Meromorphic Functions Partially Share Three Values with Their Difference Operators*

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Abstract In this paper, the authors firstly give a simple proof and strengthened version of a uniqueness theorem of meromorphic functions which partially share 0, ∞ CM and 1 IM with their difference operators. In addition, they partially solve a conjecture given by Chen-Yi (2013) and generalize some previous theorems by Chen (2018) and Chen-Xu (2022). Furthermore, the authors obtain a uniqueness result of the k-th derivative of meromorphic functions with its shift, which is a generalization of some previous theorems by Chen-Xu (2022), Qi-Li-Yang (2018) and Qi-Yang (2020).

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1 Introduction and Main Results

In this paper, the term "meromorphic function" always means meromorphic in the whole complex plane $\mathbb{C}$. We say that two meromorphic functions f and g share a constant a IM if $f - a$ and $g - a$ have the same zeros or $\overline{E}(a, f) = \overline{E}(a, g)$, and the notation $\overline{E}(a, f)$ denotes the set of all zeros of $f(z) - a$, where a zero is counted one time. In addition, we say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with the same multiplicities or $E(a, f) = E(a, g)$, and the notation $E(a, f)$ denotes the set of all zeros of $f(z) - a$, where a zero with multiplicity m is counted m times.

The paper mainly concerns the uniqueness problem of meromorphic function $f(z)$ partially sharing 2 CM + 1 IM with its difference operator $\Delta_c f = f(z + c) - f(z)$, where c is a nonzero constant. This problem was inspired by a conjecture due to Chen-Yi [5]. They considered the uniqueness problem of meromorphic function $f(z)$ sharing e_1, e_2 and ∞ CM with $\Delta_c f$ and posed the following conjecture.

Conjecture 1.1 Let f be a transcendental meromorphic function, and let $c \in \mathbb{C} \backslash \{0\}$ such that $\Delta_c f \not\equiv 0$. If $\Delta_c f$ and $f(z)$ share three distinct values e_1, e_2, ∞ CM, then $\triangle_c f = f$.

Since then, many scholars have devoted themselves to studying this conjecture. Zhang and Liao [14] affirmed the conjecture for the case that f is an entire function of finite order. Here

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and throughout this paper, the order and hyper order are defined in turn as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Later on, Lü-Lü [10] proved that the conjecture is also right when f is a meromorphic function of finite order. Some related theorems can be found in [6]. Recently, Chen and his co-workers considered the conjecture in another direction. They asked whether the shared value conditions "3 CM" in Conjecture can be weaken or not? Actually, they partially answered the question by the following two theorems, which were given by Chen in [2] and Chen-Xu in [3], respectively.

Theorem A *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and let $c \in \mathbb{C}\backslash\{0\}$ such that $\Delta_c f \not\equiv 0$. If $\Delta_c f$ and $f(z)$ share 1 CM and satisfy $E(0, f(z)) \subseteq E(0, \Delta_c f(z))$ and $E(\infty, \Delta_c f(z)) \subseteq E(\infty, f(z))$, then $\Delta_c f = f$.*

Theorem B *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and let $c \in \mathbb{C}\backslash\{0\}$ such that $\Delta_c f \not\equiv 0$. If $\Delta_c f$ and $f(z)$ share $0, \infty$ CM and 1 IM, then $\Delta_c f = f$.*

In this present paper, we still pay attention to this conjecture and give a simple proof and strengthening of a uniqueness theorem as follows.

Theorem 1.1 *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$, let $c \in \mathbb{C}\backslash\{0\}$, and let $L(f) = af(z+c) - bf(z)$, where a, b are two nonzero constants. If $E(0, f(z)) \subseteq E(0, L(f))$, $E(\infty, L(f)) \subseteq E(\infty, f(z))$ and $\overline{E}(1, f(z)) \subseteq \overline{E}(1, L(f))$, then $L(f) = f(z)$ or $f(z+c) = f(z)$.*

Obviously, Theorem 1.1 is an improvement of Theorems A and B. More generally, we deduce the following result and Theorem 1.1 is an immediate consequence of it.

Theorem 1.2 *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$. If*

$$E(0, f(z)) \subseteq E(0, f(z+c)),$$
$$E(\infty, f(z+c)) \subseteq E(\infty, f(z)),$$
$$\overline{E}(1, f(z)) \subseteq \overline{E}(A, f(z+c)),$$

where A is a nonzero constant, then $f(z+c) = Af(z)$ or $f(z+c) = f(z)$. In particular, if $A = 1$, then $f(z+c) = f(z)$.

Remark 1.1 We give the following examples to show that either $f(z+c) = Af(z)$ or $f(z+c) = f(z)$ may occur.

Example 1.1 Set $f(z) = e^{az}$, where a is a nonzero constant such that $e^{ac} = A$. Then, $f(z)$ and $f(z+c)$ satisfy all the conditions of Theorem 1.2. Clearly, $f(z+c) = Af(z)$.

Example 1.2 Set $f(z) = 1 + e^{az}$, where a is a nonzero constant such that $e^{ac} = 1$. Then, $f(z+c) = f(z)$ and 1 is a Picard value of $f(z)$. Obviously, $f(z)$ and $f(z+c)$ satisfy all the conditions of Theorem 1.2, even $A \neq 1$.

Next, we turn attention to the uniqueness problem between $f^{(k)}(z)$ and $f(z+c)$. It seems to us that the first related result was due to Qi-Li-Yang, who in [11] obtained the following theorem.

Theorem C *Let $f(z)$ be a transcendental entire function of finite order and $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share $0, a$ CM, then $f'(z) \equiv f(z+c)$.*

In 2020, Qi-Yang [12] improved Theorem C as follows.

Theorem D *Let $f(z)$ be a non-constant meromorphic function of finite order and $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $f(z+c) = 0 \to f'(z) = 0$ and $f'(z) = \infty \to f(z+c) = \infty$, then $f'(z) \equiv f(z+c)$.*

The notation $f(z) = a \to g(z) = a$ in Theorem D means that any a-point of $f(z)$ with multiplicity p is also an a-point of $g(z)$ with multiplicity at least p, where $a \in \widehat{\mathbb{C}}$. In fact, the notation $f(z) = a \to g(z) = a$ equals that $E(a, f(z)) \subseteq E(a, g(z))$.

Recently, Chen-Xu [4] generalized Theorem D from the first derivative $f'(z)$ to the k-th derivative $f^{(k)}(z)$ as follows.

Theorem E *Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, c be a non-zero finite complex number, and k be a positive integer. If $f^{(k)}(z)$ and $f(z+c)$ share $0, \infty$ CM and 1 IM, then $f^{(k)}(z) \equiv f(z+c)$.*

In this paper, using the idea of partially sharing values, we improve Theorems C–E and obtain the following result with a simple method.

Theorem 1.3 *Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, c be a non-zero finite complex number, and k be a positive integer. If*

$$E(0, f(z+c)) \subseteq E(0, f^{(k)}(z)),$$
$$E(\infty, f^{(k)}(z)) \subseteq E(\infty, f(z+c)),$$
$$\overline{E}(1, f(z+c)) = \overline{E}(1, f^{(k)}(z)),$$

then $f^{(k)}(z) \equiv f(z+c)$.

Before proceeding, we spare the reader for a moment and assume his/her familiarity with the basics of Nevanlinna's theory of meromorphic functions in $\mathbb{C}$ and the usual notations such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ and $S(r, f)$ (see [8, 13]).

2 Some Lemmas

Before the proofs of main theorems, we firstly give the following result, which is a version of the difference analogue of the logarithmic derivative lemma.

Lemma 2.1 (see [7]) *Let $f(z)$ be a non-constant meromorphic function of $\rho_2(f) < 1$, and*

let $c \in \mathbb{C} \backslash \{0\}$. *Then*

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = o\Big(\frac{T(r,f)}{r^{1-\rho_2(f)-\varepsilon}}\Big),$$

outside of an exceptional set of finite logarithmic measure.

Next, we employ a result, which is Lemma 8.3 of Halburd-Korhonen-Tohge in [7].

Lemma 2.2 *Let* $T : [0, +\infty) \to [0, +\infty)$ *be a non-decreasing continuous function and let* $s \in (0, \infty)$. *If the hyper-order of* T *is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log \log T(r)}{\log r} = \varsigma < 1$$

and $\delta \in (0, 1 - \varsigma)$, *then*

$$T(r+s) = T(r) + o\Big(\frac{T(r)}{r^\delta}\Big),$$

where r *runs to infinity outside of a set of finite logarithmic measure.*

3 Proof of Theorems

Proof of Theorem 1.2 The proof is based on some idea of Chen in [1]. On the contrary, we assume that $f(z+c) \not\equiv Af(z)$ and $f(z+c) \not\equiv f(z)$. Below, we will derive a contradiction. We firstly introduce the auxiliary function

$$H(z) = \frac{f(z+c)}{f(z)}. \tag{3.1}$$

By Lemma 2.1, one can easily get $m(r, H) = S(r, f)$. The conditions $E(0, f(z)) \subseteq E(0, f(z+c))$ and $E(\infty, f(z+c)) \subseteq E(\infty, f(z))$ yield that H is an entire function. Thus, $T(r, H) = S(r, f)$.

We claim that $\overline{N}\big(r, \frac{1}{f(z)-1}\big) = S(r, f)$.

If 1 is a Picard value of $f(z)$, then the claim is right. Below, assume that 1 is not a Picard value of $f(z)$ and z_0 is a zero of $f(z) - 1$. Then, $f(z_0 + c) = A$ and $H(z_0) = A$. If $H(z) \equiv A$, then $f(z+c) = Af(z)$, a contradiction. Therefore, $H(z) \not\equiv A$. Thus,

$$\overline{N}\Big(r, \frac{1}{f(z)-1}\Big) \le \overline{N}\Big(r, \frac{1}{H(z)-A}\Big) \le T(r,H) + O(1) = S(r,f), \tag{3.2}$$

which implies that the claim holds.

Let $T(r) = \overline{N}\big(r, \frac{1}{f(z)-1}\big)$. Clearly, $T(r)$ satisfies the conditions of Lemma 2.2. Then, by Lemma 2.2, for $\delta \in (0, 1 - \varsigma)$, we have

$$\overline{N}\Big(r+|c|, \frac{1}{f(z)-1}\Big) = T(r+|c|) = T(r) + o\Big(\frac{T(r)}{r^\delta}\Big) \le T(r) + o\Big(\frac{T(r,f)}{r^\delta}\Big)$$

$$= \overline{N}\Big(r, \frac{1}{f(z)-1}\Big) + S(r,f) = S(r,f).$$

Further,

$$\overline{N}\Big(r, \frac{1}{f(z-c)-1}\Big) \le \overline{N}\Big(r+|c|, \frac{1}{f(z)-1}\Big) = S(r,f). \tag{3.3}$$

The same argument yields that $\overline{N}\left(r, \frac{1}{f(z+c)-1}\right) = S(r, f)$. Therefore, we get

$$\overline{N}\left(r, \frac{1}{f(z-c)-1}\right) = S(r, f), \quad \overline{N}\left(r, \frac{1}{f(z+c)-1}\right) = S(r, f). \tag{3.4}$$

We rewrite (3.1) as

$$f(z+c) - 1 = H(z)\left[f(z) - \frac{1}{H(z)}\right]. \tag{3.5}$$

Then, combining (3.4) and (3.5) yields that

$$\overline{N}\left(r, \frac{1}{f(z) - \dfrac{1}{H(z)}}\right) = \overline{N}\left(r, \frac{1}{f(z+c)-1}\right) + S(r, f) = S(r, f). \tag{3.6}$$

We rewrite (3.1) as $f(z) = H(z-c)f(z-c)$. Then

$$f(z-c) - 1 = \frac{1}{H(z-c)}[f(z) - H(z-c)]. \tag{3.7}$$

The same argument leads to

$$\overline{N}\left(r, \frac{1}{f(z) - H(z-c)}\right) = \overline{N}\left(r, \frac{1}{f(z-c)-1}\right) + S(r, f) = S(r, f). \tag{3.8}$$

Assume that the functions $\frac{1}{H(z)}$, $H(z-c)$ and 1 are distinct with each other. Applying the second main theorem of Nevanlinna to $f(z)$, one gets

$$T(r, f(z)) \le \overline{N}\left(r, \frac{1}{f(z) - H(z-c)}\right) + \overline{N}\left(r, \frac{1}{f(z) - \dfrac{1}{H(z)}}\right)$$

$$+ \overline{N}\left(r, \frac{1}{f(z)-1}\right) + S(r, f) = S(r, f),$$

a contradiction. Therefore, there are at least two functions in the set $\left\{\frac{1}{H(z)}, \ H(z-c), \ 1\right\}$ which are equal identically. If $\frac{1}{H(z)} = 1$ or $H(z-c) = 1$, then $f(z) \equiv f(z+c)$, a contradiction. Therefore, it suffices to handle the case $\frac{1}{H(z)} = H(z-c)$.

We rewrite the above equation as $H(z)H(z+c) = 1$. Then, $H^2(z) = \frac{H(z)}{H(z+c)}$. Suppose that H is not a constant. Then,

$$2T(r, H(z)) = 2m(r, H(z)) = m(r, H^2(z)) = m\left(r, \frac{H(z)}{H(z+c)}\right) = S(r, H),$$

a contradiction. Thus, H is a constant and $H = \pm 1$. So, $f(z+c) = -f(z)$.

If $A = -1$, then $f(z+c) = Af(z)$, a contradiction. Thus, $A \ne -1$. Further, the condition $\overline{E}(1, f(z)) \subseteq \overline{E}(A, f(z+c))$ yields that 1 is a Picard value of $f(z)$. Meanwhile, -1 is a Picard value of $f(z+c)$, and so is $f(z)$. Thus, we can set $\frac{f(z)-1}{f(z)+1} = e^\beta$, where β is an entire function with order $\rho(\beta) < 1$. Suppose that β is a constant. Then, we get $(1 - e^\beta)f = e^\beta + 1$. Clearly, $1 - e^\beta \ne 0$. Then, $f(z) = \frac{e^\beta + 1}{1 - e^\beta}$ is a constant, which is a contradiction. So, β is a non-constant function. We rewrite the above equation as $f(z) = \frac{1+e^\beta}{1-e^\beta}$. Substituting the form of $f(z)$ into the

equation $f(z+c) = -f(z)$ yields that $e^{\beta(z+c)+\beta(z)} = 1$, which leads to $\beta(z+c) + \beta(z) = 2k\pi i$,
k is a fixed integer. Then, $\beta'(z+c) = -\beta'(z)$. Assume that β' is a constant. Then, β is a
linear function, which contradicts the equation $\beta(z+c) + \beta(z) = 2k\pi i$. So, β' is non-constant.
Further

$$\beta'(z+2c) = -\beta'(z+c) = \beta'(z),$$

which implies that β' is a periodic function with period $2c$. It is known that the order of a
non-constant periodic entire function is not less than one. So, $\rho(\beta) = \rho(\beta') \geq 1$, which is a
contradiction.

Therefore, we finish the proof of this theorem.

Proof of Theorem 1.1 We write $f(z+c)$ as

$$f(z+c) = \frac{L(f) + bf(z)}{a}.$$

Then, the conditions $E(0, f(z)) \subseteq E(0, L(f))$ and $\overline{E}(1, f(z)) \subseteq \overline{E}(1, L(f))$ yield that $E(0, f(z))$
$\subseteq E(0, f(z+c))$ and $\overline{E}(1, f) \subseteq \overline{E}\big(\frac{1+b}{a}, f(z+c)\big)$, respectively.

Set

$$f(z) = \frac{af(z+c) - L(f)}{b}.$$

Suppose that z_0 is a pole of $f(z+c)$ with multiplicity m. We assume that z_0 is a pole of
$f(z)$ with multiplicity n (It is pointed out that n maybe zero).

Next, we prove $m \leq n$. On the contrary, assume $n < m$. Then z_0 is a pole of $L(f) =$
$af(z+c) - bf(z)$ with multiplicity m. The condition $E(\infty, L(f)) \subseteq E(\infty, f(z))$ yields that z_0 is
a pole of $f(z)$ with multiplicity at least m, a contradiction. Thus $m \leq n$ and $E(\infty, f(z+c)) \subseteq$
$E(\infty, f(z))$.

The above discussions yield that $f(z)$ and $f(z+c)$ satisfy all the conditions of Theorem 1.2.
So, $f(z+c) = \frac{1+b}{a}f(z)$ or $f(z+c) = f(z)$. The first case implies that $L(f) = f$.

Therefore, the proof of Theorem 1.1 is finished.

Proof of Theorem 1.3 The proof is based on some idea of Chen-Xu in [4]. On the
contrary, we assume that $f(z+c) \not\equiv f^{(k)}(z)$. Below, we will derive a contradiction. We firstly
introduce the auxiliary function

$$H(z) = \frac{f^{(k)}(z)}{f(z+c)}. \tag{3.9}$$

By Lemma 2.1, one can easily get $m(r, H) = S(r, f)$. The conditions $E(0, f(z+c)) \subseteq$
$E(0, f^{(k)}(z))$ and $E(\infty, f^{(k)}(z)) \subseteq E(\infty, f(z+c))$ yield that H is an entire function. Thus,
$T(r, H) = S(r, f)$.

We claim that $\overline{N}\big(r, \frac{1}{f(z+c)-1}\big) = S(r, f)$.

If 1 is a Picard value of $f(z+c)$, then the claim is right. Below, assume that 1 is not a
Picard value of $f(z+c)$ and z_0 is a zero of $f(z+c) - 1$. Then, $f^{(k)}(z_0) = 1$ and $H(z_0) = 1$. If

$H(z) \equiv 1$, then $f^{(k)}(z) = f(z + c)$, a contradiction. So, $H(z) \not\equiv 1$. Thus,

$$\overline{N}\Big(r, \frac{1}{f(z + c) - 1}\Big) \leq \overline{N}\Big(r, \frac{1}{H(z) - 1}\Big) \leq T(r, H) + O(1) = S(r, f), \qquad (3.10)$$

which implies that the claim holds. By the condition $\overline{E}(1, f(z + c)) = \overline{E}(1, f^{(k)}(z))$, we have

$$\overline{N}\Big(r, \frac{1}{f^{(k)}(z) - 1}\Big) = \overline{N}\Big(r, \frac{1}{f(z + c) - 1}\Big) = S(r, f).$$

Suppose that $H(z) \equiv 0$. Then $f^{(k)}(z) \equiv 0$, which implies that f is a polynomial. The condition $E(0, f(z + c)) \subseteq E(0, f^{(k)}(z))$ yields that $\deg f \leq \deg f^{(k)}$, where $\deg g$ denotes the degree of polynomial g. It is impossible. Thus, $H(z) \not\equiv 0$. We rewrite (3.9) as

$$f^{(k)}(z) - 1 = H(z)\Big[f(z + c) - \frac{1}{H(z)}\Big], \qquad (3.11)$$

which implies that

$$\overline{N}\Big(r, \frac{1}{f(z + c) - \dfrac{1}{H(z)}}\Big) = \overline{N}\Big(r, \frac{1}{f^{(k)}(z) - 1}\Big) + S(r, f).$$

By Lemma 2.2, one gets

$$N(r, f(z)) = N(r, f(z + c)) + S(r, f) = N\Big(r, \frac{f^{(k)}(z)}{H(z)}\Big) + S(r, f)$$

$$= N(r, f^{(k)}(z)) + S(r, f) = N(r, f(z)) + k\overline{N}(r, f(z)) + S(r, f),$$

which implies that $\overline{N}(r, f(z)) = S(r, f)$.

Again by Lemma 2.2, one has $\overline{N}(r, f(z + c)) = S(r, f)$. Further, applying the second main theorem of Nevanlinna to $f(z + c)$, one gets

$$T(r, f(z + c)) \leq \overline{N}\Big(r, \frac{1}{f(z + c) - \dfrac{1}{H(z)}}\Big) + \overline{N}\Big(r, \frac{1}{f(z + c) - 1}\Big)$$

$$+ \overline{N}(r, f(z + c)) + S(r, f) = S(r, f),$$

a contradiction. Thus, we get $H(z) \equiv 1$, a contradiction.

Therefore, we finish the proof of this theorem.

4 Discussions

To conclude this paper, we give two natural further studies which are related to the main results. One is to generalize the function $L(f)$ in Theorem 1.1 to higher order linear difference polynomial $L_n(f) = \sum\limits_{j=1}^{n} a_j f(z + jc)$, where a_j's are constants, or more generally small functions with respect to $f(z)$. For special case, $L_n(f)$ becomes to the n-th difference operator $\Delta_c^n f$. The other one is to weaken the condition $E(a, f(z)) \subseteq E(a, L(f))$ in Theorem 1.1 to $E(a, f(z)) \backslash G \subseteq$

$E(a, L(f))$, where a is a constant and G is a set, which is called an exceptional set (see [9]).
One would like the exceptional set G as large as possible.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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