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Volterra Type Integration Operators from Weighted Bergman Spaces to Hardy Spaces in the Unit Ball of $\mathbb{C}^{n*}$

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Abstract The authors provide a complete characterization of the boundedness and compactness of the Volterra type integration operator T_g from weighted Bergman spaces A^p_ω, induced by doubling weights ω, to Hardy spaces H^q in the unit ball of $\mathbb{C}^n$, for all $0 < p, q < \infty$.

Keywords Bergman space, Hardy space, Volterra type integration operator
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1 Introduction

Let $\mathbb{B}_n$ denote the open unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ of $\mathbb{C}^n$, and let $\mathbb{S}_n$ denote its boundary $\{z \in \mathbb{C}^n : |z| = 1\}$. When $n = 1$, $\mathbb{B}_n$ is the open unit disc $\mathbb{D}$ of $\mathbb{C}$. The space $H(\mathbb{B}_n)$ consists of all holomorphic functions on $\mathbb{B}_n$. For $0 < p \le \infty$, the Hardy space $H^p(\mathbb{B}_n)$ (denoted H^p) is defined as the set of all functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty,$$

where $M_p(r,f) = \big(\int_{\mathbb{S}_n} |f(r\zeta)|^p \mathrm{d}\sigma(\zeta)\big)^{\frac{1}{p}}$ when $0 < p < \infty$, and $M_\infty(r,f) = \sup\limits_{|z|=r} |f(z)|$. Here, $\mathrm{d}\sigma$ denotes the normalized surface measure on $\mathbb{S}_n$.

A function ω defined on $\mathbb{B}_n$ is considered a weight if it is positive, measurable, and integrable. A weight ω is said to be radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{B}_n$. Let ω be a radial weight and define $\widehat{\omega}(z) = \int_{|z|}^1 \omega(t)\mathrm{d}t$ for all $z \in \mathbb{B}_n$. We say that ω is a doubling weight (denoted by $\omega \in \widehat{\mathcal{D}}$) if there exists a constant $C \ge 1$ such that $\widehat{\omega}(r) \le C\widehat{\omega}\big(\frac{r+1}{2}\big), 0 \le r < 1$, and ω is a reverse doubling weight (denoted by $\omega \in \check{\mathcal{D}}$) if there exist constants $C_1, C_2 > 1$ such that $\widehat{\omega}(r) \ge C_1\widehat{\omega}\big(1 - \frac{1-r}{C_2}\big)$, $0 \le r < 1$. The class of weights satisfying both conditions is denoted by $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$, and if

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$\omega \in \mathcal{D}$, we say that ω is a two-sided doubling weight. We say that ω is a regular weight (denoted by $\omega \in \mathcal{R}$) if ω is continuous and there exists a constant $C = C(\omega) > 0$ such that

$$\frac{1}{C} < \frac{\widehat{\omega}(r)}{(1-r)\omega(r)} < C, \quad 0 \le r < 1.$$

See [24–25] for additional details on these weights.

Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. The weighted Bergman space $A^p_\omega(\mathbb{B}_n)$ (denoted A^p_ω) consists of all functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{B}_n} |f(z)|^p \omega(z) \mathrm{d}v(z) < \infty,$$

where $\mathrm{d}v$ is the normalized volume measure on $\mathbb{B}_n$. A^p_ω is a Banach space when $p \ge 1$ and a complete metric space with the distance $\rho(f, g) = \|f - g\|_{A^p_\omega}$ when $0 < p < 1$. When $\omega(z) = C_\alpha(1 - |z|^2)^\alpha$ $(\alpha > -1)$, where $C_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$, the space A^p_ω coincides with the weighted Bergman space A^p_α. When $\alpha = 0$, the space A^p_ω is just the classical Bergman space A^p. For more details about these spaces, see [33–34].

Let $g \in H(\mathbb{B}_n)$. The Volterra type integration operator T_g is defined as

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{\mathrm{d}t}{t}, \quad f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n,$$

where $\Re g$ denotes the radial derivative of g, i.e., $\Re g(z) = \sum\limits_{k=1}^{n} z_k \frac{\partial g}{\partial z_k}(z),\ z \in \mathbb{B}_n$. The operator T_g in the unit disc is always denoted by J_g and was first introduced by Pommerenke [27]. The operator T_g in the unit ball was introduced by Hu [13].

Pommerenke [27] proved that J_g is bounded on $H^2(\mathbb{D})$ if and only if $g \in BMOA$, the analytic space of bounded mean oscillation. Aleman and Cima [2] characterized the boundedness of $J_g : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ for $0 < p, q < \infty$. Wu [28] described the boundedness of $J_g : A^p_\alpha(\mathbb{D}) \to H^q(\mathbb{D})$ for some $0 < p, q < \infty$ and left the case $0 < q < \min\{p, 2\}$ as an open problem. For more results about the operator J_g, see [2–3, 10, 17, 28, 30] and the references therein.

Li and Stević [18] proved that T_g is bounded on H^2 for the unit ball of $\mathbb{C}^n$ if and only if $g \in BMOA$. In [23], Pau provided a complete characterization of the boundedness and compactness of T_g from H^p to H^q. Miihkinen, Pau, Perälä, and Wang further studied the boundedness of T_g from A^p_α to H^q for $0 < p, q < \infty$ in [22]. Chen, Pau, and Wang characterized the compactness of T_g from A^p_α to H^q for all $0 < p, q < \infty$ in [7]. For more results about the operator T_g, see [7, 9, 13, 18–19, 22–23, 26, 29] and the references therein.

Motivated by [7, 22], we investigate the boundedness and compactness of the operator T_g from weighted Bergman spaces A^p_ω to Hardy spaces H^q for the entire range $0 < p, q < \infty$ and $\omega \in \mathcal{D}$. The main results are presented as follows.

Theorem 1.1 *Let $0 < p, q < \infty$, $\omega \in \mathcal{D}$ and $g \in H(\mathbb{B}_n)$. Then the following statements hold:*

(i) *If* $0 < p \le \min\{q, 2\}$ *or* $2 < p < q < \infty$, *then* $T_g : A^p_\omega \to H^q$ *is bounded if and only if*

$$\sup_{z \in \mathbb{B}_n} |\Re g(z)|(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}} \widehat{\omega}(z)^{-\frac{1}{p}} < \infty.$$

(ii) *If* $2 < p = q < \infty$, *then* $T_g : A^p_\omega \to H^q$ *is bounded if and only if*

$$|\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z)$$

is a Carleson measure.

(iii) *If* $p > \max\{q, 2\}$, *then* $T_g : A^p_\omega \to H^q$ *is bounded if and only if the function*

$$U_g(\zeta) = \Big(\int_{\Gamma_\zeta} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{p-2}{2p}}$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{S}_n)$.

(iv) *If* $0 < q < p \le 2$, *then* $T_g : A^p_\omega \to H^q$ *is bounded if and only if*

$$\int_{\mathbb{S}_n} \sup_{z \in \Gamma_\zeta} |\Re g(z)|^{\frac{pq}{p-q}} \frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \mathrm{d}\sigma(\zeta) < \infty.$$

Theorem 1.2 *Let* $0 < p, q < \infty$, $\omega \in \mathcal{D}$ *and* $g \in H(\mathbb{B}_n)$. *Then the following statements hold*:

(i) *If* $0 < p \le \min\{q, 2\}$ *or* $2 < p < q < \infty$, *then* $T_g : A^p_\omega \to H^q$ *is compact if and only if*

$$\lim_{|z| \to 1^-} |\Re g(z)|(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}} \widehat{\omega}(z)^{-\frac{1}{p}} = 0.$$

(ii) *If* $2 < p = q < \infty$, *then* $T_g : A^p_\omega \to H^q$ *is compact if and only if*

$$|\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z)$$

is a vanishing Carleson measure.

(iii) *If* $p > \max\{q, 2\}$, *then* $T_g : A^p_\omega \to H^q$ *is compact if and only if the function*

$$U_g(\zeta) = \Big(\int_{\Gamma_\zeta} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{p-2}{2p}}$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{S}_n)$.

(iv) *If* $0 < q < p \le 2$, *then* $T_g : A^p_\omega \to H^q$ *is compact if and only if*

$$\lim_{r \to 1^-} \int_{\mathbb{S}_n} \sup_{z \in \Gamma_\zeta \setminus r\mathbb{B}_n} |\Re g(z)|^{\frac{pq}{p-q}} \frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \mathrm{d}\sigma(\zeta) = 0.$$

This article is organized as follows: Section 2 provides background materials and important lemmas used in proving the main results. Section 3 provides a detailed proof for Theorem 1.1. Section 4 focuses on proving Theorem 1.2, which characterizes the compactness of $T_g : A^p_\omega \to H^q$ for all $0 < p, q < \infty$ in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2 Preliminaries

This section introduces background materials and collects important lemmas necessary for proving the main results presented in this paper.

2.1 Carleson measure

For any $\zeta \in \mathbb{S}_n$ and $\delta > 0$, the non-isotropic metric ball $Q_\delta(\zeta)$ is defined by

$$Q_\delta(\zeta) = \{z \in \mathbb{B}_n : |1 - \langle z, \zeta\rangle| < \delta\}.$$

Let $0 < s < \infty$ and μ be a positive Borel measure on $\mathbb{B}_n$. We say that μ is an s-Carleson measure if

$$\|\mu\|_{\mathcal{CM}_s} = \sup\Big\{\frac{\mu(Q_\delta(\zeta))}{\delta^{ns}}; \zeta \in \mathbb{S}_n, \delta > 0\Big\} < \infty,$$

and μ is a vanishing s-Carleson measure if

$$\lim_{\delta \to 0} \frac{\mu(Q_\delta(\zeta))}{\delta^{ns}} = 0, \quad \text{uniformlyfor } \zeta \in \mathbb{S}_n.$$

When $s = 1$, the measure μ is referred to as the (vanishing) Carleson measure. The norm $\|\mu\|_{\mathcal{CM}_1}$ can be denoted as $\|\mu\|_{\mathcal{CM}}$. In [32], Zhao and Zhu established several equivalent descriptions for s-Carleson measure (see [32, Theorem 45]). Their findings included the proof that μ is an s-Carleson measure if and only if for each (some) $t > 0$,

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1 - |a|^2)^t}{|1 - \langle z, a\rangle|^{ns+t}} \mathrm{d}\mu(z) < \infty.$$

For $0 < p < \infty$ and a positive Borel measure μ on $\mathbb{B}_n$, the Lebesgue space L^p_μ is the space of all measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|_{L^p_\mu} = \Big(\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z)\Big)^{\frac{1}{p}} < \infty.$$

Carleson [6] established that the identity operator $I_d : H^p(\mathbb{D}) \to L^p_\mu(\mathbb{D})$ is bounded if and only if μ is a Carleson measure. Later, Duren extended Carleson's theorem and proved that the identity operator $I_d : H^p(\mathbb{D}) \to L^q_\mu(\mathbb{D})$ is bounded if and only if μ is a $\frac{q}{p}$-Carleson measure when $p \le q$ (see [11]). A unit ball version of Duren's theorem is presented in [23], which we state in the following lemma.

Lemma 2.1 *Let $0 < p \le q < \infty$ and μ be a finite positive Borel measure on $\mathbb{B}_n$. Then the identity operator $I_d : H^p \to L^q_\mu$ is bounded if and only if μ is a $\frac{q}{p}$-Carleson measure. Moreover,*

$$\|I_d\|_{H^p \to L^q_\mu} \approx \|\mu\|^{\frac{1}{q}}_{\mathcal{CM}_{\frac{q}{p}}}.$$

The following Dirichlet type embedding theorem can be found in [5].

Lemma 2.2 *Let $f \in H(\mathbb{B}_n)$ with $f(0) = 0$. If $0 < p \leq 2$, then*

$$\|f\|_{H^p} \lesssim \|\Re f\|_{A^p_{p-1}}.$$

If $0 < p < q < \infty$, then

$$\|f\|_{H^q} \lesssim \|\Re f\|_{A^p_{p+\frac{np}{q}-n-1}}.$$

2.2 Equivalent norm

Let $\alpha > 1$ and $\zeta \in \mathbb{S}_n$. The admissible approach region $\Gamma_{\zeta,\alpha}$ is defined by

$$\Gamma_{\zeta,\alpha} = \Big\{z \in \mathbb{B}_n : |1 - \langle z, \zeta\rangle| < \frac{\alpha}{2}(1 - |z|^2)\Big\}.$$

For simplicity, we write $\Gamma_{\zeta,2}$ by Γ_ζ. Obviously, for any $r > 1$ and $\alpha > 1$, there exists $\beta > 1$ such that

$$\bigcup_{z\in\Gamma_{\zeta,\alpha}} D(z, r) \subset \Gamma_{\zeta,\beta}, \tag{2.1}$$

where $D(z, r) = \{a \in \mathbb{B}_n : \beta(a, z) < r\}$ denotes the Bergman metric ball with center z and radius r. Let μ be a finite positive measure and φ a positive measurable function. By Fubini's theorem,

$$\int_{\mathbb{B}_n} \varphi(z)\mathrm{d}\mu(z) \approx \int_{\mathbb{S}_n} \Big(\int_{\Gamma_{\zeta,\alpha}} \varphi(z)\frac{\mathrm{d}\mu(z)}{(1 - |z|^2)^n}\Big)\mathrm{d}\sigma(\zeta).$$

The following result is the Calderón's area theorem (see [1, 23]).

Lemma 2.3 *Let $0 < p < \infty$ and $f \in H(\mathbb{B}_n)$ with $f(0) = 0$. Then*

$$\|f\|^p_{H^p} \approx \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |\Re f(z)|^2(1 - |z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{p}{2}}\mathrm{d}\sigma(\zeta).$$

The unit disc version of the lemma below was established by Luecking [20]. The following lemma can be found in [23].

Lemma 2.4 *Let $0 < s < p < \infty$ and μ be a positive Borel measure on $\mathbb{B}_n$. Then the identity operator $I_d : H^p \to L^s_\mu$ is bounded if and only if the function*

$$\widetilde{\mu}(\zeta) = \int_{\Gamma_\zeta} \frac{\mathrm{d}\mu(z)}{(1 - |z|^2)^n}$$

belongs to $L^{\frac{p}{p-s}}(\mathbb{S}_n)$. Moreover,

$$\|I_d\|_{H^p\to L^s_\mu} \approx \|\widetilde{\mu}\|^{\frac{1}{s}}_{L^{\frac{p}{p-s}}(\mathbb{S}_n)}.$$

We also need the following integral estimate (see [4, 15]).

Lemma 2.5 *Let $0 < s < \infty$, $\theta > \max\{\frac{1}{s}, 1\}$. If μ is a positive measure, then*

$$\int_{\mathbb{S}_n} \Big(\int_{\mathbb{B}_n} \Big(\frac{1 - |z|^2}{|1 - \langle z, \zeta\rangle|}\Big)^{\theta n}\mathrm{d}\mu(z)\Big)^s\mathrm{d}\sigma(\zeta) \approx \int_{\mathbb{S}_n} \mu(\Gamma_\zeta)^s\mathrm{d}\sigma(\zeta).$$

2.3 Separated sequences and lattices

If there is a sequence $\{a_k\} \subset \mathbb{B}_n$ such that for any i, j with $i \neq j$, $\beta(a_i, a_j) \geq \delta$ for some $\delta > 0$, then this sequence $\{a_k\}$ is said to be separated. In this case, there exists an $r > 0$ such that all $D(a_k, r)$ are disjoint from each other.

According to [33, Theorem 2.23], there exists a positive integer N such that for any $0 < r < 1$, we can find a sequence $\{a_k\} \in \mathbb{B}_n$ satisfying the following conditions:

(i) $\mathbb{B}_n = \bigcup\limits_k D(a_k, r)$.

(ii) The sets $D(a_k, \frac{r}{4})$ are mutually disjoint.

(iii) Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$.

A sequence $\{a_k\}$ that satisfies the above conditions is referred to as an r-lattice in the Bergman metric. It is evident that any r-lattice sequence is separated.

2.4 Tent spaces

Tent spaces were initially introduced by Coifman, Meyer, and Stein in [8] to address problems in harmonic analysis. In [20], Luecking utilized tent spaces to investigate embedding theorems for Hardy spaces on $\mathbb{R}^n$. In this paper, we require tent spaces in the unit ball $\mathbb{B}_n$. Specifically, let $0 < p, q < \infty$ and ν be a positive Borel measure. The tent space $T^p_{q,\nu}$ consists of ν-measurable functions f on $\mathbb{B}_n$ with

$$\|f\|^p_{T^p_{q,\nu}} = \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\xi} |f(z)|^q \mathrm{d}\nu(z) \Big)^{\frac{p}{q}} \mathrm{d}\sigma(\xi) < \infty.$$

The tent space $T^p_{\infty,\nu}$ consists of ν-measurable functions f on $\mathbb{B}_n$ with

$$\|f\|^p_{T^p_{\infty,\nu}} = \int_{\mathbb{S}_n} \Big(\operatorname*{ess\,sup}_{z \in \Gamma_\xi} |f(z)| \Big)^p \mathrm{d}\sigma(\xi) < \infty.$$

We will also use a discretization scheme. Suppose $0 < p, q < \infty$, $Z = \{a_k\} \subset \mathbb{B}_n$ is a separated sequence and $\nu = \sum\limits_{k=1}^{\infty} \delta_{a_k}$, where δ_{a_k} represents the Dirac measure at a_k. Let $f(a_k) = \lambda_k$. We define $\lambda = \{\lambda_k\}$ to be in $T^p_q(Z)$, if

$$\|\lambda\|^p_{T^p_q(Z)} = \|\{\lambda_k\}\|^p_{T^p_q(Z)} = \int_{\mathbb{S}_n} \Big(\sum_{a_k \in \Gamma_\xi} |\lambda_k|^q \Big)^{\frac{p}{q}} \mathrm{d}\sigma(\xi) < \infty$$

and $\lambda = \{\lambda_k\} \in T^p_\infty(Z)$, if

$$\|\lambda\|^p_{T^p_\infty(Z)} = \|\{\lambda_k\}\|^p_{T^p_\infty(Z)} = \int_{\mathbb{S}_n} \Big(\sup_{a_k \in \Gamma_\xi} |\lambda_k| \Big)^p \mathrm{d}\sigma(\xi) < \infty.$$

The following result can be found in [4, 20].

Lemma 2.6 *Let $1 < p < \infty$ and $Z = \{a_k\}$ be a separated sequence. If $1 < q < \infty$, then the dual of $T^p_q(Z)$ is isomorphic to $T^{p'}_{q'}(Z)$ under the pairing*

$$\langle \eta, \lambda \rangle_{T^2_2(Z)} = \sum_{k=1}^{\infty} \eta_k \overline{\lambda_k} (1 - |a_k|^2)^n, \quad \eta = \{\eta_k\} \in T^p_q(Z),\ \lambda = \{\lambda_k\} \in T^{p'}_{q'}(Z),$$

where p', q' are the conjugate indices of p and q, respectively. If $0 < q \le 1$, then the dual of $T_q^p(Z)$ is isomorphic to $T_\infty^p(Z)$ under the same pairing.

The next result is the factorization of sequence tent spaces (see [22]).

Lemma 2.7 *Let $0 < p, q < \infty$ and $Z = \{a_k\}$ be an r-lattice. If $p < p_1, p_2 < \infty$, $q < q_1, q_2 < \infty$ and satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then*

$$T_q^p(Z) = T_{q_1}^{p_1}(Z) \cdot T_{q_2}^{p_2}(Z).$$

3 Boundedness

In this section, we provide a detailed proof for Theorem 1.1. We begin this section by presenting two helpful lemmas.

Lemma 3.1 *Let $0 < p, q < \infty$, $f \in H(\mathbb{B}_n)$ and $\mu \in \mathcal{R}$. Then there exists $r_0 \in (0, 1)$ so that, if $0 < r < r_0$ and $Z = \{a_k\}$ is an r-lattice, then*

$$\int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |f(z)|^q \mu(z) \mathrm{d}v(z) \Big)^{\frac{p}{q}} \mathrm{d}\sigma(\zeta) \lesssim \int_{\mathbb{S}_n} \Big(\sum_{a_k \in \Gamma_\zeta} |f(a_k)|^q \mu(a_k) (1 - |a_k|^2)^{n+1} \Big)^{\frac{p}{q}} \mathrm{d}\sigma(\zeta),$$

whenever the left-hand-side is finite.

Proof It is obvious that

$$\int_{\Gamma_\zeta} |f(z)|^q \mu(z) \mathrm{d}v(z) \lesssim \sum_{a_k \in \widetilde{\Gamma}_\zeta} \int_{D(a_k, r)} |f(z)|^q \mu(z) \mathrm{d}v(z),$$

where $\widetilde{\Gamma}_\zeta$ is an approach region with aperture larger than Γ_ζ. For z, w with $\beta(z, w) < r < \frac{1}{2}$, from [16, Lemma 2.2], we have

$$|f(z) - f(w)|^q \lesssim r^q \int_{D(w,1)} |f(u)|^q \frac{(1 - |w|^2)^{n+1}}{|1 - \langle u, w \rangle|^{2n+2}} \mathrm{d}v(u).$$

Therefore, Fubini's theorem and (2.1) yield that

$$\begin{aligned} &\sum_{a_k \in \widetilde{\Gamma}_\zeta} \int_{D(a_k, r)} |f(z) - f(a_k)|^q \mu(z) \mathrm{d}v(z) \\ &\lesssim \sum_{a_k \in \widetilde{\Gamma}_\zeta} \int_{D(a_k, r)} r^q \int_{D(a_k, 1)} |f(u)|^q \frac{(1 - |a_k|^2)^{n+1}}{|1 - \langle u, a_k \rangle|^{2n+2}} \mathrm{d}v(u) \mu(z) \mathrm{d}v(z) \\ &\lesssim r^q \sum_{a_k \in \widetilde{\Gamma}_\zeta} \int_{D(a_k, 1)} |f(u)|^q \mu(u) \mathrm{d}v(u) \lesssim r^q \int_{\widetilde{\widetilde{\Gamma}}_\zeta} |f(u)|^q \mu(u) \mathrm{d}v(u), \end{aligned}$$

where $\widetilde{\widetilde{\Gamma}}_\zeta$ is an approach region with aperture larger than $\widetilde{\Gamma}_\zeta$. Note that $\mu(z) \approx \mu(w)$ for all $z \in \mathbb{B}_n$ and $w \in D(z, r)$ when $\mu \in \mathcal{R}$. Then

$$\sum_{a_k \in \widetilde{\Gamma}_\zeta} \int_{D(a_k, r)} |f(z)|^q \mu(z) \mathrm{d}v(z)$$

$$\lesssim \sum_{a_k\in\widetilde{\Gamma}_\zeta}\int_{D(a_k,r)}|f(z)-f(a_k)|^q\mu(z)\mathrm{d}v(z)+\sum_{a_k\in\widetilde{\Gamma}_\zeta}|f(a_k)|^q\mu(a_k)(1-|a_k|^2)^{n+1}$$

$$\lesssim r^q\int_{\widetilde{\widetilde{\Gamma}}_\zeta}|f(u)|^q\mu(u)\mathrm{d}v(u)+\sum_{a_k\in\widetilde{\Gamma}_\zeta}|f(a_k)|^q\mu(a_k)(1-|a_k|^2)^{n+1}.$$

Noting that the admissible approach regions with different apertures define the same tent space with equivalent quasinorms, we obtain that

$$\begin{aligned}\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta}|f(z)|^q\mu(z)\mathrm{d}v(z)\Big)^{\frac{p}{q}}\mathrm{d}\sigma(\zeta)&\le Cr^q\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta}|f(z)|^q\mu(z)\mathrm{d}v(z)\Big)^{\frac{p}{q}}\mathrm{d}\sigma(\zeta)\\&\quad+C\int_{\mathbb{S}_n}\Big(\sum_{a_k\in\Gamma_\zeta}|f(a_k)|^q\mu(a_k)(1-|a_k|^2)^{n+1}\Big)^{\frac{p}{q}}\mathrm{d}\sigma(\zeta).\end{aligned}$$

Hence, choosing $r_0<(2C)^{-\frac{1}{q}}$, we have

$$\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta}|f(z)|^q\mu(z)\mathrm{d}v(z)\Big)^{\frac{p}{q}}\mathrm{d}\sigma(\zeta)\le 2C\int_{\mathbb{S}_n}\Big(\sum_{a_k\in\Gamma_\zeta}|f(a_k)|^q\mu(a_k)(1-|a_k|^2)^{n+1}\Big)^{\frac{p}{q}}\mathrm{d}\sigma(\zeta)$$

for any $r\in(0,r_0)$. The proof is complete.

Similarly, we obtain the following lemma.

Lemma 3.2 *Let $0<p<\infty$, $f\in H(\mathbb{B}_n)$ and $\mu\in\mathcal{R}$. Then there exists $r_0\in(0,1)$ so that, if $0<r<r_0$ and $Z=\{a_k\}$ is an r-lattice, then*

$$\int_{\mathbb{S}_n}\sup_{z\in\Gamma_\zeta}|f(z)|^p\mu(z)\mathrm{d}\sigma(\zeta)\lesssim\int_{\mathbb{S}_n}\sup_{a_k\in\Gamma_\zeta}|f(a_k)|^p\mu(a_k)\mathrm{d}\sigma(\zeta),$$

whenever the left-hand-side is finite.

Proof Note that $\mu(z)\approx\mu(w)$ for $w\in D(z,r)$ when $z\in\mathbb{B}_n$ and $\mu\in\mathcal{R}$, where $0<r<1$. By the subharmonicity of $|f|^p$, we have

$$\begin{aligned}|f(z)|^p&\lesssim\frac{1}{(1-|z|^2)^{n+1}\mu(z)}\int_{D(z,r)}|f(w)|^p\mu(w)\mathrm{d}v(w)\\&\lesssim\frac{1}{(1-|z|^2)^{n+1}\mu(z)}\int_{D(a_k,2r)}|f(w)|^p\mu(w)\mathrm{d}v(w)\end{aligned}$$

for some a_k in the lattice. Therefore, for $0<r<\frac{1}{4}$,

$$\sup_{z\in\Gamma_\zeta}|f(z)|^p\mu(z)\lesssim\sup_{a_k\in\widetilde{\Gamma}_\zeta}\frac{1}{(1-|a_k|^2)^{n+1}}\int_{D(a_k,2r)}|f(w)|^p\mu(w)dv(w).$$

Similar to the proof of Lemma 3.1, we obtain that

$$\sup_{z\in\Gamma_\zeta}|f(z)|^p\mu(z)\lesssim r^p\sup_{z\in\widetilde{\widetilde{\Gamma}}_\zeta}|f(z)|^p\mu(z)+\sup_{a_k\in\widetilde{\Gamma}_\zeta}|f(a_k)|^p\mu(a_k).$$

Integrating both sides over $\mathbb{S}_n$ deduces the desired result. The proof is complete.

Proof of Theorem 1.1 (i) Assume that $T_g : A^p_\omega \to H^q$ is bounded. Consider the test function

$$f_a(z) = \frac{1}{(1-|a|^2)^{\frac{n}{p}}\widehat{\omega}(a)^{\frac{1}{p}}}\Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|}\Big)^{\frac{\gamma+n}{p}}, \quad a \in \mathbb{B}_n,$$

where γ is large enough. From [9, Lemma 6] we see that $f_a \in A^p_\omega$. Using the standard pointwise estimate for the derivative of Hardy space functions, we get

$$|\Re(T_g f_a)(z)| = |f_a(z)||\Re g(z)| \lesssim \frac{\|T_g\|_{A^p_\omega \to H^q}\|f_a\|_{A^p_\omega}}{(1-|z|^2)^{1+\frac{n}{q}}}.$$

Therefore,

$$\sup_{a\in\mathbb{B}_n} |\Re g(a)|(1-|a|^2)^{1+\frac{n}{q}-\frac{n}{p}}\widehat{\omega}(a)^{-\frac{1}{p}} \lesssim \|T_g\|_{A^p_\omega\to H^q} < \infty.$$

Conversely, suppose that $\sup_{a\in\mathbb{B}_n} |\Re g(a)|(1-|a|^2)^{1+\frac{n}{q}-\frac{n}{p}}\widehat{\omega}(a)^{-\frac{1}{p}} < \infty$. We divide the rest of the proof into two parts.

Case 1 $p = q \le 2$. Applying the fact that $\frac{\widehat{\omega}(z)}{1-|z|^2} \in \mathcal{R}$ and Lemma 2.2, we obtain that

$$\begin{aligned}\|T_g f\|_{H^p} &\lesssim \|\Re(T_g f)\|_{A^p_{p-1}} = \Big(\int_{\mathbb{B}_n} |f(z)\Re g(z)|^p(1-|z|^2)^{p-1}\mathrm{d}v(z)\Big)^{\frac{1}{p}}\\ &\le \sup_{a\in\mathbb{B}_n}|\Re g(a)|(1-|a|^2)\widehat{\omega}(a)^{-\frac{1}{p}}\Big(\int_{\mathbb{B}_n}|f(z)|^p\frac{\widehat{\omega}(z)}{1-|z|^2}\mathrm{d}v(z)\Big)^{\frac{1}{p}}\\ &\lesssim \Big(\int_{\mathbb{B}_n}|f(z)|^p\omega(z)\mathrm{d}v(z)\Big)^{\frac{1}{p}} \le \|f\|_{A^p_\omega}.\end{aligned}$$

Case 2 $p < q$. By Lemma 2.2 and $\frac{\widehat{\omega}(z)}{1-|z|^2} \in \mathcal{R}$, we get that

$$\begin{aligned}\|T_g f\|_{H^p} &\lesssim \|\Re(T_g f)\|_{A^p_{p+\frac{pn}{q}-n-1}} = \Big(\int_{\mathbb{B}_n} |f(z)\Re g(z)|^p(1-|z|^2)^{p+\frac{pn}{q}-n-1}\mathrm{d}v(z)\Big)^{\frac{1}{p}}\\ &\le \sup_{a\in\mathbb{B}_n}|\Re g(a)|(1-|a|^2)^{1+\frac{n}{q}-\frac{n}{p}}\widehat{\omega}(a)^{-\frac{1}{p}}\Big(\int_{\mathbb{B}_n}|f(z)|^p\frac{\widehat{\omega}(z)}{1-|z|^2}\mathrm{d}v(z)\Big)^{\frac{1}{p}}\\ &\lesssim \Big(\int_{\mathbb{B}_n}|f(z)|^p\omega(z)\mathrm{d}v(z)\Big)^{\frac{1}{p}} \le \|f\|_{A^p_\omega}.\end{aligned}$$

(ii) Suppose that

$$\mathrm{d}\mu_g(z) = |\Re g(z)|^{\frac{2p}{p-2}}\frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)$$

is a Carleson measure. For any $f \in A^p_\omega$, consider the measure

$$\mathrm{d}\mu_{f,g}(z) = |f(z)|^2|\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z).$$

Employing Lemma 2.3, we have

$$\|T_g f\|^p_{H^p} \approx \int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta}|\Re g(z)|^2|f(z)|^2(1-|z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{p}{2}}\mathrm{d}\sigma(\zeta).$$

For any $h \in H^{\frac{p}{p-2}}$, by Hölder's inequality and $\frac{\widehat{\omega}(z)}{1-|z|^2} \in \mathcal{R}$, we obtain

$$\begin{aligned}
\|h\|_{L^1_{\mu_{f,g}}} &= \int_{\mathbb{B}_n} |h(z)||f(z)|^2|\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z)\\
&\lesssim \Big(\int_{\mathbb{B}_n} |f(z)|^p \frac{\widehat{\omega}(z)}{1-|z|^2}\mathrm{d}v(z)\Big)^{\frac{2}{p}} \Big(\int_{\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)\Big)^{\frac{p-2}{p}}\\
&\approx \Big(\int_{\mathbb{B}_n} |f(z)|^p \omega(z)\mathrm{d}v(z)\Big)^{\frac{2}{p}} \Big(\int_{\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} \mathrm{d}\mu_g(z)\Big)^{\frac{p-2}{p}}\\
&\lesssim \|f\|^2_{A^p_\omega} \|\mu_g\|_{\mathcal{CM}}^{\frac{p-2}{p}} \|h\|_{H^{\frac{p}{p-2}}},
\end{aligned}$$

which implies that the identity operator $I_d : H^{\frac{p}{p-2}} \to L^1_{\mu_{f,g}}$ is bounded. Therefore, using Lemmas 2.3–2.4, we get

$$\|T_g f\|^2_{H^p} \approx \Big\| \int_{\Gamma_\zeta} \frac{\mathrm{d}\mu_{f,g}(z)}{(1-|z|^2)^n} \Big\|_{L^{\frac{p}{2}}(\mathbb{S}_n)} \approx \|I_d\|_{H^{\frac{p}{p-2}} \to L^1_{\mu_{f,g}}} \lesssim \|\mu_g\|_{\mathcal{CM}}^{\frac{p-2}{p}} \|f\|^2_{A^p_\omega}.$$

Thus $T_g : A^p_\omega \to H^p$ is bounded and $\|T_g\|_{A^p_\omega \to H^p} \lesssim \|\mu_g\|_{\mathcal{CM}}^{\frac{p-2}{2p}}$.

Conversely, assume that $T_g : A^p_\omega \to H^p$ is bounded. Then for any $f \in A^p_\omega$,

$$\|I_d\|_{H^{\frac{p}{p-2}} \to L^1_{\mu_{f,g}}} \approx \|T_g f\|^2_{H^p} \lesssim \|T_g\|^2_{A^p_\omega \to H^p} \|f\|^2_{A^p_\omega}.$$

Therefore, for any $h \in H^{\frac{p}{p-2}}$, we have

$$\begin{aligned}
\|h\|_{L^1_{\mu_{f,g}}} &= \int_{\mathbb{B}_n} |h(z)||f(z)|^2|\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z)\\
&\lesssim \|T_g\|^2_{A^p_\omega \to H^p} \|f\|^2_{A^p_\omega} \|h\|_{H^{\frac{p}{p-2}}}.
\end{aligned}$$

Now define $d\upsilon_{h,g}(z) = |h(z)||\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z)$. [14, Theorem 4] gives that

$$\Big(\int_{\mathbb{B}_n} \Big(\frac{\upsilon_{h,g}(D(z,1))}{\widehat{\omega}(z)(1-|z|^2)^n}\Big)^{\frac{p}{p-2}} \frac{\widehat{\omega}(z)}{(1-|z|^2)}\mathrm{d}v(z)\Big)^{\frac{p-2}{p}} \lesssim \|I_d\|^2_{A^p_\omega \to L^2_{\upsilon_{h,g}}} \lesssim \|T_g\|^2_{A^p_\omega \to H^p} \|h\|_{H^{\frac{p}{p-2}}}.$$

By the subharmonicity of $|h||\Re g|^2$, we get

$$\upsilon_{h,g}(D(z,1)) \gtrsim |h(z)||\Re g(z)|^2(1-|z|^2)^{n+2}.$$

Therefore,

$$\begin{aligned}
\|h\|_{L^{\frac{p}{p-2}}_{\mu_g}} &= \Big(\int_{\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)\Big)^{\frac{p-2}{p}}\\
&\lesssim \|T_g\|^2_{A^p_\omega \to H^p} \|h\|_{H^{\frac{p}{p-2}}},
\end{aligned}$$

which means that μ_g is a Carleson measure and $\|\mu_g\|_{\mathcal{CM}}^{\frac{p-2}{2p}} \lesssim \|T_g\|_{A^p_\omega \to H^p}$.

(iii) Suppose first that $U_g \in L^{\frac{pq}{p-q}}(\mathbb{S}_n)$. For any $f \in A^p_\omega$, employing Lemma 2.3 and Hölder's inequality, we obtain that

$$\begin{aligned}
\|T_g f\|^q_{H^q} &\approx \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |\Re g(z)|^2 |f(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |f(z)|^p \frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}} \mathrm{d}v(z) \Big)^{\frac{q}{p}} \\
&\quad \cdot \Big(\int_{\Gamma_\zeta} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{q(p-2)}{2p}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \Big(\int_{\mathbb{S}_n} \int_{\Gamma_\zeta} |f(z)|^p \frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}} \mathrm{d}v(z) \mathrm{d}\sigma(\zeta) \Big)^{\frac{q}{p}} \\
&\quad \cdot \Big(\int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{q(p-2)}{2(p-q)}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{p-q}{p}} \\
&\lesssim \|f\|^q_{A^p_\omega} \|U_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)}. \qquad (3.1)
\end{aligned}$$

The last inequality comes from the fact that

$$\begin{aligned}
&\int_{\mathbb{S}_n} \int_{\Gamma_\zeta} |f(z)|^p \frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}} \mathrm{d}v(z) \mathrm{d}\sigma(\zeta) \\
&= \int_{\mathbb{B}_n} |f(z)|^p \frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}} \Big(\int_{\mathbb{S}_n} \chi_{\Gamma_\zeta}(z) \mathrm{d}\sigma(\zeta) \Big) \mathrm{d}v(z) \\
&\approx \int_{\mathbb{B}_n} |f(z)|^p \frac{\widehat{\omega}(z)}{1-|z|^2} \mathrm{d}v(z) \lesssim \|f\|^p_{A^p_\omega}.
\end{aligned}$$

Therefore, $T_g : A^p_\omega \to H^q$ is bounded and $\|T_g\|_{A^p_\omega \to H^q} \lesssim \|U_g\|_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)}$.

Conversely, assume that $T_g : A^p_\omega \to H^q$ is bounded. We claim that

$$\zeta \mapsto \Big(\sum_{a_k \in \Gamma_\zeta} |\Re g(a_k)|^{\frac{2p}{p-2}} \frac{(1-|a_k|^2)^{\frac{2p}{p-2}}}{\widehat{\omega}(a_k)^{\frac{2}{p-2}}} \Big)^{\frac{q(p-2)}{2p}} \in L^{\frac{p}{p-q}}(\mathbb{S}_n),$$

when $Z = \{a_k\}$ is an r-lattice and r is small enough. Then Lemma 3.1 gives the desired result. Now we only need to prove the above claim.

Let $Z = \{a_k\}$ be an r-lattice and r is small enough. Set

$$f_k(z) = \frac{1}{(1-|a_k|^2)^{\frac{n}{p}} \widehat{\omega}(a_k)^{\frac{1}{p}}} \Big(\frac{1-|a_k|^2}{1-\langle z, a_k \rangle} \Big)^{\frac{\lambda+n}{p}}$$

and consider the test functions

$$F_t(z) = \sum_{k=1}^{\infty} (1-|a_k|^2)^{\frac{n}{p}} \lambda_k r_k(t) f_k(z),$$

where $\lambda = \{\lambda_k\} \in T^p_p(Z)$ and r_k are the Rademacher functions. By [31, Theorem 3.2], we have

$$\|F_t\|^p_{A^p_\omega} \lesssim \sum_{k=1}^{\infty} (1-|a_k|^2)^n |\lambda_k|^p \approx \sum_{k=1}^{\infty} |\lambda_k|^p \int_{\mathbb{S}_n} \chi_{\Gamma_\zeta}(a_k) \mathrm{d}\sigma(\zeta)$$

$$= \int_{\mathbb{S}_n} \Big(\sum_{a_k \in \Gamma_\zeta} |\lambda_k|^p \Big) \mathrm{d}\sigma(\zeta) = \|\{\lambda_k\}\|^p_{T^p_p(Z)}.$$

Using Lemma 2.3, Fubini's theorem, Kahane's inequality (see [21, Lemma 5]), Khinchine's inequality (see [12, Appendix A]) and Lemma 2.5, we get

$$\begin{aligned}
&\int_0^1 \|T_g F_t\|^q_{H^q} \mathrm{d}t \\
&\approx \int_0^1 \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |\Re g(z) \sum_{k=1}^\infty (1-|a_k|^2)^{\frac{n}{p}} \lambda_k r_k(t) f_k(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \mathrm{d}t \\
&\approx \int_{\mathbb{S}_n} \int_0^1 \Big(\int_{\Gamma_\zeta} |\Re g(z) \sum_{k=1}^\infty (1-|a_k|^2)^{\frac{n}{p}} \lambda_k r_k(t) f_k(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}t \mathrm{d}\sigma(\zeta) \\
&\approx \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} \int_0^1 |\Re g(z) \sum_{k=1}^\infty (1-|a_k|^2)^{\frac{n}{p}} \lambda_k r_k(t) f_k(z)|^2 \mathrm{d}t (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&\approx \int_{\mathbb{S}_n} \Big(\sum_{k=1}^\infty |\lambda_k|^2 \int_{\Gamma_\zeta} |\Re g(z) (1-|a_k|^2)^{\frac{n}{p}} f_k(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&\approx \int_{\mathbb{S}_n} \Big(\sum_{k=1}^\infty |\lambda_k|^2 \int_{\mathbb{B}_n} \Big(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|} \Big)^{\theta n} |\Re g(z) (1-|a_k|^2)^{\frac{n}{p}} f_k(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta),
\end{aligned}$$

where $\theta > \max\left\{\frac{1}{s}, 1\right\}$. Next, replacing the integral over $\mathbb{B}_n$ by an integral over $D(a_k, r)$ and using the subharmonicity, we obtain that

$$\int_{\mathbb{S}_n} \Big(\sum_{k=1}^\infty |\lambda_k|^2 \Big(\frac{1-|a_k|^2}{|1-\langle a_k,\zeta\rangle|} \Big)^{\theta n} |\Re g(a_k)|^2 \frac{(1-|a_k|^2)^2}{\widehat{\omega}(a_k)^{\frac{2}{p}}} \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \lesssim \int_0^1 \|T_g F_t\|^q_{H^q} \mathrm{d}t.$$

Note that $1-|a_k|^2 \approx |1-\langle a_k,\zeta\rangle|$ for any $a_k \in \Gamma_\zeta$. Then for any $\zeta \in \mathbb{S}_n$, by summing over only the points $a_k \in \Gamma_\zeta$,

$$\begin{aligned}
&\int_{\mathbb{S}_n} \Big(\sum_{a_k \in \Gamma_\zeta} |\lambda_k|^2 |\Re g(a_k)|^2 \frac{(1-|a_k|^2)^2}{\widehat{\omega}(a_k)^{\frac{2}{p}}} \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \int_0^1 \|T_g F_t\|^q_{H^q} \mathrm{d}t \\
&\lesssim \|T_g\|^q_{A^p_\omega \to H^q} \|F_t\|^q_{A^p_\omega} \\
&\lesssim \|T_g\|^q_{A^p_\omega \to H^q} \|\lambda\|^q_{T^p_p(Z)}.
\end{aligned}$$

Let $\upsilon = \{\upsilon_k\}$, where

$$\upsilon_k = |\Re g(a_k)|^q \frac{(1-|a_k|^2)^q}{\widehat{\omega}(a_k)^{\frac{q}{p}}}.$$

Then we need to prove that $\upsilon \in T^{\frac{p}{p-q}}_{\frac{2p}{q(p-2)}}(Z)$, which is equivalent to $\upsilon^{\frac{1}{s}} \in T^{\frac{ps}{p-q}}_{\frac{2ps}{q(p-2)}}(Z)$ for any $s > 1$. For any $\eta = \{\eta_k\} \in T^{s'}_{\frac{2s}{2s-q}}(Z)$ and $\lambda = \{\lambda_k\} \in T^p_p(Z)$, let $\xi = \{\xi_k\}$, where $\xi_k = \eta_k \lambda_k^{\frac{q}{s}}$.

Applying Hölder's inequality, we have

$$
\begin{aligned}
|\langle \xi, \upsilon^{\frac{1}{s}}\rangle_{T_2^2(Z)}| &\le \sum_{k=1}^{\infty} |\eta_k \lambda_k^{\frac{q}{s}} \upsilon_k^{\frac{1}{s}}|(1-|a_k|^2)^n \approx \int_{\mathbb{S}_n} \sum_{a_k\in\Gamma_\zeta} |\eta_k \lambda_k^{\frac{q}{s}} \upsilon_k^{\frac{1}{s}}| \mathrm{d}\sigma(\zeta) \\
&\lesssim \int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} |\eta_k|^{\left(\frac{2s}{q}\right)'} \Big)^{1-\frac{q}{2s}} \Big(\sum_{a_k\in\Gamma_\zeta} |\lambda_k|^2 |\upsilon_k|^{\frac{2}{q}} \Big)^{\frac{q}{2s}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \Big(\int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} |\eta_k|^{\frac{2s}{2s-q}} \Big)^{s'\cdot\frac{2s-q}{2s}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{1}{s'}} \Big(\int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} |\lambda_k|^2 |\upsilon_k|^{\frac{2}{q}} \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{1}{s}} \\
&\lesssim \|\eta\|_{T^{s'}_{\frac{2s}{2s-q}}(Z)} \Big(\int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} |\lambda_k|^2 |\upsilon_k|^{\frac{2}{q}} \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{1}{s}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{k=1}^{\infty} |\eta_k \lambda_k^{\frac{q}{s}} \upsilon_k^{\frac{1}{s}}|(1-|a_k|^2)^n &\lesssim \|\eta\|_{T^{s'}_{\frac{2s}{2s-q}}(Z)} \Big(\int_{\mathbb{S}_n} \Big(\sum_{k=1}^{\infty} |\lambda_k|^2 |\Re g(z)|^2 \frac{(1-|a_k|^2)^2}{\widehat{\omega}(a_k)^{\frac{2}{p}}} \Big)^{\frac{q}{2}} d\sigma(\zeta) \Big)^{\frac{1}{s}} \\
&\lesssim \|\eta\|_{T^{s'}_{\frac{2s}{2s-q}}(Z)} \|T_g\|^{\frac{q}{s}}_{A^p_\omega \to H^q} \|\lambda\|^{\frac{q}{s}}_{T^p_p(Z)}.
\end{aligned}
$$

Lemmas 2.6–2.7 give that

$$
T^{\frac{ps}{p-q}}_{\frac{2ps}{q(p-2)}}(Z) = \Big(T^{(\frac{ps}{p-q})'}_{(\frac{2ps}{q(p-2)})'}(Z) \Big)^* = \Big(T^{s'}_{\frac{2s}{2s-q}}(Z) \cdot T^{\frac{ps}{q}}_{\frac{ps}{q}}(Z) \Big)^*
$$

and $\upsilon^{\frac{1}{s}} = \{\upsilon_k^{\frac{1}{s}}\} \in T^{\frac{ps}{p-q}}_{\frac{2ps}{q(p-2)}}(Z)$, that is,

$$
\|\upsilon^{\frac{1}{s}}\|_{T^{\frac{ps}{p-q}}_{\frac{2ps}{q(p-2)}}(Z)} = \Big(\int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} \Big(\frac{(1-|a_k|^2)|\Re g(a_k)|}{\widehat{\omega}(a_k)^{\frac{1}{p}}} \Big)^{\frac{2p}{p-2}} \Big)^{\frac{q(p-2)}{2(p-q)}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{p-q}{ps}} \lesssim \|T_g\|^{\frac{q}{s}}_{A^p_\omega \to H^q}.
$$

Using Lemma 3.1, we obtain

$$
\begin{aligned}
\|U_g\|^{\frac{pq}{p-q}}_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} &= \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{q(p-2)}{2(p-q)}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} \Big(\frac{(1-|a_k|^2)|\Re g(a_k)|}{\widehat{\omega}(a_k)^{\frac{1}{p}}} \Big)^{\frac{2p}{p-2}} \Big)^{\frac{q(p-2)}{2(p-q)}} \mathrm{d}\sigma(\zeta) \lesssim \|T_g\|^{\frac{pq}{p-q}}_{A^p_\omega \to H^q} < \infty.
\end{aligned}
$$

(iv) Assume that

$$
\int_{\mathbb{S}_n} \sup_{z\in\Gamma_\zeta} |\Re g(z)|^{\frac{pq}{p-q}} \frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \mathrm{d}\sigma(\zeta) < \infty.
$$

Lemma 2.3 and Hölder's inequality imply that

$$
\begin{aligned}
\|T_g f\|^q_{H^q} &\approx \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |\Re g(z)|^2 |f(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \int_{\mathbb{S}_n} V_g(\zeta)^q \Big(\int_{\Gamma_\zeta} |f(z)|^2 \widehat{\omega}(z)^{\frac{2}{p}} (1-|z|^2)^{-n-1} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta)
\end{aligned}
$$

$$\lesssim \|V_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} \Big(\int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta} |f(z)|^2 \widehat{\omega}(z)^{\frac{2}{p}} (1-|z|^2)^{-n-1} \mathrm{d}v(z) \Big)^{\frac{p}{2}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{q}{p}}.$$

Here $V_g(\zeta) = \sup_{z\in\Gamma_\zeta} |\Re g(z)| \frac{1-|z|^2}{\widehat{\omega}(z)^{\frac{1}{p}}}$. Let $\{a_k\}$ be an r-lattice. It is easy to check that

$$\begin{aligned} &\Big(\int_{\Gamma_\zeta} |f(z)|^2 \widehat{\omega}(z)^{\frac{2}{p}} (1-|z|^2)^{-n-1} \mathrm{d}v(z) \Big)^{\frac{p}{2}} \\ &\lesssim \sum_{k=1}^{\infty} \chi(\Gamma_\zeta \cap D(a_k,r)) \Big(\int_{D(a_k,r)} |f(z)|^2 \widehat{\omega}(z)^{\frac{2}{p}} (1-|z|^2)^{-n-1} \mathrm{d}v(z) \Big)^{\frac{p}{2}} \\ &\lesssim \sum_{k=1}^{\infty} \chi(\Gamma_\zeta \cap D(a_k,r)) \Big(\max_{z\in D(a_k,r)} |f(z)|^p \Big) \widehat{\omega}(a_k) \\ &\lesssim \sum_{k=1}^{\infty} \frac{\chi(\Gamma_\zeta \cap D(a_k,r))}{(1-|a_k|^2)^n} \int_{D(a_k,2r)} |f(z)|^p \frac{\widehat{\omega}(z)}{(1-|z|^2)} \mathrm{d}v(z) \end{aligned} \tag{3.2}$$

and

$$\int_{\mathbb{S}_n} \chi(\Gamma_\zeta \cap D(a_k,r)) \mathrm{d}\sigma(\zeta) \lesssim (1-|a_k|^2)^n. \tag{3.3}$$

Therefore, using the fact that $\frac{\widehat{\omega}(z)}{1-|z|^2} \in \mathcal{R}$, we have

$$\begin{aligned} \|T_g f\|^q_{H^q} &\lesssim \|V_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} \Big(\sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(z)|^p \frac{\widehat{\omega}(z)}{(1-|z|^2)} \mathrm{d}v(z) \Big)^{\frac{q}{p}} \\ &\approx \|V_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} \Big(\sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(z)|^p \omega(z) \mathrm{d}v(z) \Big)^{\frac{q}{p}} \\ &\lesssim \|V_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} \|f\|^q_{A^p_\omega}, \end{aligned}$$

which implies that $T_g : A^p_\omega \to H^q$ is bounded.

Conversely, suppose that $T_g : A^p_\omega \to H^q$ is bounded. Similar to the proof of (iii), we also have

$$\int_{\mathbb{S}_n} \Big(\sum_{a_k\in\Gamma_\zeta} |\lambda_k|^2 |\Re g(a_k)|^2 \frac{(1-|a_k|^2)^2}{\widehat{\omega}(a_k)^{\frac{2}{p}}} \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \lesssim \|T_g\|^q_{A^p_\omega \to H^q} \|\lambda\|^q_{T^p_p(Z)}$$

and

$$|\langle \xi, \upsilon^{\frac{1}{s}} \rangle_{T^2_2(Z)}| \lesssim \|\eta\|_{T^{s'}_{\frac{2s}{2s-q}}(Z)} \|T_g\|^{\frac{q}{s}}_{A^p_\omega \to H^q} \|\lambda\|^{\frac{q}{s}}_{T^p_p(Z)}.$$

Lemmas 2.6–2.7 give that

$$T^{\frac{ps}{p-q}}_\infty(Z) = \Big(T^{s'}_{\frac{2s}{2s-q}}(Z) \cdot T^{\frac{ps}{q}}_{\frac{ps}{q}}(Z) \Big)^*$$

and $\upsilon^{\frac{1}{s}} = \{\upsilon_k^{\frac{1}{s}}\} \in T^{\frac{ps}{p-q}}_\infty(Z)$. Therefore,

$$\|\upsilon^{\frac{1}{s}}\|_{T^{\frac{ps}{p-q}}_\infty(Z)} = \Big(\int_{\mathbb{S}_n} \Big(\sup_{a_k\in\Gamma_\zeta} \frac{(1-|a_k|^2)|\Re g(a_k)|}{\widehat{\omega}(a_k)^{\frac{1}{p}}} \Big)^{\frac{pq}{p-q}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{p-q}{ps}} \lesssim \|T_g\|^{\frac{q}{s}}_{A^p_\omega \to H^q}.$$

Using Lemma 3.2, we obtain

$$\int_{\mathbb{S}_n} \Big(\sup_{z\in\Gamma_\zeta} \frac{|\Re g(z)|(1-|z|^2)}{\widehat{\omega}(z)^{\frac{1}{p}}} \Big)^{\frac{pq}{p-q}} \mathrm{d}\sigma(\zeta) \lesssim \int_{\mathbb{S}_n} \Big(\sup_{a_k\in\Gamma_\zeta} \frac{|\Re g(a_k)|(1-|a_k|^2)}{\widehat{\omega}(a_k)^{\frac{1}{p}}} \Big)^{\frac{pq}{p-q}} \mathrm{d}\sigma(\zeta)$$
$$\lesssim \|T_g\|_{A^p_\omega\to H^q}^{\frac{pq}{p-q}} < \infty.$$

The proof is complete.

4 Compactness

In this section, we will prove Theorem 1.2. We begin this section with a useful lemma.

Lemma 4.1 *Let $0<p<\infty$, $f\in H(\mathbb{B}_n)$, $Z=\{a_k\}$ be an r-lattice and $\mu\in\mathcal{R}$. If*

$$\int_{\mathbb{S}_n} \sup_{z\in\Gamma_\zeta} |f(z)|^p\mu(z)\mathrm{d}\sigma(\zeta)<\infty$$

and

$$\lim_{\delta\to 1^-}\int_{\mathbb{S}_n} \sup_{a_k\in\Gamma_\zeta\backslash\delta\mathbb{B}_n} |f(a_k)|^p\mu(a_k)\mathrm{d}\sigma(\zeta)=0,$$

whenever $r>0$ is small enough, then

$$\lim_{\delta\to 1^-}\int_{\mathbb{S}_n} \sup_{z\in\Gamma_\zeta\backslash\delta\mathbb{B}_n} |f(z)|^p\mu(z)d\sigma(\zeta)=0.$$

Proof For any $\varepsilon>0$, choose $r>0$ small enough such that $r^p<\varepsilon$. Fix this r. Let

$$\widetilde{\delta}=\inf\{|a_k| : a_k\in Z, a_k\in D(z,r) \text{ for some } |z|\geq\delta\}.$$

Similar to the proof of Lemma 3.2, we have

$$\sup_{a_k\in\Gamma_\zeta\backslash\delta\mathbb{B}_n} |f(z)|^p\mu(z) \lesssim r^p \sup_{z\in\widetilde{\widetilde{\Gamma}}_\zeta} |f(z)|^p\mu(z) + \sup_{a_k\in\widetilde{\Gamma}_\zeta\backslash\widetilde{\delta}\mathbb{B}_n} |f(a_k)|^p\mu(a_k).$$

Integrating over $\mathbb{S}_n$ and noting that the admissible approach regions with different apertures define the same tent space with equivalent quasinorms, we get

$$\begin{aligned}
&\int_{\mathbb{S}_n} \sup_{z\in\Gamma_\zeta\backslash\delta\mathbb{B}_n} |f(z)|^p\mu(z)\mathrm{d}\sigma(\zeta)\\
&\lesssim r^p\int_{\mathbb{S}_n} \sup_{z\in\widetilde{\widetilde{\Gamma}}_\zeta} |f(z)|^p\mu(z)\mathrm{d}\sigma(\zeta) + \int_{\mathbb{S}_n} \sup_{a_k\in\widetilde{\Gamma}_\zeta\backslash\widetilde{\delta}\mathbb{B}_n} |f(a_k)|^p\mu(a_k)\mathrm{d}\sigma(\zeta)\\
&\lesssim \varepsilon + \int_{\mathbb{S}_n} \sup_{a_k\in\widetilde{\Gamma}_\zeta\backslash\widetilde{\delta}\mathbb{B}_n} |f(a_k)|^p\mu(a_k)\mathrm{d}\sigma(\zeta).
\end{aligned}$$

Since $\widetilde{\delta}\to 1^-$ as $\delta\to 1^-$, it is easy to obtain that

$$\limsup_{\delta\to 1^-}\int_{\mathbb{S}_n} \sup_{z\in\Gamma_\zeta\backslash\delta\mathbb{B}_n} |f(z)|^p\mu(z)\mathrm{d}\sigma(\zeta)\lesssim\varepsilon.$$

The proof is complete.

Proof of Theorem 1.2 (i) The desired result can be proved by standard modifications of the proof of Theorem 1.1(i). Here we omit the details.

(ii) Suppose that

$$\mathrm{d}\mu_g(z) = |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z)$$

is a vanishing Carleson measure. Let $\{f_k\}$ be a bounded sequence in A^p_ω and converge to 0 uniformly on any compact subset of $\mathbb{B}_n$. Next, we show that $\|T_g f_k\|_{H^p} \to 0$ as $k \to \infty$. For any $f \in A^p_\omega$, we consider the measure

$$\mathrm{d}\mu_{f,g}(z) = |f(z)|^2 |\Re g(z)|^2 (1-|z|^2) \mathrm{d}v(z).$$

For any $h \in H^{\frac{p}{p-2}}$, employing Hölder's inequality and the fact that $\frac{\widehat{\omega}(z)}{1-|z|^2} \in \mathcal{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{B}_n} |h(z)| \mathrm{d}\mu_{f_k,g}(z) \\
&= \int_{r\mathbb{B}_n} |h(z)||f_k(z)|^2 |\Re g(z)|^2 (1-|z|^2) \mathrm{d}v(z) + \int_{\mathbb{B}_n \setminus r\mathbb{B}_n} |h(z)||f_k(z)|^2 |\Re g(z)|^2 (1-|z|^2) \mathrm{d}v(z) \\
&\lesssim \Big(\int_{r\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{p-2}{p}} \Big(\int_{r\mathbb{B}_n} |f_k(z)|^p \frac{\widehat{\omega}(z)}{1-|z|^2} \mathrm{d}v(z) \Big)^{\frac{2}{p}} \\
&\quad + \Big(\int_{\mathbb{B}_n \setminus r\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} |\Re g(z)|^{\frac{2p}{p-2}} \frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}} \mathrm{d}v(z) \Big)^{\frac{p-2}{p}} \Big(\int_{\mathbb{B}_n \setminus r\mathbb{B}_n} |f_k(z)|^p \frac{\widehat{\omega}(z)}{1-|z|^2} \mathrm{d}v(z) \Big)^{\frac{2}{p}} \\
&\approx \Big(\int_{r\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} \mathrm{d}\mu_g(z) \Big)^{\frac{p-2}{p}} \Big(\int_{r\mathbb{B}_n} |f_k(z)|^p \omega(z) \mathrm{d}v(z) \Big)^{\frac{2}{p}} \\
&\quad + \Big(\int_{\mathbb{B}_n \setminus r\mathbb{B}_n} |h(z)|^{\frac{p}{p-2}} \mathrm{d}\mu_g(z) \Big)^{\frac{p-2}{p}} \Big(\int_{\mathbb{B}_n \setminus r\mathbb{B}_n} |f_k(z)|^p \omega(z) \mathrm{d}v(z) \Big)^{\frac{2}{p}} \\
&\lesssim \|h\|_{H^{\frac{p}{p-2}}} \Big(\int_{r\mathbb{B}_n} |f_k(z)|^p \omega(z) \mathrm{d}v(z) \Big)^{\frac{2}{p}} + \|h\|_{H^{\frac{p}{p-2}}} \|f_k\|^2_{A^p_\omega} \|\chi_{(\mathbb{B}_n \setminus r\mathbb{B}_n)} \mu_g\|^{\frac{p-2}{p}}_{\mathcal{CM}}.
\end{aligned}$$

From [7, Lemma 3.1], for any $\varepsilon > 0$, there exists $0 < r_0 < 1$ such that

$$\|\chi_{(\mathbb{B}_n \setminus r_0\mathbb{B}_n)} \mu_g\|^{\frac{p-2}{p}}_{\mathcal{CM}} < \varepsilon.$$

Fix this r_0. Since $\{f_k\}$ converges to 0 uniformly on $r_0\mathbb{B}_n$,

$$\Big(\int_{r_0\mathbb{B}_n} |f_k(z)|^p \omega(z) \mathrm{d}v(z) \Big)^{\frac{2}{p}} < \varepsilon \quad \text{as } k \to \infty.$$

Therefore,

$$\int_{\mathbb{B}_n} |h(z)| d\mu_{f_k,g}(z) \lesssim \varepsilon \|h\|_{H^{\frac{p}{p-2}}} \quad \text{as } k \to \infty,$$

which means that

$$\lim_{k \to \infty} \|I_{d_k}\|_{H^{\frac{p}{p-2}} \to L^1_{\mu_{f_k,g}}} = 0.$$

Employing Lemmas 2.3–2.4, we have

$$\begin{aligned}\|T_g f_k\|_{H^p}^2 &\approx \Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta}|\Re g(z)|^2|f_k(z)|^2(1-|z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{p}{2}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{2}{p}}\\ &\approx \Big\|\int_{\Gamma_\zeta}\frac{\mathrm{d}\mu_{f_k,g}(z)}{(1-|z|^2)^n}\Big\|_{L^{\frac{p}{2}}(\mathbb{S}_n)} \approx \|I_{d_k}\|_{H^{\frac{p}{p-2}}\to L^1_{\mu_{f_k,g}}} \to 0 \qquad (4.1)\end{aligned}$$

as $k\to\infty$. This implies that $T_g : A^p_\omega \to H^p$ is compact.

Conversely, assume that $T_g : A^p_\omega \to H^p$ is compact. Similar to (4.1), we have that the identity operator $I_{d_f} : H^{\frac{p}{p-2}} \to L^1_{\mu_{f,g}}$ is compact and for any $h\in H^{\frac{p}{p-2}}$,

$$\int_{\mathbb{B}_n}|h(z)|\mathrm{d}\mu_{f,g}(z) \lesssim \|T_g f\|^2_{H^p}\|h\|_{H^{\frac{p}{p-2}}}.$$

Let $B_{A^p_\omega} = \{f\in A^p_\omega : \|f\|_{A^p_\omega}\le 1\}$ be the closed unit ball of A^p_ω. Then $T_g(B_{A^p_\omega})$ is a relatively compact set in H^p. For any $\epsilon>0$, there are $f_1,\cdots,f_N\in B_{A^p_\omega}$ such that for any $f\in A^p_\omega$, there exist some $f_m,\ m=1,\cdots,N$ satisfying

$$\|T_g f - T_g f_m\|_{H^p} < \sqrt{\varepsilon}.$$

Let $\{h_k\}$ be a bounded sequence in $H^{\frac{p}{p-2}}$ and converge to 0 uniformly on any compact subset of $\mathbb{B}_n$. Since $I_{d_f} : H^{\frac{p}{p-2}} \to L^1_{\mu_{f,g}}$ is compact, for $m\in\{1,\cdots,N\}$, there exists $K_m>0$ such that $k>K_m$ satisfying

$$\int_{\mathbb{B}_n}|h_k(z)|\mathrm{d}\mu_{f_m,g}(z) < \varepsilon.$$

Now we define

$$\mathrm{d}\upsilon_{h_k,g}(z) = |h_k(z)||\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z).$$

Let $f\in B_{A^p_\omega}$ and $k>K=\max\{K_1,\cdots,K_N\}$. Then there exists $m\in\{1,\cdots,N\}$ such that

$$\begin{aligned}\int_{\mathbb{B}_n}|f(z)|^2 d\upsilon_{h_k,g}(z) &\le \int_{\mathbb{B}_n}|f(z)-f_m(z)|^2\mathrm{d}\upsilon_{h_k,g}(z) + \int_{\mathbb{B}_n}|f_m(z)|^2\mathrm{d}\upsilon_{h_k,g}(z)\\ &= \int_{\mathbb{B}_n}|h_k(z)|\mathrm{d}\mu_{f-f_m,g}(z) + \int_{\mathbb{B}_n}|h_k(z)|\mathrm{d}\mu_{f_m,g}(z)\\ &\lesssim \|T_g(f-f_m)\|^2_{H^p}\|h_k\|_{H^{\frac{p}{p-2}}} + \int_{\mathbb{B}_n}|h_k(z)|\mathrm{d}\mu_{f_m,g}(z)\\ &\lesssim \varepsilon,\end{aligned}$$

which means that these identity operators $I_{d_k} : A^p_\omega \to L^2_{\upsilon_{h_k,g}}$ are bounded and

$$\lim_{k\to\infty}\|I_{d_k}\|^2_{A^p_\omega\to L^2_{\upsilon_{h_k,g}}} = 0.$$

From the proof of Theorem 1.1, we have

$$\|h_k\|_{L^{\frac{p}{p-2}}_{\mu_g}} = \Big(\int_{\mathbb{B}_n}|h_k(z)|^{\frac{p}{p-2}}|\Re g(z)|^{\frac{2p}{p-2}}\frac{(1-|z|^2)^{\frac{2+p}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)\Big)^{\frac{p-2}{p}} \lesssim \|I_{d_k}\|^2_{A^p_\omega\to L^2_{\upsilon_{k,g}}}.$$

Thus,

$$\lim_{k\to\infty}\int_{\mathbb{B}_n}|h_k(z)|^{\frac{p}{p-2}}\mathrm{d}\mu_g(z)=0,$$

which implies that $I_d : H^{\frac{p}{p-2}} \to L^{\frac{p}{p-2}}_{\mu_g}$ is compact and μ_g is a vanishing Carleson measure.

(iii) The necessity is obvious. Now we only need to consider the sufficiency. Suppose that the function $U_g \in L^{\frac{pq}{p-q}}(\mathbb{S}_n)$. Then for any $\varepsilon > 0$, there exists $\delta_0 \in (0,1)$ such that

$$\Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|\Re g(z)|^{\frac{2p}{p-2}}\frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)\Big)^{\frac{q(p-2)}{2(p-q)}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{p-q}{pq}} < \varepsilon^{\frac{1}{q}}.$$

Let $\{f_k\}$ be a bounded sequence in A^p_ω and converge to 0 uniformly on any compact subset of $\mathbb{B}_n$. Next, we show that $\|T_gf_k\|_{H^q} \to 0$ as $k\to\infty$. Similar to getting (3.1), by Lemma 2.3 and Hölder's inequality, we have

$$\begin{aligned}
\|T_gf_k\|^q_{H^q} &\approx \int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|\Re g(z)|^2|f_k(z)|^2(1-|z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{q}{2}}\mathrm{d}\sigma(\zeta)\\
&\quad+\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|\Re g(z)|^2|f_k(z)|^2(1-|z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{q}{2}}\mathrm{d}\sigma(\zeta)\\
&\lesssim \Big(\int_{\mathbb{S}_n}\int_{\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|f_k(z)|^p\frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}}\mathrm{d}v(z)\mathrm{d}\sigma(\zeta)\Big)^{\frac{q}{p}}\\
&\quad\cdot\Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|\Re g(z)|^{\frac{2p}{p-2}}\frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)\Big)^{\frac{q(p-2)}{2(p-q)}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{p-q}{p}}\\
&\quad+\Big(\int_{\mathbb{S}_n}\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|f_k(z)|^p\frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}}\mathrm{d}v(z)\mathrm{d}\sigma(\zeta)\Big)^{\frac{q}{p}}\\
&\quad\cdot\Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|\Re g(z)|^{\frac{2p}{p-2}}\frac{(1-|z|^2)^{\frac{(1-n)p+2(n+1)}{p-2}}}{\widehat{\omega}(z)^{\frac{2}{p-2}}}\mathrm{d}v(z)\Big)^{\frac{q(p-2)}{2(p-q)}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{p-q}{p}}\\
&\lesssim \|U_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)}\Big(\int_{\mathbb{S}_n}\int_{\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|f_k(z)|^p\frac{\widehat{\omega}(z)}{(1-|z|^2)^{n+1}}\mathrm{d}v(z)\mathrm{d}\sigma(\zeta)\Big)^{\frac{q}{p}}+\varepsilon\|f_k\|^q_{A^p_\omega}\\
&\lesssim \|U_g\|^q_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)}\Big(\int_{\delta_0\mathbb{B}_n}|f_k(z)|^p\omega(z)\mathrm{d}v(z)\Big)^{\frac{q}{p}}+\varepsilon\|f_k\|^q_{A^p_\omega}\to 0
\end{aligned}$$

as $k\to\infty$. This means that $T_g : A^p_\omega \to H^q$ is compact.

(iv) We first consider the sufficiency. Assume that

$$\lim_{r\to 1^-}\int_{\mathbb{S}_n}\sup_{z\in\Gamma_\zeta\setminus r\mathbb{B}_n}|\Re g(z)|^{\frac{pq}{p-q}}\frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}}\mathrm{d}\sigma(\zeta)=0.$$

Then for any $\varepsilon>0$, there exists $\delta_0\in(0,1)$ such that

$$\int_{\mathbb{S}_n}\sup_{z\in\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|\Re g(z)|^{\frac{pq}{p-q}}\frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}}\mathrm{d}\sigma(\zeta)<\varepsilon^{\frac{p}{p-q}}.$$

Let $\{f_k\}$ be a bounded sequence in A^p_ω and converge to 0 uniformly on any compact subset of $\mathbb{B}_n$. Next, we show that $\|T_gf_k\|_{H^q}\to 0$ as $k\to\infty$. Note that there exists $N\in\mathbb{N}$ such that

$|f_k| < \varepsilon^{\frac{1}{q}}$ on $\delta_0\mathbb{B}_n$ for all $k > N$. Using Lemma 2.3, (3.2)–(3.3) and Hölder's inequality, we have that

$$\begin{aligned}
\|T_g f_k\|_{H^q}^q &\approx \int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|\Re g(z)|^2|f_k(z)|^2(1-|z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{q}{2}}\mathrm{d}\sigma(\zeta)\\
&\quad+\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|\Re g(z)|^2|f_k(z)|^2(1-|z|^2)^{1-n}\mathrm{d}v(z)\Big)^{\frac{q}{2}}\mathrm{d}\sigma(\zeta)\\
&\lesssim \Big(\int_{\mathbb{S}_n}\sup_{z\in\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|\Re g(z)|^{\frac{pq}{p-q}}\frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{p-q}{p}}\\
&\quad\cdot\Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\cap\delta_0\mathbb{B}_n}|f_k(z)|^2\widehat{\omega}(z)^{\frac{2}{p}}(1-|z|^2)^{-n-1}\mathrm{d}v(z)\Big)^{\frac{p}{2}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{q}{p}}\\
&\quad+\Big(\int_{\mathbb{S}_n}\sup_{z\in\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|\Re g(z)|^{\frac{pq}{p-q}}\frac{(1-|z|^2)^{\frac{pq}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{p-q}{p}}\\
&\quad\cdot\Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|f_k(z)|^2\widehat{\omega}(z)^{\frac{2}{p}}(1-|z|^2)^{-n-1}\mathrm{d}v(z)\Big)^{\frac{p}{2}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{q}{p}}\\
&\lesssim \varepsilon+\varepsilon\cdot\Big(\int_{\mathbb{S}_n}\Big(\int_{\Gamma_\zeta\setminus\delta_0\mathbb{B}_n}|f_k(z)|^2\widehat{\omega}(z)^{\frac{2}{p}}(1-|z|^2)^{-n-1}\mathrm{d}v(z)\Big)^{\frac{p}{2}}\mathrm{d}\sigma(\zeta)\Big)^{\frac{q}{p}}\\
&\lesssim \varepsilon+\varepsilon\|f_k\|_{A^p_\omega}^q,
\end{aligned}$$

which implies that $T_g : A^p_\omega \to H^q$ is compact.

Now we consider the necessity. Suppose that $T_g : A^p_\omega \to H^q$ is compact. Let

$$V_{g,\delta}(\zeta) = \sup_{z\in\Gamma_\zeta\setminus\delta\mathbb{B}_n}|\Re g(z)|\frac{1-|z|^2}{\widehat{\omega}(z)^{\frac{1}{p}}}.$$

Theorem 1.1(iv) gives that $V_{g,0} \in L^{\frac{pq}{p-q}}(\mathbb{S}_n)$. Next, we show that

$$\lim_{\delta\to 1^-}\|V_{g,\delta}\|_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} = 0.$$

By Lemma 4.1, it is sufficient to prove that

$$\zeta \mapsto \sup_{a_k\in\Gamma_\zeta\setminus\delta\mathbb{B}_n}|\Re g(a_k)|^q\frac{(1-|a_k|^2)^q}{\widehat{\omega}(a_k)^{\frac{q}{p}}}$$

converges to 0 in $L^{\frac{p}{p-q}}(\mathbb{S}_n)$ as $\delta \to 1^-$, where $Z = \{a_k\}$ is an r-lattice for r small enough. Let $B_{T_p^p(Z)} = \{\lambda \in T_p^p(Z) : \|\lambda\|_{T_p^p(Z)} \le 1\}$ be the closed unit ball of $T_p^p(Z)$. For any $\lambda = \{\lambda_k\} \in B_{T_p^p(Z)}$, define

$$S(\lambda)(z) = \sum_{k=1}^{\infty}(1-|a_k|^2)^{\frac{n}{p}}\lambda_k f_k(z), \quad z \in \mathbb{B}_n,$$

where

$$f_k(z) = \frac{1}{(1-|a_k|^2)^{\frac{n}{p}}\widehat{\omega}(a_k)^{\frac{1}{p}}}\Big(\frac{1-|a_k|^2}{1-\langle z,a_k\rangle}\Big)^{\frac{\lambda+n}{p}}.$$

Then we have that $S(\lambda) \in A^p_\omega$ with $\|S(\lambda)\|_{A^p_\omega} \lesssim 1$ for any $\lambda \in B_{T_p^p(Z)}$. Since $T_g : A^p_\omega \to H^q$ is compact, it is obvious that $T_g \circ S(B_{T_p^p(Z)})$ is a relatively compact set in H^q. Using Lemma 2.3, there exists $\delta_0 \in (0,1)$ such that

$$\begin{aligned}
&\int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta \backslash \delta_0 \mathbb{B}_n} |\Re g(z) S(\lambda)(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&= \int_{\mathbb{S}_n} \Big(\int_{\Gamma_\zeta \backslash \delta_0 \mathbb{B}_n} |\Re g(z) \sum_{k=1}^{\infty} (1-|a_k|^2)^{\frac{n}{p}} \lambda_k f_k(z)|^2 (1-|z|^2)^{1-n} \mathrm{d}v(z) \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \varepsilon^q
\end{aligned}$$

for any $\lambda \in B_{T_p^p(Z)}$. Similar to the proof of Theorem 1.1(iv), we have

$$\int_{\mathbb{S}_n} \Big(\sum_{a_k \in \Gamma_\zeta \backslash \delta_0' \mathbb{B}_n} |\lambda_k|^2 |\Re g(a_k)|^2 \frac{(1-|a_k|^2)^2}{\widehat{\omega}(a_k)^{\frac{2}{p}}} \Big)^{\frac{q}{2}} \mathrm{d}\sigma(\zeta) \lesssim \varepsilon^q \|\lambda\|^q_{T_p^p(Z)},$$

where $\delta_0' = \inf\{|a_k| : D(a_k, r) \subset \mathbb{B}_n \backslash \delta_0 \mathbb{B}_n\}$. Let $\upsilon_\delta = \{\upsilon_{k,\delta}\}$, where

$$\upsilon_{k,\delta} = |\Re g(a_k)|^q \frac{(1-|a_k|^2)^q}{\widehat{\omega}(a_k)^{\frac{q}{p}}} \chi_{(\mathbb{B}_n \backslash \delta \mathbb{B}_n)}(a_k).$$

Then we need to prove that

$$\lim_{\delta \to 1^-} \|\upsilon_\delta\|_{T_\infty^{\frac{p}{p-q}}(Z)} = 0,$$

which is equivalent to

$$\lim_{\delta \to 1^-} \|\upsilon_\delta^{\frac{1}{s}}\|_{T_\infty^{\frac{ps}{p-q}}(Z)} = 0$$

for any $s > 1$. Similar to the proof of Theorem 1.2(iii), we get

$$\sum_{k=1}^{\infty} |\xi_k \upsilon_\delta^{\frac{1}{s}}| (1-|a_k|^2)^n \lesssim \varepsilon^{\frac{q}{s}} \|\eta\|_{T^{s'}_{\frac{2s}{2s-q}}(Z)} \|\lambda\|^{\frac{q}{s}}_{T_p^p(Z)},$$

whenever $\delta > \delta_0'$, where η and ξ_k are defined as Theorem 1.2(iii). Lemmas 2.6–2.7 give that

$$T_\infty^{\frac{ps}{p-q}}(Z) = \Big(T^{s'}_{\frac{2s}{2s-q}}(Z) \cdot T^{\frac{ps}{q}}_{\frac{ps}{q}}(Z) \Big)^*$$

and $\upsilon_\delta^{\frac{1}{s}} = \{\upsilon_{k,\delta}^{\frac{1}{s}}\} \in T_\infty^{\frac{p}{p-q}}(Z)$. Therefore,

$$\|\upsilon_\delta^{\frac{1}{s}}\|_{T_\infty^{\frac{ps}{p-q}}(Z)} = \Big(\int_{\mathbb{S}_n} \Big(\sup_{a_k \in \Gamma_\zeta \backslash \delta \mathbb{B}_n} \frac{(1-|a_k|^2)|\Re g(a_k)|}{\widehat{\omega}(a_k)^{\frac{1}{p}}} \Big)^{\frac{pq}{p-q}} \mathrm{d}\sigma(\zeta) \Big)^{\frac{p-q}{ps}} \lesssim \varepsilon^{\frac{q}{s}}.$$

Using Lemma 3.2, we obtain

$$\begin{aligned}
\|V_{g,\delta}\|^{\frac{pq}{p-q}}_{L^{\frac{pq}{p-q}}(\mathbb{S}_n)} &= \int_{\mathbb{S}_n} \Big(\sup_{z \in \Gamma_\zeta \backslash \delta \mathbb{B}_n} \frac{|\Re g(z)|(1-|z|^2)}{\widehat{\omega}(z)^{\frac{1}{p}}} \Big)^{\frac{pq}{p-q}} \mathrm{d}\sigma(\zeta) \\
&\lesssim \int_{\mathbb{S}_n} \Big(\sup_{a_k \in \Gamma_\zeta \backslash \delta \mathbb{B}_n} \frac{|\Re g(a_k)|(1-|a_k|^2)}{\widehat{\omega}(a_k)^{\frac{1}{p}}} \Big)^{\frac{pq}{p-q}} \mathrm{d}\sigma(\zeta) \lesssim \varepsilon^{\frac{pq}{p-q}}.
\end{aligned}$$

The proof is complete.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Ahern, P. and Bruna, J., Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of $\mathbb{C}^n$, *Rev. Mat. Iberoamericana*, **4**(1), 1988, 123–153.

[2] Aleman, A. and Cima, J., An integral operator on H^p and Hardy's inequality, *J. Anal. Math.*, **85**, 2001, 157–176.

[3] Aleman, A. and Siskakis, A., Integration operators on Bergman spaces, *Indiana Univ. Math. J.*, **46**, 1997, 337–356.

[4] Arsenović, M., Embedding derivatives of $\mathcal{M}$-harmonic functions into L^p-spaces, *Rocky Mountain J. Math.*, **29**(1), 1999, 61–76.

[5] Beatrous, F. and Burbea, J., Holomorphic Sobolev spaces on the ball, *Dissertationes Math.*, **276**, 1989, 60 pp.

[6] Carleson, L., An interpolation problem for bounded analytic functions, *Amer. J. Math.*, **80**, 1958, 921–930.

[7] Chen, J., Pau, J. and Wang, M., Essential norms and Schatten(-Herz) classes of integration operators from Bergman spaces to Hardy spaces, *Results Math.*, **76**(2), 2021, Paper No. 88, 33 pp.

[8] Coifman, R., Meyer, Y. and Stein, E., Some new function spaces and their applications to harmonic analysis, *J. Funct. Anal.*, **62**, 1985, 304–335.

[9] Du, J., Li, S., Liu, X. and Shi, Y., Weighted Bergman spaces induced by doubling weights in the unit ball of $\mathbb{C}^n$, *Anal. Math. Phys.*, **10**(4), 2020, Paper No. 64, 41 pp.

[10] Duan, Y., Wang, S. and Wang, Z., Volterra-type integration operators between weighted Bergman spaces and Hardy spaces, *Comput. Meth. Funct. Theory*, **23**(4), 2023, 589–627.

[11] Duren, P., Extension of a theorem of Carleson, *Bull. Amer. Math. Soc.*, **75**, 1969, 143–146.

[12] Duren, P., Theory of H^p Spaces, Academic Press, New York, London, 1970.

[13] Hu, Z., Extended Cesàro operators on mixed norm spaces, *Proc. Amer. Math. Soc.*, **131**(7), 2003, 2171–2179.

[14] Hu, Z., Extended Cesàro operators on Bergman spaces, *J. Math. Anal. Appl.*, **296**(2), 2004, 435–454.

[15] Jevtić, M., Embedding derivatives of $\mathcal{M}$-harmonic Hardy spaces H^p into Lebesgue spaces, $0 < p < 2$, *Rocky Mountain J. Math.*, **26**(1), 1996, 175–187.

[16] Koo, H. and Wang, M., Joint Carleson measure and the difference of composition operators on $A^p_\alpha(\mathbb{B}_n)$, *J. Math. Anal. Appl.*, **419**(2), 2014, 1119–1142.

[17] Li, P., Liu, J. and Lou, Z., Integral operators on analytic Morrey spaces, *Sci. China Math.*, **57**, 2014, 1961–1974.

[18] Li, S. and Stević, S., Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin*, **14**(4), 2007, 621–628.

[19] Li, S. and Stević, S., Riemann-Stieltjes-type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Var. Elliptic Equ.*, **52**(6), 2007, 495–517.

[20] Luecking, D., Embedding derivatives of Hardy spaces into Lebesgue spaces, *Proc. London Math. Soc.*, **63**(3), 1991, 595–619.

[21] Luecking, D., Embedding theorems for spaces of analytic functions via Khinchine's inequality, *Michigan Math. J.*, **40**(2), 1993, 333–358.

[22] Miihkinen, S., Pau, J., Perälä, A. and Wang, M., Volterra type integration operators from Bergman spaces to Hardy spaces, *J. Funct. Anal.*, **279**(4), 2020, 108564, 32 pp.

[23] Pau, J., Integration operators between Hardy spaces on the unit ball of $\mathbb{C}^n$, *J. Funct. Anal.*, **270**(1), 2016, 134–176.

[24] Peláez, J., Small Weighted Bergman Spaces, Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, University of Eastern Finland, Kuopio, 2016, 29–98.

[25] Peláez, J. and Rättyä, J., Weighted Bergman spaces induced by rapidly increasing weights, *Mem. Amer. Math. Soc.*, **227**(1066), 2014, 3155774.

[26] Peng, R. and Ouyang, C., Riemann-Stieltjes operators and multipliers on Q_p spaces in the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.*, **377**(1), 2011, 180–193.

[27] Pommerenke, C., Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation, *Comment. Math. Helv.*, **52**, 1977, 591–602.

[28] Wu, Z., Volterra operator, area integral and Carleson measures, *Sci. China Math.*, **54**(11), 2011, 2487–2500.

[29] Xiao, J., Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, *J. London Math. Soc.*, **70**(2), 2004, 199–214.

[30] Yang, Y. and Liu, J., Integral operators on Bergman-Morrey spaces, *J. Geom. Anal.*, **32**(6), 2022, Paper No. 181, 23 pp.

[31] Zhang, X., Xi, L., Fan, H. and Li, J., Atomic decomposition of μ-Bergman space in $\mathbb{C}^n$, *Acta Math. Sci. Ser.*, **34**(3), 2014, 779–789.

[32] Zhao, R. and Zhu, K., Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$, *Mém. Soc. Math. Fr.* (*N. S.*), **115**, 2008, 2537698.

[33] Zhu, K., Spaces of Holomorphic Functions in the Unit Ball, Grad. Texts in Math., **226**, Springer-Verlag, New York, 2005.

[34] Zhu, K., Operator Theory in Function Spaces (2nd ed.), American Mathematical Society, Providence, RI, 2007.