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# Inverse Elastic Scattering from a Cavity in Homogeneous Medium*

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**Abstract** The paper discusses direct and inverse elastic scattering from a cavity in a homogeneous medium with both Dirichlet and Neumann boundary conditions. Regarding direct scattering, the existence and uniqueness are derived using a variational approach. In the case of inverse scattering, the Fréchet derivatives of the solution operators are investigated, which provides a local stability for the Dirichlet condition.

**Keywords** Elastic cavity, Variational formulation, The Navier equation, Existence, Uniqueness
**2020 MR Subject Classification** 35P25, 35A15, 35R30

## 1 Introduction

In this paper, we consider a time-harmonic plane elastic wave incident on a cavity which can be regarded as a local perturbation below the plane. Elastic scattering has wide applications in seismology and geophysics such as in [1, 18, 23] and attracts more attention recently. However, elastic scattering problems have not been studied intensively due to their inherent difficulties arising from the governing equations and boundary conditions. Arens used integral equation methods to prove the well-posedness of elastic scattering from periodic and non-periodic surfaces in [5–9]. Elschner and Hu applied variational approaches to get similar results for periodic and non-periodic surfaces in [15–16]. They (see [17]) also considered the solvability in weighted Sobolev spaces and proved the existence and uniqueness of elastic scattering from unbounded rough surface with an incident wave. Moreover, Hu, Yuan and Zhao [20] combined the variational formula and the integral equation to prove the uniqueness and existence of two-dimensional local perturbed surface scattering problem with Lipschitz graph.

For cavity problems, there are considerable research results in the literature. For instance, Ammari, Bao and Wood proved the uniqueness and existence of electromagnetic cavity prob-

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lems for TE and TM polarization by variational approaches in [4], and by integral equation methods in [2]. They also showed similar results for Maxwell's equations in [3] using variational approaches. Recently, Bao et al. established a stability result explicitly dependent on the wave number for TE polarization based on Fourier analysis in energy spaces in [14], and for TM polarization in [13]. Numerical methods for large cavities can be found in [12, 21, 24]. For elastic scattering from cavities, Hu, Li and Zhao considered elastic scattering from a cavity in three dimensions and proved the uniqueness under Dirichlet boundary conditions in [19].

The inverse cavity problem is typically formulated to reconstruct the shape of a cavity by measured far-field data on artificial boundaries. In general, to reconstruct a shape, the Fréchet derivative is commonly used. For instance, Bao, Gao and Li considered TE and TM polarizations in [10], proved the uniqueness for lossy media and studied the domain derivative and local stability for the inverse electromagnetic cavity. A similar result for Maxwell's equations was extended in [22]. Bao and Lai [11] proposed an optimization scheme to design a cavity for radar cross section reduction. Recently, Hu, Yuan and Zhao considered the domain derivative for the inverse elastic scattering by a locally perturbed rough surface in [20]. To the authors' best knowledge, research on inverse elastic scattering from cavities is still in its infancy stage.

In this paper, we consider a time-harmonic plane wave incident on a cavity, which is shown in Figure 1, denoted by $D$ with boundary $\Gamma \cup S$, where $\Gamma \subset \{x_2 = 0\}$ is bounded and $S$ is a Lipschitz curve, i.e., there exists an invertible map $f : [0,1] \to S$ satisfying

$$|f(t_1) - f(t_2)| \le L|t_1 - t_2|, \quad t_1, t_2 \in [0,1],$$

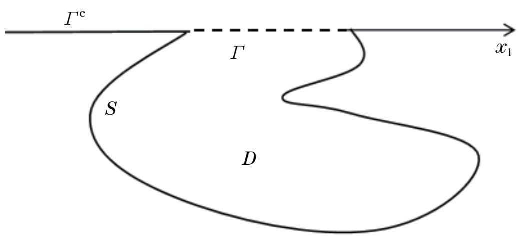


Figure 1 Geometry of the cavity problem.

with the Lipschitz constant $L > 0$ and

$$|f^{-1}(p_1) - f^{-1}(p_2)| \le L'|p_1 - p_2|, \quad p_1, p_2 \in S,$$

with the Lipschitz constant $L' > 0$. We assume if $x = (x_1, x_2) \in S$, then $x_2 \le 0$. It is noted that $S$ is not necessarily a graph of some function. In addition, denote $\Gamma^c = \{x \in \mathbb{R}^2 : x_2 = 0\} \backslash \Gamma$. The medium is assumed to be homogeneous. Suppose the incident time-harmonic plane wave takes the form

$$u_i = c_p \vec{d} \exp(\mathrm{i}k_p(x_1 \sin\theta - x_2 \cos\theta)) + c_s \vec{d}^{\perp} \exp(\mathrm{i}k_s(x_1 \sin\theta - x_2 \cos\theta)),$$

where

$$c_p, c_s \in \mathbb{C}, \quad \vec{d} = (\sin\theta, -\cos\theta), \quad \vec{d}^{\perp} = (\cos\theta, \sin\theta), \quad \theta \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

Define the compressible and shear wave numbers by

$$k_p := \frac{\omega}{\sqrt{2\mu + \lambda}}, \quad k_s := \frac{\omega}{\sqrt{\mu}}.$$

For simplicity, throughout this paper, we assume $c_p = k_p$, $c_s = 0$. Consider the total wave field $u$ which satisfies the following Navier equation with Lamé constants $\lambda > 0$, $\mu > 0$ and frequency $\omega > 0$

$$\mu\Delta u + (\mu+\lambda)\nabla(\nabla\cdot u) + \omega^2 u = 0 \quad \text{in } D \cup \{x_2 > 0\}. \tag{1.1}$$

For convenience, denote by $\Delta^* u := \mu\Delta u + (\mu+\lambda)\nabla(\nabla\cdot u)$. This paper considers two kinds of boundary conditions, i.e., the Dirichlet condition

$$u = 0 \quad \text{on } S \cup \Gamma^c \tag{1.2}$$

and the Neumann condition

$$Tu = 0 \quad \text{on } S \cup \Gamma^c, \tag{1.3}$$

where the differential operator $T$ is defined by $Tu := \mu\partial_n u + (\lambda+\mu)\vec{n}\,\mathrm{div}\, u$. Consider the Helmholtz decomposition for $u$, i.e.,

$$u = -\mathrm{i}(\mathrm{grad}\,\phi + \overrightarrow{\mathrm{curl}}\,\psi), \tag{1.4}$$

with

$$\phi = -\frac{\mathrm{i}}{k_p^2}\mathrm{div}\, u, \quad \psi = \frac{\mathrm{i}}{k_s^2}\mathrm{curl}\, u, \tag{1.5}$$

where $\overrightarrow{\mathrm{curl}}\, u = (\partial_2 u_1, -\partial_1 u_2)^{\mathrm{T}}$ and $\mathrm{curl}\, u = \partial_1 u_2 - \partial_2 u_1$. Then, $\phi$ and $\psi$ satisfy the Helmholtz equations

$$(\Delta + k_p^2)\phi = 0, \quad (\Delta + k_s^2)\psi = 0, \quad x_2 > 0. \tag{1.6}$$

The total wave field consists of the incident wave $u_i$, the reflected wave $u_r$ and the scattered wave $u_s$, i.e., $u = u_i + u_r + u_s$. Here, the reflected wave $u_r$ satisfies

$$\begin{aligned} \Delta^* u_r + \omega^2 u_r = 0 \quad & \text{in } \{x_2 > 0\}, \\ u_r + u_i = 0 \quad & \text{on } \{x_2 = 0\}, \qquad (1.7) \\ \text{or} \quad T(u_i + u_r) = 0 \quad & \text{on } \{x_2 = 0\}. \qquad (1.8) \end{aligned}$$

For the scattering wave $u_s$, it is required to satisfy the following half-plane Kupradze-Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{\frac{1}{2}}(\partial_r\phi(x) - \mathrm{i}k_p\phi(x)) = 0 \quad \text{in } \{x \in \mathbb{R}^2 : x_2 > 0\} \tag{1.9}$$

and

$$\lim_{r\to\infty} r^{\frac{1}{2}}(\partial_r\psi(x) - \mathrm{i}k_s\psi(x)) = 0 \quad \text{in } \{x \in \mathbb{R}^2 : x_2 > 0\}, \tag{1.10}$$

with $r = |x|$ for $\phi = -\frac{\mathrm{i}}{k_p^2}\mathrm{div}\, u_s$ and $\psi = \frac{\mathrm{i}}{k_s^2}\mathrm{curl}\, u_s$.

Now we can state the following two problems, respectively.

**Dirichlet Problem** Find $u$ which satisfies (1.1)–(1.2), where $u_s$ satisfies the Kupradze-Sommerfeld radiation condition (1.9)–(1.10).

**Neumann Problem** Find $u$ which satisfies (1.1) and (1.3), where $u_s$ satisfies the Kupradze-Sommerfeld radiation condition (1.9)–(1.10).

**Remark 1.1** Obviously, the reflected waves exist and we can give the following examples. In fact, we can choose functions

$$\begin{aligned} u_1 &= (\alpha, \beta)^{\mathrm{T}} \exp(\mathrm{i}(\alpha x_1 - \beta x_2)), \\ u_2 &= (\alpha, -\beta)^{\mathrm{T}} \exp(\mathrm{i}(\alpha x_1 + \beta x_2)), \\ u_3 &= (\eta, -\alpha)^{\mathrm{T}} \exp(\mathrm{i}(\alpha x_1 + \eta x_2)), \end{aligned}$$

with $\alpha = k_p \sin\theta, \beta = k_p \cos\theta$ and $\eta = \sqrt{k_s^2 - \alpha^2}$, which are linearly independent solutions to the Navier equation. For the Dirichlet boundary condition (1.7), we can take the reflected wave

$$u_{r,1} = u_1 - \frac{\alpha^2 - \beta\eta}{\alpha^2 + \beta\eta} u_2 - \frac{2\alpha\beta}{\alpha^2 + \beta\eta} u_3,$$

and take the reflected wave

$$\begin{aligned} u_{r,2} = {} & \frac{(\lambda+\mu)k_p^2 + \mu\beta^2\eta + \mu\alpha^2\beta}{2\mu\alpha^2\beta} u_1 \\ & + \frac{-(\lambda+\mu)k_p^2 - \mu\beta^2\eta + \mu\alpha^2\beta}{2\mu\alpha^2\beta} u_2 + \frac{(\mu+\lambda)k_p^2 + \mu\beta^2}{\mu\alpha\eta} u_3 \end{aligned}$$

for the Neumann boundary condition (1.8).

This paper will give the well-posedness of the above Dirichlet problem and Neumann problem. By employing transparent boundary conditions (TBCs for short) given by the Dirichlet to Neumann (DtN for short) and the Neumann to Dirichlet (NtD for short) operators, the original problems can be reduced to a boundary value problem in a bounded domain. Then the variational approach is used to prove the existence and uniqueness of the elastic scattering problem in two dimensions with the Dirichlet condition or the Neumann condition. For the Dirichlet problem, the existence in two dimensions is actually trivial compared to three dimensions in [19]. However, we can obtain the uniqueness for two dimensions while it is still open for three dimensions. One of the main novelties in this paper is to prove the uniqueness for the Neumann problem which is not a trivial extension of the Dirichlet problem. In fact, the main difference between the Dirichlet problem and the Neumann problem is the transparent boundary condition which is given by an DtN operator for the Dirichlet problem, but given by a NtD operator for the Neumann problem. Hence, more complicated discussions are necessary to give the uniqueness for the Neumann problem. Another contribution of this paper is giving the Fréchet derivative for the Neumann problem. In fact, this paper investigates the Fréchet derivative for both the Dirichlet problem and the Neumann problem. But again, the Neumann case is more complicated because the boundary value problem reduced to a bounded domain in Section 2 has an inhomogeneous Neumann boundary condition by directly applying NtD operator. Hence, in order to take advantage of the TBC deduced by a DtN operator, we have to use an additional artificial boundary to rewrite the boundary value problem for the Neumann case. Moreover, the Fréchet derivative for the Neumann problem can only be represented by a variational problem, unlike the Fréchet derivative for the Dirichlet case which can be represented by a boundary value problem.

The paper is outlined as follows. In Section 2, the direct problems are reduced to variational problems in a bounded domain by the DtN operator and the NtD operator. In Section 3 and

Section 4, the existence and the uniqueness for the Dirichlet problem and the Neumann problem are established, respectively. Moreover, the solutions to variational problems can be extended to satisfy the Kupradze-Sommerfeld radiation condition and the Navier equation in upper half-space. In Section 5, the Fréchet derivatives for the Dirichlet problem and the Neumann problem are given respectively and a local stability result is given for the Dirichlet problem. Conclusions are given in Section 6.

## 2 Variational Formulation

To reformulate the two problems in Section 1 in an unbounded domain, it is necessary to introduce transparent boundary conditions to reduce the model problems to a bounded domain. By the Helmholtz decomposition, the wave fields $u_s$, $u_p$ satisfy (1.4) with corresponding $\phi$, $\psi$ satisfying (1.5)–(1.6). Applying the Fourier transform to (1.6) with respect to $x_1$ and using the radiation condition gives

$$\widehat{\phi} = P(\xi)\exp(\mathrm{i}x_2\gamma_p(\xi)), \quad \widehat{\psi} = S(\xi)\exp(\mathrm{i}x_2\gamma_s(\xi))$$

with

$$\gamma_p(\xi) = \sqrt{k_p^2 - \xi^2}, \quad \gamma_s(\xi) = \sqrt{k_s^2 - \xi^2}.$$

Denote by $\widehat{u}(\xi) = \mathcal{F}u(\xi)$ the Fourier transform of $u$ with respect to $x_1$. Then the Fourier transform of $u$ is given by

$$\begin{pmatrix} \widehat{u}_{s,1} \\ \widehat{u}_{s.2} \end{pmatrix} = \begin{pmatrix} \xi & -\mathrm{i}\partial_2 \\ -\mathrm{i}\partial_2 & -\xi \end{pmatrix} \begin{pmatrix} P(\xi)\exp(\mathrm{i}x_2\gamma_p) \\ S(\xi)\exp(\mathrm{i}x_2\gamma_s) \end{pmatrix}. \tag{2.1}$$

Let $x_2 = 0$,

$$\widehat{u}_s(\xi, 0) = \begin{pmatrix} \widehat{u}_{s,1}(\xi, 0) \\ \widehat{u}_{s,2}(\xi, 0) \end{pmatrix} = \begin{pmatrix} \xi & \gamma_s \\ \gamma_p & -\xi \end{pmatrix} \begin{pmatrix} P(\xi) \\ S(\xi) \end{pmatrix}, \tag{2.2}$$

which implies

$$\begin{pmatrix} P(\xi) \\ S(\xi) \end{pmatrix} = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \xi & \gamma_s \\ \gamma_p & -\xi \end{pmatrix} \begin{pmatrix} \widehat{u}_{s,1}(\xi, 0) \\ \widehat{u}_{s,2}(\xi, 0) \end{pmatrix}. \tag{2.3}$$

Inserting (2.3) into (2.1) arrives at the following representation for $u_s$:

$$u_s = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\exp(\mathrm{i}x_2\gamma_p(\xi))M_p(\xi) + \exp(\mathrm{i}x_2\gamma_s(\xi))M_s(\xi))\widehat{u}_s(\xi, 0)\exp(\mathrm{i}x_1\xi)\mathrm{d}\xi \tag{2.4}$$

in $\{x_2 > 0\}$ with

$$M_p(\xi) = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \xi^2 & \gamma_s\xi \\ \gamma_p\xi & \gamma_p\gamma_s \end{pmatrix}, \quad M_s(\xi) = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \gamma_p\gamma_s & -\gamma_s\xi \\ -\gamma_p\xi & \xi^2 \end{pmatrix}.$$

Recalling the definition of differential operator $T$, we have

$$Tu_s := \mu\partial_n u_s + (\lambda + \mu)\vec{n}\,\mathrm{div}\, u_s \quad \text{on } \Gamma. \tag{2.5}$$

Combining (2.4)–(2.5), direct calculation implies that

$$Tu_s = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} M(\xi)\widehat{u}_s(\xi, 0)\exp(\mathrm{i}x\xi)\mathrm{d}\xi, \tag{2.6}$$

where

$$M(\xi) = \frac{\mathrm{i}}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \omega^2\gamma_p & -\xi\omega^2 + \xi\mu(\xi^2 + \gamma_p\gamma_s) \\ \xi\omega^2 - \xi\mu(\xi^2 + \gamma_p\gamma_s) & \omega^2\gamma_s \end{pmatrix}. \tag{2.7}$$

Then the Dirichlet to Neumann operator can be defined by

$$\mathcal{T}f := \mathcal{F}^{-1}(M\widehat{f}), \quad f \in H^{\frac{1}{2}}(\mathbb{R})^2.$$

So $Tu_s = \mathcal{T}u_s$ on $\Gamma$. Noting that $u_i + u_r = 0$ on $\{x_2 = 0\}$ and $u = 0$ on $\Gamma^c$, then TBC for the Dirichlet problem can be given by

$$Tu = \mathcal{T}u + g \quad \text{on } \Gamma \quad \text{with } g = T(u_i + u_r).$$

By TBC, the Dirichlet problem can be reformulated as

$$\begin{aligned} \Delta^* u + \omega^2 u &= 0 \quad \text{in } D, \\ u &= 0 \quad \text{on } S, \\ Tu &= \mathcal{T}u + g \quad \text{on } \Gamma. \end{aligned}$$

Denote $V = H^1_S(D)^2 = \{u \in H^1(D)^2 : u|_S = 0\}$. Then for $u, v \in V$, the Betti formula gives

$$0 = -\int_D (\Delta^* + \omega^2)u \cdot \overline{v}\mathrm{d}x = \int_D \mathcal{E}(u, \overline{v}) - \omega^2 u \cdot \overline{v}\mathrm{d}x - \int_\Gamma (\mathcal{T}u + g) \cdot \overline{v}\mathrm{d}s,$$

where

$$\mathcal{E}(u, v) = \mu(\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2) + (\lambda + \mu)(\nabla \cdot u)(\nabla \cdot v). \tag{2.8}$$

Define a sesquilinear form $B : V \times V \to \mathbb{C}$ by

$$B(u, v) = \int_D \mathcal{E}(u, \overline{v}) - \omega^2 u \cdot \overline{v}\mathrm{d}x - \int_\Gamma \mathcal{T}u \cdot \overline{v}\mathrm{d}s.$$

Now we obtain the variational formula of Dirichlet boundary value problem.

**Variational Problem 2.1** Find $u \in V$ such that $B(u, v) = (g, v)_\Gamma, \forall v \in V$.

It is similar to give TBC and the variational formula corresponding to the Neumann problem. Notice that in this case, it is not $u_s$ but $Tu_s$, that has compact support on $\Gamma$. So $\widehat{u}_s(\xi, 0)$ may not be integrable with respect to $\xi \in \mathbb{R}$. Thus, (2.3) and (2.6) may not hold in this case. However, $\widehat{u}_s(\xi, 0)$ can be represented by $\widehat{Tu_s}(\xi, 0)$ similarly. According to (2.4), we obtain

$$\widehat{Tu_s}(\xi, x_2) = \begin{pmatrix} \mu\partial_2 & 0 \\ \mathrm{i}(\lambda + \mu)\xi & (\lambda + 2\mu)\partial_2 \end{pmatrix} \begin{pmatrix} \widehat{u}_{s,1} \\ \widehat{u}_{s,2} \end{pmatrix}. \tag{2.9}$$

Combining (2.1) and (2.9) yields

$$\widehat{Tu_s}(\xi, x_2) = \begin{pmatrix} \mu\partial_2 & 0 \\ \mathrm{i}(\lambda + \mu)\xi & (\lambda + 2\mu)\partial_2 \end{pmatrix} \begin{pmatrix} \xi & -\mathrm{i}\partial_2 \\ -\mathrm{i}\partial_2 & -\xi \end{pmatrix} \begin{pmatrix} P(\xi)\exp(\mathrm{i}x_2\gamma_p) \\ S(\xi)\exp(\mathrm{i}x_2\gamma_s) \end{pmatrix}.$$

Take $x_2 = 0$, then $P$ and $S$ can be represented by $\widehat{Tu_s}(\xi, 0)$. Inserting the representation in (2.1), we have

$$\widehat{u}_s(\xi, x_2) = (\exp(\mathrm{i}x_2\gamma_p(\xi))M_p(\xi) + \exp(\mathrm{i}x_2\gamma_s(\xi))M_s(\xi))M^{-1}\widehat{Tu_s}(\xi, 0).$$

Taking the inverse Fourier transformation and $x_2 = 0$ gives

$$u_s(\xi, 0) = \mathcal{F}^{-1}(M^{-1}\widehat{u}_s(\xi, 0)).$$

Hence the NtD operator $\Lambda$ can be defined by

$$\Lambda(f) := \mathcal{F}^{-1}(M^{-1}\widehat{f}), \quad f \in H^{-\frac{1}{2}}(\mathbb{R})^2,$$

which implies

$$\Lambda(Tu_s) = u_s \quad \text{on } \Gamma.$$

Denote $h = u_r + u_i$ and $g = -(\Delta^* + \omega^2)h$. Then the Neumann problem can be reformulated as

$$\begin{aligned} \Delta^* u_s + \omega^2 u_s &= 0 \quad \text{in } D, \\ Tu_s &= -Th \quad \text{on } S, \\ u_s &= \Lambda(Tu_s) \quad \text{on } \Gamma. \end{aligned} \tag{2.10}$$

Define the function space

$$W = \{u \in H^1(D)^2 : Tu \in H_0^{-\frac{1}{2}}(\Gamma)^2, u|_\Gamma = \Lambda(Tu|_\Gamma)\},$$

where

$$H_0^{-\frac{1}{2}}(\Gamma)^2 = \{u \in H^{-\frac{1}{2}}(\Gamma)^2 : \Lambda(\widetilde{u}) \in H^{\frac{1}{2}}(K)^2, \forall K \subset\subset \mathbb{R}^2\},$$

and $\widetilde{u}$ is zero extension of $u$. In fact, it is clear that $H_0^{-\frac{1}{2}}(\Gamma)^2$ is the dual space of $H^{\frac{1}{2}}(\Gamma)^2$ in [7]. The sesquilinear form $\mathcal{B}$: $W \times W \to \mathbb{C}$ is defined by

$$\mathcal{B}(v, w) = \int_D \mathcal{E}(v, \overline{w}) - \omega^2 v \cdot \overline{w}\mathrm{d}x - \int_\Gamma Tu \cdot \overline{\Lambda(Tw)}\mathrm{d}s.$$

Now we can give the following variational formula by the Betti formula.

**Variational Problem 2.2** Find $u_s \in W$ such that

$$\mathcal{B}(u_s, w) = \int_S -Th \cdot \overline{w}\mathrm{d}s - \int_D g \cdot \overline{w}\mathrm{d}x, \quad \forall w \in W.$$

In the next two sections, we consider the existence and uniqueness of solutions to the above two variational problems. Moreover, they can be extended to $\{x_2 > 0\}$ satisfying the Kupradze-Sommerfeld radiation condition.

## 3 Existence and Uniqueness for the Dirichlet Problem

Since this problem is in a bounded domain with Lipschitz continuous boundary, we have the well-known compact embedding result that $H^1(D)$ is a compact subset of $L^2(D)$. Hence the uniqueness of the variational problem and Fredholm alternative yield existence directly. To this end, we show that the continuity of the DtN operator results in the continuity of the sesquilinear form $B$, which is stated in the following lemma with the proof omitted.

**Lemma 3.1** *The* DtN *operator* $\mathcal{T}$ *is a continuous linear operator from* $H_0^{\frac{1}{2}}(\Gamma)^2$ *to* $H^{-\frac{1}{2}}(\Gamma)^2$.

In order to use the Fredholm alternative, it is necessary to show the sesquilinear form satisfies the Gårding inequality. Take the real part of $B$,

$$\Re B(u,u) = \int_D \mathcal{E}(u,\overline{u}) - \omega^2 u\cdot\overline{u}\mathrm{d}x - \Re\int_\Gamma \mathcal{T}u\cdot\overline{u}\mathrm{d}s.$$

Considering (2.8) gives

$$\int_D \mathcal{E}(u,\overline{u})\mathrm{d}x = \mu\|\nabla u\|^2_{L^2(D)^2} + (\lambda+\mu)\|\nabla\cdot u\|^2_{L^2(D)^2} \ge \mu\|\nabla u\|^2_{L^2(D)^2}.$$

It turns out that

$$\Re B(u,u) \ge \mu\|\nabla u\|^2_{L^2(D)^2} - \omega^2\|u\|^2_{L^2(D)^2} - \Re\langle\mathcal{T}u,u\rangle_\Gamma. \tag{3.1}$$

By the Plancherel identity,

$$-\Re\langle\mathcal{T}u,u\rangle_\Gamma = -\langle\Re M\mathcal{F}u,\mathcal{F}u\rangle_\Gamma,$$

where $\Re M = \frac{M+\overline{M}^{\mathrm{T}}}{2}$. However, the sign of $-\Re\langle\mathcal{T}u,u\rangle_\Gamma$ is undetermined. But fortunately, for sufficiently large $|\xi|$, we know $-\Re M(\xi)$ is positive definite, as given in Lemma 3.2.

**Lemma 3.2** $-\Re M(\xi) > 0$ *for* $|\xi| > k_s$.

The proof is also omitted here as it is trivial compared to the results in [16]. Noting that

$$\langle M\mathcal{F}u,\mathcal{F}u\rangle = \int_{|\xi|>k_s} M(\xi)|\widehat{u}|^2\mathrm{d}\xi + \int_{|\xi|\le k_s} M(\xi)|\widehat{u}|^2\mathrm{d}\xi,$$

and using Lemma 3.2, we have

$$-\langle\Re M\mathcal{F}u,\mathcal{F}u\rangle \ge -\int_{|\xi|\le k_s}\Re M(\xi)|\widehat{u}|^2\mathrm{d}\xi.$$

Therefore, it suffices to estimate the intergal in $\{|\xi|\le k_s\}$. The following inequality is similar to the three-dimensional case in [20].

**Lemma 3.3** *For any* $u\in V$, *we have* $\Re B(u,u) > C_1\|\nabla u\|^2_{L^2(D)^2} - C_2\|u\|^2_{L^2(D)^2}$, *where* $C_1, C_2$ *are positive constants which depend on* $\lambda,\mu,\omega$.

Next, the uniqueness and existence of the variational problem can be derived by Lemma 3.3 and the Fredholm alternative.

**Theorem 3.1** *Variational Problem* 2.1 *admits a unique solution* $u\in V$.

**proof** Assume $g=0$. We have

$$0 = \Im\langle g,u\rangle_\Gamma = -\Im B(u,u) = \Im\langle\mathcal{T}u,u\rangle_\Gamma = \Im\langle M\widehat{u},\widehat{u}\rangle_{\mathbb{R}}.$$

Combining (2.2) and (2.7) gives

$$\langle M\widehat{u},\widehat{u}\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \mathrm{i}\begin{pmatrix} P\\ S\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\mu\xi\gamma_p & \omega^2-\mu\xi^2\\ \omega^2-\mu\xi^2 & -\mu\xi\gamma_s\end{pmatrix}\begin{pmatrix}\xi & \overline{\gamma_s}\\ \overline{\gamma_p} & -\xi\end{pmatrix}\begin{pmatrix}\overline{P}\\ \overline{S}\end{pmatrix}\mathrm{d}\xi.$$

Denote the matrix by

$$A = \begin{pmatrix} \mu\xi\gamma_p & \omega^2 - \mu\xi^2 \\ \omega^2 - \mu\xi^2 & -\mu\xi\gamma_s \end{pmatrix} \begin{pmatrix} \xi & \overline{\gamma_s} \\ \overline{\gamma_p} & -\xi \end{pmatrix}.$$

Taking the real part implies

$$\Re A = \begin{cases} 0, & \text{if } |\xi| > k_s, \\ \omega^2 \mathrm{diag}(0, \gamma_s), & \text{if } k_p < |\xi| \le k_s, \\ \omega^2 \mathrm{diag}(\gamma_p, \gamma_s), & \text{if } |\xi| \le k_p. \end{cases}$$

So we have

$$\Im\langle \mathcal{T} u, u\rangle_\Gamma = \omega^2 \Big( \int_{|\xi|\le k_p} \gamma_p(\xi)|P(\xi)|^2 \mathrm{d}\xi + \int_{|\xi|\le k_s} \gamma_s(\xi)|S(\xi)|^2 \mathrm{d}\xi \Big) = 0. \tag{3.2}$$

Hence

$$P(\xi) = S(\xi) = 0 \quad \text{for } |\xi| \le k_p,$$

which implies

$$\widehat{u}(\xi, 0) = 0 \quad \text{for } |\xi| \le k_p.$$

Since $u$ has compact support on $\{y = 0\}$, $\widehat{u}(\xi, 0)$ is analytic with respect to $\xi$. By unique continuation, $\widehat{u}(\xi, 0) = 0$ for $\xi \in \mathbb{R}$. So $u(x, 0) = 0$ and $Tu = \mathcal{T}u = 0$ on $\Gamma$. By the Holmgren uniqueness theorem, $u = 0$ in $D$ which implies uniqueness. Finally, combining Lemma 3.3 and the Fredholm alternative yields existence.

**Remark 3.1** Theorem 3.1 shows that there exists a unique solution $u$ to Variational Problem 2.1 in a bounded domain $D$. We can extend $u_s = u - u_i - u_r$ to $\{x_2 > 0\}$ by (2.4), though it may not satisfy the Kupradze-Sommerfeld radiation condition. In fact, the condition that $u_s$ satisfies (2.4) is weaker than the half-plane Kupradze-Sommerfeld radiation condition (1.9)–(1.10) (see [8]). But we can extend $u_s$ by

$$u_s = \int_\Gamma T_z G_D(x, y) u_s(y) \mathrm{d}s(y), \quad x \in \{x_2 > 0\}, \tag{3.3}$$

where $G_D(x, y)$ is the half-space Green tensor for the Dirichlet problem. According to the property of Green tensor, we can verify that the corresponding $u$ is the solution to the Dirichlet problem in $D \cup \{x_2 \ge 0\}$ (see [8]). Then by representation theorem we know that any solution to Dirichlet problem in $D \cup \{x_2 \ge 0\}$ satisfies (3.3), which means that the extension is unique.

## 4 Existence and Uniqueness for the Neumann Problem

Recalling the two variational problems in Section 2, we can find that the main difference between the Dirichlet problem and the Neumann one is that $M$ in (2.7) is replaced by $M^{-1}$, i.e.,

$$M^{-1} = \frac{\rho(\xi)}{\mathrm{i}d(\xi)} \begin{pmatrix} \omega^2\gamma_s & \xi\omega^2 - \xi\mu\rho \\ -\xi\omega^2 + \xi\mu\rho & \omega^2\gamma_p \end{pmatrix}, \tag{4.1}$$

where

$$\rho = \xi^2 + \gamma_s\gamma_p, \quad d = \omega^4\gamma_s\gamma_p + (\xi\omega^2 - \xi\mu\rho)^2.$$

This section shows that $M^{-1}$ has similar properties to $M$ though it is more complicated. Before proceeding to the uniqueness, we give the following lemma to show that $\Lambda$ is continuous, which implies that the sesquilinear form is also continuous.

**Lemma 4.1** *The* NtD *operator $\Lambda$ is a bounded linear operator from $H_0^{-\frac{1}{2}}(\Gamma)^2$ to $H^{\frac{1}{2}}(\Gamma)^2$.*

**Proof**

$$\|\Lambda\|^2 = \sup_{u\in W, u\neq 0} \frac{\|\Lambda u\|^2_{H^{\frac{1}{2}}(\Gamma)^2}}{\|u\|^2_{H^{-\frac{1}{2}}(\Gamma)^2}} = \sup_{u\in W, u\neq 0} \frac{\displaystyle\int_{\mathbb{R}} |M^{-1}\widehat{u}|^2(1+\xi^2)^{\frac{1}{2}}\mathrm{d}\xi}{\displaystyle\int_{\mathbb{R}} |\widehat{u}|^2(1+\xi^2)^{-\frac{1}{2}}\mathrm{d}\xi}. \tag{4.2}$$

When $|\xi|\to+\infty$, it is easy to calculate

$$\gamma_p \sim \mathrm{i}|\xi|, \quad \gamma_s \sim \mathrm{i}|\xi|, \quad (\xi^2+\gamma_s\gamma_p) \sim \frac{k_p^2+k_s^2}{2}.$$

It turns out that

$$[\xi\omega^2 - \xi\mu(\xi^2+\gamma_s\gamma_p)] \sim \xi\Big(\omega^2 - \mu\frac{k_s^2+k_p^2}{2}\Big) \tag{4.3}$$

and

$$[\omega^4\gamma_s\gamma_p + (\xi\omega^2 - \xi\mu(\xi^2+\gamma_s\gamma_p))^2] \sim \xi^2\Big[\omega^4 - \Big(\omega^2 - \mu\frac{k_s^2+k_p^2}{2}\Big)^2\Big]. \tag{4.4}$$

Combining (4.1) and (4.3)–(4.4) gives an estimate of $\|M^{-1}\|$,

$$\|M^{-1}\| \le C(1+\xi^2)^{-1}.$$

Inserting it into (4.2) yields $\|\Lambda\|\le C$, where $C$ depends on $\omega,\mu$ and $\lambda$.

Then we consider $-\Re M^{-1}$ similarly to in Section 2.

**Lemma 4.2** $-\Re M^{-1}(\xi)>0$ *for sufficiently large* $|\xi|$.

**Proof** Taking the real part of $M^{-1}$ for $|\xi|>k_s$, we have

$$\Re M^{-1} = \frac{\rho(\xi)}{d(\xi)}\begin{pmatrix} \omega^2|\gamma_s| & -\mathrm{i}(\xi\omega^2-\xi\mu\rho) \\ \mathrm{i}(\xi\omega^2-\xi\mu\rho) & \omega^2|\gamma_p| \end{pmatrix},$$

with

$$\rho(\xi) = \xi^2 - |\gamma_s||\gamma_p| > 0$$

and

$$d(\xi) = -\omega^4|\gamma_s||\gamma_p| + (\xi\omega^2-\xi\mu\rho)^2.$$

In order to prove $-\Re M^{-1}(\xi)>0$, it suffices to verify $\frac{\rho(\xi)}{d(\xi)}\omega^2|\gamma_s|<0$ and $\det(-\Re M^{-1})>0$. Recall that when $|\xi|\to\infty$, $\gamma_s\sim\mathrm{i}|\xi|, \gamma_p\sim\mathrm{i}|\xi|$ and $\rho\sim\frac{k_s^2+k_p^2}{2}$. It turns out that

$$d \sim \xi^2\Big(\Big(\omega^2-\mu\frac{k_s^2+k_p^2}{2}\Big)^2 - \omega^4\Big) = \omega^2\Big(\frac{\mu}{4\mu+2\lambda}-\frac{3}{2}\Big) < 0,$$

which means that $\frac{\rho(\xi)}{d(\xi)}\omega^2|\gamma_s| < 0$ for sufficiently large $|\xi|$. Direct calculation gives

$$\det(-\Re M^{-1}) = \frac{\rho^2}{d^2}(\omega^4|\gamma_s||\gamma_p| - (\xi\omega^2 - \xi\mu\rho)^2) > 0$$

for sufficiently large $|\xi|$. So $-\Re M^{-1}$ is positive definite for sufficiently large $|\xi|$.

Using the above lemma, we can estimate $\Re\mathcal{B}(u,u)$ and then get the following inequality.

**Lemma 4.3** *For any $u \in W$, we have $\Re\mathcal{B}(u,u) > C_1\|\nabla u\|^2_{L^2(D)^2} - C_2\|u\|^2_{L^2(D)^2}$, where $C_1, C_2$ are positive constants and they both depend on $\lambda, \mu$ and $\omega$.*

**Proof** Similar to Section 3, for $u \in W$, we have

$$\Re\mathcal{B}(u,u) \geq \mu\|\nabla u\|^2_{L^2(D)^2} - \omega^2\|u\|^2_{L^2(D)^2} - \Re\langle Tu, \Lambda(Tu)\rangle_\Gamma. \tag{4.5}$$

By Lemma 4.2, there exists $k_0 > 0$ such that $-\Re M^{-1}(\xi) > 0$ for $|\xi| > k_0$. So it can be deduced that

$$-\Re\langle Tu, \Lambda(Tu)\rangle_\Gamma \geq -\Re\int_{|\xi|\leq k_0} \widehat{Tu}\cdot\overline{M^{-1}\widehat{Tu}}\mathrm{d}\xi \geq -C\int_{|\xi|\leq k_0} \|M^{-1}\||\widehat{Tu}|^2\mathrm{d}\xi. \tag{4.6}$$

Noting for $|\xi| \leq k_0$,

$$\|M^{-1}\| \leq C(1+\xi^2)^{-\frac{1}{2}} \leq C(1+\xi^2)^{-1}.$$

So we have

$$\int_{|\xi|\leq k_0} \|M^{-1}\||\widehat{Tu}|^2\mathrm{d}\xi \leq C\int_{|\xi|\leq k_0} (1+\xi^2)^{-1}|\widehat{Tu}|^2\mathrm{d}\xi \leq C\|Tu\|^2_{H^{-1}(\Gamma)}. \tag{4.7}$$

By the trace theorem and the interpolation inequality, we obtain

$$\|Tu\|^2_{H^{-1}(\Gamma)^2} \leq C\|u\|^2_{L^2(\Gamma)^2} \leq C\|u\|^2_{H^{\frac{1}{2}}(D)^2} \leq \varepsilon\|u\|^2_{H^1(D)^2} + C(\varepsilon)\|u\|^2_{L^2(D)^2}. \tag{4.8}$$

Here $C > 0$ are different constants depending on $\mu, \lambda$ and $\omega$, $\varepsilon > 0$ is sufficiently small and $C(\varepsilon) > 0$ is dependent on $\mu, \lambda, \omega$ and $\varepsilon$. By combining (4.5)–(4.8), we obtain the inequality

$$\Re\mathcal{B}(u,u) \geq \mu\|\nabla u\|^2_{L^2(D)^2} - \omega^2\|u\|^2_{L^2(D)^2} - C\varepsilon\|u\|^2_{H^1(D)^2} - C(\varepsilon)\|u\|^2_{L^2(D)^2}.$$

As $\varepsilon$ is sufficiently small, it turns out that

$$\Re\mathcal{B}(u,u) \geq C_1\|\nabla u\|^2_{L^2(D)^2} - C_2\|u\|^2_{L^2(D)^2}.$$

Next, we can proceed to the uniqueness for the Neumann problem. Though $M^{-1}$ is more complicated than $M$, it is difficult to directly give the representation of the imaginary part of the sesquilinear form like (3.2). But the following proof shows that it is not necessary to get the exact representation of the imaginary part of the sesquilinear form.

**Theorem 4.1** *Variational Problem* 2.2 *admits a unique solution $u \in W$.*

**Proof** Similar to Section 3, we will prove the uniqueness, which yields existence. Assume $h = 0$ (i.e., $u = u_s$) which implies $g = 0$. Taking the imaginary part of the variational formula gives

$$\Im\mathcal{B}(u,u) = -\Im\langle Tu, \Lambda(Tu)\rangle_\Gamma = -\Im\langle\widehat{Tu}, M^{-1}\widehat{Tu}\rangle_\Gamma = \langle\Im M^{-1}\widehat{Tu}, \widehat{Tu}\rangle_\Gamma = 0.$$

For $|\xi| > k_s$, we have

$$\rho = \xi^2 - |\gamma_s||\gamma_p|, \quad d = -\omega^4|\gamma_s||\gamma_p| + (\xi\omega^2 - \xi\mu\rho)^2.$$

Then take the imaginary part of (4.1) to get $\Im M^{-1} = 0$. For $|\xi| \le k_p$, we have

$$\rho = \xi^2 + |\gamma_s||\gamma_p| > 0, \quad d = \omega^4|\gamma_s||\gamma_p| + (\xi\omega^2 - \xi\mu\rho)^2 > 0.$$

Simple calculation gives

$$\Im M^{-1} = -\frac{\rho(\xi)}{d(\xi)}\begin{pmatrix} \omega^2|\gamma_s| & 0 \\ 0 & \omega^2|\gamma_p| \end{pmatrix}.$$

It turns out that $\Im M^{-1} < 0$. For $k_p < |\xi| \le k_s$, we have

$$\rho = \xi^2 + \mathrm{i}|\gamma_s||\gamma_p|, \quad d = \omega^4|\gamma_s||\gamma_p|\mathrm{i} + (\xi\omega^2 - \xi\mu\rho)^2.$$

Direct calculation gives

$$\Im M^{-1} = \frac{1}{\zeta^4 + \mu^4\xi^4|\gamma_p|^2|\gamma_s|^2}\begin{pmatrix} a_1 & b \\ -b & a_2 \end{pmatrix},$$

where

$$a_1 = -\zeta^2\omega^2|\gamma_s|, \quad a_2 = -\mu^2\xi^2\omega^2|\gamma_s||\gamma_p|^2,$$
$$b = -\mathrm{i}[-\zeta^2\xi\mu|\gamma_p||\gamma_s| - (\xi\omega^2 - \xi^3\mu)\mu^2\xi^2|\gamma_p||\gamma_s|],$$

with

$$\zeta = \omega^2 - \xi^2\mu.$$

Obviously $a_1 < 0$, then it suffices to verify $\det(\Im M^{-1}) = 0$. In fact, we have

$$\begin{aligned} &(\zeta^4 + \mu^4\xi^4|\gamma_p|^2|\gamma_s|^2)^2 \det(\Im M^{-1}) \\ &= a_1a_2 + b^2 \\ &= \mu^2\xi^2\zeta^2\omega^4|\gamma_p|^2|\gamma_s|^2 - (-\mu^2\xi^3\omega^2 + \xi^5\mu^3 - \zeta^2\xi\mu)^2|\gamma_p|^2|\gamma_s|^2 \\ &= \mu^2\xi^2\zeta^2\omega^4|\gamma_p|^2|\gamma_s|^2 - |\gamma_p|^2|\gamma_s|^2\mu^2\xi^2\zeta^2(\zeta + \mu\xi^2)^2 \\ &= \mu^2\xi^2\zeta^2|\gamma_p|^2|\gamma_s|^2(\omega^4 - \omega^4) = 0. \end{aligned}$$

Therefore it turns out that $\det(\Im M^{-1}) = 0$, and thus $\Im M^{-1} \le 0$.

In summary,

$$\Im M^{-1} \begin{cases} = 0, & \text{if } |\xi| > k_s, \\ \le 0, & \text{if } k_p < |\xi| \le k_s, \\ < 0, & \text{if } |\xi| \le k_p. \end{cases}$$

Hence $\widehat{Tu} = 0$ for $|\xi| \le k_p$. Similarly to Section 3, by unique continuation and the Holmgren's uniqueness theorem, $u = 0$ in $D$ which implies uniqueness. Combining Lemma 4.3 and the Fredholm alternative yields existence.

**Remark 4.1** Similar to Section 2, we can extend $u_s$ to $\{x_2 > 0\}$ by

$$u_s = -\int_\Gamma G_N(x,y)Tu_s(y)\mathrm{d}s(y), \quad x \in \{x_2 > 0\}, \tag{4.9}$$

where $G_N(x,y)$ is the half-space Green tensor for the Neumann problem. Obviously by representation theorem, this extension is unique. Hence we get the unique solution to the Neumann problem in $D \cup \{x_2 \ge 0\}$.

## 5 Local Stability of Inverse Cavity Problems

In this section, inverse cavity problems, i.e., to reconstruct the unknown cavity $S$ from $u|_\Gamma$, are considered by investigating the Fréchet derivative of the solution operator for the two cases. For the Dirichlet problem, the Fréchet derivative implies a local stability result of inverse cavity problem directly. This can be deduced by the fact that when the Dirichlet data of Fréchet derivative on the boundary of the cavity vanishes, i.e., the solution $u$ to the original problem is zero. However, for the Neumann case, the Fréchet derivative satisfies different boundary value condition, which results in major difference from the Dirichlet case.

### 5.1 Dirichlet case

Let $S$ be $C^2$ and $h(x) \in C^2(S,\mathbb{R})^2$ satisfying $h(x)|_{S\cup\Gamma} = 0$. Denote by $S_h = \{x + h(x) : x \in S\}$ and $D_h$ the domain with boundary $S_h \cup \Gamma$. Similarly denote $V_h = \{u \in H^1(D_h)^2 : u = 0, \text{on } S_h\}$. Define the solution operator $\mathcal{U} : C^2(S,\mathbb{R}^2) \to H^{\frac{1}{2}}(\Gamma)^2$ by

$$\mathcal{U}(h) = u_h|_\Gamma,$$

where $u_h$ is the solution to the variational problem for the Dirichlet boundary condition, i.e.,

$$B_h(u_h, v_h) = (g, v_h)_\Gamma, \quad \forall v_h \in V_h \tag{5.1}$$

with

$$B_h(u_h, v_h) = \int_{D_h} \mathcal{E}(u_h, \overline{v}_h) - \omega^2 u_h \cdot \overline{v}_h \mathrm{d}x - \int_\Gamma \mathcal{T} u_h \cdot \overline{v}_h \mathrm{d}s. \tag{5.2}$$

Extend $h(x)$ to $C^2(\overline{D},\mathbb{R}^2)$ and still denote it by $h$ such that $h(x)|_\Gamma = 0$ and

$$\|h\|_{H^{1,\infty}(D)^2} \le C\|h\|_{H^{1,\infty}(S)^2}. \tag{5.3}$$

Denote $\mathcal{H}_h$ by $\mathcal{H}_h(y) = y + h(y),\ y \in D$. Obviously, it is invertible for sufficiently small $\|h\|_{1,\infty}$. Then taking the variable transform $x = \mathcal{H}_h(y)$ implies

$$\begin{aligned} B_h(u_h, v_h) = &\mu \int_D \sum_{j=1}^2 \nabla \widetilde{u}_{h_j} \mathcal{J}_{\mathcal{H}_h^{-1}} \mathcal{J}^{\mathrm{T}}_{\mathcal{H}_h^{-1}} \nabla \overline{\widetilde{v}}_{h_j} \det \mathcal{J}_{\mathcal{H}_h} \mathrm{d}y \\ &+ (\lambda+\mu) \int_D (\nabla \widetilde{u}_h : \mathcal{J}_{\mathcal{H}_h^{-1}})(\nabla \overline{\widetilde{v}}_h : \mathcal{J}^{\mathrm{T}}_{\mathcal{H}_h^{-1}}) \det \mathcal{J}_{\mathcal{H}_h} \mathrm{d}y \\ &- \omega^2 \int_D \widetilde{u}_h \cdot \overline{\widetilde{v}}_h \det \mathcal{J}_{\mathcal{H}_h} \mathrm{d}y - \int_\Gamma \mathcal{T} \widetilde{u}_h \cdot \overline{\widetilde{v}}_h \mathrm{d}s(y) := B^h(\widetilde{u}_h, \widetilde{v}_h), \end{aligned}$$

where $\widetilde{u}_h = u_h \circ \mathcal{H}_h \in V$, $\widetilde{v}_h = v_h \circ \mathcal{H}_h \in V$. Hence $B^h$ is a sesquilinear form on $V \times V$. Thus the variational formula (5.1) becomes

$$B^h(\widetilde{u}_h, v) = (g, v)_\Gamma, \quad \forall v \in V. \tag{5.4}$$

Next we proceed to derive the Fréchet derivative of the solution operator $\mathcal{U}$. Let $\mathcal{U}'(0)$: $C^2(S,\mathbb{R})^2 \to H^{\frac{1}{2}}(\Gamma)^2$ be the Fréchet derivative of $\mathcal{U}$ at 0. Notice that $C^2(S,\mathbb{R})^2$ is equipped with norm $\|\cdot\|_{H^{1,\infty}(S)^2}$. The following theorem shows for $h \in C^2(S,\mathbb{R})^2, \mathcal{U}'(0)h$ satisfies the homogeneous Navier equation in $D$, the homogeneous TBC on $\Gamma$ and an inhomogeneous Dirichlet condition (5.7) on $S$.

**Theorem 5.1** *Suppose $h \in C^2(S,\mathbb{R})^2$ satisfying $h = 0$ on $S \cup \Gamma$. Let $u^*$ be the solution to the problem*

$$(\Delta^* + \omega^2)u^* = 0 \quad \text{in } D, \tag{5.5}$$

$$Tu^* = \mathcal{T}u^* \quad \text{on } \Gamma, \tag{5.6}$$

$$u^* = -(n \cdot h)\partial_n u \quad \text{on } S. \tag{5.7}$$

*Then $\mathcal{U}'(0)h = u^*|_\Gamma$.*

**Proof** Let $w_h = h \cdot \nabla u$. Then $h|_\Gamma = 0$ implies $w_h|_\Gamma = 0$. So $w_h + u^* \in V$ and $(w_h + u^*)|_\Gamma = u^*|_\Gamma$. By the trace theorem,

$$\|u_h - u - u^*\|_{H^{\frac{1}{2}}(\Gamma)^2} \leq C\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(D)^2} \tag{5.8}$$

with the constant $C > 0$. For convenience, we denote $\|h\|_{1,\infty} = \|h\|_{H^{1,\infty}(D)^2}$. Then combining (5.3) and (5.8) yields that we only need to prove

$$\lim_{\|h\|_{1,\infty}\to 0} \frac{\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(D)^2}}{\|h\|_{1,\infty}} = 0.$$

For any $v \in V$,

$$B(\widetilde{u}_h - u - u^* - w_h, v) = B(\widetilde{u}_h - u, v) - B(u^* + w_h, v).$$

Firstly consider

$$B(\widetilde{u}_h - u, v) = B(\widetilde{u}_h, v) - (g, v)_\Gamma = B(\widetilde{u}_h, v) - B^h(\widetilde{u}_h, v) = \sum_{i=1}^{3} B_i(\widetilde{u}_h, v),$$

where

$$B_1(\widetilde{u}_h, v) = \mu \int_D \sum_{j=1}^{2} \nabla \widetilde{u}_{h,j}(I_2 - \mathcal{J}_{\mathcal{H}_h^{-1}}\mathcal{J}^{\mathrm{T}}_{\mathcal{H}_h^{-1}} \det \mathcal{J}_{\mathcal{H}_h})\nabla \overline{v}_j \mathrm{d}x, \tag{5.9}$$

$$B_2(\widetilde{u}_h, v) = (\lambda + \mu) \int_D (\nabla \cdot \widetilde{u}_h)(\nabla \cdot \overline{v}) - (\nabla \widetilde{u}_h : \mathcal{J}_{\mathcal{H}_h^{-1}})(\nabla \overline{v} : \mathcal{J}^{\mathrm{T}}_{\mathcal{H}_h^{-1}}) \det \mathcal{J}_{\mathcal{H}_h} \mathrm{d}x, \tag{5.10}$$

$$B_3(\widetilde{u}_h, v) = \omega^2 \int_D \widetilde{u}_h \cdot \overline{v}(\det \mathcal{J}_{\mathcal{H}_h} - 1)\mathrm{d}x. \tag{5.11}$$

Consider the Jacobi matrix

$$\mathcal{J}_{\mathcal{H}_h} = I_2 + \nabla h.$$

Direct calculation implies

$$\det \mathcal{J}_{\mathcal{H}_h} = 1 + \nabla \cdot h + O(\|h\|^2_{1,\infty}\|v\|_{H^1(D)^2}) \tag{5.12}$$

and

$$\mathcal{J}_{\mathcal{H}_h^{-1}} = I_2 - \nabla h + O(\|h\|^2_{1,\infty}\|v\|_{H^1(D)^2}). \tag{5.13}$$

Combining (5.12)–(5.13) gives

$$\mathcal{J}^{\mathrm{T}}_{\mathcal{H}_h^{-1}}\mathcal{J}_{\mathcal{H}_h^{-1}}\det\mathcal{J}_{\mathcal{H}_h}=I_2-(\nabla h+\nabla h^{\mathrm{T}})-(\nabla\cdot h)I_2+O(\|h\|^2_{1,\infty}\|v\|_{H^1(D)^2}). \tag{5.14}$$

Insert (5.12)–(5.14) into (5.9)–(5.11),

$$\sum_{i=1}^{3}B_i(\widetilde{u}_h,v)=\sum_{i=1}^{3}g_i(h)(\widetilde{u}_h,v)+O(\|h\|^2_{1,\infty}\|v\|_{H^1(D)^2}), \tag{5.15}$$

where

$$g_1(h)(\widetilde{u}_h,v)=\mu\int_D\sum_{j=1}^{2}\nabla\widetilde{u}_{h,j}(\nabla h+\nabla h^{\mathrm{T}}-(\nabla\cdot h)I_2)\nabla\overline{v}_j\mathrm{d}x, \tag{5.16}$$

$$\begin{aligned}g_2(h)(\widetilde{u}_h,v)=(\lambda+\mu)\int_D&(\nabla\cdot\widetilde{u}_h)(\nabla\overline{v}:\nabla h^{\mathrm{T}})+(\nabla\cdot\overline{v})(\nabla\widetilde{u}_h:\nabla h^{\mathrm{T}})\\&-(\nabla\cdot h)(\nabla\cdot\overline{v})(\nabla\cdot\widetilde{u}_h)\mathrm{d}x,\end{aligned} \tag{5.17}$$

$$g_3(h)(\widetilde{u}_h,v)=\omega^2\int_D(\widetilde{u}_h\cdot\overline{v})(\nabla\cdot h)\mathrm{d}x. \tag{5.18}$$

It is easy to verify

$$|g_i(h)(\widetilde{u}_h,v)-g_i(h)(u,v)|=O(\|h\|_{1,\infty}\|v\|_{H^1(D)^2}\|\widetilde{u}_h-u\|_{H^1(D)^2}). \tag{5.19}$$

Insert (5.19) into (5.15)–(5.18),

$$\begin{aligned}B(\widetilde{u}_h-u,v)=g_1(h)(u,v)+g_2(h)(u,v)+g_3(h)(u,v)+O(\|h\|^2_{1,\infty}\|v\|_{H^1(D)^2})\\+O(\|h\|_{1,\infty}\|v\|_{H^1(D)^2}\|\widetilde{u}_h-u\|_{H^1(D)^2}).\end{aligned} \tag{5.20}$$

For $g_1(h)(u,v)$, applying the identity

$$\begin{aligned}\nabla u(\nabla h+\nabla h^{\mathrm{T}}-(\nabla\cdot h)I_2)\nabla v=\nabla\cdot\{h\cdot(\nabla u)\nabla v+(h\cdot\nabla v)\nabla u-(\nabla u\cdot\nabla v)h\}\\-(h\cdot\nabla v)\Delta u-(h\cdot\nabla u)\Delta v,\end{aligned}$$

and the divergence theorem gives

$$\begin{aligned}g_1(h)(u,v)=-\mu\sum_{j=1}^{2}\Big\{&\int_D(h\cdot\nabla u_j)\Delta\overline{v}_j+(h\cdot\nabla\overline{v}_j)\Delta u_j\mathrm{d}x\\&-\int_S(h\cdot\nabla u_j)(n\cdot\nabla\overline{v}_j)+(h\cdot\nabla\overline{v}_j)(n\cdot\nabla u_j)-(h\cdot n)(\nabla\overline{v}_j\cdot\nabla u_j)\mathrm{d}s\Big\}.\end{aligned} \tag{5.21}$$

Considering $(\Delta^*+\omega^2)u=0$, integration by parts gives

$$\begin{aligned}\sum_{j=1}^{2}\mu\int_D(h\cdot\nabla\overline{v}_j)\Delta u_j\mathrm{d}x=(\lambda+\mu)\int_D(\nabla\cdot u)\nabla\cdot(h\cdot\nabla\overline{v})\mathrm{d}x\\-(\lambda+\mu)\int_S(\nabla\cdot u)(n\cdot(h\cdot\nabla\overline{v}))\mathrm{d}s-\omega^2\int_D(h\cdot\nabla\overline{v})\cdot u\mathrm{d}x.\end{aligned} \tag{5.22}$$

Integration by parts again gives

$$\sum_{j=1}^{2}\mu\int_D(h\cdot\nabla u_j)\Delta\overline{v}_j\mathrm{d}x=-\mu\int_D\nabla(h\cdot\nabla u):\nabla\overline{v}\mathrm{d}x+\mu\int_S(h\cdot\nabla u)\cdot(n\nabla\overline{v})\mathrm{d}s. \tag{5.23}$$

Denote $n = (n_1, n_2)^{\mathrm{T}}$ and $\tau = (-n_2, n_1)^{\mathrm{T}}$. Notice $v|_S = 0$ implies $\partial_\tau v|_S = 0$. This turns out that

$$\sum_{i=1}^{2} \int_S (h \cdot \nabla \overline{v}_j)(n \cdot \nabla u_j) - (h \cdot n)(\nabla u_j \cdot \nabla \overline{v}_j) \mathrm{d}s = 0. \tag{5.24}$$

Applying the divergence theorem again and noticing $h|_\Gamma = 0$, $v|_S = 0$, we obtain

$$\begin{aligned} \int_D (\nabla \cdot h)(u \cdot \overline{v}) + u \cdot (h \cdot \nabla \overline{v}) \mathrm{d}x &= \int_D \nabla \cdot ((u \cdot \overline{v})h) - (h \cdot \nabla u) \cdot \overline{v} \mathrm{d}x \\ &= -\int_D (h \cdot \nabla u) \cdot \overline{v} \mathrm{d}x. \end{aligned} \tag{5.25}$$

Inserting (5.21)–(5.25) into (5.16) and (5.18) yields

$$\begin{aligned} g_1(h)(u,v) + g_3(h)(u,v) &= \mu \int_D \nabla(h \cdot \nabla u) : \nabla \overline{v} \mathrm{d}x - (\lambda + \mu) \int_D (\nabla \cdot u) \nabla \cdot (h \cdot \nabla \overline{v}) \mathrm{d}x \\ &\quad - \omega^2 \int_D (h \cdot \nabla u) \cdot \overline{v} \mathrm{d}x + (\lambda + \mu) \int_S (\nabla \cdot u)(n \cdot (h \cdot \nabla \overline{v})) \mathrm{d}s. \end{aligned} \tag{5.26}$$

For $g_2(h)(u,v)$, direct calculation gives

$$\int_D (\nabla \cdot u)(\nabla \overline{v} : \nabla h^{\mathrm{T}}) \mathrm{d}x = \int_D (\nabla \cdot u) \nabla \cdot (h \cdot \nabla \overline{v}) - (\nabla \cdot u)(h \cdot (\nabla \cdot \nabla \overline{v}^{\mathrm{T}})) \mathrm{d}x \tag{5.27}$$

and similarly

$$\int_D (\nabla \cdot \overline{v})(\nabla u : \nabla h^{\mathrm{T}}) \mathrm{d}x = \int_D (\nabla \cdot \overline{v}) \nabla \cdot (h \cdot \nabla u) - (\nabla \cdot \overline{v})(h \cdot (\nabla \cdot \nabla u^{\mathrm{T}})) \mathrm{d}x. \tag{5.28}$$

Then applying the divergence theorem yields

$$\begin{aligned} &\int_D (\nabla \cdot \overline{v})(h \cdot (\nabla \cdot \nabla u^{\mathrm{T}})) \mathrm{d}x + \int_D (\nabla \cdot u)(h \cdot (\nabla \cdot \nabla \overline{v}^{\mathrm{T}})) \mathrm{d}x \\ &+ \int_D (\nabla \cdot u)(\nabla \cdot \overline{v})(\nabla \cdot h) \mathrm{d}x = \int_D \nabla \cdot (h(\nabla \cdot u)(\nabla \cdot \overline{v})) \mathrm{d}x \\ = &\int_S (h \cdot n)(\nabla \cdot u)(\nabla \cdot \overline{v}) \mathrm{d}s. \end{aligned} \tag{5.29}$$

Recalling $\partial_\tau v|_S = 0$, we have

$$\int_S (h \cdot n)(\nabla \cdot u)(\nabla \cdot \overline{v}) \mathrm{d}s = \int_S (\nabla \cdot u)(n \cdot (h \cdot \nabla \overline{v})) \mathrm{d}s. \tag{5.30}$$

Together with (5.27)–(5.30) implies

$$\begin{aligned} g_2(h)(u,v) &= (\lambda + \mu) \int_D (\nabla \cdot u) \nabla \cdot (h \cdot \nabla \overline{v}) + \nabla \cdot (h \cdot \nabla u)(\nabla \cdot \overline{v}) \mathrm{d}x \\ &\quad - (\lambda + \mu) \int_S (\nabla \cdot u)(n \cdot (h \cdot \nabla \overline{v})) \mathrm{d}s. \end{aligned} \tag{5.31}$$

Now recalling $w_h = h \cdot \nabla u$, $w_h|_\Gamma = 0$ and combining the representations of $g_i(h)(u,v)$ in (5.26) and (5.31) implies

$$\sum_{i=1}^{3} g_i(h)(u,v)$$

$$
\begin{aligned}
&= \mu \int_D \nabla(h\cdot\nabla u) : \nabla \overline{v} \mathrm{d}x + (\lambda+\mu)\int_D \nabla\cdot(h\cdot\nabla u)(\nabla\cdot\overline{v})\mathrm{d}x - \omega^2 \int_D (h\cdot\nabla u)\cdot\overline{v}\mathrm{d}x \\
&= B(w_h, v). \qquad (5.32)
\end{aligned}
$$

Notice $u^*$ satisfies (5.5)–(5.6), so it satisfies the variational formula $B(u^*, v) = 0$ for any $v \in V$. Hence by (5.20) and (5.32), we can obtain

$$
\frac{B(\widetilde{u}_h - u - u^* - w_h, v)}{\|h\|_{1,\infty}\|v\|_{H^1(D)^2}} = O(\|h\|_{1,\infty}) + O(\|\widetilde{u}_h - u\|_{H^1(D)^2}). \tag{5.33}
$$

The Dirichlet boundary condition (5.7) and $u, \widetilde{u}_h \in V$ deduce $(\widetilde{u}_h - u - u^* - w_h)|_S = 0$ so that $\widetilde{u}_h - u - u^* - w_h \in V$. Then using Theorem 3.1, we know the sesquilinear form $B$ generates an invertible bounded linear operator which is still denoted by $B$. By Banach open mapping theorem, $B^{-1} : V^* \to V$ is also a bounded linear operator which implies

$$
\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(D)^2} \le C \sup_{v\in V, v\neq 0} \frac{|B(\widetilde{u}_h - u - u^* - w_h, v)|}{\|v\|_{H^1(D)^2}}. \tag{5.34}
$$

Then combining (5.33)–(5.34) gives

$$
\frac{\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(D)^2}}{\|h\|_{1,\infty}} = O(\|h\|_{1,\infty}) + O(\|\widetilde{u}_h - u\|_{H^1(D)^2}).
$$

Hence we only need to prove when $\|h\|_{1,\infty} \to 0$, $\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(D)^2} \to 0$. Since $B^h(u, v) - B(u, v) = B_1(u, v) + B_2(u, v) + B_3(u, v)$, inserting (5.12)–(5.14) gives

$$
|B^h(u, v) - B(u, v)| \le C\|h\|_{1,\infty}\|u\|_{H^1(D)^2}\|v\|_{H^1(D)^2}.
$$

Since

$$
B^h(\widetilde{u}_h, v) = (g, v)_\Gamma = B(u, v),
$$

we obtain

$$
\|\widetilde{u}_h - u\|_{H^1(D)^2} \le C \sup_{u\in V, u\neq 0}\sup_{v\in V, v\neq 0} \frac{|B^h(u, v) - B(u, v)|}{\|v\|_{H^1(D)^2}\|u\|_{H^1(D)^2}} \le C\|h\|_{1,\infty} \to 0.
$$

Hence

$$
\lim_{\|h\|_{1,\infty}\to 0} \frac{\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(D)^2}}{\|h\|_{1,\infty}} = 0,
$$

which completes the proof.

Next we consider a local stability result for inverse cavity problem. For two domains $\Omega_1, \Omega_2$ in $\mathbb{R}^2$, define the Hausdorff distance between $\Omega_1$ and $\Omega_2$ by

$$
\mathrm{dist}(\Omega_1, \Omega_2) = \max\{\rho(\Omega_1, \Omega_2), \rho(\Omega_2, \Omega_1)\},
$$

where

$$
\rho(\Omega_1, \Omega_2) = \sup_{x\in\Omega_1}\inf_{y\in\Omega_2} |x - y|.
$$

Take $h(x) = kp(x)n$, where $k \in \mathbb{R}$ and $p(x) \in C^2(S, \mathbb{R})$. Our goal is to give the following local stability result.

**Theorem 5.2** *Given $p \in C^2(S, \mathbb{R})$ and $k > 0$ is sufficiently small, then*

$$\mathrm{dist}(D_k, D) \le C\|u_k - u\|_{H^{\frac{1}{2}}(\Gamma)^2},$$

*where $D_k = D_{kpn}$, $u_k = u_{kpn}$ and $C > 0$ is a constant independent of $k$.*

**Proof** For $p(x) \equiv 0$, it is obvious that this conclusion is true. So we can assume $p \not\equiv 0$.

We firstly prove $\mathrm{dist}(D, D_k) = O(k)$. For any fixed $x_0 \in S$, denote $y_0 = x_0 + kp(x_0)n(x_0)$. Then

$$\inf_{y \in D_k} |x_0 - y| \le |x_0 - y_0| \le Ck,$$

where $C$ is a constant only dependent on $p$. Hence we obtain $\rho(D, D_k) \le Ck$. And similarly we have $\rho(D_k, D) \le Ck$, which gives

$$\mathrm{dist}(D, D_k) \le Ck. \tag{5.35}$$

Since $p \not\equiv 0$, without loss of generality, let $p(x_0) \ne 0$. Then we know

$$\mathrm{dist}(D, D_k) \ge \rho(D_k, D) \ge \inf_{x \in D} |y_0 - x| = |y_0 - x_0| = |p(x_0)|k. \tag{5.36}$$

Combining (5.35)–(5.36) gives $\mathrm{dist}(D, D_k) = O(k)$.

Now we prove the conclusion. Assume the conclusion is not true. There exists a $p(x) \in C^2(S, \mathbb{R})$ such that

$$\Big\|\frac{u_{k'} - u}{k'}\Big\|_{H^{\frac{1}{2}}(\Gamma)^2} \to 0, \quad \text{when } k' \to 0, \tag{5.37}$$

where $\{u_{k'}\}$ is some subsequence of $\{u_k\}$. Applying Theorem 5.1 to $h(x) = kp(x)n$ gives

$$\lim_{k \to 0} \Big\|\frac{u_k - u - u^*}{k}\Big\|_{H^{\frac{1}{2}}(\Gamma)^2} = 0, \tag{5.38}$$

where $u^*$ satisfies

$$\begin{aligned} (\Delta^* + \omega^2)u^* = 0 &\quad \text{in } D, \\ Tu^* = \mathcal{T}u^* &\quad \text{on } \Gamma, \\ u^* = -hp\partial_n u &\quad \text{on } S. \end{aligned}$$

Let $v^* = \frac{u^*}{h}$, then $v^*$ satisfies

$$\begin{aligned} (\Delta^* + \omega^2)v^* = 0 &\quad \text{in } D, \\ Tv^* = \mathcal{T}v^* &\quad \text{on } \Gamma, \\ v^* = -p\partial_n u &\quad \text{on } S. \end{aligned}$$

Then by (5.38), we obtain

$$\lim_{h \to 0} \Big\|\frac{u_h - u}{h} - v^*\Big\|_{H^{\frac{1}{2}}(\Gamma)^2} = \lim_{h \to 0} \Big\|\frac{u_k - u}{k} - v^*\Big\|_{H^{\frac{1}{2}}(\Gamma)^2} = 0. \tag{5.39}$$

Combining (5.37) and (5.39) yields $\|v^*\|_{H^{\frac{1}{2}}(\Gamma)^2} = 0$. Hence

$$Tv^* = \mathcal{T}v^* = 0 \quad \text{on } \Gamma.$$

By the unique continuation,

$$v^* = 0 \quad \text{in } D.$$

Recalling that

$$v^* = -p\partial_n u \quad \text{on } S,$$

$p \not\equiv 0$ and $p \in C^2(S, \mathbb{R})$, there must exist a continuous part $S' \subset S$ such that

$$p \neq 0 \quad \text{on } S',$$

which implies

$$\partial_n u = 0 \quad \text{on } S'.$$

Then by the unique continuation, $u = 0$ in $D$, which is a contradiction to $Tu = \mathcal{T}u + g$ on $\Gamma$.

### 5.2 Neumann case

Next, consider the Neumann problem. Notice that the boundary value problem deduced by the NtD operator has the inhomogeneous boundary condition (2.10), so it is difficult to directly calculate the Fréchet derivative like the Dirichlet problem. Here, we still take advantage of the TBC given by a DtN operator, though its definition on $\Gamma$ is difficult. Instead, an upper half-circle artificial boundary is taken (see Figure 2).

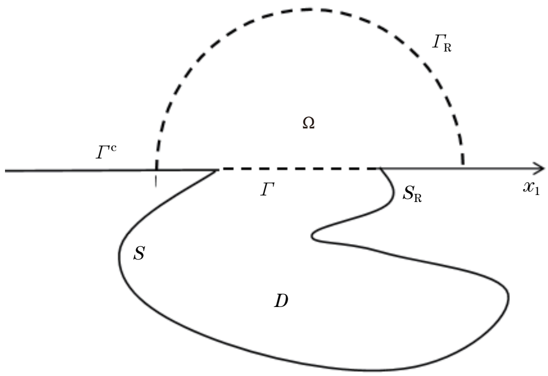


Figure 2 The Neumann case.

Take $R > 0$ such that $\{x_2 = 0, |x| > R\} \subset \Gamma^c$ in Figure 2. Denote $\Omega = D \cup \Gamma \cup \{0 < x_2, |x| < R\}$, $\Gamma_R = \{x_2 \geq 0, |x| = R\}$ and $S_R = S \cup (\Gamma^c \cap \{|x| \leq R\})$. Let $x \in \mathbb{R} \cap \{|x| > R\}$.

Recalling that the scattering wave $u_s$ of the Neumann problem satisfies $Tu_s|_{\Gamma^c} = 0$, the Green's representation theorem yields

$$u_s = \int_{\Gamma_R} T_y G_N(x, y) \cdot u_s(y) - G_N(x, y) T_y u_s(y) \mathrm{d}s(y),$$

where $T_y$ is the differential operator $T$ with respect to $y$. Taking $x \to \Gamma_R$ gives

$$\frac{1}{2} u_s - \int_{\Gamma_R} T_y G_N(x, y) \cdot u_s(y) + G_N(x, y) \cdot T_y u_s(y) = 0.$$

Define the single-layer operator $\mathcal{S}$ by

$$\mathcal{S}v(x) = \int_{\Gamma} G_N(x,y) \cdot v(x)\mathrm{d}s(y), \quad v \in H^{\frac{1}{2}}(\Gamma_R)$$

and double-layer operator $\mathcal{D}$ by

$$\mathcal{D}v(x) = \int_{\Gamma_R} T_y G_N(x,y) v(y)\mathrm{d}s(y), \quad v \in H^{-\frac{1}{2}}(\Gamma_R).$$

We can choose $R$ such that $\omega^2$ is not the interior Dirichlet eigenvalue of $\Delta^*$ so that $\mathcal{S}$ is invertible. Then define the DtN operator by

$$\mathscr{T} := -\mathcal{S}^{-1}\Big(\frac{1}{2}\mathcal{I} - \mathcal{D}\Big),$$

where $\mathcal{I}$ is the identity operator. It is easy to verify that this DtN operator implies the TBC

$$Tu = \mathscr{T}u + g \quad \text{on } \Gamma_R$$

with $g = T(u_i + u_r) - (u_i + u_r)$.

Then rewrite the boundary value problem as

$$\begin{aligned} (\Delta^* + \omega^2)u = 0 &\quad \text{in } \Omega, \\ Tu = 0 &\quad \text{on } S_R, \\ Tu = \mathscr{T}u + g &\quad \text{on } \Gamma_c. \end{aligned}$$

The variational formula is given by

$$\mathscr{B}(u,v) = (g,v)_{\Gamma_R}, \quad \forall v \in H^1(\Omega)^2, \tag{5.40}$$

where $B_c : H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{C}$ is defined by

$$\mathscr{B}(u,v) = \int_{\Omega} \mathcal{E}(u,\overline{v}) - \omega^2 u \cdot \overline{v}\mathrm{d}x - \int_{\Gamma_R} \mathscr{T}u \cdot \overline{v}\mathrm{d}s.$$

The Dirichlet data on $\Gamma_R$ is directly given from the Neumann data on $\Gamma$ by (4.9). Hence it is enough to consider the reconstruction from the Dirichlet data on $\Gamma_R$.

Let $\Omega_h = D_h \cup \Gamma \cup \{0 < x_2, |x| < R\}$. Define the solution operator $\mathcal{U}_N : C^2(S,\mathbb{R}^2) \to H^{\frac{1}{2}}(\Gamma_R)^2$ by

$$\mathcal{U}_N(h) = u_h|_{\Gamma_R},$$

where $u_h$ is solution to

$$\mathscr{B}_h(u_h,v_h) = (g,v_h)_{\Gamma_R}, \quad \forall v_h \in H^1(\Omega_h)^2 \tag{5.41}$$

with

$$\mathscr{B}_h(u_h,v_h) = \int_{\Omega_h} \mathcal{E}(u_h,\overline{v}_h) - \omega^2 u_h \cdot \overline{v}_h\mathrm{d}x - \int_{\Gamma_R} \mathscr{T}u_h \cdot \overline{v}_h\mathrm{d}s. \tag{5.42}$$

Extend $h(x)$ to $C^2(\Omega,\mathbb{R}^2)$ and still denote it by $h$ such that

$$h(x) = 0 \quad \text{on } \Gamma_R$$

and

$$\|h\|_{H^{1,\infty}(\Omega)^2} \le C\|h\|_{H^{1,\infty}(S)^2}$$

with constant $C > 0$. Define $\mathcal{H}_h$ by

$$\mathcal{H}_h(y) = y + h(y), \quad y \in \Omega.$$

It is invertible for sufficiently small $\|h\|_{H^{1,\infty}(\Omega)^2}$.

Then taking the variable transform $x = \mathcal{H}_h(y)$ in (5.42) implies

$$\mathscr{B}_h(u_h, v_h) = \mathscr{B}^h(\widetilde{u}_h, \widetilde{v}_h).$$

Here $\mathscr{B}^h$ is the similarly defined as $B^h$ in Section 5.1 except that $D_h$, $\Gamma$ and $\mathcal{T}$ are replaced by $\Omega_h$, $\Gamma_R$ and $\mathscr{T}$, respectively. Since $\widetilde{u}_h, \widetilde{v}_h \in H^1(\Omega)^2$, $\mathscr{B}^h$ is a sesquilinear form on $H^1(\Omega)^2 \times H^1(\Omega)^2$. So the variational formula (5.41) becomes

$$\mathscr{B}^h(\widetilde{u}_h, v) = (g, v)_{\Gamma_R}, \quad \forall v \in H^1(\Omega_h)^2.$$

Let $\mathcal{U}_N'(0) : C^2(S, \mathbb{R})^2 \to H^{\frac{1}{2}}(\Gamma_R)^2$ be the Fréchet derivative of $\mathcal{U}_N$ at 0. Then result similar to Theorem 5.1 holds for the Neumann case.

**Theorem 5.3** *Suppose $h \in C^2(S, \mathbb{R})^2$ satisfying $h = 0$ on $S \cap \Gamma$. Let $u^*$ be solution to the variational problem*

$$B(u^*, v) = \mathcal{A}(v), \quad \forall v \in H^1(\Omega)^2, \tag{5.43}$$

*where $\mathcal{A} \in (H(\Omega)^2)^*$ is a functional defined by*

$$\begin{aligned}\mathcal{A}(\overline{v}) = {} & \omega^2 \int_{S_R} (u \cdot \overline{v})(h \cdot n)\mathrm{d}s \\ & - (\lambda + \mu) \int_{S_R} (\nabla \cdot u)(\nabla \cdot \overline{v})(n \cdot h)\mathrm{d}s - \mu \int_{S_R} (h \cdot n)(\nabla u : \nabla \overline{v})\mathrm{d}s.\end{aligned}$$

*Then $\mathcal{U}_N'(0)h = u^*|_{\Gamma_c}$.*

**Proof** Let $w_h = h \cdot \nabla u$. Then $h|_{\Gamma_R} = 0$ implies $w_h|_{\Gamma_R}$. So $w_h + u^* \in H^1(\Omega)^2$ and $(w_h + u^*)|_{\Gamma_R} = u^*|_{\Gamma_R}$. Similarly to Theorem 5.1, we only need to prove

$$\lim_{\|h\|_{1,\infty} \to 0} \frac{\|\widetilde{u}_h - u - u^* - w_h\|_{H^1(\Omega)^2}}{\|h\|_{1,\infty}} = 0,$$

where $\|\cdot\|_{1,\infty} = \|\cdot\|_{H^1(\Omega)^2}$. For any $v \in H^1(\Omega)^2$,

$$\mathscr{B}(\widetilde{u}_h - u - u^* - w_h, v) = \mathscr{B}(\widetilde{u}_h - u, v) - \mathscr{B}(u^* + w_h, v).$$

Like Section 5.1, we have

$$\begin{aligned}\mathscr{B}(\widetilde{u}_h - u, v) = {} & g_1(h)(u, v) + g_2(h)(u, v) + g_3(h)(u, v) \\ & + O(\|h\|_{1,\infty}^2 \|v\|_{H^1(\Omega)^2}) + O(\|h\|_{1,\infty} \|v\|_{H^1(\Omega)^2} \|\widetilde{u}_h - u\|_{H^1(\Omega)^2}),\end{aligned} \tag{5.44}$$

where $g_i(h)$ is defined the same as (5.16)–(5.18) except that $D$ is replaced by $\Omega$.

For $g_1(h)(u,v)$, similarly to (5.21), we obtain

$$g_1(h)(u,v) = -\mu\sum_{j=1}^{2}\Big\{\int_\Omega (h\cdot\nabla u_j)\Delta\overline{v}_j + (h\cdot\nabla\overline{v}_j)\Delta u_j \mathrm{d}x - \int_{S_R}(h\cdot\nabla u_j)(n\cdot\nabla\overline{v}_j) + (h\cdot\nabla\overline{v}_j)(n\cdot\nabla u_j) - (h\cdot n)(\nabla\overline{v}_j\cdot\nabla u_j)\mathrm{d}s\Big\}. \tag{5.45}$$

By $(\Delta^* + \omega^2)u = 0$ and integration by parts, we have

$$\begin{aligned}&\sum_{j=1}^{2}\mu\int_\Omega (h\cdot\nabla\overline{v}_j)\Delta u_j\mathrm{d}x\\ &= (\lambda+\mu)\int_\Omega(\nabla\cdot u)\nabla\cdot(h\cdot\nabla\overline{v})\mathrm{d}x\\ &\quad - (\lambda+\mu)\int_{S_R}(\nabla\cdot u)(n\cdot(h\cdot\nabla\overline{v}))\mathrm{d}s - \omega^2\int_\Omega(h\cdot\nabla\overline{v})\cdot u\mathrm{d}x.\end{aligned} \tag{5.46}$$

Integration by parts again gives

$$\begin{aligned}&\sum_{j=1}^{2}\mu\int_\Omega (h\cdot\nabla u_j)\Delta\overline{v}_j\mathrm{d}x\\ &= -\mu\int_\Omega \nabla(h\cdot\nabla u):\nabla\overline{v}\mathrm{d}x + \mu\int_{S_R}(h\cdot\nabla u)\cdot(n\cdot\nabla\overline{v})\mathrm{d}s.\end{aligned} \tag{5.47}$$

Combining (5.45)–(5.47) and $(\lambda+\mu)(\nabla\cdot u)n + \mu\partial_n u = 0$ on $S_R$ gives

$$\begin{aligned}g_1(h)(u,v) &= \mu\int_\Omega\nabla(h\cdot\nabla u):\nabla\overline{v}\mathrm{d}x - (\lambda+\mu)\int_\Omega(\nabla\cdot u)\nabla\cdot(h\cdot\nabla\overline{v})\mathrm{d}x\\ &\quad + \omega^2\int_\Omega(h\cdot\nabla\overline{v})\cdot u\mathrm{d}x - \mu\int_{S_R}(h\cdot n)(\nabla u:\nabla\overline{v})\mathrm{d}s.\end{aligned} \tag{5.48}$$

Apply the divergence theorem again, we obtain

$$\begin{aligned}\int_\Omega(\nabla\cdot h)(u\cdot\overline{v}) + u\cdot(h\cdot\nabla\overline{v})\mathrm{d}x &= \int_\Omega\nabla\cdot((u\cdot\overline{v})h) - (h\cdot\nabla u)\cdot\overline{v}\mathrm{d}x\\ &= \int_{S_R}(u\cdot\overline{v})(h\cdot n)\mathrm{d}s - \int_\Omega(h\cdot\nabla u)\cdot\overline{v}\mathrm{d}x.\end{aligned} \tag{5.49}$$

Combining (5.48)–(5.49) yields

$$\begin{aligned}&g_1(h)(u,v) + g_3(h)(u,v)\\ &= \mu\int_\Omega\nabla(h\cdot\nabla u):\nabla\overline{v}\mathrm{d}x - (\lambda+\mu)\int_\Omega(\nabla\cdot u)\nabla\cdot(h\cdot\nabla\overline{v})\mathrm{d}x\\ &\quad - \omega^2\int_\Omega(h\cdot\nabla u)\cdot\overline{v}\mathrm{d}x + \omega^2\int_{S_R}(u\cdot\overline{v})(h\cdot n)\mathrm{d}s - \mu\int_{S_R}(h\cdot n)(\nabla u:\nabla\overline{v})\mathrm{d}s.\end{aligned} \tag{5.50}$$

For $g_2(h)(u,v)$, a similar discussion as in (5.27)–(5.29) shows

$$g_2(h)(u,v) = (\lambda+\mu)\int_\Omega(\nabla\cdot u)\nabla\cdot(h\cdot\nabla\overline{v}) + \nabla\cdot(h\cdot\nabla u)(\nabla\cdot\overline{v})\mathrm{d}x$$

$$- (\lambda+\mu)\int_{S_R}(\nabla\cdot u)(\nabla\cdot\overline{v})(n\cdot h)\mathrm{d}s. \tag{5.51}$$

Combining $w_h|_{\Gamma_R}=0$, (5.43) and (5.50)–(5.51) gives

$$\begin{aligned}\sum_{i=1}^{3} g_i(h)(u,v) &= \mu\int_\Omega \nabla(h:\nabla u)\nabla\overline{v}\mathrm{d}x\\ &\quad+(\lambda+\mu)\int_\Omega \nabla\cdot(\nabla u\cdot h)(\nabla\cdot\overline{v})\mathrm{d}x-(\lambda+\mu)\int_{S_R}(\nabla\cdot u)(\nabla\cdot\overline{v})(n\cdot h)\mathrm{d}s\\ &\quad-\mu\int_{S_R}(h\cdot n)(\nabla u:\nabla\overline{v})\mathrm{d}s+\omega^2\int_{S_R}(u\cdot\overline{v})(h\cdot n)\mathrm{d}s\\ &= B(w_h+u^*,v).\end{aligned}$$

Then we arrive at

$$B(\widetilde{u}_h-u-u^*-w_h,v)=O(\|h\|_{1,\infty}^2\|v\|_{H^1(\Omega)^2})+O(\|\widetilde{u}_h-u\|_{H^1(\Omega)^2}\|v\|_{H^1(\Omega)^2}\|h\|_{1,\infty}).$$

It completes the proof.

**Remark 5.1** We cannot get local stability from the Fréchet derivative like the Dirichlet problem. Let $h(x)=kp(x)n$ with $k>0$ and $p(x)\in C^2(S,\mathbb{R})$. Assume $p(x)\not\equiv 0$. For any $v\in H^1(\Omega)^2$, the equality

$$-(\lambda+\mu)\int_{S_R}p(\nabla\cdot u)(\nabla\cdot\overline{v})\mathrm{d}s-\mu\int_{S_R}p(\nabla u:\nabla\overline{v})\mathrm{d}s+\omega^2\int_{S_R}p(u\cdot\overline{v})\mathrm{d}s=0$$

does not imply $u|_{S_R}=0$. In fact, we can see in [10] for electromagnetic scattering, the local stability holds only for lossy medium, i.e., non-real wave number.

## 6 Conclusions

In this paper, the elastic cavity problem with the Dirichlet or Neumann condition is reduced to a bounded domain by the DtN or NtD operator, respectively. The variational approach is utilized to prove the uniqueness and existence of the boundary value problem in a bounded domain. For the inverse cavity problem, the Fréchet derivative is given for shape reconstruction and a local stability result is given for the Dirichlet problem. The stability results explicitly dependent on with frequency for large elastic cavities as in [13–14] remain unsolved, which will be discussed in a forthcoming paper.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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