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Order Preserving and Order Reversing Operators on the Class of L^0-Convex Functions in Complete Random Normed Modules*

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Abstract Based on both the fundamental theorem of affine geometry in regular L^0-modules and the recent progress in random convex analysis, this paper characterizes the stable and fully order preserving and order reversing operators acting on the class of proper lower semicontinuous L^0-convex functions in complete random normed modules.

Keywords Order preserving operators, Order reversing operators, Fenchel conjugation, L^0-convex functions, Complete random normed modules, Stability
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1 Introduction and Main Results

In 2009, Artstein-Avidan and Milman [1] established the characterizations of the fully order preserving and the fully order reversing operators acting on the class of lower semicontinuous convex functions defined on the n-dimensional Euclidean space $\mathbb{R}^n$. In 2015, Iusem, Reem and Svaiter [19] successfully generalized the work of [1] to Banach spaces in a nontrivial manner. In 2021, Cheng and Luo [2] solved several difficult problems presented in [19] on fully order-preserving and order-reversing operators in the case of Banach spaces. The purpose of this paper is to generalize the work of [19] to complete random normed modules, which are a random generalization of Banach spaces. Random functional analysis is just based on the idea of randomizing the traditional space theory of functional analysis and has been deeply and widely developed (cf. [8, 13]), in particular random functional analysis has also been successfully applied to dynamic risk measures (cf. [9]) and nonsmooth differential geometry (cf. [5, 20–21]). We will generalize the main results of [2] to random normed modules in the forthcoming

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work since this generalization will unavoidably involve the complicated theory of weakly stable compactness which is under development.

To introduce the main results of this paper, let us first recall some usual terminologies and notations frequently used in random functional analysis.

Throughout this paper, $(\Omega, \mathcal{F}, P)$ always denotes a given probability space, $\mathbb{K}$ is the scalar field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, $L^0(\mathcal{F}, \mathbb{K})$ is the algebra of equivalence classes of $\mathbb{K}$-valued random variables on $(\Omega, \mathcal{F}, P)$ (as usual, two random variables that are equal almost surely (a.s. for short) are said to be equivalent), $L^0(\mathcal{F}) := L^0(\mathcal{F}, \mathbb{R})$, and $\overline{L}^0(\mathcal{F})$ is the set of equivalence classes of extended real-valued random variables on $(\Omega, \mathcal{F}, P)$.

$\overline{L}^0(\mathcal{F})$ is usually partially ordered via $\xi \leq \eta$ if $\xi^0(\omega) \leq \eta^0(\omega)$ for almost all ω in Ω, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. The nice properties of $(\overline{L}^0(\mathcal{F}), \leq)$ are surveyed as follows.

Proposition 1.1 (cf. [3]) *$(\overline{L}^0(\mathcal{F}), \leq)$ is a complete lattice. $\vee H$ and $\wedge H$ stand for the supremum and infimum of H, respectively, for any nonempty subset H of $\overline{L}^0(\mathcal{F})$. Specially, the following statements are true*:

(1) *There are two sequences $\{a_n, n \in \mathbb{N}\}$ and $\{b_n, n \in \mathbb{N}\}$ in H for any nonempty subset H of $\overline{L}^0(\mathcal{F})$ such that $\bigvee_{n\geq 1} a_n = \vee H$ and $\bigwedge_{n\geq 1} b_n = \wedge H$.*

(2) *If H in* (1) *is directed upwards* (*downwards*), *i.e., there exists $h_3 \in H$ for any two elements h_1 and h_2 in H such that $h_1 \vee h_2 \leq h_3$ ($h_3 \leq h_1 \wedge h_2$), then $\{a_n, n \in \mathbb{N}\}$* (*correspondingly, $\{b_n, n \in \mathbb{N}\}$*) *can be chosen as nondecreasing* (*nonincreasing*).

(3) *$(L^0(\mathcal{F}), \leq)$ is a Dedekind complete lattice, i.e., every nonempty subset with an upper bound has a supremum.*

As usual, for any two elements ξ and η in $\overline{L}^0(\mathcal{F})$, $\xi < \eta$ means $\xi \leq \eta$ but $\xi \neq \eta$, whereas, for any $A \in \mathcal{F}$ with $P(A) > 0$, $\xi < \eta$ on A means $\xi^0(\omega) < \eta^0(\omega)$ for almost all ω in A, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively.

In this paper, the following two notations are always employed:

$$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) : \xi \geq 0\},$$
$$L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) : \xi > 0 \text{ on } \Omega\}.$$

Definition 1.1 (cf. [8]) *Let E be a left module over the algebra $L^0(\mathcal{F}, \mathbb{K})$* (*$L^0(\mathcal{F}, \mathbb{K})$-module for short*). *A mapping $\|\cdot\|$ from E to $L^0_+(\mathcal{F})$ is an L^0-seminorm on E if the following conditions are satisfied*:

(1) *$\|\xi x\| = |\xi|\|x\|$ for all $\xi \in L^0(\mathcal{F}, \mathbb{K})$ and $x \in E$;*

(2) *$\|x + y\| \leq \|x\| + \|y\|$ for all x and y in E.*

If, in addition, $\|x\| = 0$ implies $x = \theta$ (*the null of E*), *then the L^0-seminorm is an L^0-norm on E. In this case, $(E, \|\cdot\|)$ is a random normed module* (*RN module for short*) *over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. If a family $\mathcal{P}$ of L^0-seminorms on E satisfies that $\vee\{\|x\| : \|\cdot\| \in \mathcal{P}\} = 0$ implies $x = \theta$, then the ordered pair $(E, \mathcal{P})$ is a random locally convex module* (*RLC module for short*) *over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$.*

When $(\Omega, \mathcal{F}, P)$ is trivial, namely $\mathcal{F} = \{\Omega, \emptyset\}$, an RN or RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ just reduces to an ordinary normed or locally convex space over $\mathbb{K}$, and thus an RN or RLC module is a random generalization of an ordinary normed or locally convex space. The simplest RN module is $L^0(\mathcal{F}, \mathbb{K})$ endowed with the L^0-norm $|\cdot|$ (namely the absolute value mapping). It is well known that $L^0(\mathcal{F}, \mathbb{K})$ is a metrizable linear topological space (in fact, also a topological algebra) in the topology of convergence in probability, and the topology of convergence in probability is often nonlocally convex, for example, $L^0(\mathcal{F}, \mathbb{K})$ has the trivial topological dual when $\mathcal{F}$ is atomless. The (ε, λ)-topology defined in Proposition 1.2 below is exactly a natural generalization of the topology of convergence in probability.

Proposition 1.2 (cf. [6, 8, 11]) *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. Denote by $\mathcal{P}_f$ the family of nonempty finite subsets of $\mathcal{P}$, for each $Q \in \mathcal{P}_f$, the L^0-seminorm $\|\cdot\|_Q$ is defined by $\|x\|_Q = \vee\{\|x\| : \|\cdot\| \in Q\}$ for any $x \in E$. For any given positive numbers ε and λ with $0 < \lambda < 1$, and any $Q \in \mathcal{P}_f$, let $\mathcal{U}_\theta(Q, \varepsilon, \lambda) = \{x \in E : P\{\omega \in \Omega \mid \|x\|_Q(\omega) < \varepsilon\} > 1-\lambda\}$. Then $\{\mathcal{U}_\theta(Q, \varepsilon, \lambda) : Q \in \mathcal{P}_f, \varepsilon > 0, 0 < \lambda < 1\}$ forms a local base of some Hausdorff linear topology for E, called the (ε, λ)-topology induced by $\mathcal{P}$, denoted by $\mathcal{T}_{\varepsilon,\lambda}$. Furthermore, the following statements hold*:

(1) *$(E, \mathcal{T}_{\varepsilon,\lambda})$ is a Hausdorff topological module over the topological algebra $L^0(\mathcal{F}, \mathbb{K})$.*

(2) *$\mathcal{T}_{\varepsilon,\lambda}$ is metrizable when $(E, \mathcal{P})$ is an RN module $(E, \|\cdot\|)$, namely $\mathcal{P}$ reduces to a single L^0-norm $\|\cdot\|$.*

In this paper, we always assume that an RLC module is endowed with its (ε, λ)-topology. It is easy to see that in an RLC module $(E, \mathcal{P})$ over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, a net $\{x_\alpha, \alpha \in \Gamma\}$ converges in the (ε, λ)-topology to x iff $\{\|x_\alpha - x\|, \alpha \in \Gamma\}$ converges in probability to 0 for each $\|\cdot\| \in \mathcal{P}$. In particular, in an RN module $(E, \|\cdot\|)$ a sequence $\{x_n, n \in \mathbb{N}\}$ converges in the (ε, λ)-topology to x iff $\{\|x_n - x\|, n \in \mathbb{N}\}$ converges in probability to 0.

The (ε, λ)-topology is essentially a kind of nonlocally convex topology. The theory of traditional conjugate spaces universally fails for the development of random locally convex modules. The idea of random conjugate spaces is introduced and has been systematically developed in order to overcome the above-stated obstacle, see [8] for details. Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. Then the set $(E, \mathcal{P})^*$ of continuous module homomorphisms from $(E, \mathcal{P})$ to $L^0(\mathcal{F}, \mathbb{K})$ is the random conjugate space of $(E, \mathcal{P})$. $(E, \mathcal{P})^*$ forms an $L^0(\mathcal{F}, \mathbb{K})$-module in a natural way, denoted by E^* if no confusion occurs.

Let $L(E, E_1)$ be the set of continuous module homomorphisms from an RN module $(E, \|\cdot\|)$ to another RN module $(E_1, \|\cdot\|_1)$, where $(E, \|\cdot\|)$ and $(E_1, \|\cdot\|_1)$ are both over the scalar field $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. Then it is well known from [10] that $T \in L(E, E_1)$ if and only if T is an a.s. bounded linear operator, namely T is linear and there exists $\xi \in L^0_+(\mathcal{F})$ such that $\|T(x)\|_1 \le \xi\|x\|$ for all $x \in E$, and $\|T\| := \wedge\{\xi \in L^0_+(\mathcal{F}) \mid \|T(x)\|_1 \le \xi\|x\| \text{ for all } x \in E\}$ is the L^0-norm of T. It is also known from [10] that $(L(E, E_1), \|\cdot\|)$ forms an RN module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, which is still complete if $(E_1, \|\cdot\|_1)$ is complete. In particular, one can see by taking $E_1 = L^0(\mathcal{F}, \mathbb{K})$ that E^* is complete for any RN module E. For any given $T \in L(E, E_1)$, $T^* : E_1^* \to E^*$ defined by $T^*(f)(x) = f(T(x))$ for any $f \in E_1^*$ and $x \in E$, is the random conjugate operator of T. It is also known from [10] that $T^* \in L(E_1^*, E^*)$ and $\|T^*\| = \|T\|$.

Random convex analysis has been developed to provide a useful analytical tool for the study of conditional convex risk measures, see [8–9, 15–18] and the references therein.

In the sequel of this paper, all the RLC modules occurring in this paper are assumed to be over the real field $\mathbb{R}$ with base $(\Omega, \mathcal{F}, P)$.

Definition 1.2 (cf. [9, 15–18]) *Let $(E, \mathcal{P})$ be an RLC module with base $(\Omega, \mathcal{F}, P)$. A mapping $f : E \to \overline{L}^0(\mathcal{F})$ is a proper lower semicontinuous L^0-convex function if the following are satisfied:*

(1) $f(x) > -\infty$ *on* Ω *for all* $x \in E$;

(2) $\mathrm{dom}(f) := \{x \in E \mid f(x) < +\infty \text{ on } \Omega\}$ *is nonempty*;

(3) $f(\xi x + (1-\xi)y) \le \xi f(x) + (1-\xi) f(y)$ *for any* $x, y \in E$ *and* $\xi \in L^0_+(\mathcal{F})$ *with* $0 \le \xi \le 1$ (*here, we make the convention that* $0 \cdot (+\infty) = 0$);

(4) $\mathrm{epi}(f) := \{(x, r) \in E \times L^0(\mathcal{F}) \mid f(x) \le r\}$ *is closed, where* $E \times L^0(\mathcal{F})$ *naturally forms an RLC module with the family* $\mathcal{P}'$ *of* L^0*-seminorms consisting of* $\{\|\cdot\| + |\cdot| : \|\cdot\| \in \mathcal{P}\}$ $((\|\cdot\| + |\cdot|)(x, r) := \|x\| + |r|$ *for any* $(x, r) \in E \times L^0(\mathcal{F})$ *and any* $\|\cdot\| \in \mathcal{P})$.

Let $(E, \mathcal{P})$ be an RLC module with base $(\Omega, \mathcal{F}, P)$. For any $x \in E$, define the L^0-seminorm $|\langle \cdot, x\rangle| : E^* \to L^0_+(\mathcal{F})$ by $|\langle \cdot, x\rangle|(f) = |f(x)|$ for any $f \in E^*$. Then $\sigma(E^*, E) := \{|\langle \cdot, x\rangle| : x \in E\}$ is a family of L^0-seminorms such that $(E^*, \sigma(E^*, E))$ is an RLC module with base $(\Omega, \mathcal{F}, P)$, whose (ε, λ)-topology is the random w^*-topology for E^*. A proper lower semicontinuous L^0-convex function from $(E^*, \sigma(E^*, E))$ to $\overline{L}^0(\mathcal{F})$ is a proper random w^*-lower semicontinuous L^0-convex function on E^*. If $f : E \to \overline{L}^0(\mathcal{F})$ is a proper lower semicontinuous L^0-convex function, as usual, we can define the random (Fenchel) conjugate f^* of f by $f^*(u) = \vee\{u(x) - f(x) : x \in E\}$ for any $u \in E^*$. Then f^* is just a proper random w^*-lower semicontinuous L^0-convex function on E^*. Similarly, $f^{**}_E : E \to \overline{L}^0(\mathcal{F})$ is defined by $f^{**}_E(x) = \vee\{u(x) - f^*(u) \mid u \in E^*\}$ for any $x \in E$. Then one can have $f^{**}_E = f$, namely the random Fenchel-Moreau duality theorem, which was established in [18]. When E is an RN module with base $(\Omega, \mathcal{F}, P)$, we denote the random conjugate space of E^* by E^{**}, and further define $f^{**} : E^{**} \to \overline{L}^0(\mathcal{F})$ by $f^{**}(g) = \vee\{g(u) - f^*(u) : u \in E^*\}$ for any $g \in E^{**}$. Then it is clear that $f^{**}_E = f^{**}|_E$ (the restriction of f^{**} to E).

In the sequel of this section, $(E, \|\cdot\|)$ always denotes a complete RN module with base $(\Omega, \mathcal{F}, P)$ such that $(E, \|\cdot\|)$ has full support (see the related discussion in Section 2), $\mathscr{C}(E)$ is the set of proper lower semicontinuous L^0-convex functions on E, and $\mathscr{C}_{w^*}(E^*)$ is the set of proper random w^*-lower semicontinuous L^0-convex functions on E^*. We observe that $\mathscr{C}(E)$ is stable in the sense that $\widetilde{I}_A f_1 + \widetilde{I}_{A^c} f_2 \in \mathscr{C}(E)$ for any $A \in \mathcal{F}, f_1$ and f_2 in $\mathscr{C}(E)$, where $A^c = \Omega \setminus A$, I_A stands for the characteristic function of A and $\widetilde{I}_A$ is the equivalence class of I_A, and $\widetilde{I}_A f_1 + \widetilde{I}_{A^c} f_2 : E \to \overline{L}^0(\mathcal{F})$ is defined by $[\widetilde{I}_A f_1 + \widetilde{I}_{A^c} f_2](x) = \widetilde{I}_A f_1(x) + \widetilde{I}_{A^c} f_2(x), \forall x \in E$. Similarly, one can understand that $\mathscr{C}_{w^*}(E^*)$ is stable.

We can now define the order preserving (reversing) operators acting on $\mathscr{C}(E)$ and then state the main results of this paper. As usual, for any f_1 and f_2 in $\mathscr{C}(E)$ (g_1 and g_2 in $\mathscr{C}_{w^*}(E^*)$), $f_1 \le f_2$ means $f_1(x) \le f_2(x)$ for any $x \in E$ ($g_1 \le g_2$ means $g_1(u) \le g_2(u)$ for any $u \in E^*$).

Definition 1.3 *An operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is*

(1) *order preserving if $f \le g$ implies $T(f) \le T(g)$;*

(2) *fully order preserving if $f \le g$ if and only if $T(f) \le T(g)$, and additionally T is onto;*

(3) *stable if $T(\widetilde{I}_A f + \widetilde{I}_{A^c} g) = \widetilde{I}_A T(f) + \widetilde{I}_{A^c} T(g)$ for any $A \in \mathcal{F}$ and any f and g in $\mathscr{C}(E)$;*

(4) *an involution if $T(T(f)) = f$ for any $f \in \mathscr{C}(E)$.*

Our first main result is as follows.

Theorem 1.1 *An operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is stable and fully order preserving if and only if there exist $c \in E, w \in E^*, \beta \in L^0(\mathcal{F}), \tau \in L^0_{++}(\mathcal{F})$ and a continuous module automorphism H of E such that $T(f)(x) = \tau f(H(x) + c) + w(x) + \beta$ for any $x \in E$ and $f \in \mathscr{C}(E)$.*

Corollary 1.1 *An operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is a stable order preserving involution if and only if there exist $c \in E, w \in E^*$ and a continuous module automorphism H of E, satisfying $H^2 = ID_E, c \in \mathrm{Ker}(H + ID_E), w \in \mathrm{Ker}(H^* + ID_{E^*})$ such that*

$$T(f)(x) = f(H(x) + c) + w(x) - \frac{1}{2} w(c)$$

for all $x \in E$, where ID_E and ID_{E^*} stand for the identity operators on E and E^*, respectively.

Definition 1.4 *An operator $S : \mathscr{C}(E) \to \mathscr{C}_{w^*}(E^*)$ is*

(1) *order reversing if $f \le g$ implies $S(f) \ge S(g)$;*

(2) *fully order reversing whenever $f \le g$ if and only if $S(f) \ge S(g)$, and additionally S is onto;*

(3) *stable if $S(\widetilde{I}_A f + \widetilde{I}_{A^c} g) = \widetilde{I}_A S(f) + \widetilde{I}_{A^c} S(g)$ for any $A \in \mathcal{F}$ and f and g in $\mathscr{C}(E)$.*

Our second main result is as follows.

Theorem 1.2 *An operator $S : \mathscr{C}(E) \to \mathscr{C}_{w^*}(E^*)$ is stable and fully order reversing if and only if there exist $v \in E^*, y \in E, \rho \in L^0(\mathcal{F}), \tau \in L^0_{++}(\mathcal{F})$ and a continuous module automorphism H of E such that $S(f)(u) = \tau f^*(H^*(u) + v) + u(y) + \rho$ for all $f \in \mathscr{C}(E)$ and all $u \in E^*$.*

The main results of this paper, whether from their formulations or from the ideas of their proofs, are considerably motivated by the work in [1, 19], but we would like to mention here the main differences between the work in [1, 19] and our work.

The foundation for the work in [1, 19] is different from that for our work. The work in [1, 19] depends on classical convex analysis, which is based on the theory of conjugate spaces for locally convex spaces, and the fundamental theorem of affine geometry in linear spaces, whereas our work depends on random convex analysis, which is based on the theory of random conjugate spaces for random locally convex modules, and has just been developed for the last ten years (cf. [15, 17–18]), and the fundamental theorem of affine geometry in regular L^0-modules, which was just established in [23].

We mention now the second difference between our approach and the one in [19]. Let $\mathscr{A}(E)$ be the L^0-module of continuous L^0-affine functions on a complete RN module E with base $(\Omega, \mathcal{F}, P)$. As in [19] for a fully order preserving operator T from $\mathscr{C}(X)$ to $\mathscr{C}(X)$ when X

is a Banach space, we will first prove that for a stable and fully order preserving operator T from $\mathscr{C}(E)$ to $\mathscr{C}(E)$ in the setting of Theorem 1.1, $T|_{\mathscr{A}(E)}$ is a stable bijection from $\mathscr{A}(E)$ onto $\mathscr{A}(E)$, and hence induces a stable bijection $\widehat{T}$ from $E^* \times L^0(\mathcal{F})$ onto $E^* \times L^0(\mathcal{F})$ by identifying $\mathscr{A}(E)$ with $E^* \times L^0(\mathcal{F})$. Then we will directly prove that $\widehat{T}$ is L^0-affine as a whole by applying the fundamental theorem of affine geometry in regular L^0-modules established in [23], not so in coordinate component manner as in [19]. Our approach has an advantage that many complicated discussions on the rank of L^0-modules can be avoided.

The third difference is that the operators in Theorem 1.1, Corollary 1.1 and Theorem 1.2 are additionally assumed to be stable. The additional assumption is natural and necessary from the process of establishing the fundamental theorem of affine geometry in regular L^0-modules in [23]. When $(\Omega, \mathcal{F}, P)$ is trivial, namely $\mathcal{F} = \{\Omega, \emptyset\}$, a complete RN module E with base $(\Omega, \mathcal{F}, P)$ just reduces to a Banach space, in which case an order preserving (or reversing) operator (in fact any operator) is automatically stable, that is to say, the stability assumption automatically disappears. On the other hand, when $(\Omega, \mathcal{F}, P)$ is a general probability space, $(L^0(\mathcal{F}), \leq)$ is a partially ordered set rather than a totally ordered set like $\mathbb{R}$, which also makes the proofs of the main results of this paper more involved than those of [19].

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries mainly on L^0-affine functions. In Section 3, we treat the stable order preserving operators and prove Theorem 1.1 and Corollary 1.1. Finally, in Section 4 we treat the stable order reversing operator and prove Theorem 1.2.

2 Preliminaries

For each $A \in \mathcal{F}$, $\widetilde{A} = \{B \in \mathcal{F} : P(B \triangle A) = 0\}$ denotes the equivalence class of A, and we also use $I_{\widetilde{A}}$ to stand for $\widetilde{I}_A$. Given $\xi \in L^0(\mathcal{F})$, the notation $[\xi > 0]$ stands for the equivalence class of A, where $A = \{\omega \in \Omega \mid \xi^0(\omega) > 0\}$ and ξ^0 is an arbitrarily chosen representative of ξ. Thus $I_{[\xi>0]} = \widetilde{I}_A$. Some other notation like $[\xi = 0]$ and so on are understood in a similar way.

Besides, we denote $\mathcal{F}_+ = \{A \in \mathcal{F} : P(A) > 0\}$.

In our main results, Theorem 1.1, Corollary 1.1 and Theorem 1.2, the RN module is required to have full support. Now we recall the notion of support for an RN module. Let $(E, \|\cdot\|)$ be an RN module with base $(\Omega, \mathcal{F}, P)$. Denote $\xi = \vee\{\|x\| : x \in E\}$ (generally $\xi \in \overline{L}^0(\mathcal{F})$). Clearly $[\xi > 0] = [\xi = +\infty]$. Any representative A of $[\xi > 0]$ is called a support of $(E, \|\cdot\|)$. If Ω is a support of $(E, \|\cdot\|)$, then $(E, \|\cdot\|)$ is called having full support. Except for the trivial case $E = \{\theta\}$, a support A of $(E, \|\cdot\|)$ must have positive probability. If $(E, \|\cdot\|)$ does not have full support, then A^c has positive probability and $\widetilde{I}_{A^c} x = \theta$ for every $x \in E$. Therefore A^c is indeed redundant for $(E, \|\cdot\|)$, in which case we define a probability measure $P_A : \mathcal{F} \cap A \to [0, 1]$ by $P_A(B \cap A) = \frac{P(B \cap A)}{P(A)}, \forall B \in \mathcal{F}$. Since $\|x\| = \widetilde{I}_A \|x\|$ for every $x \in E$, we can always take $(E, \|\cdot\|)$ as an RN module with base the smaller probability space $(A, \mathcal{F} \cap A, P_A)$, in which case $(E, \|\cdot\|)$ has full support. We see that the requirement of full support is not an excessive restriction.

In the sequel of this paper, an element x of an RN module $(E, \|\cdot\|)$ is said to have full support if $\|x\| \neq 0$ on Ω, in other words, $\|x\| \in L^0_{++}(\mathcal{F})$. For any $x \in E$ and $u \in E^*$, we write

$\langle u, x\rangle$ for $u(x)$. As usual, $E^{**} = (E^*)^*$, and we can regard E as a subset of E^{**} by the natural embedding.

Proposition 2.1 *Let $(E, \|\cdot\|)$ be an RN module and E^* its random conjugate space. Suppose that $(E, \|\cdot\|)$ is complete and has full support. Then the following statements are true:*

(1) *There exists $x_0 \in E$ such that $\|x_0\| = 1$, in addition, there exists $u_0 \in E^*$ such that $\langle u_0, x_0\rangle = 1$ and $\|u_0\| = 1$.*

(2) *If $u \in E^*$ has full support, then for any $\xi \in L^0(\mathcal{F})$, there exists $x_0 \in E$ such that $\langle u, x_0\rangle = \xi$.*

Proof (1) The first part is shown in [7, Lemma 3.1]. The second part is given in [6, Corollary 3.2], see also [7, Remark 2.2].

(2) Consider the family $\{|\langle u, x\rangle| : x \in E, \|x\| \le 1\}$. Using the same technique as in the proof of [7, Lemma 3.1], there exists $x_u \in E$ such that $\langle u, x_u\rangle = \vee\{|\langle u, x\rangle| : x \in E, \|x\| \le 1\} = \|u\|$. If u has full support, then $\|u\|$ is invertible in $L^0(\mathcal{F})$. As a result $x_0 = \xi\|u\|^{-1}x_u$ satisfies $\langle u, x_0\rangle = \xi$.

Now we discuss the indicator function.

For any $x_0 \in E$, we define the indicator function $\delta_{x_0} : E \to \overline{L}^0(\mathcal{F})$ by

$$\delta_{x_0}(x) = 0 \cdot I_{[\|x-x_0\|=0]} + \infty \cdot I_{[\|x-x_0\|\neq 0]}, \quad \forall x \in E.$$

Note that $\delta_{x_0}(x_0) = 0$. However if $x \neq x_0$, $\delta_{x_0}(x)$ may not be equal to $+\infty$, since it is possible that $\|x - x_0\| = 0$ on some $A \in \mathcal{F}_+$ so that $\delta_{x_0}(x) = 0$ on A.

By simple analysis and calculation, we can obtain the following result.

Lemma 2.1 *For any $x_0 \in E$, the indicator function δ_{x_0} is a member of $\mathscr{C}(E)$ with* $\mathrm{dom}(\delta_{x_0}) = \{x_0\}$. *The random conjugate $\delta^*_{x_0} : E^* \to \overline{L}^0(\mathcal{F})$ is given by $\delta^*_{x_0}(u) = \langle u, x_0\rangle$, $\forall u \in E^*$, and the bi-random conjugate $\delta^{**}_{x_0} : E^{**} \to \overline{L}^0(\mathcal{F})$ is given by $\delta^{**}_{x_0}(x^{**}) = 0 \cdot I_{[\|x^{**}-x_0\|=0]} + \infty \cdot I_{[\|x^{**}-x_0\|\neq 0]} = \delta_{x_0}(x^{**}), \forall x^{**} \in E^{**}$.*

Now, we turn to L^0-affine functions.

A function $h : E \to L^0(\mathcal{F})$ is called an L^0-affine function if there exists a pair $(u, \alpha) \in E^* \times L^0(\mathcal{F})$ such that $h(x) = \langle u, x\rangle + \alpha, \forall x \in E$. From now on, such an L^0-affine function h will be written as $h_{u,\alpha}$. Denote by $\mathscr{A}(E)$ the set of L^0-affine functions on E. Obviously, $\mathscr{A}(E)$ is a subset of $\mathscr{C}(E)$.

Given a family of functions $f_i : E \to \overline{L}^0(\mathcal{F})$, $i \in I$, define $\bigvee_{i\in I} f_i : E \to \overline{L}^0(\mathcal{F})$ by $\big[\bigvee_{i\in I} f_i\big](x) = \vee\{f_i(x) : i \in I\}, \forall x \in E$. Specially, we write $f \vee g$ for $\vee\{f, g\}$.

For $f \in \mathscr{C}(E)$, define $\mathscr{A}(f) \subset \mathscr{A}(E)$ as $\mathscr{A}(f) = \{h \in \mathscr{A}(E) : h \le f\}$.

We will present a series of useful properties for L^0-affine functions in the sequel of this section.

Corollary 2.1 below is indeed obtained in the proof of random Fenchel-Moreau duality theorem (namely, [18, Theorem 5.1], see also [15]). Now we deduce it from the random Fenchel-Moreau duality theorem.

Fix $f \in \mathscr{C}(E)$. For any $u \in E^*$, it is easy to verify that

$$\vee\{\langle u, x\rangle - f(x) : x \in E\} = \vee\{\langle u, x\rangle - f(x) : x \in \mathrm{dom}(f)\},$$

and therefore

$$f^*(u) = \vee\{\langle u, x\rangle - f(x) : x \in \mathrm{dom}(f)\}, \quad \forall u \in E^*.$$

Similarly, we have

$$f_E^{**}(x) = \vee\{\langle u, x\rangle - f^*(u) : u \in \mathrm{dom}(f^*)\}, \quad \forall x \in E.$$

By the random Fenchel-Moreau duality theorem, namely $f_E^{**} = f$, we can see the following result.

Corollary 2.1 *For any $f \in \mathscr{C}(E)$, $\mathscr{A}(f)$ is nonempty and $f = \vee\{h : h \in \mathscr{A}(f)\}$.*

Using L^0-affine functions, we can give a useful characterization for the bi-random conjugate f^{**}, which is an analogue of [19, Proposition 10(ii)].

Proposition 2.2 *For any $f \in \mathscr{C}(E)$ and every $a \in E^{**}$, $f^{**}(a) = \vee\{\langle a, u\rangle + \alpha : h_{u,\alpha} \in \mathscr{A}(f)\}$.*

Proof By definition, $f^{**}(a) = \vee\{\langle a, u\rangle - f^*(u) : u \in E^*\} = \vee\{\langle a, u\rangle - f^*(u) : u \in \mathrm{dom}(f^*)\}$, also $f^*(u) = \vee\{\langle u, x\rangle - f(x) : x \in E\}$, so that for each $u \in \mathrm{dom}(f^*)$, $h_{u,-f^*(u)}(x) = \langle u, x\rangle - f^*(u)$ for all $x \in E$, and hence $h_{u,-f^*(u)}$ belongs to $\mathscr{A}(f)$ for all $u \in \mathrm{dom}(f^*)$. Next, we prove that if $h_{u,\alpha}$ belongs to $\mathscr{A}(f)$ then $u \in \mathrm{dom}(f^*)$ and $\alpha \leq -f^*(u)$. Indeed, if $h_{u,\alpha} \in \mathscr{A}(f)$, then for all $x \in E$, $\langle u, x\rangle + \alpha \leq f(x)$. Therefore $\alpha \leq \wedge\{f(x) - \langle u, x\rangle : x \in E\} \in L^0(\mathcal{F})$. Then $f^*(u) = \vee\{\langle u, x\rangle - f(x) : x \in E\} = -\wedge\{f(x) - \langle u, x\rangle : x \in E\} \in L^0(\mathcal{F})$ and $-f^*(u) \geq \alpha$. We conclude that $\vee\{\langle a, u\rangle + \alpha : h_{u,\alpha} \in \mathscr{A}(f)\} = \vee\{\langle a, u\rangle - f^*(u) : u \in \mathrm{dom}(f^*)\} = f^{**}(a)$.

We present a comparison theorem for two L^0-affine functions as follows.

Proposition 2.3 *Suppose that $h_{u,\alpha}$ and $h_{v,\beta}$ are two L^0-affine functions on E such that $h_{v,\beta} \leq h_{u,\alpha}$. Then $v = u$ and $\beta \leq \alpha$.*

Proof According to definition, $h_{v,\beta} \leq h_{u,\alpha}$ means that $\langle v, x\rangle + \beta \leq \langle u, x\rangle + \alpha, \forall x \in E$. Taking $x = \theta$ yields that $\beta \leq \alpha$. For any $n \in \mathbb{N}$, replacing x with nx, we can deduce that $\langle v, x\rangle + \frac{\beta}{n} \leq \langle u, x\rangle + \frac{\alpha}{n}, \forall x \in E$. Letting n tend to ∞, we obtain that $\langle v, x\rangle \leq \langle u, x\rangle, \forall x \in E$. Then replacing x with $-x$, we get $-\langle v, x\rangle \leq -\langle u, x\rangle, \forall x \in E$. Therefore $\langle v, x\rangle = \langle u, x\rangle, \forall x \in E$, namely $v = u$.

A useful characterization for $f \in \mathscr{C}(E)$ to belong to $\mathscr{A}(E)$ is given as follows.

Proposition 2.4 *Given $f \in \mathscr{C}(E)$, then f belongs to $\mathscr{A}(E)$ if and only if there exists a unique pair $(u, \alpha) \in E^* \times L^0(\mathcal{F})$ such that $\mathscr{A}(f) = \{h_{u,\beta} : \beta \leq \alpha\}$.*

Proof (1) Necessity. If $f \in \mathscr{A}(E)$, then there exists a unique pair $(u, \alpha) \in E^* \times L^0(\mathcal{F})$ such that $f = h_{u,\alpha}$. If $h_{v,\beta} \in \mathscr{A}(E)$ satisfies that $h_{v,\beta} \leq h_{u,\alpha}$, then it follows from Proposition 2.3 that $v = u$ and $\beta \leq \alpha$, thus $\mathscr{A}(f) = \{h_{u,\beta} : \beta \leq \alpha\}$.

(2) Sufficiency. According to Corollary 2.1, $\mathscr{A}(f)$ is nonempty and $f = \vee\{h : h \in \mathscr{A}(f)\}$. If there exists a unique pair $(u, \alpha) \in E^* \times L^0(\mathcal{F})$ such that $\mathscr{A}(f) = \{h_{u,\beta} : \beta \leq \alpha\}$, then $f = \vee\{h_{u,\beta} : \beta \leq \alpha\} = h_{u,\alpha} \in \mathscr{A}(E)$.

Corollary 2.2 *Assume that $h_{u,\alpha} \in \mathscr{A}(E)$ and $f \in \mathscr{C}(E)$. If $f \leq h_{u,\alpha}$, then $f = h_{u,\beta}$ for some $\beta \leq \alpha$.*

Proof If $h_{u',\alpha'} \in \mathscr{A}(f)$, then $h_{u',\alpha'} \leq f \leq h_{u,\alpha}$, i.e., $h_{u',\alpha'} \in \mathscr{A}(h_{u,\alpha})$. By Proposition 2.3, $u' = u$ and $\alpha' \leq \alpha$. Let $\beta = \vee\{\alpha' : h_{u',\alpha'} \in \mathscr{A}(f)\}$. Then $\beta \leq \alpha$, and thus $\beta \in L^0(\mathcal{F})$. It is easy to see that $f = h_{u,\beta}$.

Proposition 2.5 below is substantial in Step 2 of the proof of the following Proposition 3.7, which is the foundation of our proof of Theorem 1.1. To prove Proposition 2.5, we are forced to use a separation theorem that relies on the random w^*-closedness of an L^0-convex subset of E^*. Lemma 2.2 below is an auxiliary result which says that the L^0-convex hull of any finite subset of an RLC module is closed. Since L^0-convex sets play a crucial role in random convex analysis on RLC modules, Lemma 2.2 is a by-product which may be interesting on its own.

Lemma 2.2 *Let $(E, \mathcal{P})$ be an RLC module with base $(\Omega, \mathcal{F}, P)$. Then for any finite subset $\{x_1, x_2, \cdots, x_d\}$ of E, its L^0-convex hull*

$$\mathrm{Conv}_{L^0}\{x_1, x_2, \cdots, x_d\} := \Big\{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_d x_d : \lambda_i \in L^0_+(\mathcal{F}), i = 1, 2, \cdots, d, \sum_{i=1}^{d} \lambda_i = 1\Big\}$$

is a complete and thus closed subset of E.

Proof Consider the submodule M of E generated by $\{x_1, x_2, \cdots, x_d\}$, namely,

$$M = \mathrm{span}_{L^0}\{x_1, x_2, \cdots, x_d\} := \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_d x_d : \lambda_i \in L^0_+(\mathcal{F}), i = 1, 2, \cdots, d\}$$

inherited the topology from E.

By [12, Theorem 1.1], there exists a finite partition $\{A_0, A_1, \cdots, A_d\}$ of Ω in $\mathcal{F}$ such that $\widetilde{I}_{A_i} M$ is a free module of rank i over the algebra $\widetilde{I}_{A_i} L^0(\mathcal{F})$ for each $i \in \{0, 1, \cdots, d\}$ satisfying $P(A_i) > 0$, in which case $M = \bigoplus_{i=0}^{d} \widetilde{I}_{A_i} M$. We can assume, without loss of generality, that $P(A_i) > 0$ for all $i \in \{1, 2, \cdots, d\}$. According to [11, Lemma 3.4(1)], for each $i \in \{1, 2, \cdots, d\}$, $\widetilde{I}_{A_i} M$ and $\widetilde{I}_{A_i} L^0(\mathcal{F}, \mathbb{R}^i)$ are isomorphic in the sense of topological modules, and hence it is also a complete subset of E.

For each $i \in \{1, 2, \cdots, d\}$, suppose that T_i is the topological module isomorphism from $\widetilde{I}_{A_i} M$ to $\widetilde{I}_{A_i} L^0(\mathcal{F}, \mathbb{R}^i)$. By regarding $L^0(\mathcal{F}, \mathbb{R}^i)$ as the submodule $L^0(\mathcal{F}, \mathbb{R}^i) \times \{0\}^{d-i}$ of $L^0(\mathcal{F}, \mathbb{R}^d)$ for each $i \in \{1, \cdots, d-1\}$, the mapping $T : M \to L^0(\mathcal{F}, \mathbb{R}^d)$ given by

$$Tx = T\Big(\sum_{i=0}^{n} \widetilde{I}_{A_i} x\Big) = \sum_{i=1}^{d} \widetilde{I}_{A_i} T_i(\widetilde{I}_{A_i} x), \quad \forall x \in M$$

is a topological module homomorphism such that M and TM are topological module isomorphic. From now on, we identify M with TM by regarding $x_1, x_2, \cdots, x_d$ as elements of $L^0(\mathcal{F}, \mathbb{R}^d)$

for simplicity of notation. It suffices to show that $\mathrm{Conv}_{L^0}\{x_1, x_2, \cdots, x_d\}$ is a complete subset of $L^0(\mathcal{F}, \mathbb{R}^d)$.

For any Cauchy sequence $\{y_n, n \in \mathbb{N}\}$ in $\mathrm{Conv}_{L^0}\{x_1, x_2, \cdots, x_d\}$, where each y_n can be expressed as $y_n = \lambda_n^1 x_1 + \cdots + \lambda_n^d x_d$ for some $\lambda_n^i \in L^0_+(\mathcal{F}),\ i = 1, 2, \cdots, d,\ \sum\limits_{i=1}^{d} \lambda_n^i = 1$. Due to the completeness of $L^0(\mathcal{F}, \mathbb{R}^d)$, y_n converges in probability to some $y \in L^0(\mathcal{F}, \mathbb{R}^d)$. It remains to show $y \in \mathrm{Conv}_{L^0}\{x_1, x_2, \cdots, x_d\}$. By passing to a suitable subsequence, we may assume without loss of generality that y_n converges P-a.s. to y. Since $\{\lambda_n = (\lambda_n^1, \cdots, \lambda_n^d), n \in \mathbb{N}\}$ is a.s. bounded in $L^0(\mathcal{F}, \mathbb{R}^d)$, by a variant of Komlos' principle of subsequences (see [4, Lemma 1.70]), there exists a sequence of convex combinations

$$\mu_n \in \mathrm{Conv}\{\lambda_n, \lambda_{n+1}, \cdots\},$$

which converges P-a.s. to some $\mu \in L^0(\mathcal{F}, \mathbb{R}^d)$. For each n, assume that $\mu_n^i = t_0^n \lambda_n^i + \cdots + t_{l(n)}^n \lambda_{n+l(n)}^i$, $i = 1, 2, \cdots, d$, where $l(n)$ is some positive integer and $t_0^n, \cdots, t_{l(n)}^n$ are some non-negative numbers such that $\sum\limits_{k=0}^{l(n)} t_k^n = 1$. Clearly $\mu_n^i \in L^0_+(\mathcal{F}),\ i = 1, \cdots, d$ and $\mu_n^1 + \cdots + \mu_n^d = 1$. Combining with the fact that $\mu_n = (\mu_n^1, \cdots, \mu_n^d)$ converges P-a.s. to $\mu = (\mu^1, \cdots, \mu^d)$, we deduce that $\mu^i \in L^0_+(\mathcal{F}),\ i = 1, \cdots, d$ and $\mu^1 + \cdots + \mu^d = 1$. Since y_n converges P-a.s. to y, the forward convex combination sequence $z_n := \sum\limits_{k=0}^{l(n)} t_k^n y_{n+k}$ also converges P-a.s. to y. By simple calculation, we see that $z_n = \mu_n^1 x_1 + \cdots + \mu_n^d x_d$. Then the fact that μ_n converges P-a.s. to μ implies that z_n converges P-a.s. to $\mu^1 x_1 + \cdots + \mu^d x_d$. We conclude that $y = \mu^1 x_1 + \cdots + \mu^d x_d$ belongs to $\mathrm{Conv}_{L^0}\{x_1, x_2, \cdots, x_d\}$.

Remark 2.1 We point out that the proof of Lemma 2.2 is similar to that of [22, Theorem 3.5], which states that for any finite subset $\{x_1, x_2, \cdots, x_d\}$ of an RLC module E, the finitely generated L^0-convex cone

$$\mathrm{Cone}_{L^0}\{x_1, \cdots, x_d\} := \{\lambda_1 x_1 + \cdots + \lambda_d x_d : \lambda_i \in L^0_+(\mathcal{F}),\ i = 1, \cdots, d\}$$

is a complete and thus closed subset of E.

We now state and prove Proposition 2.5. We remind the reader that for any ξ and η in $L^0(\mathcal{F})$, $\xi > \eta$ means $\xi \geq \eta$ and $\xi \neq \eta$.

Proposition 2.5 *Let $(E, \|\cdot\|)$ be an RN module with base $(\Omega, \mathcal{F}, P)$, and E^* be its random conjugate space. Suppose that $h_{u,\alpha}, h_{u_1,\alpha_1}$ and h_{u_2,α_2} are L^0-affine functions on E such that $h_{u,\alpha} \leq h_{u_1,\alpha_1} \vee h_{u_2,\alpha_2}$. Then $\langle u, x\rangle \leq (\langle u_1, x\rangle \vee \langle u_2, x\rangle), \forall x \in E$, and there exists $\mu \in L^0(\mathcal{F})$ with $0 \leq \mu \leq 1$ such that $u = \mu u_1 + (1-\mu) u_2$.*

Proof By definition, $h_{u,\alpha} \leq h_{u_1,\alpha_1} \vee h_{u_2,\alpha_2}$ means that $\langle u, x\rangle + \alpha \leq (\langle u_1, x\rangle + \alpha_1) \vee (\langle u_2, x\rangle + \alpha_2), \forall x \in E$.

For any $n \in \mathbb{N}$, replacing x with nx, we can deduce that $\langle u, x\rangle + \frac{\alpha}{n} \leq \big(\langle u_1, x\rangle + \frac{\alpha_1}{n}\big) \vee \big(\langle u_2, x\rangle + \frac{\alpha_2}{n}\big), \forall x \in E$. Letting n tend to ∞, we obtain that

$$\langle u, x\rangle \leq (\langle u_1, x\rangle \vee \langle u_2, x\rangle), \quad \forall x \in E. \tag{2.1}$$

Consider the RLC module $(E^*, \sigma(E^*, E))$. It follows from Lemma 2.2 that $\mathrm{Conv}_{L^0}\{u_1, u_2\}$ is an L^0-convex and closed subset of $(E^*, \sigma(E^*, E))$. We prove $u \in \mathrm{Conv}_{L^0}\{u_1, u_2\}$ by contradiction. If $u \notin \mathrm{Conv}_{L^0}\{u_1, u_2\}$, then by the separation theorem in RLC modules (see [14, Theorem 3.1], also [8, Theorem 3.6]), there exists $g \in (E^*, \sigma(E^*, E))^*$ such that

$$\langle u, g\rangle > \vee\{\langle w, g\rangle : w \in \mathrm{Conv}_{L^0}\{u_1, u_2\}\}. \tag{2.2}$$

(2.2) also means that there exists some $B \in \mathcal{F}$ with $P(B) > 0$ such that

$$\langle u, g\rangle > \vee\{\langle w, g\rangle : w \in \mathrm{Conv}_{L^0}\{u_1, u_2\}\} \text{ on } B. \tag{2.3}$$

Since $(E^*, \sigma(E^*, E))$ is endowed with the (ε, λ)-topology in this paper, it is well known that the random conjugate space of $(E^*, \sigma(E^*, E))$ under the locally L^0-convex topology is E (see [8, 18] for the notion of the locally L^0-convex topology and the related random conjugate space). Then by [18, Proposition 2.6], the random conjugate space under the (ε, λ)-topology of $(E^*, \sigma(E^*, E))$ is $H_{cc}(E)$, where $H_{cc}(E)$ stands for the countable concatenation hull of E in the random conjugate space under the (ε, λ)-topology of $(E^*, \sigma(E^*, E))$. Thus there exists a sequence $\{x_n, n \in \mathbb{N}\}$ in E and a countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ such that $\langle u, g\rangle = \sum\limits_{n=1}^{\infty} \widetilde{I}_{A_n}\langle u, x_n\rangle$, which must imply by (2.3) that there exists some $n_0 \in \mathbb{N}$ such that $P(A_{n_0} \cap B) > 0$ and $\langle u, x_{n_0}\rangle > \vee\{\langle w, x_{n_0}\rangle : w \in \mathrm{Conv}_{L^0}\{u_1, u_2\}\}$ on $A_{n_0} \cap B$. Specially $\langle u, x_{n_0}\rangle > (\langle u_1, x_{n_0}\rangle) \vee (\langle u_2, x_{n_0}\rangle)$ on $A_{n_0} \cap B$, which is a contradiction to (2.1). We conclude that $u \in \mathrm{Conv}_{L^0}\{u_1, u_2\}$, namely there exists $\mu \in L^0(\mathcal{F})$ with $0 \le \mu \le 1$ such that $u = \mu u_1 + (1-\mu)u_2$.

3 The Order Preserving Case

In this section, we prove Theorem 1.1 and Corollary 1.1. For this purpose, we need some preparations. We will gradually establish a series of characteristic properties for a stable and fully order preserving operator.

Proposition 3.1 *If $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is a stable and fully order preserving operator, then so is its inverse T^{-1}.*

Proof It is obvious.

Proposition 3.2 *If $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is stable and fully order preserving, then for any family $\{f_i\}_{i\in I} \subset \mathscr{C}(E)$ such that $\bigvee\limits_{i\in I} f_i$ belongs to $\mathscr{C}(E)$, we have that $T\big(\bigvee\limits_{i\in I} f_i\big) = \bigvee\limits_{i\in I} T(f_i)$.*

The proof is omitted, since it is a word-by-word copy of the proof of [19, Proposition 1(ii)]. Note that in the proof, the assumption that T is stable is not used.

We then show that each stable and fully order preserving operator maps any $L^0(\mathcal{F})$-valued function to an $L^0(\mathcal{F})$-valued function. Precisely, we have the following result.

Proposition 3.3 *If $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is stable and fully order preserving and $f \in \mathscr{C}(E)$ with $f(x) \in L^0(\mathcal{F})$ for every $x \in E$, then $[T(f)](x) \in L^0(\mathcal{F})$ for every $x \in E$.*

Proof We prove this proposition by contradiction. Suppose that there exist $x_0 \in E$ and $A \in \mathcal{F}_+$ such that $[T(f)](x_0) = \infty$ on A. According to Lemma 2.1, the indicator function δ_{x_0} is a member of $\mathscr{C}(E)$. Since T is onto, there exists $f_0 \in \mathscr{C}(E)$ such that $T(f_0) = \delta_{x_0}$. Due to the assumption on f we see that $f_1 = f \vee f_0$ belongs to $\mathscr{C}(E)$. Then using Proposition 3.2 we get $T(f_1) = T(f) \vee T(f_0) = T(f) \vee \delta_{x_0}$. If $\widetilde{x} \in \mathrm{dom}(T(f_1))$, then it follows from $\delta_{x_0}(\widetilde{x}) \leq [T(f_1)](\widetilde{x}) \in L^0(\mathcal{F})$ that $\widetilde{x} = x_0$, subsequently, $[T(f_1)](\widetilde{x}) \geq [T(f)](\widetilde{x}) = [T(f)](x_0) = \infty$ on A. This implies that $\mathrm{dom}(T(f_1))$ must be empty and thus $T(f_1) \notin \mathscr{C}(E)$, which is a contradiction.

Since we have known from Corollary 2.1 that every $f \in \mathscr{C}(E)$ can be expressed as $f = \vee\{h : h \in \mathscr{A}(E), h \leq f\}$, according to Proposition 3.2, every stable and fully order preserving operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is fully determined by its action on $\mathscr{A}(E)$. Hence, we will analyze the behavior of the restriction of a stable and fully order preserving operator to $\mathscr{A}(E)$.

Next, we prove that each stable and fully order preserving operator maps any L^0-affine function to an L^0-affine function. Compared with the proof of [19, Proposition 3], our proof must use the stable property of T.

Proposition 3.4 *If $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is stable and fully order preserving, then we have*

(1) $T(h) \in \mathscr{A}(E)$ for all $h \in \mathscr{A}(E)$.

(2) If $T(f) \in \mathscr{A}(E)$ for some $f \in \mathscr{C}(E)$, then $f \in \mathscr{A}(E)$.

Proof (1) Let $h = h_{u,\alpha}$. We will use Proposition 2.4 in order to establish L^0-affineness of $T(h)$. Take $h_{v,\delta}$ and $h_{v',\delta'}$ in $\mathscr{A}(T(h))$, i.e., $h_{v,\delta} \leq T(h)$ and $h_{v',\delta'} \leq T(h)$. Since T is onto, there exist g and g' in $\mathscr{C}(E)$ such that $h_{v,\delta} = T(g)$ and $h_{v',\delta'} = T(g')$, so that $T(g) \leq T(h)$ and $T(g') \leq T(h)$. Since T is fully order preserving, we get $g \leq h$ and $g' \leq h$. By Proposition 2.3, there exist η and η' in $L^0(\mathcal{F})$ such that $g = h_{u,\eta}$ and $g' = h_{u,\eta'}$. Let $A = [\eta \geq \eta']$ and denote $\eta'' = \widetilde{I}_A\eta + \widetilde{I}_{A^c}\eta'$. Then $\eta \leq \eta''$ and $\eta' \leq \eta''$. Denote $g'' = \widetilde{I}_A g + \widetilde{I}_{A^c} g' = h_{u,\eta''}$. Then $g \leq g''$ and $g' \leq g''$ and by the assumption that T is stable, $T(g'') = \widetilde{I}_A T(g) + \widetilde{I}_{A^c} T(g') = h_{v'',\delta''}$, where $v'' = \widetilde{I}_A v + \widetilde{I}_{A^c} v'$ and $\delta'' = \widetilde{I}_A \delta + \widetilde{I}_{A^c}\delta'$. Since T is order preserving, $h_{v,\delta} = T(g) \leq T(g'') = h_{v'',\delta''}$ and $h_{v',\delta'} = T(g') \leq T(g'') = h_{v'',\delta''}$. It follows from Proposition 2.3 that $v = v' = v''$. We have proved that there exists a unique $v \in E^*$ such that $h_{v,\delta} \in \mathscr{A}(T(h))$ for some $\delta \in L^0(\mathcal{F})$, and hence $T(h)$ is L^0-affine by Proposition 2.4.

(2) By Proposition 3.1, T's inverse T^{-1} is also stable and fully order preserving. It follows that T^{-1} also maps an L^0-affine function to an L^0-affine function, which implies the result.

We have seen that any stable and fully order preserving operator maps $\mathscr{A}(E)$ to $\mathscr{A}(E)$. For a stable and fully order preserving operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$, denote by $\widehat{T} : \mathscr{A}(E) \to \mathscr{A}(E)$ the restriction of T to $\mathscr{A}(E)$. Note that $\widehat{T}$ inherits from T the properties of being onto and fully order preserving on $\mathscr{A}(E)$.

For the following discussion, we recall some notions from [23]. Let E_1 and E_2 be two $L^0(\mathcal{F})$-modules, and $R : E_1 \to E_2$ a mapping. R is said to be L^0-linear, if R is a module homomorphism, namely $R(x+y) = R(x) + R(y)$ for any $x, y \in E_1$, and $R(\xi x) = \xi R(x)$ for any $\xi \in L^0(\mathcal{F})$ and any $x \in E_1$; R is said to be L^0-affine, if $R(\lambda x + (1-\lambda)y) = \lambda R(x) + (1-\lambda)R(y)$ for any $\lambda \in L^0(\mathcal{F})$ and any $x, y \in E_1$, equivalently, $R(\cdot) - R(\theta)$ is L^0-linear; R is said to be stable, if $R(\widetilde{I}_A x + \widetilde{I}_{A^c} y) = \widetilde{I}_A R(x) + \widetilde{I}_{A^c} R(y)$ for all $x, y \in E_1$ and $A \in \mathcal{F}$.

By associating $h_{u,\alpha}$ to the pair (u, α), we can identify $\mathscr{A}(E)$ with $E^* \times L^0(\mathcal{F})$, and hence we take $\widehat{T}$ as an operator acting on $E^* \times L^0(\mathcal{F})$ and write $\widehat{T}(u, \alpha)$ instead of $\widehat{T}(h_{u,\alpha})$. We will prove next that $\widehat{T}$ is L^0-affine.

Proposition 3.5 $\widehat{T} : E^* \times L^0(\mathcal{F}) \to E^* \times L^0(\mathcal{F})$ *is bijective and stable.*

Proof It is obvious that $\widehat{T}$ is a bijection. We only need to show that $\widehat{T}$ is stable. For any (u, α) and (v, β) in $E^* \times L^0(\mathcal{F})$ and any A in $\mathcal{F}$, since

$$h_{\widetilde{I}_A u + \widetilde{I}_{A^c} v, \widetilde{I}_A \alpha + \widetilde{I}_{A^c} \beta} = \widetilde{I}_A h_{u,\alpha} + \widetilde{I}_{A^c} h_{v,\beta},$$

it follows from the stable property of T that

$$T(h_{\widetilde{I}_A u + \widetilde{I}_{A^c} v, \widetilde{I}_A \alpha + \widetilde{I}_{A^c} \beta}) = \widetilde{I}_A T(h_{u,\alpha}) + \widetilde{I}_{A^c} T(h_{v,\beta}),$$

namely,

$$\widehat{T}(\widetilde{I}_A u + \widetilde{I}_{A^c} v, \widetilde{I}_A \alpha + \widetilde{I}_{A^c} \beta) = \widetilde{I}_A \widehat{T}(u, \alpha) + \widetilde{I}_{A^c} \widehat{T}(v, \beta).$$

Thus $\widehat{T}$ is stable.

Lemma 3.1 below tells us that for the mapping $\widehat{T} : (u, \alpha) \mapsto (v, \beta)$, v only depends on u.

Lemma 3.1 *Suppose that* $\widehat{T}(u, \alpha_1) = (v_1, \beta_1)$ *and* $\widehat{T}(u, \alpha_2) = (v_2, \beta_2)$. *Then we have* $v_1 = v_2$. *Conversely, if* $\widehat{T}(u_1, \alpha_1) = (v, \beta_1)$ *and* $\widehat{T}(u_2, \alpha_2) = (v, \beta_2)$, *then we have* $u_1 = u_2$.

Proof Arbitrarily choose an $\alpha \in L^0(\mathcal{F})$ such that $\alpha \geq \alpha_1 \vee \alpha_2$ and suppose that $\widehat{T}(u, \alpha) = (v, \beta)$. Since $h_{u,\alpha} \geq h_{u,\alpha_1}$ and T is fully order preserving, we have $T(h_{u,\alpha}) \geq T(h_{u,\alpha_1})$, namely $h_{v,\beta} \geq h_{v_1,\beta_1}$, which implies $v_1 = v$ by Proposition 2.3. Similarly, we have $v_2 = v$, and thus $v_1 = v_2$. By considering the inverse of $\widehat{T}$ which is obviously $\widehat{T}^{-1}$, the converse part immediately follows.

Proposition 3.6 *Assume that* $\widehat{T}(u_1, \alpha_1) = (v_1, \beta_1)$ *and* $\widehat{T}(u_2, \alpha_2) = (v_2, \beta_2)$. *If* $u_1 - u_2$ *has full support, then* $v_1 - v_2$ *has full support.*

Proof We prove the conclusion by contradiction. Suppose there exists $A \in \mathcal{F}_+$ such that $\widetilde{I}_A(v_1 - v_2) = 0$, equivalently, $\widetilde{I}_A v_1 = \widetilde{I}_A v_2$. Let $u = \widetilde{I}_A u_1 + \widetilde{I}_{A^c} u_2$ and $\alpha = \widetilde{I}_A \alpha_1 + \widetilde{I}_{A^c} \alpha_2$. Since $\widehat{T}$ is stable, we have $\widehat{T}(u, \alpha) = \widetilde{I}_A \widehat{T}(u_1, \alpha_1) + \widetilde{I}_{A^c} \widehat{T}(u_2, \alpha_2) = \widetilde{I}_A(v_1, \beta_1) + \widetilde{I}_{A^c}(v_2, \beta_2) = (v_2, \widetilde{I}_A \beta_1 + \widetilde{I}_{A^c} \beta_2)$. It follows from Lemma 3.1 that $u = u_2$, so $\widetilde{I}_A u = \widetilde{I}_A u_1 = \widetilde{I}_A u_2$, which contradicts the assumption that $u_1 - u_2$ has full support.

Let us recall the notion of L^0-line segments (cf. [23]): For any two elements x and y of an $L^0(\mathcal{F})$-module, $[x, y] := \{\lambda x + (1 - \lambda) y : \lambda \in L^0_+(\mathcal{F}), 0 \leq \lambda \leq 1\}$, called the L^0-line segment linking x and y.

To establish the L^0-affineness of $\widehat{T}$, we need to clarify $\widehat{T}$'s action on L^0-line segments. Please bear in mind that for any h_1 and h_2 in $\mathscr{A}(E)$, $h_1 < h_2$ means $h_1 \leq h_2$ and $h_1 \neq h_2$.

Proposition 3.7 $\widehat{T} : E^* \times L^0(\mathcal{F}) \to E^* \times L^0(\mathcal{F})$ *maps any* L^0*-line segment to an* L^0*-line segment. Namely,* $\widehat{T}([z_1, z_2]) = [\widehat{T} z_1, \widehat{T} z_2]$ *for any* z_1 *and* z_2 *in* $E^* \times L^0(\mathcal{F})$.

Proof Observe that if we can prove $\widehat{T}$ maps any L^0-line segment into an L^0-line segment, namely $\widehat{T}([z_1, z_2]) \subset [\widehat{T}z_1, \widehat{T}z_2]$ for any z_1 and z_2 in $E^* \times L^0(\mathcal{F})$, by considering $\widehat{T}^{-1}$ we can see that $\widehat{T}$ would also map any L^0-line segment onto an L^0-line segment, namely, $\widehat{T}([z_1, z_2]) = [\widehat{T}z_1, \widehat{T}z_2]$ for any z_1 and z_2 in $E^* \times L^0(\mathcal{F})$. Thus we only need to show that $\widehat{T}$ maps any L^0-line segment into an L^0-line segment.

Fix two elements (u_1, α_1) and (u_2, α_2) in $E^* \times L^0(\mathcal{F})$. Assume $\widehat{T}(u_1, \alpha_1) = (v_1, \beta_1)$ and $\widehat{T}(u_2, \alpha_2) = (v_2, \beta_2)$. Fix a $\lambda \in L^0(\mathcal{F})$ with $0 \leq \lambda \leq 1$, let $u = \lambda u_1 + (1 - \lambda)u_2$ and $\alpha = \lambda\alpha_1 + (1 - \lambda)\alpha_2$, and suppose $\widehat{T}(u, \alpha) = (v, \beta)$. We shall show that there exists $\mu \in L^0(\mathcal{F})$ with $0 \leq \mu \leq 1$ such that $v = \mu v_1 + (1 - \mu)v_2$ and $\beta = \mu\beta_1 + (1 - \mu)\beta_2$. The proof is divided into 3 steps.

Step 1 If $u_1 = u_2$, then the conclusion holds.

From the construction we see that $u = u_1 = u_2$. Then it follows from Lemma 3.1 that $v = v_1 = v_2$. We only need to find $\mu \in L^0(\mathcal{F})$ with $0 \leq \mu \leq 1$ such that $\beta = \mu\beta_1 + (1 - \mu)\beta_2$.

By Proposition 3.2 we obtain

$$\widehat{T}(u, \alpha_1 \vee \alpha_2) = (v, \beta_1 \vee \beta_2).$$

Since $\widehat{T}$ is fully order preserving, we have

$$\widehat{T}(u, \alpha_1 \wedge \alpha_2) \leq (\widehat{T}(u, \alpha_1) \wedge \widehat{T}(u, \alpha_2)) = (v, \beta_1 \wedge \beta_2).$$

Similarly, by considering the fully order preserving operator $\widehat{T}^{-1}$, we get

$$\widehat{T}^{-1}(v, \beta_1 \wedge \beta_2) \leq (u, \alpha_1 \wedge \alpha_2).$$

Then it follows that

$$(v, \beta_1 \wedge \beta_2) \leq \widehat{T}(u, \alpha_1 \wedge \alpha_2).$$

Thus we obtain

$$\widehat{T}(u, \alpha_1 \wedge \alpha_2) = (v, \beta_1 \wedge \beta_2).$$

Note that $(\alpha_1 \wedge \alpha_2) \leq \alpha \leq (\alpha_1 \vee \alpha_2)$. We thus deduce

$$(\beta_1 \wedge \beta_2) \leq \beta \leq (\beta_1 \vee \beta_2).$$

Then there must exist $\mu_0 \in L^0(\mathcal{F})$ with $0 \leq \mu_0 \leq 1$ such that

$$\beta = \mu_0(\beta_1 \wedge \beta_2) + (1 - \mu_0)(\beta_1 \vee \beta_2).$$

Denote $A = [\beta_1 \geq \beta_2]$. Then $(\beta_1 \wedge \beta_2) = I_{A^c}\beta_1 + I_A\beta_2$ and $(\beta_1 \vee \beta_2) = I_A\beta_1 + I_{A^c}\beta_2$. Let $\mu = \mu_0 I_{A^c} + (1 - \mu_0)I_A$. We see that $0 \leq \mu \leq 1$ and $\beta = \mu\beta_1 + (1 - \mu)\beta_2$.

Step 2 If $u_1 - u_2$ has full support, then the conclusion holds.

Note by Proposition 3.6, $v_1 - v_2$ also has full support.

Since

$$h_{u,\alpha} = \lambda h_{u_1,\alpha_1} + (1 - \lambda)h_{u_2,\alpha_2} \leq h_{u_1,\alpha_1} \vee h_{u_2,\alpha_2},$$

and T is fully order preserving, we obtain

$$T(h_{u,\alpha}) \leq T(h_{u_1,\alpha_1} \vee h_{u_2,\alpha_2}).$$

From Proposition 3.2, $T(h_{u_1,\alpha_1} \vee h_{u_2,\alpha_2}) = T(h_{u_1,\alpha_1}) \vee T(h_{u_2,\alpha_2})$, and thus we get $h_{v,\beta} \leq h_{v_1,\beta_1} \vee h_{v_2,\beta_2}$, namely,

$$\langle v, x\rangle + \beta \leq (\langle v_1, x\rangle + \beta_1) \vee (\langle v_2, x\rangle + \beta_2), \quad \forall x \in E, \tag{3.1}$$

by Proposition 2.5,

$$\langle v, x\rangle \leq (\langle v_1, x\rangle \vee \langle v_2, x\rangle), \quad \forall x \in E, \tag{3.2}$$

and there exists $\mu \in L^0(\mathcal{F})$ with $0 \leq \mu \leq 1$ such that $v = \mu v_1 + (1-\mu)v_2$. It remains to show that $\beta = \mu\beta_1 + (1-\mu)\beta_2$ holds for this μ. To this end, we first show that there exists an $x_0 \in E$ such that

$$\langle v, x_0\rangle + \beta = \langle v_1, x_0\rangle + \beta_1 = \langle v_2, x_0\rangle + \beta_2. \tag{3.3}$$

Since $v_1 - v_2$ has full support, it follows from Proposition 2.1(2) that there exists $x_0 \in E$ such that $\langle v_1 - v_2, x_0\rangle = \beta_2 - \beta_1$, equivalently, $\langle v_1, x_0\rangle + \beta_1 = \langle v_2, x_0\rangle + \beta_2$. Let $\epsilon = (\langle v_1, x_0\rangle + \beta_1) - (\langle v, x_0\rangle + \beta)$. From (3.1) we see that $\epsilon \geq 0$. To show that (3.3) holds for the x_0, it suffices to show that $\epsilon = 0$.

For any $x \in E$, it follows from (3.2) that

$$\langle v, x - x_0\rangle \leq (\langle v_1, x - x_0\rangle) \vee (\langle v_2, x - x_0\rangle),$$

and thus,

$$\begin{aligned}
&\langle v, x\rangle + \beta + \epsilon \\
&= \langle v, x - x_0\rangle + \langle v, x_0\rangle + \beta + \epsilon \\
&= \langle v, x - x_0\rangle + (\langle v_1, x_0\rangle + \beta_1) \\
&\leq (\langle v_1, x - x_0\rangle) \vee (\langle v_2, x - x_0\rangle) + (\langle v_1, x_0\rangle + \beta_1) \\
&= (\langle v_1, x - x_0\rangle + \langle v_1, x_0\rangle + \beta_1) \vee (\langle v_2, x - x_0\rangle + \langle v_1, x_0\rangle + \beta_1) \\
&= (\langle v_1, x - x_0\rangle + \langle v_1, x_0\rangle + \beta_1) \vee (\langle v_2, x - x_0\rangle + \langle v_2, x_0\rangle + \beta_2) \\
&= (\langle v_1, x\rangle + \beta_1) \vee (\langle v_2, x\rangle + \beta_2),
\end{aligned}$$

namely, we get

$$h_{v,\beta+\epsilon} \leq h_{v_1,\beta_1} \vee h_{v_2,\beta_2}. \tag{3.4}$$

We can now show $\epsilon = 0$ by contradiction.

If $\epsilon > 0$, then $h_{v,\beta} < h_{v,\beta+\epsilon}$. Due to our assumption that $T(h_{u,\alpha}) = h_{v,\beta}$, from Lemma 3.1 we can assume that $T(h_{u,\alpha'}) = h_{v,\beta+\epsilon}$. Since T is fully order preserving, we must have $\alpha' > \alpha$. Besides, noting that $T(h_{u_1,\alpha_1}) = h_{v_1,\beta_1}$ and $T(h_{u_2,\alpha_2}) = h_{v_2,\beta_2}$, from (3.4) we have

$$h_{u,\alpha'} \leq h_{u_1,\alpha_1} \vee h_{u_2,\alpha_2}. \tag{3.5}$$

Using the assumption that $u_1 - u_2$ has full support, by Proposition 2.1(2) there exists some $x_1 \in E$ such that $\langle u_1 - u_2, x_1\rangle = \alpha_2 - \alpha_1$, equivalently, $\langle u_1, x_1\rangle + \alpha_1 = \langle u_2, x_1\rangle + \alpha_2$, namely $h_{u_1,\alpha_1}(x_1) = h_{u_2,\alpha_2}(x_1)$. Then using $h_{u,\alpha} = \lambda h_{u_1,\alpha_1} + (1-\lambda)h_{u_2,\alpha_2}$, we get

$$\begin{aligned}h_{u,\alpha'}(x_1) &= h_{u,\alpha}(x_1) + (\alpha' - \alpha)\\ &= \lambda h_{u_1,\alpha_1}(x_1) + (1-\lambda)h_{u_2,\alpha_2}(x_1) + (\alpha' - \alpha)\\ &= h_{u_1,\alpha_1}(x_1) + (\alpha' - \alpha)\\ &> h_{u_1,\alpha_1}(x_1)\\ &= h_{u_1,\alpha_1}(x_1) \vee h_{u_2,\alpha_2}(x_1),\end{aligned}$$

this contradicts (3.5).

Now that we have shown that (3.3) holds for some x_0. Then for this x_0,

$$\begin{aligned}\langle v, x_0\rangle + \beta &= \mu(\langle v_1, x_0\rangle + \beta_1) + (1-\mu)(\langle v_2, x_0\rangle + \beta_2)\\ &= \mu\langle v_1, x_0\rangle + (1-\mu)\langle v_2, x_0\rangle + \mu\beta_1 + (1-\mu)\beta_2\\ &= \langle \mu v_1 + (1-\mu)v_2, x_0\rangle + \mu\beta_1 + (1-\mu)\beta_2\\ &= \langle v, x_0\rangle + \mu\beta_1 + (1-\mu)\beta_2,\end{aligned}$$

and thus we have $\beta = \mu\beta_1 + (1-\mu)\beta_2$.

Step 3 Generally, the conclusion holds.

Denote $A = [\|u_1 - u_2\| = 0]$. From Proposition 2.1(1), there exists $u_0 \in E^*$ with $\|u_0\| = 1$. Let $w = I_A u_0 + u_1$ and $u_3 = I_A w + I_{A^c} u_2$. Then $\|u_3 - u_1\| = I_A\|u_0\| + I_{A^c}\|u_1 - u_2\| \neq 0$ on Ω, namely $u_3 - u_1$ has full support.

Now according to Step 1, there exists $\mu_1 \in L^0(\mathcal{F})$ with $0 \le \mu_1 \le 1$ such that

$$\widehat{T}(u_1, \alpha) = \mu_1\widehat{T}(u_1, \alpha_1) + (1-\mu_1)\widehat{T}(u_1, \alpha_2);$$

according to Step 2, there exists $\mu_2 \in L^0(\mathcal{F})$ with $0 \le \mu_2 \le 1$ such that

$$\widehat{T}(\lambda u_1 + (1-\lambda)u_3, \alpha) = \mu_2\widehat{T}(u_1, \alpha_1) + (1-\mu_2)\widehat{T}(u_3, \alpha_2).$$

Since

$$\begin{aligned}&I_A u_1 + I_{A^c}(\lambda u_1 + (1-\lambda)u_3)\\ &= I_A\lambda u_1 + I_A(1-\lambda)u_1 + I_{A^c}\lambda u_1 + I_{A^c}(1-\lambda)u_3\\ &= I_A\lambda u_1 + I_A(1-\lambda)u_2 + I_{A^c}\lambda u_1 + I_{A^c}(1-\lambda)u_2\\ &= \lambda u_1 + (1-\lambda)u_2\\ &= u,\end{aligned}$$

by the stable property of $\widehat{T}$, we get

$$\begin{aligned}&\widehat{T}(u, \alpha)\\ &= I_A\widehat{T}(u_1, \alpha) + I_{A^c}\widehat{T}(\lambda u_1 + (1-\lambda)u_3, \alpha)\end{aligned}$$

$$= I_A(\mu_1\widehat{T}(u_1,\alpha_1) + (1-\mu_1)\widehat{T}(u_1,\alpha_2)) + I_{A^c}(\mu_2\widehat{T}(u_1,\alpha_1) + (1-\mu_2)\widehat{T}(u_3,\alpha_2))$$
$$= (I_A\mu_1 + I_{A^c}\mu_2)\widehat{T}(u_1,\alpha_1) + I_A(1-\mu_1)\widehat{T}(u_2,\alpha_2) + I_{A^c}(1-\mu_2)\widehat{T}(u_2,\alpha_2)$$
$$= (I_A\mu_1 + I_{A^c}\mu_2)\widehat{T}(u_1,\alpha_1) + [I_A(1-\mu_1) + I_{A^c}(1-\mu_2)]\widehat{T}(u_2,\alpha_2),$$

if we set $\mu = I_A\mu_1 + I_{A^c}\mu_2$, then $\mu \in L^0(\mathcal{F})$ with $0 \le \mu \le 1$ and

$$\widehat{T}(u,\alpha) = \mu\widehat{T}(u_1,\alpha_1) + (1-\mu)\widehat{T}(u_2,\alpha_2).$$

This finally completes the proof.

Now we can give an explicit form of $\widehat{T}$.

Proposition 3.8 *$\widehat{T} : E^* \times L^0(\mathcal{F}) \to E^* \times L^0(\mathcal{F})$ is invertible and L^0-affine. Precisely, there exist $w \in E^*$, $\tau \in L^0_{++}(\mathcal{F})$, $\beta \in L^0(\mathcal{F})$, a module automorphism $D : E^* \to E^*$ and an L^0-linear function $d : E^* \to L^0(\mathcal{F})$ such that for each $(u,\alpha) \in E^* \times L^0(\mathcal{F})$,*

$$\widehat{T}(u,\alpha) = (Du + w, d(u) + \tau\alpha + \beta).$$

Proof According to Proposition 3.5, $\widehat{T}$ is invertible. By Proposition 2.1(1), there exists $u_0 \in E^*$ such that $\|u_0\| = 1$. If ξ and η belong to $L^0(\mathcal{F})$ and $\xi(u_0,0)+\eta(0,1) = (\xi u_0,\eta) = (\theta,0)$, then clearly $\xi = \eta = 0$, and thus $(u_0,0)$ and $(0,1)$ are L^0-independent in $E^* \times L^0(\mathcal{F})$. By Proposition 3.7, $\widehat{T} : E^* \times L^0(\mathcal{F}) \to E^* \times L^0(\mathcal{F})$ maps any L^0-line segment to an L^0-line segment. Invoking the fundamental theorem of affine geometry in regular L^0-modules (see [23, Theorem 1.1]), $\widehat{T}$ must be L^0-affine. In view of Lemma 3.1, we can denote $\widehat{T}(u,\alpha) = (y(u),\gamma(u,\alpha))$ for each $(u,\alpha) \in E^* \times L^0(\mathcal{F})$. Then both $y : E^* \to E^*$ and $\gamma : E^* \times L^0(\mathcal{F}) \to L^0(\mathcal{F})$ are L^0-affine, namely there exist $w \in E^*$, $\tau \in L^0(\mathcal{F})$, $\beta \in L^0(\mathcal{F})$, a module homomorphism $D : E^* \to E^*$ and an L^0-linear function $d : E^* \to L^0(\mathcal{F})$ such that $y(u) = Du + w$, $\gamma(u,\alpha) = d(u) + \tau\alpha + \beta$. Since $\widehat{T}$ is bijective and preserves the order of the second variable, we can see that D must be invertible and $\tau \in L^0_{++}(\mathcal{F})$.

Remark 3.1 In [19], the affineness of $\widehat{T} : X^* \times \mathbb{R} \to X^* \times \mathbb{R}$ is established separately for the mapping $y : X^* \to X^*$ (see [19, Corollary 4]) and the mapping $\gamma : X^* \times \mathbb{R} \to \mathbb{R}$ (see [19, Proposition 7]). Since the affineness of y relies on the fundamental theorem of affine geometry, Iusem, Reem and Svaiter had to impose the assumption that the Banach space X in their main results, Theorems 1–2, has dimension not less than 2. As a comparison, we establish the L^0-affineness of $\widehat{T}$ as a whole so that we do not need an additional assumption that E contains a free submodule with rank 2.

We proceed to verify the a.s. boundedness of D and d.

Proposition 3.9 *Both D and d in Proposition 3.8 are a.s. bounded and thus continuous. Specially, $d \in E^{**}$.*

Proof We first prove d is a.s. bounded. Since $g = \|\cdot\| \in \mathscr{C}(E)$, we denote $g_1 = T(g)$. Let B be the random closed unit ball of E^*, i.e., $B = \{u \in E^* : \|u\| \le 1\}$. Since $\|x\| = \vee\{h_{u,0}(x) = \langle u,x\rangle : u \in B\}$ for every $x \in E$, we have

$$g_1(x) = [T(g)](x)$$

$$= \vee\{T(h_{u,0})(x) : u \in B\}$$
$$= \vee\{h_{\widehat{T}(u,0)}(x) : u \in B\}$$
$$= \vee\{\langle Du + w, x\rangle + d(u) + \beta : u \in B\}.$$

Therefore, $g_1(0) = \vee\{d(u) + \beta : u \in B\} = \vee\{d(u) : u \in B\} + \beta$. By Proposition 3.3, we have $\vee\{d(u) : u \in B\} = g_1(0) - \beta \in L^0(\mathcal{F})$, and thus d is a.s. bounded with $\|d\| = g_1(0) - \beta$.

We then prove that D is a.s. bounded. By Proposition 3.3,

$$\vee\{\langle Du + w, x\rangle + d(u) + \beta : u \in B\} = g_1(x) \in L^0(\mathcal{F}), \quad \forall x \in E,$$

and thus

$$\langle Du, x\rangle \leq g_1(x) - \langle w, x\rangle - d(u) - \beta \leq g_1(x) - \langle w, x\rangle + \|d\| - \beta$$

for every $u \in B$ and $x \in E$. Then

$$\vee\{|\langle Du, x\rangle| : u \in B\} = \vee\{\langle Du, x\rangle : u \in B\} \leq g_1(x) - \langle w, x\rangle + \|d\| - \beta \in L^0(\mathcal{F})$$

for every $x \in E$. Using uniform boundedness principle in RN modules (see [10, Theorem 29]), we obtain that $\{Du : u \in B\}$ is a.s. bounded, and therefore D is a.s. bounded.

Now we can give the proof of Theorem 1.1, which is just a copy of the proof of [19, Theorem 1], except that we should be more careful in dealing with the indicator functions.

Proof of Theorem 1.1 Sufficiency can be easily shown by straightforward verification. We only prove the necessity.

From Corollary 2.1, for any $f \in \mathscr{C}(E)$ we have that $f = \vee\{h_{u,\alpha} : h_{u,\alpha} \leq f\}$. Then according to Proposition 3.2, $T(f) = \vee\{Th_{u,\alpha} : h_{u,\alpha} \leq f\}$. By Propositions 3.8–3.9, there exist $w \in E^*$, $d \in E^{**}$, $\tau \in L^0_{++}(\mathcal{F}), \beta \in L^0(\mathcal{F})$ and a continuous module automorphism D of E^* such that

$$\widehat{T}(u, \alpha) = (Du + w, \langle d, u\rangle + \tau\alpha + \beta),$$

and thus for each $x \in E$,

$$\begin{aligned}[T(f)](x) &= \vee\{\langle Du + w, x\rangle + \langle d, u\rangle + \tau\alpha + \beta : h_{u,\alpha} \leq f\} \\ &= \vee\{\langle Du, x\rangle + \langle d, u\rangle + \tau\alpha : h_{u,\alpha} \leq f\} + \langle w, x\rangle + \beta \\ &= \tau \vee \{\langle Cu, x\rangle + \langle c, u\rangle + \alpha : h_{u,\alpha} \leq f\} + \langle w, x\rangle + \beta \\ &= \tau \vee \{\langle C^*x + c, u\rangle + \alpha : h_{u,\alpha} \leq f\} + \langle w, x\rangle + \beta,\end{aligned}$$

where $C = \tau^{-1}D,\ c = \tau^{-1}d$ and $C^* : E^{**} \to E^{**}$ is the conjugate of C, so that C^*x must be understood through the natural embedding $E \subset E^{**}$. It follows from Proposition 2.2 that

$$[T(f)](x) = \tau f^{**}(C^*x + c) + \langle w, x\rangle + \beta, \quad \forall x \in E. \tag{3.6}$$

Define $Z \subset E^{**}$ as $Z = \{C^*x + c : x \in E\}$. We will prove that $Z = E$.

We first show that $E \subset Z$. Assume that there exists $\widetilde{x} \in E \setminus Z$. Consider the indicator function $\delta_{\widetilde{x}} : E \to \overline{L}^0(\mathcal{F})$. From Lemma 2.1, $\delta_{\widetilde{x}} \in \mathscr{C}(E)$ and $\delta_{\widetilde{x}}^{**}(x^{**}) = \delta_{\widetilde{x}}(x^{**})$ for every $x^{**} \in E^{**}$, where $\widetilde{x}$ is seen as an element in E^{**}. It follows from (3.6) that

$$[T(\delta_{\widetilde{x}})](x) = \tau\delta_{\widetilde{x}}(C^*x + c) + \langle w, x\rangle + \beta, \quad \forall x \in E.$$

However, since $\widetilde{x} \notin Z$, we have $C^*x + c \neq \widetilde{x}$ for all $x \in E$. Thus

$$\delta^{**}_{\widetilde{x}}(C^*x + c) = \infty \quad \text{ on } [C^*x + c \neq \widetilde{x}] \in \mathcal{F}_+$$

for all $x \in E$, which implies that $\mathrm{dom}(T(\delta_{\widetilde{x}})) = \emptyset$. This is a contradiction.

We then show that $Z \subset E$. Suppose now that there exists $\check{x} \in Z \setminus E$. Since $\check{x} \in Z$, there exists $x' \in E$ such that $C^*x' + c = \check{x}$. Consider the indicator function $\delta_{x'} : E \to \overline{L}^0(\mathcal{F})$. Since T is a bijection, there exists $g \in \mathscr{C}(E)$ such that $T(g) = \delta_{x'}$. Then it follows from (3.6) that

$$\delta_{x'}(x) = \tau g^{**}(C^*x + c) + \langle w, x\rangle + \beta, \quad \forall x \in E.$$

Now that $\mathrm{dom}(\delta_{x'}) = \{x'\}$, we have $Z \cap \mathrm{dom}(g^{**}) = \{C^*x' + c\} = \{\check{x}\}$. Since $E \subset Z$ and $\check{x} \notin E$, we obtain that $E \cap \mathrm{dom}(g^{**}) = \emptyset$. From the random Fenchel-Moreau duality theorem we see that $g^{**}|_E = g$, and thus $\mathrm{dom}(g) = E \cap \mathrm{dom}(g^{**}) = \emptyset$. This is a contradiction.

We have completed the proof of $E = Z$. We observe now that, since $C^*x + c \in E$ for all $x \in E$, by taking $x = 0$, we obtain $c \in E$. As a consequence, $H := C^*|_E$, the restriction of C^* to E, is a continuous module automorphism of E.

Using $f^{**}|_E = f$, we obtain that

$$[T(f)](x) = \tau f^{**}(C^*x + c) + \langle w, x\rangle + \beta = \tau f(Hx + c) + \langle w, x\rangle + \beta, \quad \forall x \in E.$$

This completes the proof.

Then we give the proof of Corollary 1.1.

Proof of Corollary 1.1 This is just a copy of the proof of [19, Corollary 6].

As a complement of Theorem 1.1, we give an example showing that a fully order preserving operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$ is not necessarily stable.

Example 3.1 Let the probability space $(\Omega, \mathcal{F}, P)$ be $([0, 1), \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel σ-algebra of $[0, 1)$ and m is the Lebesgue measure. Set $(E, \|\cdot\|) = (L^0(\mathcal{F}), |\cdot|)$. Define a mapping $\theta : [0, 1) \to [0, 1)$ as $\theta(\omega) = \omega + \frac{1}{2}$ for $\omega \in \left[0, \frac{1}{2}\right)$ and $\theta(\omega) = \omega - \frac{1}{2}$ for $\omega \in \left[\frac{1}{2}, 1\right)$. Then θ is an involution, namely θ is a bijection with inverse $\theta^{-1} = \theta$. Since θ is obviously measure preserving, θ is an automorphism of $([0, 1), \mathcal{B}, m)$. θ induces an involution $\sigma : \overline{L}^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ by sending each $x \in \overline{L}^0(\mathcal{F})$ to the equivalence class of $x^0(\theta(\cdot))$, where $x^0(\cdot)$ is a representative of x. It is easy to check that σ is fully order preserving: For every $x, y \in \overline{L}^0(\mathcal{F})$, $x \le y$ iff $\sigma(x) \le \sigma(y)$.

Now for every $f \in \mathscr{C}(E)$, define $T(f) : E \to \overline{L}^0(\mathcal{F})$ by $[T(f)](x) = \sigma[f(\sigma(x))], \forall x \in E$. We then check that $T(f) \in \mathscr{C}(E)$.

(1) For any $x \in E$, it is obvious that $[T(f)](x) > -\infty$ on Ω.

(2) For any $x \in \mathrm{dom}(f)$, we see that $\sigma(x) \in \mathrm{dom}[T(f)]$, and thus $\mathrm{dom}[T(f)]$ is nonempty.

(3) For any $x, y \in E$ and $\lambda \in L^0(\mathcal{F})$ with $0 \le \lambda \le 1$, we have $f(\lambda x + (1 - \lambda)y) \le \lambda f(x)$ $+(1 - \lambda)f(y)$ since f is L^0-convex. As a result

$$[T(f)](\lambda x + (1 - \lambda)y) = \sigma(f[\sigma(\lambda x + (1 - \lambda)y)])$$

$$
\begin{aligned}
&= \sigma(f[\sigma(\lambda)\sigma(x) + \sigma(1-\lambda)\sigma(y)]) \\
&\leq \sigma[\sigma(\lambda)f(\sigma(x)) + \sigma(1-\lambda)f(\sigma(y))] \\
&= \lambda\sigma(f(\sigma(x))) + (1-\lambda)\sigma(f(\sigma(y))) \\
&= \lambda[T(f)](x) + (1-\lambda)[T(f)](y),
\end{aligned}
$$

and thus $T(f)$ is L^0-convex.

(4) Since $\mathrm{epi}(f)$ is a closed subset of $E \times L^0(\mathcal{F})$ and

$$\mathrm{epi}(T(f)) = \{(\sigma(x), \sigma(r)) : (x, r) \in \mathrm{epi}(f)\},$$

we see that $\mathrm{epi}(T(f))$ is also a closed subset of $E \times L^0(\mathcal{F})$.

After this verification, by $f \mapsto T(f)$ we obtain an operator $T : \mathscr{C}(E) \to \mathscr{C}(E)$. It is easy to see that T is order preserving and $T(T(f)) = f$ for every $f \in \mathscr{C}(E)$ since σ^2 is the identity. Thus T is an order preserving involution.

We claim that T is not stable. In fact, let $A = \left[0, \frac{1}{2}\right)$ and $B = A^c = \left[\frac{1}{2}, 1\right)$. Then $\theta(A) = B$ and $\theta(B) = A$, equivalently, $\sigma(\widetilde{I}_A) = \widetilde{I}_B$ and $\sigma(\widetilde{I}_B) = \widetilde{I}_A$. Choose $f_0 \in \mathscr{C}(E)$ as $f_0(x) = x, \forall x \in E$. Then by construction,

$$[T(f_0)](x) = \sigma[f_0(\sigma(x))] = \sigma(\sigma(x)) = x, \quad \forall x \in E,$$

namely, $[T(f_0)] = f_0$, and thus

$$[\widetilde{I}_A T(f_0)](x) = \widetilde{I}_A f_0(x) = \widetilde{I}_A x, \quad \forall x \in E$$

and

$$[T(\widetilde{I}_A f_0)](x) = \sigma[\widetilde{I}_A \sigma(x)] = \sigma(\widetilde{I}_A)\sigma(\sigma(x)) = \widetilde{I}_B x, \quad \forall x \in E.$$

We see that $\widetilde{I}_A T(f_0) \neq T(\widetilde{I}_A f_0)$. Noting that $T(0) = 0$, we have

$$T(\widetilde{I}_A f_0 + \widetilde{I}_B \cdot 0) = T(\widetilde{I}_A f_0) \neq \widetilde{I}_A T(f_0) = \widetilde{I}_A T(f_0) + \widetilde{I}_B T(0),$$

which implies that T is not stable.

4 The Order Reversing Case

This section gives the proof of Theorem 1.2.

Proposition 4.1 below is indeed implied in the proof of random Fenchel-Moreau duality theorem (see [15, 18, Theorem 5.1]).

Proposition 4.1 *The random conjugate transform $f \mapsto f^*$ is a bijection from $\mathscr{C}(E)$ to $\mathscr{C}_{w^*}(E^*)$.*

We give the proof of Theorem 1.2 as follows.

Proof of Theorem 1.2 Sufficiency can be easily shown by straightforward verification. We only prove the necessity.

For an operator $S : \mathscr{C}(E) \to \mathscr{C}_{w^*}(E^*)$, it follows from Proposition 4.1 that there exists a unique $g \in \mathscr{C}(E)$ for each $f \in \mathscr{C}(E)$ such that $S(f) = g^*$. We can define an operator $T : \mathscr{C}(E) \mapsto \mathscr{C}(E)$ by $T(f) = g$ such that $S(f) = g^*$ for each $f \in \mathscr{C}(E)$. If S is a stable and fully order reversing operator, then it is easily verified that T is a stable and fully order preserving operator. Thus according to Theorem 1.1, there exists a continuous module automorphism $H_1 : E \to E$, and $c \in E$, $w \in E^*$, and $\tau \in L^0_{++}(\mathcal{F})$, $\rho_1 \in L^0(\mathcal{F})$ such that

$$[T(f)](x) = \tau f(H_1 x + c) + \langle w, x\rangle + \rho_1, \quad \forall f \in \mathscr{C}(E),\ x \in E.$$

Consequently, for each $u \in E^*$,

$$\begin{aligned}
[S(f)](u) &= [T(f)]^*(u) \\
&= \vee\{\langle u, x\rangle - \tau f(H_1 x + c) - \langle w, x\rangle - \rho_1 : x \in E\} \\
&= \vee\{\langle u - w, H_1^{-1}(z - c)\rangle - \tau f(z) - \rho_1 : z \in E\} \\
&= \vee\{\langle u - w, H_1^{-1} z\rangle - \tau f(z) : z \in E\} - \langle u - w, H_1^{-1} c\rangle - \rho_1 \\
&= \tau \vee \{\langle u - w, Hz\rangle - f(z) : z \in E\} + \langle u, -H_1^{-1} c\rangle + \langle w, H_1^{-1} c\rangle - \rho_1 \\
&= \tau f^*(H^* u + v) + \langle u, y\rangle + \rho,
\end{aligned}$$

where $H = \frac{1}{\tau} H_1^{-1}$, $v = -H^* w$, $y = -H_1^{-1} c$ and $\rho = \langle w, H_1^{-1} c\rangle - \rho_1$.

We see that $H : E \to E$ is a continuous module automorphism, $v \in E^*$, $y \in E$ and $\rho \in L^0(\mathcal{F})$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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