sum_{n+1 \le i < \frac{n+m}{2}} \chi_A(i)\chi_A(n+m-i) \\
&= \sum_{0 \le i < \frac{n}{2}} 1 + \sum_{n+1 \le i < \frac{n+m}{2}} 1 - |A| + \chi_A\Big(\frac{n}{2}\Big) + \chi_A\Big(\frac{n+m}{2}\Big) + R_A(\overline{n}).
\end{aligned}$$

This completes the proof of Lemma 2.2.

Lemma 2.3 *Let $m \ge 2$, k and d be three integers such that $2 \mid m$ and $d = \frac{m}{(k,m)}$. Let A, B be two sets with $A \cup B = (\mathbb{Z}/m\mathbb{Z})$, $A \cap B = \overline{k}\mathbb{Z}$. If $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in (\mathbb{Z}/m\mathbb{Z})$, then*

$$\frac{d}{2} = \sum_{i=0}^{d-1} \chi_A(n - (k,m)i), \quad (k,m) \nmid n,\ 2 \nmid n,$$

$$\frac{d}{2} - 1 + \chi_A\Big(\frac{n}{2}\Big) + \chi_A\Big(\frac{n+m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(n - (k,m)i), \quad (k,m) \nmid n,\ 2 \mid n.$$

Proof By Lemma 2.2, we have

$$R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) = \frac{m}{2} - |A| + R_A(\overline{n}), \quad \text{if } n \text{ is odd}, \tag{2.1}$$

$$R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) = \frac{m}{2} - 1 - |A| + \chi_A\Big(\frac{n}{2}\Big) + \chi_A\Big(\frac{n+m}{2}\Big) + R_A(\overline{n}), \quad \text{if } n \text{ is even}. \tag{2.2}$$

For any integer a, there exists an integer $i \in [0, d-1]$ such that

$$ka \equiv (k,m)i \pmod{m}.$$

It follows that

$$\overline{k}\mathbb{Z} \subseteq \{\overline{(k,m)i} : i = 0, 1, \cdots, d-1\}.$$

On the other hand, for any integer $a \in [0, d-1]$, there exists an integer i such that

$$(k,m)a \equiv ki \pmod{m}.$$

It follows that

$$\overline{k}\mathbb{Z} \supseteq \{\overline{(k,m)i} : i = 0, 1, \cdots, d-1\}.$$

Therefore

$$\overline{k}\mathbb{Z} = \{\overline{(k,m)i} : i = 0, 1, \cdots, d-1\}.$$

Noting that

$$B = ((\mathbb{Z}/m\mathbb{Z}) \setminus A) \cup \{\overline{(k,m)i} : i = 0, 1, \cdots, d-1\},$$
$$((\mathbb{Z}/m\mathbb{Z}) \setminus A) \cap \{\overline{(k,m)i} : i = 0, 1, \cdots, d-1\} = \emptyset,$$

we have

$$\begin{aligned} R_B(\overline{n}) &= R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) + R_{(\mathbb{Z}/m\mathbb{Z})\setminus A,\{\overline{(k,m)i}:i=0,1,\cdots,d-1\}}(\overline{n}) + R_{\{\overline{(k,m)i}:i=0,1,\cdots,d-1\}}(\overline{n}) \\ &= R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) + \sum_{i=0}^{d-1}(1-\chi_A(n-(k,m)i)) + R_{\{\overline{(k,m)i}:i=0,1,\cdots,d-1\}}(\overline{n}) \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{Z}/m\mathbb{Z}$, which implies that

$$R_B(\overline{n}) = R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) + \sum_{i=0}^{d-1}(1-\chi_A(n-(k,m)i)), \quad \text{if } (k,m) \nmid n, \tag{2.4}$$

$$R_B(\overline{n}) = R_{(\mathbb{Z}/m\mathbb{Z})\setminus A}(\overline{n}) + R_{\{\overline{(k,m)i}:i=0,1,\cdots,d-1\}}(\overline{n}), \quad \text{if } (k,m) \mid n. \tag{2.5}$$

For $(k,m) \nmid n$, let $n = (k,m)t + r, 1 \le r \le (k,m)-1$. If n is odd, then by (2.1) and (2.4), we have

$$R_B(\overline{n}) = \frac{m}{2} - |A| + R_A(\overline{n}) + \sum_{i=0}^{d-1}(1-\chi_A(n-(k,m)i)). \tag{2.6}$$

Hence, by Lemma 2.1, we have

$$\frac{d}{2} = \sum_{i=0}^{d-1}\chi_A(n-(k,m)i), \quad (k,m) \nmid n,\ 2 \nmid n. \tag{2.7}$$

If n is even, then by (2.2) and (2.4), we have

$$\begin{aligned} R_B(\overline{n}) = \frac{m}{2} &- 1 - |A| + R_A(\overline{n}) + \chi_A\Big(\frac{n}{2}\Big) + \chi_A\Big(\frac{n+m}{2}\Big) \\ &+ \sum_{i=0}^{d-1}(1-\chi_A(n-(k,m)i)). \end{aligned} \tag{2.8}$$

Hence, by Lemma 2.1, we have

$$\frac{d}{2} - 1 + \chi_A\Big(\frac{n}{2}\Big) + \chi_A\Big(\frac{n+m}{2}\Big) = \sum_{i=0}^{d-1}\chi_A(n-(k,m)i), \quad (k,m) \nmid n,\ 2 \mid n.$$

This completes the proof of Lemma 2.3.

3 Proof of Theorem 1.1

It is clear that if m is even and $B = A + \overline{\frac{m}{2}}$, then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$. Suppose that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}/m\mathbb{Z}$. Let $d = \frac{m}{(k,m)}$. If m is odd, then for any t with $(k,m) \nmid t$, we have $(k,m) \nmid 2t$. Then by Lemma 2.1, Lemma 2.2 and (2.4) with $n = 2t$, we have

$$\frac{d-1}{2} + \chi_A(t) = \sum_{i=0}^{d-1} \chi_A(2t - (k,m)i). \tag{3.1}$$

Since $m \nmid k$, it follows that $d > 1$. So there exist different integers $i_1, i_2 \in [0, d-1]$ such that

$$\chi_A(2t - (k,m)i_1) + \chi_A(2t - (k,m)i_2) = 1. \tag{3.2}$$

Noting that $(k,m) \nmid 2t - (k,m)i_1$ and $(k,m) \nmid 2t - (k,m)i_2$, it follows from (3.1) that

$$\frac{d-1}{2} + \chi_A(2t - (k,m)i_1) = \sum_{i=0}^{d-1} \chi_A(4t - 2(k,m)i_1 - (k,m)i) = \sum_{i=0}^{d-1} \chi_A(4t - (k,m)i),$$

$$\frac{d-1}{2} + \chi_A(2t - (k,m)i_2) = \sum_{i=0}^{d-1} \chi_A(4t - 2(k,m)i_2 - (k,m)i) = \sum_{i=0}^{d-1} \chi_A(4t - (k,m)i),$$

which implies that

$$\chi_A(2t - (k,m)i_1) = \chi_A(2t - (k,m)i_2).$$

This is a contradiction with (3.2). Hence, m is even and $2 \mid d$ from Lemma 2.1.

To prove $B = A + \overline{\frac{m}{2}}$, it suffices to prove that $\chi_A(t) + \chi_A(t + \frac{m}{2}) = 1$ for any $\overline{t} \in \mathbb{Z}/m\mathbb{Z}$, $(k,m) \nmid t$.

Case 1 k is odd. For any $r \in [1, (k,m) - 1]$, taking $n = (k,m)(r+1) + r$, then n is odd, by Lemma 2.3, we have

$$\frac{d}{2} = \sum_{i=0}^{d-1} \chi_A(n - (k,m)i) = \sum_{i=0}^{d-1} \chi_A(r - (k,m)i). \tag{3.3}$$

For any $\overline{t} \in \mathbb{Z}/m\mathbb{Z}$, $(k,m) \nmid t$, let $2t = (k,m)s + r$, $1 \le r \le (k,m) - 1$. Then by Lemma 2.3, we have

$$\frac{d}{2} - 1 + \chi_A(t) + \chi_A\Big(t + \frac{m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(2t - (k,m)i) = \sum_{i=0}^{d-1} \chi_A(r - (k,m)i).$$

By (3.3), we have

$$\chi_A(t) + \chi_A\Big(t + \frac{m}{2}\Big) = 1.$$

Case 2 k is even. For any odd n, $\frac{(k,m)}{2} \nmid n$, then $(k,m) \nmid n$ and $(k,m) \nmid 2n$, by Lemma 2.3, we have

$$\frac{d}{2} = \sum_{i=0}^{d-1} \chi_A(n - (k,m)i) \tag{3.4}$$

and

$$\frac{d}{2} - 1 + \chi_A(n) + \chi_A\Big(n + \frac{m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(2n - (k,m)i). \tag{3.5}$$

Noting that $\frac{(k,m)}{2} \nmid n$, we have $\frac{(k,m)}{2} \nmid n + j(k,m)$, it follows that $(k,m) \nmid 2n + 2j(k,m)$ for any $j \in \left[0, \frac{d}{2} - 1\right]$, by Lemma 2.3, we have

$$\frac{d}{2} - 1 + \chi_A(n + j(k,m)) + \chi_A\Big(n + j(k,m) + \frac{m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(2n + 2j(k,m) - (k,m)i).$$

It follows that

$$\begin{aligned}&\sum_{j=0}^{\frac{d}{2}-1} \Big(\frac{d}{2} - 1 + \chi_A(n + j(k,m)) + \chi_A\Big(n + j(k,m) + \frac{m}{2}\Big)\Big)\\ &= \sum_{j=0}^{\frac{d}{2}-1} \sum_{i=0}^{d-1} \chi_A(2n + 2j(k,m) - (k,m)i),\end{aligned}$$

that is

$$\sum_{j=0}^{\frac{d}{2}-1} \Big(\frac{d}{2} - 1\Big) + \sum_{j=0}^{d-1} \chi_A(n - j(k,m)) = \sum_{j=0}^{\frac{d}{2}-1} \sum_{i=0}^{d-1} \chi_A(2n - (k,m)i),$$

from (3.4),

$$\frac{d}{2} = \sum_{i=0}^{d-1} \chi_A(2n - (k,m)i),$$

which implies from (3.5) that

$$\chi_A(n) + \chi_A\Big(n + \frac{m}{2}\Big) = 1. \tag{3.6}$$

For any integer $t \geq 0$ and odd n, $\frac{(k,m)}{2} \nmid n$, we define $a_{t,n} = 2^t n$. Next we will use induction on t to prove the following conclusion: For any integer $t \geq 0$ and odd n, $\frac{(k,m)}{2} \nmid n$, we have

$$\chi_A(a_{t,n}) + \chi_A\Big(a_{t,n} + \frac{m}{2}\Big) = 1. \tag{3.7}$$

The result is true for $t = 0$ from (3.6). Assume that the result is true for all positive integers less than t, we will prove that the result is true for integer t. Since $a_{t,n}$ and $a_{t+1,n}$ are even, $(k,m) \nmid a_{t+1,n}$ and $(k,m) \nmid a_{t+1,n}$, it follows from Lemma 2.3 that

$$\frac{d}{2} - 1 + \chi_A(a_{t-1,n}) + \chi_A\Big(a_{t-1,n} + \frac{m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(a_{t,n} - (k,m)i) \tag{3.8}$$

and

$$\frac{d}{2} - 1 + \chi_A(a_{t,n}) + \chi_A\Big(a_{t,n} + \frac{m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(a_{t+1,n} - (k,m)i). \tag{3.9}$$

By induction hypothesis and (3.8), we have

$$\frac{d}{2} = \sum_{i=0}^{d-1} \chi_A(a_{t,n} - (k,m)i). \tag{3.10}$$

Noting that $(k,m) \nmid a_{t+1,n}$, by Lemma 2.3, we have

$$\frac{d}{2} - 1 + \chi_A(a_{t,n} + j(k,m)) + \chi_A\Big(a_{t,n} + j(k,m) + \frac{m}{2}\Big) = \sum_{i=0}^{d-1} \chi_A(a_{t+1,n} + 2j(k,m) - (k,m)i).$$

It follows that

$$\sum_{j=0}^{\frac{d}{2}-1}\Big(\frac{d}{2}-1+\chi_A(a_{t,n}+j(k,m))+\chi_A\Big(a_{t,n}+j(k,m)+\frac{m}{2}\Big)\Big)$$
$$=\sum_{j=0}^{\frac{d}{2}-1}\sum_{i=0}^{d-1}\chi_A(a_{t+1,n}-(k,m)i),$$

that is

$$\sum_{j=0}^{\frac{d}{2}-1}\Big(\frac{d}{2}-1\Big)+\sum_{j=0}^{d-1}\chi_A(a_{t,n}-j(k,m))=\sum_{j=0}^{\frac{d}{2}-1}\sum_{i=0}^{d-1}\chi_A(a_{t+1,n}-(k,m)i),$$

from (3.10),

$$\frac{d}{2}=\sum_{i=0}^{d-1}\chi_A(a_{t+1,n}-(k,m)i),$$

which implies from (3.9) that

$$\chi_A(a_{t,n})+\chi_A\Big(a_{t,n}+\frac{m}{2}\Big)=1.$$

For any $\overline{h}\in\mathbb{Z}/m\mathbb{Z}$, $\frac{(k,m)}{2}\nmid h$, there exist integer $t\geq 0$ and odd n such that $h=2^t n$. Noting that $\frac{(k,m)}{2}\nmid h$, we have $\frac{(k,m)}{2}\nmid n$. It follows from (3.7) that

$$\chi_A(h)+\chi_A\Big(h+\frac{m}{2}\Big)=1. \tag{3.11}$$

For any $\overline{h}\in\mathbb{Z}/m\mathbb{Z}$, $\frac{(k,m)}{2}\big|h$, we have $(k,m)\big|2h$. By (2.2) and (2.5), we have

$$\chi_A(h)+\chi_A\Big(h+\frac{m}{2}\Big)=1,\quad \text{if } (k,m)\nmid h. \tag{3.12}$$

Hence, by (3.11)–(3.12), we have

$$\chi_A(h)+\chi_A\Big(h+\frac{m}{2}\Big)=1$$

for any $\overline{h}\in\mathbb{Z}/m\mathbb{Z},(k,m)\nmid h$.

This completes the proof of Theorem 1.1.

4 Example

Based on Theorem 1.1 and Remark 1.1, we provide the following concrete example.

Let $m=8$ and $k=2$. Then all sets A and B that satisfy $A\cup B=\mathbb{Z}/8\mathbb{Z}$, $A\cap B=\overline{2}\mathbb{Z}$ and $R_A(\overline{n})=R_B(\overline{n})$ for all $\overline{n}\in\mathbb{Z}/8\mathbb{Z}$ are as follows:

$A=\{\overline{1},\overline{3}\}\cup\{\overline{0},\overline{2},\overline{4},\overline{6}\},\quad B=\{\overline{5},\overline{7}\}\cup\{\overline{0},\overline{2},\overline{4},\overline{6}\},$

$R_A(\overline{0})=R_B(\overline{0})=1,\quad R_A(\overline{1})=R_B(\overline{1})=2,\quad R_A(\overline{2})=R_B(\overline{2})=2,\quad R_A(\overline{3})=R_B(\overline{3})=2,$

$R_A(\overline{4})=R_B(\overline{4})=2,\quad R_A(\overline{5})=R_B(\overline{5})=2,\quad R_A(\overline{6})=R_B(\overline{6})=2,\quad R_A(\overline{7})=R_B(\overline{7})=2.$

$A=\{\overline{1},\overline{7}\}\cup\{\overline{0},\overline{2},\overline{4},\overline{6}\},\quad B=\{\overline{3},\overline{5}\}\cup\{\overline{0},\overline{2},\overline{4},\overline{6}\},$

$R_A(\overline{0})=R_B(\overline{0})=2,\quad R_A(\overline{1})=R_B(\overline{1})=2,\quad R_A(\overline{2})=R_B(\overline{2})=2,\quad R_A(\overline{3})=R_B(\overline{3})=2,$

$R_A(\overline{4}) = R_B(\overline{4}) = 1, \quad R_A(\overline{5}) = R_B(\overline{5}) = 2, \quad R_A(\overline{6}) = R_B(\overline{6}) = 2, \quad R_A(\overline{7}) = R_B(\overline{7}) = 2.$

$A = \{\overline{3}, \overline{5}\} \cup \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}, \quad B = \{\overline{7}, \overline{1}\} \cup \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\},$

$R_A(\overline{0}) = R_B(\overline{0}) = 2, \quad R_A(\overline{1}) = R_B(\overline{1}) = 2, \quad R_A(\overline{2}) = R_B(\overline{2}) = 2, \quad R_A(\overline{3}) = R_B(\overline{3}) = 2,$

$R_A(\overline{4}) = R_B(\overline{4}) = 1, \quad R_A(\overline{5}) = R_B(\overline{5}) = 2, \quad R_A(\overline{6}) = R_B(\overline{6}) = 2, \quad R_A(\overline{7}) = R_B(\overline{7}) = 2.$

$A = \{\overline{5}, \overline{7}\} \cup \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}, \quad B = \{\overline{1}, \overline{3}\} \cup \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\},$

$R_A(\overline{0}) = R_B(\overline{0}) = 1, \quad R_A(\overline{1}) = R_B(\overline{1}) = 2, \quad R_A(\overline{2}) = R_B(\overline{2}) = 2, \quad R_A(\overline{3}) = R_B(\overline{3}) = 2,$

$R_A(\overline{4}) = R_B(\overline{4}) = 2, \quad R_A(\overline{5}) = R_B(\overline{5}) = 2, \quad R_A(\overline{6}) = R_B(\overline{6}) = 2, \quad R_A(\overline{7}) = R_B(\overline{7}) = 2.$

Acknowledgement We would like to thank the anonymous referee for his/her helpful comments.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Chen, S. Q. and Chen, Y. G., Integer sets with identical representation functions, II, *European J. Combin.*, **94**, 2021, 103293.

[2] Chen, Y. G. and Lev, V. F., Integer sets with identical representation functions, *Integers*, **16**, 2016, Paper No. A36.

[3] Dombi, G., Additive properties of certain sets, *Acta Arith.*, **103**, 2002, 137–146.

[4] Kiss, S. Z. and Sándor, C., On the multiplicativity of the linear combination of additive representation functions, *Ramanujan J.*, **44**, 2017, 385–399.

[5] Kiss, S. Z. and Sándor, C., Partitions of the set of nonnegative integers with the same representation functions, *Discrete Math.*, **340**, 2017, 1154–1161.

[6] Lev, V. F., Representation of elements of a sequence by sumsets, *Combinatorica,* **16**, 1996, 587–590.

[7] Lev, V. F., Reconstructing integer sets from their representation functions, *Electron. J. Combin.*, **11**, 2004, Paper No. R78.

[8] Nathanson, M. B., Representation functions of sequences in additive number theory, *Proc. Amer. Math. Soc.*, **72**, 1978, 16–20.

[9] Sándor, C., Partitions of natural numbers and their representation functions, *Integers*, **4**, 2004, Paper No. A18.

[10] Tang, M., Partitions of the set of natural numbers and their representation functions, *Discrete Math.*, **308**, 2008, 2614–2616.

[11] Yang, Q. H. and Chen, F. J., Partitions of $\mathbb{Z}_m$ with the same representation functions, *Australas. J. Combin.*, **53**, 2012, 257–262.

[12] Yang, Q. H. and Tang, M., Representation functions on finite sets with extreme symmetric differences, *J. Number Theory*, **180**, 2017, 73–85.

[13] Yu, W. and Tang, M., A note on partitions of natural numbers and their representation functions, *Integers*, **12**, 2012, Paper No. A53.