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An L^p-Imprimitivity Theorem for the Group of Integers*

Zhen WANG[1] Sen ZHU[2]

Abstract In this paper, the authors obtain an L^p-version of Green's Imprimitivity Theorem for the group $\mathbb{Z}$ of integers. More precisely, for $X = \mathbb{Z}/n\mathbb{Z}$ and the action α of $\mathbb{Z}$ on X by translation, it is proved that the full L^p-operator crossed product $F^p(\mathbb{Z}, X, \alpha)$ is isometrically isomorphic to the spatial L^p-operator tensor product of M_n^p and the reduced group L^p-operator algebra $F_\lambda^p(\mathbb{Z})$. This solves the L^p-imprimitivity problem raised by Phillips for the group of integers. They also prove that $F^p(\mathbb{Z}, X, \alpha)$ is isometrically isomorphic to $C(S^1, M_n^p)$ precisely when $p = 2$, where S^1 denotes the unit circle in the complex plane $\mathbb{C}$. Moreover, they determine the K-theory groups of $F^p(\mathbb{Z}, X, \alpha)$.

Keywords L^p-operator crossed products, L^p-operator algebras, Gelfand transform, K-theory
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1 Introduction

Green's Imprimitivity Theorem (see [14]) is an important result concerning crossed products of C^*-algebras, which says that $C_0(G/H) \rtimes_{\mathrm{lt}} G$ is isomorphic to $C^*(H) \otimes \mathcal{K}(L^2(G/H, \mu_{G/H}))$, where H is a closed subgroup of a locally compact group G. The result determines which representations of a locally compact group G are induced from its closed subgroups. When $H = \{e\}$ is the trivial subgroup of G, Green's Imprimitivity Theorem turns out to be the Stone-von Neumann Theorem, which is useful in proving the Takai Duality Theorem (see [32, Lemma 7.5]).

The aim of this paper is to consider the Green type imprimitivity for L^p-operator crossed products. To proceed, we give a brief introduction to L^p-operator algebras.

Given $p \in [1, \infty)$, a Banach algebra A is called an L^p-operator algebra if it is isometrically isomorphic to a norm closed subalgebra of the algebra of all bounded operators on some L^p-space. The study concerning L^p-operator algebras dates back to Herz's influential works [16–18] on harmonic analysis of group algebras on L^p-spaces in the 1970s. People's revived interest in L^p-operator algebras is chiefly inspired by the works of Phillips [21–24], which provided new

Manuscript received October 11, 2023. Revised February 24, 2024.
[1]School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China.
E-mail: wangzhen@hznu.edu.cn
[2]Corresponding author. Department of Mathematics, Jilin University, Changchun 130012, China.
E-mail: zhusen@jlu.edu.cn
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ideas, examples and techniques from operator algebras. This leads to progress on some long standing questions. For example, it was shown in [10] that the class of L^p-operator algebras ($p \neq 2$) is not closed under quotients, answering an old open question asked by Le Merdy [19].

Recently there has been growing interest in L^p-operator algebras. People introduce L^p-analogs of classical operator algebras on Hilbert spaces and study whether one can establish L^p-analogues of important results from the theory of C^*-algebras. Along this line, group algebras (see [8, 13]), crossed products (see [11, 15, 31]), groupoid algebras (see [3, 7]), AF algebras (see [6, 28]), graph algebras (see [4]) and the l^p-Toeplitz algebra (see [29]) were studied. Also, some recent works [1, 9, 12] showed that there is a rich general theory waiting to be discovered. The reader is referred to [5] for more historical comments and recent developments in L^p-operator algebras.

L^p-operator crossed products were introduced by Phillips [24] as analogues of C^*-crossed products in the setting of L^p-operator algebras. So it is natural to ask whether one can establish Green's Imprimitivity Theorem for L^p-operator crossed products.

We consider the crossed product of $C(\mathbb{Z}/n\mathbb{Z})$ by the group $\mathbb{Z}$ of integers as a special case of Green's Imprimitivity Theorem. Let $X = \mathbb{Z}/n\mathbb{Z}$ and α be the action of $\mathbb{Z}$ on X by translation. It follows immediately from Green's Imprimitivity Theorem that $C(X) \rtimes_\alpha \mathbb{Z} \cong C(S^1, M_n)$, where $\cong$ denotes isometric isomorphism and S^1 denotes the unit circle in the complex plane $\mathbb{C}$. The reader is referred to Phillips' book [27, Example 10.11] for another direct proof. Furthermore, Phillips raised the following L^p-imprimitivity problem.

Problem 1.1 (see [25, Problem 8.4]) Let $X = \mathbb{Z}/n\mathbb{Z}$ and α be the action of $\mathbb{Z}$ on X by translation, where n is a positive integer. Identify $F^p(\mathbb{Z}, X, \alpha)$ and determine whether it is isometrically isomorphic to $C(S^1, M_n^p)$.

Here $F^p(\mathbb{Z}, X, \alpha)$ is the full L^p-operator crossed product of $C(X)$ by $\mathbb{Z}$ and M_n^p denotes the algebra of bounded linear operators on $l^p(\{1, 2, \cdots, n\})$.

In this paper we solve Phillips' problem by proving a Green type imprimitivity theorem for the L^p-operator crossed product of $C(\mathbb{Z}/n\mathbb{Z})$ by $\mathbb{Z}$. The main result of this paper is the following theorem.

Theorem 1.1 *Let $X = \mathbb{Z}/n\mathbb{Z}$ and α be the action of $\mathbb{Z}$ on X by translation, where n is a positive integer. Then*

(i) *$F^p(\mathbb{Z}, X, \alpha)$ is isometrically isomorphic to $M_n^p \otimes_p F_\lambda^p(\mathbb{Z})$ for $p \in [1, \infty)$;*

(ii) *$F^p(\mathbb{Z}, X, \alpha)$ is isometrically isomorphic to $C(S^1, M_n^p)$ precisely when $p = 2$.*

Here $\otimes_p$ is the spatial L^p-operator tensor product of L^p-operator algebras.

Remark 1.1 (i) Theorem 1.1(i) can be viewed as an L^p-version of Green's Imprimitivity Theorem for the group $\mathbb{Z}$ of integers. Given a locally compact group G and a closed subgroup

H, it is natural to ask whether or not

$$F^p(G, C_0(G/H), \mathrm{lt}) \cong F^p(H) \otimes_p \mathcal{K}(L^p(G/H, \mu_{G/H})) \tag{1.1}$$

for $p \in [1, \infty)$. Gardella and Thiel [8, Theorem 4.3] proved that (1.1) holds when G is a discrete group and $H = \{e\}$ is the trivial subgroup of G. We do not know whether (1.1) holds in the general case.

(ii) For $p \in [1, \infty)\backslash\{2\}$, we remark that $F^p(\mathbb{Z}, X, \alpha)$ can be identified with a dense proper subalgebra of $C(S^1, M_n^p)$. In fact, by [8, Proposition 3.13, Corollary 3.20], the Gelfand transform $\Gamma_p \colon F_\lambda^p(\mathbb{Z}) \to C(S^1)$ is injective with dense range and not surjective. Then $\mathrm{id}_{M_n^p} \otimes \Gamma_p \colon M_n^p \otimes_p F_\lambda^p(\mathbb{Z}) \to M_n^p \otimes_p C(S^1)$ is injective with dense range and not surjective.

It is fundamental to compute the K-theory groups of an L^p-operator crossed product and determine whether they are independent of p (see [25, Problem 11.2]). By [30, Corollary 3.2], we have $K_i(F_\lambda^p(\mathbb{Z})) \cong K_i(C(S^1)) \cong \mathbb{Z}$ for $i = 1, 2$. Hence the following corollary follows immediately from Theorem 1.1(i).

Corollary 1.1 *Let $X = \mathbb{Z}/n\mathbb{Z}$ and α be the action of $\mathbb{Z}$ on X by translation. Then $K_i(F^p(\mathbb{Z}, X, \alpha)) \cong \mathbb{Z}$ for $i = 1, 2$.*

The preceding result shows that the K-theory groups of $F^p(\mathbb{Z}, X, \alpha)$ are independent of the choice of p.

2 Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we first introduce some notations and terminology concerning L^p-operator crossed products.

2.1 Preliminaries

An isometric representation $\pi \colon A \to \mathcal{B}(L^p(X, \mu))$ is said to be σ-finite if μ is σ-finite.

Throughout the following, discrete groups will always be implicitly endowed with the counting measure.

An L^p-operator algebra dynamical system is a triple (G, A, α) consisting of a locally compact group G, an L^p-operator algebra A and a continuous homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of isometric automorphisms of A. A contractive covariant representation of (G, A, α) on an L^p-space E is a pair (π, v) consisting of a nondegenerate contractive homomorphism $\pi \colon A \to \mathcal{B}(E)$ and an isometric group representation $v \colon G \to \mathcal{B}(E)$, satisfying the covariance condition

$$v_t \pi(a) v_{t^{-1}} = \pi(\alpha_t(a))$$

for all $t \in G$ and $a \in A$. A covariant representation (π, v) of (G, A, α) is said to be σ-finite if π is σ-finite.

Let G be a discrete group. Denote by $C_c(G, A, \alpha)$ the vector space of compactly supported continuous functions from G to A, made into an algebra over $\mathbb{C}$ with product given by twisted convolution, that is,

$$(f * g)(t) := \sum_{s \in G} f(s)\alpha_s(g(s^{-1}t))$$

for all $f, g \in C_c(G, A, \alpha)$ and $t \in G$. The integrated form of (π, v) is the nondegenerate contractive homomorphism $\pi \rtimes v\colon C_c(G, A, \alpha) \to \mathcal{B}(E)$ given by

$$(\pi \rtimes v)(f)(\xi) := \sum_{t \in G} \pi(f(t))v_t(\xi)$$

for all $f \in C_c(G, A, \alpha)$ and for all $\xi \in E$.

Denote by $\mathrm{Rep}_p(G, A, \alpha)$ the class of all nondegenerate σ-finite contractive covariant representations of (G, A, α) on L^p-spaces. The full L^p-operator crossed product $F^p(G, A, \alpha)$ is defined as the completion of $C_c(G, A, \alpha)$ in the norm

$$\|f\|_{F^p(G,A,\alpha)} := \sup\{\|(\pi \rtimes v)(f)\|\colon (\pi, v) \in \mathrm{Rep}_p(G, A, \alpha)\}.$$

Let (G, A, α) be an L^p-operator algebra dynamical system. Given a nondegenerate σ-finite contractive representation $\pi_0\colon A \to \mathcal{B}(E_0)$ on an L^p-space E_0, its associated regular covariant representation is the pair $(\pi, \lambda_p^{E_0})$ on $l^p(G) \otimes_p E_0 \cong l^p(G, E_0)$ given by

$$\pi(a)(\xi)(s) := \pi_0\left(\alpha_{s^{-1}}(a)\right)(\xi(s))$$

and

$$\lambda_p^{E_0}(s)(\xi)(t) := \xi(s^{-1}t)$$

for all $a \in A$, $\xi \in l^p(G, E_0)$ and $s, t \in G$. We denote by $\mathrm{RegRep}_p(G, A, \alpha)$ the class consisting of nondegenerate σ-finite contractive regular covariant representations of (G, A, α), which is clearly a subclass of $\mathrm{Rep}_p(G, A, \alpha)$. The reduced L^p-operator crossed product $F^p_\lambda(G, A, \alpha)$ is defined as the completion of $C_c(G, A, \alpha)$ in the norm

$$\|f\|_{F^p_\lambda(G,A,\alpha)} := \sup\{\|(\pi \rtimes v)(f)\|\colon (\pi, v) \in \mathrm{RegRep}_p(G, A, \alpha)\}.$$

By [26, Theorem 7.1], if G is amenable, then the identity map on $C_c(G, A, \alpha)$ can be extended to an isometric isomorphism from $F^p(G, A, \alpha)$ onto $F^p_\lambda(G, A, \alpha)$. If $A = \mathbb{C}$, then it is easy to see that $F^p(G, A, \mathrm{id})$ is the full group L^p operator algebra $F^p(G)$ and $F^p_\lambda(G, A, \mathrm{id})$ is the reduced group L^p operator algebra $F^p_\lambda(G)$, where id is the trivial action of G on $\mathbb{C}$. If $A = C(X)$ for some compact Hausdorff space X, we simply write $F^p(G, X, \alpha)$ for $F^p(G, C(X), \alpha)$ and write $F^p_\lambda(G, X, \alpha)$ for $F^p_\lambda(G, C(X), \alpha)$.

2.2 Proof of Theorem 1.1(i)

We first make some preparation on isometric representations of reduced L^p-operator crossed products.

Given a C^*-algebra A and a discrete group G, it is well known that the reduced crossed product $A \rtimes_{\alpha,r} G$ does not depend on the choice of faithful representation of A, since the C^*-norm on $M_n(A)$ is unique (see [2, Proposition 8.4]). For L^p-operator algebras with unique L^p-operator matrix norms (see [28, Definition 4.1]), the following analogous result holds.

Lemma 2.1 (see [31]) *Let (G, A, α) be an L^p-operator algebra dynamical system, where G is a countable discrete group and A is a separable L^p-operator algebra with unique L^p-operator matrix norms. Let $\pi_0\colon A \to \mathcal{B}(E_0)$ be a nondegenerate σ-finite isometric representation of A on an L^p-space E_0, and $(\pi, \lambda_p^{E_0})$ be its associated regular covariant representation. Then the integrated form $\pi \rtimes \lambda_p^{E_0}$ is an isometric representation of $F_\lambda^p(G, A, \alpha)$.*

Here we recall some basic facts on spatial L^p-operator tensor product of L^p-operator algebras. The reader is referred to [24] for the details of spatial L^p operator tensor products. Note that $M_n^p = \mathcal{B}(l^p(\{1, 2, \cdots, n\}))$ and $F_\lambda^p(\mathbb{Z}) \subset \mathcal{B}(l^p(\mathbb{Z}))$. We denote by $M_n^p \otimes_p F_\lambda^p(\mathbb{Z})$ the Banach subalgebra of $\mathcal{B}(l^p(\{1, 2, \cdots, n\}) \otimes_p l^p(\mathbb{Z}))$ generated by all $T \otimes a$ for $T \in M_n^p$ and $a \in F_\lambda^p(\mathbb{Z})$ (see [24, Example 1.15]).

Let μ denote the normalized arc-length measure on S^1. Then the map sending $f \in C(S^1)$ to its multiplication operator M_f on $L^p(S^1, \mu)$ is an isometric representation of $C(S^1)$ on $L^p(S^1, \mu)$. So we can construct the spatial L^p-operator tensor product $M_n^p \otimes_p C(S^1)$ as above. In fact, the norm on $M_n^p \otimes_p C(S^1)$ is independent of the choice of isometric representation of $C(S^1)$ (see [28, Proposition 4.6]). Moreover, $M_n^p \otimes_p C(S^1)$ is isometrically isomorphic to $C(S^1, M_n^p)$.

Lemma 2.2 *Let* id *be the trivial action of $\mathbb{Z}$ on M_n^p. Then*

$$F^p(\mathbb{Z}, M_n^p, \mathrm{id}) \cong F_\lambda^p(\mathbb{Z}, M_n^p, \mathrm{id}) \cong M_n^p \otimes_p F_\lambda^p(\mathbb{Z}),$$

where $\cong$ denotes isometric isomorphism.

Proof Since the integer group $\mathbb{Z}$ is amenable, $F^p(\mathbb{Z}, M_n^p, \mathrm{id})$ is isometrically isomorphic to $F_\lambda^p(\mathbb{Z}, M_n^p, \mathrm{id})$ (see [26, Theorem 7.1]). Let $E_0 = l^p(\{1, 2, \cdots, n\})$. We consider the canonical isometric representation $\pi_0\colon M_n^p \to \mathcal{B}(E_0)$ and its associated regular covariant representation $(\pi, \lambda_p^{E_0})$ on $l^p(\{1, 2, \cdots, n\}) \otimes_p l^p(\mathbb{Z})$. By Lemma 2.1, the integrated form $\pi \rtimes \lambda_p^{E_0}$ is an isometric representation of $F_\lambda^p(\mathbb{Z}, M_n^p, \mathrm{id})$ on $l^p(\{1, 2, \cdots, n\}) \otimes_p l^p(\mathbb{Z})$. Obviously, $M_n^p \otimes_p F_\lambda^p(\mathbb{Z})$ can be isometrically represented on $l^p(\{1, 2, \cdots, n\}) \otimes_p l^p(\mathbb{Z})$. Since the action of $\mathbb{Z}$ on M_n^p is trivial, it follows that $F_\lambda^p(\mathbb{Z}, M_n^p, \mathrm{id})$ is isometrically isomorphic to $M_n^p \otimes_p F_\lambda^p(\mathbb{Z})$.

Proof of Theorem 1.1(i) By Lemma 2.2, it suffices to prove that $F^p(\mathbb{Z}, X, \alpha)$ is isometrically isomorphic to $F^p(\mathbb{Z}, M_n^p, \mathrm{id})$. We will prove (i) by exploiting the universal property of full L^p-operator crossed products. The proof is divided into three steps.

Step 1 The construction of an isometric isomorphism $\Phi\colon F^p(\mathbb{Z}, X, \alpha) \to F^p(\mathbb{Z}, M_n^p, \mathrm{id})$.

Let $\{\delta_i\}_{i \in \mathbb{Z}}$ be the canonical basis of $l^1(\mathbb{Z})$. For each $T \in M_n^p$ and $i \in \mathbb{Z}$, we denote by $T \otimes \delta_i$

the function from $\mathbb{Z}$ to M_n^p given by

$$T \otimes \delta_i(j) = \begin{cases} T, & j = i, \\ 0, & j \neq i. \end{cases}$$

Then $T \otimes \delta_i \in C_c(\mathbb{Z}, M_n^p, \mathrm{id}) \subset F^p(\mathbb{Z}, M_n^p, \mathrm{id})$.

Let $\{e_{j,k}\}_{1 \leq j,k \leq n}$ be the standard matrix units of M_n^p. We identify $X = \mathbb{Z}/n\mathbb{Z}$ with $\{1, 2, \cdots, n\}$ by taking the residue $\mathrm{mod}\, n$ plus 1. For $m \in \mathbb{Z}$ and $k, r \in \{1, 2, \cdots, n\}$, recall that $\alpha_m(\chi_{\{k\}}) = \chi_{\{r\}}$ whenever $m + k - r \in n\mathbb{Z}$. Here $\chi_{\{k\}}$ denotes the characteristic function of $\{k\}$.

We define a homomorphism $\varphi\colon C(X) \to F^p(\mathbb{Z}, M_n^p, \mathrm{id})$ by

$$\varphi(\chi_{\{k\}}) = e_{k,k} \otimes \delta_0, \quad 1 \leq k \leq n.$$

We define $v \in C_c(\mathbb{Z}, M_n^p, \mathrm{id})$ as

$$v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & \delta_1 \\ \delta_0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \delta_0 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \delta_0 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & \delta_0 & 0 \end{pmatrix}.$$

It is clear that

$$v^{-1} = \begin{pmatrix} 0 & \delta_0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \delta_0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \delta_0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \delta_0 \\ \delta_{-1} & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

Obviously, v and v^{-1} are isometric operators on $l^p(\{1, 2, \cdots, n\}) \otimes_p l^p(\mathbb{Z})$.

For each $m \in \mathbb{Z}$, we define $\nu(m) = v^m$. Then an easy computation shows that

$$\varphi(\alpha_m(f)) = \nu(m)\varphi(f)\nu(-m), \quad \forall f \in C(X),\ m \in \mathbb{Z}.$$

Let I_X be the unit of $C(X)$. For each $m \in \mathbb{Z}$, we define a map w_m from $\mathbb{Z}$ to $C(X)$ by

$$w_m(j) = \begin{cases} I_X, & j = m, \\ 0, & j \neq m. \end{cases}$$

We also write $w_m = I_X \delta_m$. Clearly, $w_m \in C_c(\mathbb{Z}, C(X), \alpha) \subset F^p(\mathbb{Z}, X, \alpha)$. By the universal property of $F^p(\mathbb{Z}, X, \alpha)$ (see [24, Theorem 3.6]), there is a contractive homomorphism $\Phi\colon F^p(\mathbb{Z}, X, \alpha) \to F^p(\mathbb{Z}, M_n^p, \mathrm{id})$ such that $\Phi|_{C(X)} = \varphi$ and $\Phi(w_m) = v^m$.

Step 2 The verification of the density of the range of Φ.

Let 1 be the unit of M_n^p and denote by ranΦ the range of Φ. To prove ranΦ is dense, it suffices to show ranΦ contains $1\otimes\delta_1$, $1\otimes\delta_{-1}$ and $e_{j,k}\otimes\delta_0$ for all $1\le j,k\le n$. Since $\Phi(w_n)=v^n=1\otimes\delta_1$ and $\Phi(w_{-n})=v^{-n}=1\otimes\delta_{-1}$, it follows that $1\otimes\delta_1, 1\otimes\delta_{-1}\in \mathrm{ran}\Phi$. Therefore both

$$e_{j+1,j}\otimes\delta_0=(e_{j+1,j+1}\otimes\delta_0)v(e_{j,j}\otimes\delta_0)$$

and

$$e_{j,j+1}\otimes\delta_0=(e_{j,j}\otimes\delta_0)v^{-1}(e_{j+1,j+1}\otimes\delta_0)$$

belong to ranΦ for $1\le j\le n-1$. It follows readily that $e_{j,k}\otimes\delta_0\in\mathrm{ran}\Phi$ for all $1\le j,k\le n$.

Step 3 The verification of the isometry property of Φ.

We will construct a contractive homomorphism $\Psi\colon F^p(\mathbb{Z},M_n^p,\mathrm{id})\to F^p(\mathbb{Z},X,\alpha)$ such that $\Psi\circ\Phi$ is the identity map on $F^p(\mathbb{Z},X,\alpha)$. Clearly, this implies that Φ is isometric.

Let ψ_0 be the inclusion map of $C(X)$ into $F^p(\mathbb{Z},X,\alpha)$. For each $j\in\{1,2,\cdots,n\}$, we define $f_{j,j}=\psi_0(\chi_{\{j\}})$. For $1\le j,k\le n$, we define

$$f_{j,k}=f_{j,j}w_{j-k}f_{k,k}.$$

Claim 1 $\{f_{j,k}\}_{1\le j,k\le n}$ is a system of matrix units in $F^p(\mathbb{Z},X,\alpha)$.

For $k,j,s,t\in\{1,2,\cdots,n\}$, if $k\ne s$, then

$$f_{j,k}f_{s,t}=f_{j,j}w_{j-k}f_{k,k}f_{s,s}w_{s-t}f_{t,t}=0;$$

if $k=s$, then

$$\begin{aligned}f_{j,k}f_{s,t}&=f_{j,j}w_{j-k}f_{k,k}w_{k-t}f_{t,t}\\&=f_{j,j}f_{j,j}w_{j-k}w_{k-t}f_{t,t}=f_{j,j}w_{j-t}f_{t,t}.\end{aligned}$$

It is easy to check that $\sum\limits_{j=1}^{n}f_{j,j}=I_X\delta_0$, which is the unit of $F^p(\mathbb{Z},X,\alpha)$. Hence $\{f_{j,k}\}_{1\le j,k\le n}$ is a system of matrix units. This proves Claim 1.

Claim 2 Both w_n and w_{-n} commute with $f_{j,k}$ for $1\le j,k\le n$.

An easy computation shows that

$$\begin{aligned}w_nf_{j,k}&=w_nf_{j,j}w_{j-k}f_{k,k}=w_nf_{j,j}w_{-n}w_nw_{j-k}f_{k,k}\\&=f_{j,j}w_{j-k}w_nf_{k,k}=f_{j,j}w_{j-k}f_{k,k}w_n=f_{j,k}w_n.\end{aligned}$$

Similarly, we have $w_{-n}f_{j,k}=f_{j,k}w_{-n}$. This proves Claim 2.

Clearly, by Claim 2, we can construct a homomorphism $\Psi\colon C_c(\mathbb{Z},M_n^p,\mathrm{id})\to F^p(\mathbb{Z},X,\alpha)$ satisfying

$$\Psi(1\otimes\delta_1)=w_n,\quad \Psi(1\otimes\delta_{-1})=w_{-n},\quad \Psi(e_{j,k}\otimes\delta_0)=f_{j,k}$$

for $1 \le j, k \le n$. Since w_n and w_{-n} commute with $f_{j,k}$'s, it follows that

$$\Psi(1 \otimes \delta_i)\Psi(e_{j,k} \otimes \delta_0)\Psi(1 \otimes \delta_{-i}) = \Psi(e_{j,k} \otimes \delta_0)$$

for all $i \in \mathbb{Z}$. By the universal property of $F^p(\mathbb{Z}, M_n^p, \mathrm{id})$ (see [24, Theorem 3.6]), Ψ can be extended to a contractive homomorphism from $F^p(\mathbb{Z}, M_n^p, \mathrm{id})$ to $F^p(\mathbb{Z}, X, \alpha)$, still denoted by Ψ.

Now we shall show that $\Psi \circ \Phi$ is the identity map on $F^p(\mathbb{Z}, X, \alpha)$. It is easy to see that $(\Psi \circ \Phi)(f) = \psi_0(f)$ for all $f \in C(X)$. Since $w_m = (w_1)^m$ for all $m \in \mathbb{Z}$, it remains to show $(\Psi \circ \Phi)(w_1) = w_1$.

Observe that

$$\Phi(w_1) = \Phi(I_X \delta_1) = v = (1 \otimes \delta_1)(e_{1,n} \otimes \delta_0) + \sum_{j=1}^{n-1} e_{j+1,j} \otimes \delta_0.$$

By the definition of $f_{j,k}$'s, one can see $f_{j+1,j+1} = w_1 f_{j,j} w_{-1}$ for $1 \le j \le n-1$ and $f_{1,1} = w_{-(n-1)} f_{n,n} w_{n-1}$. Hence

$$\begin{aligned}(\Psi \circ \Phi)(w_1) = \Psi(v) &= w_n f_{1,n} + \sum_{j=1}^{n-1} f_{j+1,j} \\ &= w_n f_{1,1} w_{-(n-1)} f_{n,n} + \sum_{j=1}^{n-1} f_{j+1,j+1} w_1 f_{j,j} \\ &= w_1 f_{n,n} + \sum_{j=1}^{n-1} w_1 f_{j,j} = w_1.\end{aligned}$$

Hence $\Psi \circ \Phi$ is the identity map on $F^p(\mathbb{Z}, X, \alpha)$. Since Φ and Ψ are contractive homomorphisms, it follows that Φ is an isometric isomorphism from $F^p(\mathbb{Z}, X, \alpha)$ onto $F^p(\mathbb{Z}, M_n^p, \mathrm{id})$.

2.3 Proof of Theorem 1.1(ii)

We first make some preparations.

Definition 2.1 (see [20]) *Let A be a unital Banach algebra in which $\|1\| = 1$. The numerical range $W(a)$ of an element a in A is the set of all numbers $w(a) \in \mathbb{C}$ with w being a continuous linear functional on A with $\|w\| = w(1) = 1$.*

Definition 2.2 (see [28]) *Let A be a unital Banach algebra in which $\|1\| = 1$. An element $a \in A$ is said to be hermitian if $W(a) \subset \mathbb{R}$.*

Definition 2.3 (see [3]) *Let $p \in [1, \infty)$ and A be a unital L^p-operator algebra. Denote by A_h the set of hermitian elements in A. We call the algebra* $\mathrm{core}(A) := A_h + \mathrm{i}A_h$ *the C^*-core of A.*

By [3, Theorem 2.9], the C^*-core of A is the largest unital C^*-subalgebra of A, and it is commutative when $p \in [1, \infty) \setminus \{2\}$. By [3, Proposition 2.13], if $\varphi\colon A \to B$ is an isometric isomorphism between two L^p-operator algebras A and B, then $\varphi\colon \mathrm{core}(A) \to \mathrm{core}(B)$ is a $*$-isomorphism.

We conclude this paper by giving the proof of Theorem 1.1(ii).

Proof of Theorem 1.1(ii) If $p = 2$, then it is clear that $M_n^p \otimes_p F_\lambda^p(\mathbb{Z}) \cong M_n^p \otimes_p C(S^1)$.

Now we assume $p \in [1, \infty) \setminus \{2\}$. Since $\mathbb{Z}$ is amenable, by [3, Corollary 2.20], it follows that $\mathrm{core}(F^p(\mathbb{Z}, X, \alpha)) = \mathrm{core}(F_\lambda^p(\mathbb{Z}, X, \alpha)) = C(X)$. By [3, Example 2.11], we have $\mathrm{core}(C(S^1, M_n^p)) = C(S^1, C(X))$. Since it is clear that $C(X)$ is not isomorphic to $C(S^1, C(X))$, we conclude that $F^p(\mathbb{Z}, X, \alpha)$ is not isometrically isomorphic to $C(S^1, M_n^p)$. This completes the proof.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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