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Second Main Theorem and Uniqueness Problem of Meromorphic Functions with Finite Growth Index Sharing Five Small Functions on a Complex Disc*

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Abstract This paper has a twofold purpose. The first is to establish a second main theorem for meromorphic functions on the complex disc $\Delta(R_0) \subset \mathbb{C}$ with finite growth index and small functions, where the counting functions are truncated to level 1 and the small term is more detailedly estimated. The second is to prove a generalization and improvement of the five values theorem of Nevanlinna for the case of five small functions on the complex disc $\Delta(R_0)$.

Keywords Nevanlinna theory, Second main theorem, Meromorphic function, Small function
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1 Introduction and Results

In 1926, Nevanlinna [1] showed that two distinct non-constant meromorphic functions f and g on $\mathbb{C}$ cannot have the same inverse images for five distinct values. This theorem is called the five values theorem. Later on, many authors have improved and generalized this theorem by replacing the five values with five small functions and by weakening the condition "having the same inverse images". For related results, we refer the readers to some important papers such as [1, 5–6, 8] and the references therein. All of the proofs of these results are based on the second main theorem of Yi and Yang [7] or the second main theorem of Yamanoi [4] for meromorphic functions and small functions on $\mathbb{C}$.

Recently, Ru and Sibony [3] gave a definition of a class of meromorphic functions on the complex disc $\Delta(R_0) = \{z \in \mathbb{C}; |z| < R_0\}$ with finite growth index, which is larger than the class of transcendental meromorphic functions. They also established a new second main theorem for meromorphic functions in this class with fixed values. Motivated by their work, in this paper we will give a truncated second main theorem for meromorphic functions of finite growth index on $\Delta(R_0)$ with small functions, which generalizes the second main theorem of Yi and Yang mentioned above.

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In order to state our result, we recall the following notations. Let $\Delta(R_0) = \{z \in \mathbb{C}; |R| < R_0\}$ $(0 < R_0 \leq +\infty)$ be a disc in $\mathbb{C}$. Let ν be a divisor on $\Delta(R_0)$, which is regarded as a function on $\Delta(R_0)$ with values in $\mathbb{Z}$ such that $\mathrm{Supp}\,(\nu) := \{z; \nu(z) \neq 0\}$ is a discrete subset of $\Delta(R_0)$. The counting function of ν is defined by

$$n(t) := \sum_{|z| \leq t} \nu(z) \quad 0 \leq t \leq R_0 \quad \text{and} \quad N(r, \nu) := \int_0^r \frac{n(t) - n(0)}{t} \mathrm{d}t.$$

Let $f : \Delta(R_0) \to \mathbb{C} \cup \{\infty\}$ be a non-constant meromorphic function. We denote by ν_f^0 (resp. ν_f^∞) the divisor of zeros (resp. divisor of poles) of f. For a meromorphic function a, we define

$$N(r, a, f) := N(r, \nu_{f-a}^0) \quad \text{and} \quad \overline{N}(r, a, f) := N(r, \min\{1, \nu_{f-a}^0\}).$$

Here, if $a \equiv \infty$ then we regard $\nu_{f-\infty}^0$ as ν_f^∞. We say that the function $N(r, a, f)$ (resp. $\overline{N}(r, a, f)$) is the counting function with multiplicity (resp. the counting function without multiplicity) of the divisor of all a-points of f.

The proximity function and the characteristic function of f are defined respectively by

$$m(r, f) := \int_0^{2\pi} \log^+ |f(re^{\mathrm{i}\theta})| \mathrm{d}\theta$$

and

$$T(r, f) := m(r, f) + N(r, \infty, f).$$

A meromorphic function a is said to be small with respect to f if $T(r, a) = o(T(r, f))$ as $r \to R_0$. The first main theorem states that

$$T(r, f) = T\Big(r, \frac{1}{f - a}\Big) + o(T(r, f))$$

for every small function a $\big($here, we regard $\frac{1}{f-\infty}$ as $f\big)$.

According to Ru and Sibony [3], the growth index of f is defined by

$$c_f = \inf\Big\{c > 0; \int_0^R \exp(cT(r, f))\mathrm{d}r = +\infty\Big\}.$$

For convenience, we will set $c_f = +\infty$ if $\big\{c > 0; \int_0^R \exp(cT(r, f))\mathrm{d}r = +\infty\big\} = \emptyset$.

For the case where the domain $\Delta(R_0)$ is $\mathbb{C}$, Yi and Yang proved a second main theorem for small functions as follows.

Theorem A (see [7, p. 185]) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, \cdots, a_5$ be five distinct small functions* (*with respect to f*)*. Then we have*

$$\| \; 2T(r, f) \leq \sum_{i=1}^{5} \overline{N}(r, a_i, f) + o(T(r, f)).$$

Here, the notation "$\| P$" means the assertion P holds for all $r \in (0, +\infty)$ outside a set of finite Borel measure. Hence, for q distinct small functions $a_1, \cdots, a_q$ ($q \geq 5$), from the above second main theorem we get

$$\Big\| \ \frac{2q}{5} T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r, a_i, f) + o(T(r,f)).$$

We note that, Yamanoi in [4] improved the above result by increasing the coefficient $\frac{2q}{5}$ to $(q-2)$. The result of Yamanoi is believed to be sharp for the case of functions on $\mathbb{C}$. However, the method of Yamanoi in [4] does not work for the case of functions on complex discs with finite growth index.

Our first purpose in this paper is to generalize Theorem A to the case of meromorphic functions on $\Delta(R_0)$ with finite growth index. Namely, we will prove the following.

Theorem 1.1 *Let f be a non-constant meromorphic function on $\Delta(R_0)$ and let $a_1, \cdots, a_5$ be five distinct small functions (with respect to f). Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R_0)$ with $\int_0^{R_0} \gamma(r)\mathrm{d}r = +\infty$. Then, for every $\varepsilon > 0$,*

$$\|_E \ 2T(r,f) \leq \sum_{i=1}^{5} \overline{N}(r, a_i, f) + 19((1+\varepsilon)\log\gamma(r) + \varepsilon \log r) + o(T(r,f)).$$

Here and later on, we use the notation "$\|_E P$" to say that the assertion P holds for all $r \in (0; R_0)$ outside a subset E of $(0; R_0)$ with $\int_E \gamma(r)\mathrm{d}r < +\infty$.

Remark 1.1 (1) If the characteristic function $T(r,f)$ grows rapidly enough $\big($more than the order of $\log \frac{1}{R_0 - r}\big)$, then Theorem 1.1 is no more special than Theorem A. But for the general case, Theorem 1.1 will be more interesting since the error term in the second main theorem may not be a small term $o(T(r,f))$.

(2) If we assume that f is of finite growth index (i.e., $c_f < +\infty$), then in Theorem 1.1 we may take $\gamma(r) = \exp{(c_f + \varepsilon)T(r,f)}$.

(3) For the case of $R_0 = +\infty$, we may take $c_f = 0$ (with a non-constant meromorphic function f) and Theorem 1.1 will give back Theorem A.

Our second aim in this paper is to apply the above second main theorem to investigate the uniqueness problem of meromorphic functions on $\Delta(R_0)$ with finite growth index sharing some small functions ignoring multiplicity.

Now, for two meromorphic functions h, a and a positive integer k (k may be $+\infty$), we denote by $\overline{E}_{k)}(a,h)$ the set of all zeros of the function $h - a$ with multiplicity not exceeding k (if $a \equiv \infty$ then $\overline{E}_{k)}(a,h)$ is regarded as the set of all poles of h with multiplicity not exceeding k). Our uniqueness theorem is stated as follows.

Theorem 1.2 *Let f and g be non-constant meromorphic functions on $\Delta(R_0)$ with finite growth indices c_f, c_g. Let $a_1, \cdots, a_q$ be q ($q \geq 5$) distinct small functions (with respect to f and g) and let k be a positive integer or $k = +\infty$. Assume that*

$$\overline{E}_{k)}(a_i, f) = \overline{E}_{k)}(a_i, g), \quad 1 \leq i \leq q.$$

If

$$c_f + c_g < \frac{k(2q-8) - 3(q+4)}{k(19q - \frac{117}{2}) + 19(q+4)},$$

then $f = g$.

Remark 1.2 (1) If $c_f = c_g = 0$, then the condition of the above theorem is fulfilled with

$$k > \frac{3(q+4)}{2q-8}.$$

Moreover, for the case of $q = 5$, we only need $k \geq 14$. Therefore, our result improves the result of Yao in [5] (for $q = 5, k \geq 22$) and generalizes the result of Yi in [6] (for $q = 5$, $k \geq 14$).

(2) If $k = +\infty$ and $q = 5$, then the condition of the above theorem is fulfilled with

$$c_f + c_g < \frac{4}{73}.$$

Then, we have a generalization of the classical five values theorem of Nevanlinna.

(3) In the proof of Theorem 1.2, we use the method of Yi in [6]. However, the most difficulty comes from the fact that all error terms occurring are not small with respect to these functions. Therefore, we will have to control such terms at every step in the proof.

2 Truncated Second Main Theorem for Meromorphic Functions and Small Functions on the Complex Disc

In this section, we will prove Theorem 1.1. Firstly, we need the following lemma on logarithmic derivative due to Ru and Sibony [3].

Lemma 2.1 (Lemma on logarithmic derivative) (see [3, Theorem 5.1]) *Let* $0 < R \leq +\infty$ *and let* $\gamma(r)$ *be a non-negative measurable function defined on* $(0, R_0)$ *with* $\int_0^{R_0} \gamma(r)\mathrm{d}r = \infty$. *Let* $f(z)$ *be a meromorphic function on* $\Delta(R_0)$. *Then, for* $\varepsilon > 0$, *we have*

$$\Big\|_E\, m\Big(r, \frac{f'}{f}\Big) \leq (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + O(\log T(r,f)).$$

Then, for any small function a (with respect to f) we also have

$$\Big\|_E\, m\Big(r, \frac{a'}{a}\Big) \leq (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + o(T(r,f)). \tag{2.1}$$

This implies that

$$\begin{aligned}\Big\|_E\, N\Big(r, 0, \frac{a'}{a}\Big) &\leq T\Big(r, \frac{a'}{a}\Big) = N\Big(r, \infty, \frac{a'}{a}\Big) + m\Big(r, \frac{a'}{a}\Big)\\ &\leq \overline{N}(r,0,a) + \overline{N}(r,\infty,a) + (1+\varepsilon)\log\gamma(r) + \varepsilon\log r + o(T(r,f))\\ &= (1+\varepsilon)\log\gamma(r) + \varepsilon\log r + o(T(r,f)). \end{aligned} \tag{2.2}$$

Ru and Sibony proved the following second main theorem for fixed values.

Theorem 2.1 (see [3, Theorem 1.7]) *Let f be a non-constant meromorphic function on $\Delta(R_0)$ $(0 < R_0 \leq +\infty)$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R_0)$ with $\int_0^{R_0} \gamma(r)\mathrm{d}r = \infty$ and let $c_1, \cdots, c_q$ be q distinct values in $\mathbb{C} \cup \{\infty\}$. Then, for every $\varepsilon > 0$, we have*

$$\|_E \ (q-n-1)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r, c_i, f) + (1+\varepsilon)\log\gamma(r) + \varepsilon \log r + o(T(r,f)).$$

Proof of Theorem 1.1 We set $S(r) = (1+\varepsilon)\log\gamma(r) + \varepsilon\log r$. Replacing f by its quasi-Möbius transformation

$$g = \frac{f-a_2}{f-a_1} \cdot \frac{a_3-a_1}{a_3-a_2},$$

if necessary, we may assume that $a_1 = \infty, a_2 = 0, a_3 = 1$ and $a_4 = b_1, a_5 = b_2$, where b_1, b_2 are two small functions with respect to f.

If b_1 or b_2 is constant, then we get the conclusion from the second main theorem for four values (Theorem 2.1). Then, we may assume that both b_1 and b_2 are not constant. We define

$$F = \begin{vmatrix} ff' & f' & f^2-f \\ b_1b_1' & b_1' & b_1^2-b_1 \\ b_2b_2' & b_2' & b_2^2-b_2 \end{vmatrix}. \tag{2.3}$$

We consider the following two cases.

Case 1 $F(z) \equiv 0$. From (2.3), we get

$$\Big(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}\Big)\Big(\frac{f'}{f-1} - \frac{b_2'}{b_2-1}\Big) \equiv \Big(\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}\Big)\Big(\frac{f'}{f} - \frac{b_2'}{b_2}\Big). \tag{2.4}$$

We distinguish the following four subcases.

Subcase 1 $\frac{b_1'}{b_1} \equiv \frac{b_2'}{b_2}$. We have $\frac{b_1'}{b_1-1} \not\equiv \frac{b_2'}{b_2-1}$, since if otherwise b_1 and b_2 are constants. Therefore, from (2.4), we have $\frac{f'}{f} \equiv \frac{b_2'}{b_2}$, and hence $f = cb_2$ with a constant c. This implies that $T(r,f) = o(T(r,f))$ and we get a contradiction.

Subcase 2 $\frac{b_1'}{b_1-1} \equiv \frac{b_2'}{b_2-1}$. By the same arguments as in Subcase 1, we get again a contradiction.

Subcase 3 $\frac{b_1'}{b_1} \not\equiv \frac{b_2'}{b_2}, \frac{b_1'}{b_1-1} \not\equiv \frac{b_2'}{b_2-1}, \frac{b_1'}{b_1} - \frac{b_2'}{b_2} \equiv \frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}$. The identity (2.4) implies that

$$\frac{f'}{f-1} - \frac{f'}{f} \equiv \frac{b_2'}{b_2-1} - \frac{b_2'}{b_2}.$$

Then, we have

$$\frac{f-1}{f} \equiv c \cdot \frac{b_2-1}{b_2},$$

where c is a constant. Therefore

$$f = \frac{b_2}{(c-1)b_2 - c},$$

and hence

$$T(r,f) = o(T(r,f)).$$

This is a contradiction.

Subcase 4 $\frac{b_1'}{b_1} \not\equiv \frac{b_2'}{b_2}, \frac{b_1'}{b_1-1} \not\equiv \frac{b_2'}{b_2-1}, \frac{b_1'}{b_1} - \frac{b_2'}{b_2} \not\equiv \frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}$. The identity (2.4) may be rewritten as

$$\Big(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}\Big)\frac{f'}{f-1} - \Big(\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}\Big)\frac{f'}{f} \equiv \frac{b_1'b_2'}{b_1(b_2-1)} - \frac{b_2'b_1'}{b_2(b_1-1)}. \tag{2.5}$$

From (2.5), we see that each zero of $(f-1)$ must be a zero or a 1-point or a pole of b_j $(j=1,2)$ or a zero of $\frac{b_1'}{b_1} - \frac{b_2'}{b_2}$. Therefore,

$$\min\{1, \nu^0_{f-1}\} \leq \sum_{i=1,2}\sum_{a=0,1,\infty} \min\{1, \nu^0_{b_i-a}\} + \min\Big\{1, \nu^0_{\frac{b_1'}{b_1} - \frac{b_2'}{b_2}}\Big\}. \tag{2.6}$$

Similarly, we have

$$\min\{1, \nu^0_{f}\} \leq \sum_{i=1,2}\sum_{a=0,1,\infty} \min\{1, \nu^0_{b_i-a}\} + \min\Big\{1, \nu^0_{\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}}\Big\}. \tag{2.7}$$

From (2.7), we also see that each pole of f must be a zero or a 1-point or a pole of b_j $(j=1,2)$ or a zero of $(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}) - (\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1})$. Therefore, by the similar arguments, we have

$$\min\{1, \nu^\infty_{f}\} \leq \sum_{i=1,2}\sum_{a=0,1,\infty} \min\{1, \nu^0_{b_i-a}\} + \min\Big\{1, \nu^0_{(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}) - (\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1})}\Big\}. \tag{2.8}$$

Combining (2.6)–(2.8), we have

$$\begin{aligned}\sum_{a=0,1,\infty} \min\{1, \nu^0_{f-a}\} \leq & \sum_{i=1,2}\sum_{a=0,1,\infty} \min\Big\{1, \nu^0_{b_i-a}\} + \min\{1, \nu^0_{\frac{b_1'}{b_1} - \frac{b_2'}{b_2}}\Big\} \\ & + \min\Big\{1, \nu^0_{\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}}\Big\} + \min\Big\{1, \nu^0_{(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}) - (\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1})}\Big\}. \end{aligned} \tag{2.9}$$

By using the second main theorem (Theorem 2.1), we get

$$\begin{aligned} \|_E\ T(r,f) &\leq \overline{N}(r,0,f) + \overline{N}(r,1,f) + \overline{N}(r,\infty,f) + S(r) \\ &\leq \sum_{i=1,2}\sum_{a=0,1,\infty} \overline{N}(r,0,b_i-a) + \overline{N}\Big(r,0,\frac{b_1'}{b_1} - \frac{b_2'}{b_2}\Big) + \overline{N}\Big(r,0,\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}\Big) \\ &\quad + \overline{N}\Big(r,0,\Big(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}\Big) - \Big(\frac{b_1'}{b_1-1} - \frac{b_2'}{b_2-1}\Big)\Big) + S(r) \\ &\leq 4S(r) \quad \text{(by (2.2))}. \end{aligned}$$

Then we have the desired second main theorem in this case.

Case 2 $F(z) \not\equiv 0$. We set

$$\begin{aligned} &\delta(z) = \min\{1, |b_1(z)|, |b_2(z)|, |b_1(z)-1|, |b_2(z)-1|, |b_1(z)-b_2(z)|\}, \\ &\theta_j(r) = \{\theta : |f(re^{i\theta}) - b_j(re^{i\theta})| \leq \delta(re^{i\theta})\}, \quad j=1,2, \\ &\theta_3(r) = \{\theta : |f(re^{i\theta})| \leq \delta(re^{i\theta})\}, \\ &\theta_4(r) = \{\theta : |f(re^{i\theta}) - 1| \leq \delta(re^{i\theta})\}. \end{aligned}$$

It is clear that the sets $\theta_i(r) \cap \theta_j(r)$ $(i \neq j, i, j = 1, 2, 3, 4)$ have at most finitely many points. Then, we easily see that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\delta(re^{i\theta})} d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} \log\max\Big\{1, \frac{1}{|b_1|}, \frac{1}{|b_2|}, \frac{1}{|b_1-1|}, \frac{1}{|b_2-1|}, \frac{1}{|b_1-b_2|}\Big\} d\theta \\
&\leq m\Big(r, \frac{1}{b_1}\Big) + m\Big(r, \frac{1}{b_2}\Big) + m\Big(r, \frac{1}{b_1-1}\Big) \\
&\quad + m\Big(r, \frac{1}{b_2-1}\Big) + m\Big(r, \frac{1}{b_1-b_2}\Big).
\end{aligned}$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\delta(re^{i\theta})} d\theta = o(T(r, f)). \tag{2.10}$$

We also see that

$$\begin{aligned}
ff' &= (f-b_1)(f'-b_1') + b_1'(f-b_1) + b_1(f'-b_1') + b_1 b_1', \\
f' &= (f'-b_1') + b_1', \\
f^2 - f &= (f-b_1)^2 + (2b_1-1)(f-b_1) + b_1^2 - b_1.
\end{aligned}$$

Substituting these functions (on the right hand side) into (2.3), by the properties of the Wronskian, we have

$$F = \begin{vmatrix} \varphi & f'-b_1' & \psi \\ b_1 b_1' & b_1' & b_1'-b_1 \\ b_2 b_2' & b_2' & b_2'-b_2 \end{vmatrix}, \tag{2.11}$$

where

$$\begin{aligned}
\varphi &= (f-b_1)(f'-b_1') + b_1'(f-b_1) + b_1(f'-b_1'), \\
\psi &= (f-b_1)^2 + (2b_1-1)(f-b_1).
\end{aligned}$$

From (2.11) we see that each zero with multiplicity p $(p > 1)$ of $f - b_1$ which is neither a pole of b_1 nor a pole of b_2 must be a zero of F with multiplicity at least $p-1$. Similarly, each zero with multiplicity p $(p > 1)$ of $f - b_2$ which is neither a pole of b_1 nor a pole of b_2 must be a zero of F with multiplicity at least $p-1$. Moreover, from (2.3) one sees that each zero of multiplicity p $(p > 1)$ of f or $f - 1$ which is neither a pole of b_1 nor a pole of b_2 must be a zero of F with multiplicity at least $p-1$. This implies that

$$\sum_{i=2}^{5} (N(r, a_i, f) - \overline{N}(r, a_i, f)) \leq N(r, 0, F). \tag{2.12}$$

On the other hand, since $|f(re^{i\theta}) - b_1(re^{i\theta})| \leq \delta(re^{i\theta}) \leq 1 + |b_1(re^{i\theta})|$ for every $\theta \in \theta_1(r)$,

$$\Big\|_E \ \frac{1}{2\pi}\int_{\theta_1(r)} \log^+ \Big|\frac{F}{f-b_1}\Big| d\theta \leq \frac{1}{2\pi}\int_{\theta_1(r)} \log\Big(1 + \Big|\frac{F}{f-b_1}\Big|\Big) d\theta$$

$$\leq \frac{1}{2\pi}\int\limits_{\theta_1(r)} \log\Big\{(1+|b_1'|)(1+|b_1|)\Big(1+\Big|\frac{f'-b_1'}{f-b_1}\Big|\Big)(1+|f-b_1|)^2\Big\}\mathrm{d}\theta$$
$$+\frac{1}{2\pi}\int\limits_{\theta_1(r)} \log\Big\{(1+|b_1|)(1+|b_1'|)\Big\}\mathrm{d}\theta$$
$$+\frac{1}{2\pi}\int\limits_{\theta_1(r)} \log\Big\{(1+|b_2|)(1+|b_2'|)\Big\}\mathrm{d}\theta+O(1)$$
$$\leq m\Big(r,\frac{f'-b_1'}{f-b_1}\Big)+2m\Big(r,\frac{b_1'}{b_1}\Big)+m\Big(r,\frac{b_2'}{b_2}\Big)+o(T(r,f))$$
$$\leq 4S(r)+o(T(r,f)). \tag{2.13}$$

Therefore, from (2.9) and (2.13), we have

$$\Big\|_E\ m\Big(r,\frac{1}{f-b_1}\Big) \leq \frac{1}{2\pi}\int\limits_{\theta_1(r)} \log^+\Big|\frac{1}{f-b_1}\Big|\mathrm{d}\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log\frac{1}{\delta(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}\theta$$
$$\leq \frac{1}{2\pi}\int\limits_{\theta_1(r)} \log^+\Big|\frac{F}{f-b_1}\Big|\mathrm{d}\theta+\frac{1}{2\pi}\int\limits_{\theta_1(r)} \log^+\frac{1}{|F|}\mathrm{d}\theta+o(T(r,f))$$
$$=\frac{1}{2\pi}\int\limits_{\theta_1(r)} \log^+\frac{1}{|F|}\mathrm{d}\theta+4S(r)+o(T(r,f)). \tag{2.14}$$

Similarly, we get

$$\Big\|_E\ m\Big(r,\frac{1}{f-b_2}\Big) \leq \frac{1}{2\pi}\int\limits_{\theta_2(r)} \log^+\frac{1}{|F|}\mathrm{d}\theta+4S(r)+o(T(r,f)), \tag{2.15}$$
$$\Big\|_E\ m\Big(r,\frac{1}{f}\Big) \leq \frac{1}{2\pi}\int\limits_{\theta_3(r)} \log^+\frac{1}{|F|}\mathrm{d}\theta+4S(r)+o(T(r,f)), \tag{2.16}$$
$$\Big\|_E\ m\Big(r,\frac{1}{f-1}\Big) \leq \frac{1}{2\pi}\int\limits_{\theta_4(r)} \log^+\frac{1}{|F|}\mathrm{d}\theta+4S(r)+o(T(r,f)). \tag{2.17}$$

Combining (2.14)–(2.17), we have

$$\Big\|_E\ \sum_{i=2}^{5} m\Big(r,\frac{1}{f-a_i}\Big) \leq m\Big(r,\frac{1}{F}\Big)+16S(r)+o(T(r,f)).$$

Therefore

$$\|_E\ 4T(r,f)\leq \sum_{i=2}^{5} N(r,a_i,f)-N(r,0,F)+T(r,F)+16S(r)+o(T(r,f)).$$

Combining this inequality with (2.12), we obtain

$$\|_E\ 4T(r,f)\leq \sum_{i=2}^{5} \overline{N}(r,a_i,f)+T(r,F)+16S(r)+o(T(r,f)). \tag{2.18}$$

Also, from (2.3), we have

$$m(r,F)\le 2m(r,f)+m\Big(r,\frac{f'}{f}\Big)+m\Big(r,\frac{b_1'}{b_1}\Big)+m\Big(r,\frac{b_2'}{b_2}\Big)+o(T(r,f))$$

and

$$N(r,\infty,F)\le 2N(r,\infty,f)+\overline{N}(r,\infty,f)+o(T(r,f)).$$

These yield that

$$\|_E\ T(r,F)\le 2T(r,f)+\overline{N}(r,\infty,f)+3S(r)+o(T(r,f)). \tag{2.19}$$

From (2.18)–(2.19), we have the conclusion of the theorem.

3 Uniqueness Theorem for Meromorphic Functions with Finite Growth Index Sharing Five Small Functions

In this section, we will prove Theorem 1.2. Firstly, we prove the following lemma.

Lemma 3.1 *Let f and g be two distinct meromorphic functions on $\Delta(R_0)$ with finite growth indices c_f and c_g, respectively. Let $a_1,\cdots,a_q$ $(q\ge 5)$ be distinct small functions with respect to f and g. Suppose that*

$$\overline{E}(a_i,k,f)=\overline{E}(a_i,k,g),\quad 1\le i\le q.$$

Let ε be a positive real number. Setting

$$T(r)=T(r,f)+T(r,g),\quad \gamma(r)=\mathrm{e}^{(\varepsilon+\max\{c_f,c_g\})T(r)}$$

and

$$S(r)=(1+\varepsilon)\log\gamma(r)+\varepsilon\log r,$$

then we have

$$\Big\|_E\ \sum_{i=5}^{q}\overline{N}(r,a_i,f,g)\le\sum_{i=1}^{4}(\overline{N}_{(k+1}(r,a_i,f)+\overline{N}_{(k+1}(r,a_i,g))+7S(r)+o(T(r)).$$

Here, $\overline{N}(r,a,f,g)$ denotes the counting function without multiplicity which counts all common zeros of $(f-a)$ and $(g-a)$, and $\overline{N}_{(k+1}(r,a,f)$ denotes the counting function without multiplicity which counts zeros of $(f-a)$ with multiplicity at least $k+1$.

Proof We set $S=\bigcup\limits_{1\le i<j\le q}\{z;a_i(z)=a_j(z)\}$. Then we see that S is a discrete subset of $\Delta(R_0)$ and $\overline{N}(r,S)=o(T(r))$.

By considering two functions $L(f)$ and $L(g)$ instead of f and g if necessary, where L is the quasi-Möbius transformation

$$L(w)=\frac{(\omega-a_1)(a_3-a_2)}{(\omega-a_2)(a_3-a_1)},$$

we may assume that $a_1 = 0, a_2 = \infty, a_3 = 1$ and $a_4 = a$ with $a \notin \{0, 1, \infty\}$. Here, we note that the quasi-Möbius transformation L only makes the counting functions in the inequality of the lemma change up to small terms $o(T(r))$.

Consider the function

$$H = \frac{f'(a'g - ag')(f - g)}{(f(f - 1)g(g - a))} - \frac{g'(a'f - af')(f - g)}{g(g - 1)f(f - a)}. \tag{3.1}$$

We rewrite H as

$$H = \frac{(f - g)Q}{f(f - 1)(f - a)g(g - 1)(g - a)}, \tag{3.2}$$

where

$$\begin{aligned} Q &= f'(a'g - ag')(f - a)(g - 1) - g'(a'f - af')(g - a)(f - 1) \\ &= a'ff'g^2 - a'ff'g - a(a - 1)ff'g' - aa'f'g^2 + aa'f'g \\ &\quad - a'f^2gg' + a'fgg' + a(a - 1)f'gg' + aa'f^2g' - aa'fg'. \end{aligned} \tag{3.3}$$

Case 1 Suppose that $H \equiv 0$. Then from (3.1), we have

$$\frac{f'(a'g - ag')}{(f - 1)(g - a)} \equiv \frac{g'(a'f - af')}{(g - 1)(f - a)}.$$

This implies that

$$\begin{aligned} \frac{(f - g)(1 - a)}{(g - 1)(f - a)} &= \frac{(f - 1)(g - a)}{(g - 1)(f - a)} - 1 = \frac{f'(a'g - ag')}{g'(a'f - af')} - 1 \\ &= \frac{a'[(f' - g')g - (f - g)g']}{g'(a'f - af')}. \end{aligned}$$

It yields that

$$\frac{f' - g'}{f - g} = \frac{(1 - a)g'(a'f - af')}{a'g(g - 1)(f - a)} + \frac{g'}{g}. \tag{3.4}$$

Hence, if there exists a point $z_0 \notin S$ which is a common zero of $(f - a_i)$ and $(g - a_i)$ $(5 \le i \le q)$, then it must be a pole of the left hand side of (3.4) but not a pole of the right hand side. This is a contradiction. Therefore each common zero of $(f - a_i)$ and $(g - a_i)$ $(5 \le i \le q)$ must belong to S. Thus

$$\sum_{i=5}^{q} \overline{N}(r, a_i, f, g) \le (q - 4)\overline{N}(r, S) = o(T(r)),$$

where $\overline{N}(r, S)$ is the counting function without multiplicity which counts all points in S. Hence, the lemma is proved in this case.

Case 2 Suppose that $H \not\equiv 0$. From (3.1)–(3.2), we easily see that if $z \notin S$ is a common zero of $(f - a_i)$ and $(g - a_i)$ $(5 \le i \le q)$ then it is a zero of $(f - g)$ and is not a pole of

$$\frac{Q}{f(f - 1)(f - a)g(g - 1)(g - a)},$$

and hence it is a zero of H. Therefore we have

$$\begin{aligned}\sum_{i=5}^{q}\overline{N}(r,a_1,f,g) &\le \overline{N}(r,0,H)+(q-4)\overline{N}(r,S)\\ &\le T(r,H)+o(T(r))\\ &\le m(r,H)+N(r,\infty,H)+o(T(r)).\end{aligned}\tag{3.5}$$

We now estimate the proximity function $m(r,H)$. Firstly, we have

$$\begin{aligned}H =&\frac{f'}{f-1}\frac{a'g-ag'}{g(g-a)}-\Big(\frac{f'}{f-1}-\frac{f'}{f}\Big)\frac{a'g-ag'}{g-a}\\ &-\frac{g'}{g-1}\frac{a'f-af'}{f(f-a)}-\Big(\frac{g'}{g-1}-\frac{g'}{g}\Big)\frac{a'f-af'}{f-a}\\ =&\frac{f'}{f-1}\Big(\frac{g'}{g}-\frac{g'-a'}{g-a}\Big)-\Big(\frac{f'}{f-1}-\frac{f'}{f}\Big)\Big(a'-a\frac{g'-a'}{g-a}\Big)\\ &-\frac{g'}{g-1}\Big(\frac{f'}{f}-\frac{f'-a'}{f-a}\Big)-\Big(\frac{g'}{g-1}-\frac{g'}{g}\Big)\Big(a'-a\frac{f'-a'}{f-a}\Big).\end{aligned}\tag{3.6}$$

By the lemma on logarithmic derivatives, we get

$$\begin{aligned}m(r,H)&\le m\Big(r,\frac{f'}{f}\Big)+m\Big(r,\frac{g'}{g}\Big)+m\Big(r,\frac{f'}{f-1}\Big)+m\Big(r,\frac{g'}{g-1}\Big)\\ &\quad+m\Big(r,\frac{(f-a)'}{f-a}\Big)+m\Big(r,\frac{(g-a)'}{g}\Big)+m\Big(r,\frac{a'}{a}\Big)+o(T(r))\\ &\le 7S(r)+o(T(r)).\end{aligned}\tag{3.7}$$

We now estimate the counting function $N(r,\infty,H)$. From (3.6), we see that each pole of H must be a zero of some functions $(f-a_i)$ or $(g-a_i)$ $(i=1,\cdots,4)$ $\big($here we regard $f-\infty=\frac{1}{f}\big)$ or a pole of a' or a. We consider the following four subcases.

Subcase 1 z is a pole of a' or a. Hence z must be a pole of a. We note that each pole of every meromorphic function of the form $\frac{h'}{h}$ has multiplicity at most 1. Therefore, from (3.6) we have

$$\nu_H^{\infty}(z)\le\nu_a^{\infty}(z)+2\le 3\nu_a^{\infty}(z).$$

Subcase 2 z is not a pole of a and z is a common zero of $(f-u)$ and $(g-u)$ for a function $u\in\{0,1,a\}$. From (3.2), we rewrite H as follows

$$H=(f-g)\Big[\Big(\frac{f'}{f-1}-\frac{f'}{f}\Big)\Big(\frac{g'}{g}-\frac{g'-a'}{g-a}\Big)-\Big(\frac{g'}{g-1}-\frac{g'}{g}\Big)\Big(\frac{f'}{f}-\frac{f'-a'}{f-a}\Big)\Big]=(f-g)P,$$

where

$$P=\Big[\frac{f'}{f-1}\frac{g'}{g}-\frac{f'}{f-1}\frac{g'-a'}{g-a}+\frac{f'}{f}\frac{g'-a'}{g-a}-\frac{g'}{g-1}\frac{f'}{f}+\frac{g'}{g-1}\frac{f'-a'}{f-a}-\frac{f'-a'}{f-a}\frac{g'}{g}\Big].$$

Hence, z is a zero of $(f-g)$ and is a simple pole of P. Therefore z is not a pole of H.

Subcase 3 z is not a pole of a and is a common pole of f and g. From (3.2)–(3.3), we easily see that z is not a pole of H.

Subcase 4 z is not a pole of a and z is either a zero of $f-a_i$ or a zero of $g-a_i$ for some $i \in \{1, \cdots, 4\}$. From (3.6), we see that H has the following form

$$H = \sum_{\substack{u,v \in \{0,1,a\} \\ u \neq v}} a_{uv} \frac{(f-u)'}{(f-u)} \frac{(g-v)'}{(g-v)},$$

where a_{uv} are constants or $\pm a'$ or $\pm a$. Hence

$$\begin{aligned} \nu_H^\infty(z) &\le \max_{\substack{u,v \in \{0,1,a\} \\ u \neq v}} (\nu^\infty_{\frac{(f-u)'}{(f-u)}}(z) + \nu^\infty_{\frac{(g-v)'}{(g-v)}}(z)) \\ &\le \sum_{i=1}^4 (\min\{1, \nu^0_{f-a_i}(z)\} + \min\{1, \nu^0_{g-a_i}(z)\} - \min\{1, \nu^0_{f-a_i}(z) \cdot \nu^0_{g-a_i}(z)\}). \end{aligned}$$

From the above four cases, we have

$$\nu_H^\infty \le 3\nu_a^\infty + \sum_{i=1}^4 (\min\{1, \nu^0_{f-a_i}\} + \min\{1, \nu^0_{g-a_i}\} - \min\{1, \nu^0_{f-a_i} \cdot \nu^0_{g-a_i}\}).$$

This yields that

$$\begin{aligned} N(r, \infty, H) &\le \sum_{i=1}^4 (\overline{N}(r, a_i, f) + \overline{N}(r, a_i, f) - \overline{N}(r, a_i, f, g)) + o(T(r)) \\ &\le \sum_{i=1}^4 (\overline{N}_{(k+1}(r, a_i, f) + \overline{N}_{(k+1}(r, a_i, g)) + o(T(r)). \end{aligned}$$

Combining the above inequality and (3.5) and (3.7), we get

$$\sum_{i=5}^q \overline{N}(r, a_1, f, g) \le \sum_{i=1}^4 (\overline{N}_{(k+1}(r, a_i, f) + \overline{N}_{(k+1}(r, a_i, g)) + 7S(r) + o(T(r)).$$

Then the lemma is proved in this case.

Proof of Theorem 1.2 Suppose contrarily that $f \not\equiv g$. We define $T(r), \gamma(r)$ and $S(r)$ as in Lemma 3.1. By Lemma 3.1, for every subset $\{i_1, i_2, i_3, i_4\}$ of $\{1, \cdots, q\}$ we have

$$\begin{aligned} &\sum_{i=1}^q \overline{N}(r, a_i, f, g) - \sum_{j=1}^4 \overline{N}(r, a_{i_j}, f, g) \\ &\le \sum_{j=1}^4 (\overline{N}_{(k+1}(r, a_{i_j}, f) + \overline{N}_{(k+1}(r, a_{i_j}, g)) + 7S(r) + o(T(r)). \end{aligned}$$

Summing up both sides of the above inequality over all subsets $\{i_1, i_2, i_3, i_4\}$, we obtain

$$(q-4) \sum_{i=1}^q \overline{N}(r, a_i, f, g) \le 4 \sum_{i=1}^q (\overline{N}_{(k+1}(r, a_i, f) + \overline{N}_{(k+1}(r, a_i, g)) + 7qS(r) + o(T(r)).$$

Since

$$\overline{N}(r, a_i, f) + \overline{N}(r, a_i, g) \le \overline{N}(r, a_i, f, g) + \overline{N}_{(k+1}(r, a_i, f) + \overline{N}_{(k+1}(r, a_i, g), \quad 1 \le i \le q,$$

the above inequality implies that

$$\begin{aligned}
&(q-4)\sum_{i=1}^{q}(\overline{N}(r,a_i,f)+\overline{N}(r,a_i,g))\\
&\le (q+4)\sum_{i=1}^{q}(\overline{N}_{(k+1}(r,a_i,f)+\overline{N}_{(k+1}(r,a_i,g))+7qS(r)+o(T(r))\\
&\le \frac{q+4}{k}\sum_{i=1}^{q}(k\overline{N}_{(k+1}(r,a_i,f)+\overline{N}_{(k+1}(r,a_i,g))+7qS(r)+o(T(r)).
\end{aligned}$$

Since $k\overline{N}_{(k+1}(r,a_i,h)+\overline{N}(r,a_i,h)\le N(r,a_i,h)$ $(i=1,\cdots,q;h=f,g)$, we easily have

$$\begin{aligned}
&\Big(q-4+\frac{q+4}{k}\Big)\sum_{i=1}^{q}(\overline{N}(r,a_i,f)+\overline{N}(r,a_i,g))\\
&\le \frac{q+4}{k}\sum_{i=1}^{q}(N(r,a_i,f)+N(r,a_i,g))+7qS(r)+o(T(r))\\
&\le \frac{(q+4)q}{k}T(r)+7qS(r)+o(T(r)).
\end{aligned}$$

By the second main theorem, we have

$$\Big\|_E \Big(q-4+\frac{q+4}{k}\Big)\frac{q}{5}(2T(r)-38S(r))\le \frac{(q+4)q}{k}T(r)+7qS(r)+o(T(r)),$$

i.e.,

$$\Big\|_E \ \frac{k(2q-8)-3(q+4)}{k(38q-117)+38(q+4)}T(r)\le S(r)+o(T(r)).$$

Letting $\varepsilon\to 0$, and then letting $r\to R_0$ $(r\notin E)$, we obtain

$$c_f+c_g\ge 2\min\{c_f,c_g\}\ge \frac{k(2q-8)-3(q+4)}{k(19q-\frac{117}{2})+19(q+4)}.$$

This contradiction completes the proof of the theorem.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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