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On Pure Silting Complexes*

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Abstract The authors introduce the concept of pure silting complexes and study their main properties. This concept generalizes silting complexes in pure derived category. They also show Bazzoni's characterization of the pure silting complexes and characterize 2-term pure silting complexes by the connections with the t-structure and torsion pair. Furthermore, they give the Brenner-Butler Theorem about 2-term pure silting complexes.

Keywords Pure silting complex, Pure derived category, t-structure, Torsion pair
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1 Introduction

Tilting theory is very vital in the representation theory of Artin algebras. It gives a powerful tool to study the triangulated categories. Happel and Ringel [1] introduced the classical tilting theory in the context of finitely generated modules over Artin algebras. Colby and Fuller [2] studied the finitely generated tilting modules over arbitrary rings. Moreover, Angeleri-Hügel and Coelho [3] investigated the infinitely generated tilting and cotilting modules over arbitrary associative rings. Miyashita [4] generalized the tilting modules to the finite projective dimension case and obtained many fundamental results, especially the generalization of the Brenner-Butler Theorem. Bazzoni [5] gave a characterization of n-cotilting and n-tilting modules.

Silting complexes were first introduced by Keller and Vossieck [6] to study t-structures in the bounded derived category of representations of Dynkin quivers. They can be understood as derived analogues of tilting and cotilting modules. Wei [7] described the semi-tilting (silting) complexes and gave Bazzoni's characterization. Moreover, 2-term silting complexes have attracted many scholars' great interest. Hoshino, Kato and Miyachi [8] studied the relation between the 2-term silting complexes and torsion pairs in the category of modules. They characterized the 2-term presilting complexes as a direct summand completion of the 2-term silting complexes, which is the analogue of the Bongartz completion of a classical tilting module. Later, Koga [9] generalized presilting complexes of finite-length in the bounded homotopy category of finitely generated projectives. Buan and Zhou [10] regarded the silting theory as the relative

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version of tilting theory in the level of derived category, and gave a generalization of the classical tilting theorem of Brenner and Butler in terms of 2-term silting complexes, which is said to be the silting theorem. Marks and Vitória [11] showed every silting or cosilting complex is associated with a t-structure and co-t-structure in a derived category.

Relative homological algebra provides a new idea for comprehensive understanding of tilting modules. Eilenberg and Moore [12] first considered the relative homological theory in the 1960s. Enochs and Jenda [13] gave many fundamental works on the study of relative homological algebra. Auslander and Solberg [14] introduced the concept of relative (co)tilting modules, which was limited to finitely generated modules over Artin algebras. Wei [15] gave a nice characterization for relative n-tilting modules over Artin algebras. Yan, Li and Ouyang [16] showed an interesting case, which generalized Auslander-Solberg relative notions, that they defined the m-Gorenstein tilting and cotilting modules over an n-Gorenstein ring in the context of Gorenstein homological algebra. Moradifar and Yassemi [17] furthermore advanced the theory of infinitely generated Gorenstein tilting modules through the development of Gorenstein tilting approximations.

Purity is the basis of a relative homological theory, which leads to concepts such as pure-projective modules, pure projective resolutions, pure derived functors $\mathrm{Pext}(-,-)$ and pure-projective dimension, etc., see [18–22]. Zheng and Huang introduced the pure derived category $D_{\mathrm{pur}}(R)$ as the derived category of the exact category in [23]. They investigated the properties of pure derived categories of module categories, and showed that pure derived categories share many nice properties of classical derived categories.

This paper aims to introduce pure silting complexes and study their main properties. We mainly show Bazzoni's characterization of pure silting complexes and investigate the 2-term pure silting complexes by the connections with t-structure and torsion pair. Then, we propose the Brenner-Butler Theorem of 2-term pure silting complexes.

This paper is organized as follows: In Section 2, we state the main results that will be used in the subsequent sections. In Section 3, we introduce the notion of pure silting complexes and give the characterization of them in bounded pure derived categories. In Section 4, we focus on the 2-term pure silting complexes, and show the relation between 2-term pure silting complexes with t-structure and torsion pair.

2 Preliminaries

Let R denote an associative ring with an identity and R-Mod (R-mod) be the category of (finitely generated) left R-modules. As usual, we use $C(R)$ and $K(R)$ to denote the category of complexes and the homotopy category of R-Mod, respectively. For any $X^\bullet \in C(R)$, we write

$$X^\bullet := \cdots \to X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots .$$

We regard an R-module M as a stalk complex, that is, a complex concentrated in degree 0.

Let $X^\bullet \in C(R)$. If $X^i = 0$ for $i \gg 0$, then $X^\bullet$ is called bounded above (or bounded on the right). If $X^i = 0$ for $i \ll 0$, then $X^\bullet$ is called bounded below (or bounded on the left).

$X^\bullet$ is called a quasi-isomorphism if it induces isomorphic homology groups. And f is called a homotopy equivalence if there exists a cochain map $g: Y^\bullet \to X^\bullet$ such that there exist homotopies $g \circ f \sim \mathrm{Id}X^\bullet$ and $f \circ g \sim \mathrm{Id}Y^\bullet$. For $\mathrm{Con}(f)$, we mean the mapping cone of a cochain map f. Let $X^\bullet, Y^\bullet \in C(R)$. We use $\mathrm{Hom}_R(X^\bullet, Y^\bullet)$ to denote the total complex, that is, a complex of $\mathbb{Z}$-modules (where $\mathbb{Z}$ is the additive group of integers). Now we write $C^b(R), K^b(R)$ and $D^b(R)$ for the bounded complex category, bounded homotopy category and bounded derived category of R-Mod, respectively.

Let $\mathcal{C}$ be an abelian category, $\mathcal{X}$ and $\mathcal{Y}$ be the full additive subcategories of $\mathcal{C}$. Let Z be an object of $\mathcal{C}$. A morphism $f: X \to Z$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation of Z if any morphism from an object in $\mathcal{X}$ to Z factors through f.

Let $\mathcal{C}$ be a triangulated category. A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of the triangulated category $\mathcal{C}$ is called a t-structure if it satisfies the following conditions:

(1) $\mathcal{X}[1] \subset \mathcal{X}$ and $\mathcal{Y}[-1] \subset \mathcal{Y}$;

(2) $\mathrm{Hom}_{\mathcal{C}}(X, Y[-1]) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;

(3) For any $C \in \mathcal{C}$, there is a distinguished triangle

$$X \to C \to Y[-1] \to X[1]$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of R-mod is called a torsion pair if it satisfies the following conditions:

(1) $\mathcal{T} \cap \mathcal{F} = 0$;

(2) If $T \to Y \to 0$ is exact with $T \in \mathcal{T}$ then $Y \in \mathcal{T}$;

(3) If $0 \to Y \to F$ is exact with $F \in \mathcal{F}$ then $Y \in \mathcal{F}$;

(4) For each R-module X, there is an exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

We list some notations and basic facts on pure derived categories.

Definition 2.1 (cf. [22]) *A short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in R-Mod *is called pure exact if for any right R-module M, the induced sequence*

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is exact. In this case, f is called pure monic and g is called pure epic.

Remark 2.1 Using Cohn's theorem (cf. [24]), we have that a short exact sequence $0 \to A \to B \to C \to 0$ in R-Mod is pure exact if and only if $0 \to \mathrm{Hom}_R(F, A) \to \mathrm{Hom}_R(F, B) \to \mathrm{Hom}_R(F, C) \to 0$ is exact for any finitely presented R-module F.

Definition 2.2 (cf. [23]) *Let $X^\bullet \in C(R)$ and $n \in \mathbb{Z}$. Then $X^\bullet$ is called pure exact at n, if the differentials d_X^{n-1} and d_X^n can decompose as above, and the sequence*

$$0 \to K^n \to X^n \to C^{n-1} \to 0$$

is pure exact, where $K^n = \mathrm{Ker} d_X^n$ *and* $C^{n-1} = \mathrm{Coker} d_X^{n-1}$. $X^\bullet$ *is called pure exact if it is pure exact at* n *for all* n.

Definition 2.3 (cf. [23]) *A module* $M \in R$-Mod *is called pure projective* (*resp. pure injective*) *if it is projective* (*resp. injective*) *with respect to every pure exact complex.*

Let $\mathcal{PP}$ (resp. $\mathcal{P}$proj, $\mathcal{PI}$) be the class of all pure projective (resp. finitely generated pure projective, pure injective) R-modules. We denote $K^-(\mathcal{PP})$ (resp. $K^+(\mathcal{PI})$) as the bounded above (resp. below) homotopy category of $\mathcal{PP}$ (resp. $\mathcal{PI}$). For $K^b(\mathcal{PP})$ (resp. $K^b(\mathcal{P}\mathrm{proj})$), we mean the corresponding bounded homotopy category of pure projective modules (resp. finitely generated projective modules).

Definition 2.4 (cf. [22]) *A cochain map* $f : X \to Y$ *in* $C(R)$ *is called a pure quasi-isomorphism if* f *satisfies one of the following conditions*:

(1) *its mapping cone* $\mathrm{Con}(f)$ *is a pure exact complex*;

(2) $M \otimes_R f : M \otimes_R X \to M \otimes_R Y$ *is a quasi-isomorphism for any right* R*-module* M;

(3) $\mathrm{Hom}_R(P, f) : \mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R(P, Y)$ *is a quasi-isomorphism for any* $P \in \mathcal{PP}$;

(4) $\mathrm{Hom}_R(f, I) : \mathrm{Hom}_R(Y, I) \to \mathrm{Hom}_R(X, I)$ *is a quasi-isomorphism for any* $I \in \mathcal{PI}$.

Lemma 2.1 (cf. [25]) (1) *Let* $X^\bullet \in C(R)$. *Then* $X^\bullet$ *is pure exact if and only if* $\mathrm{Hom}_R(P^\bullet, X^\bullet)$ *is exact for any* $P^\bullet \in K^-(\mathcal{PP})$, *and if and only if* $\mathrm{Hom}_R(X^\bullet, I^\bullet)$ *is exact for any* $I^\bullet \in K^+(\mathcal{PI})$.

(2) *A cochain map* f *in* $C(R)$ *is a pure quasi-isomorphism if and only if* $\mathrm{Hom}_R(P^\bullet, f)$ *is a quasi-isomorphism for any* $P^\bullet \in K^-(\mathcal{PP})$, *and if and only if* $\mathrm{Hom}_R(f, I^\bullet)$ *is a quasi-isomorphism for any* $I^\bullet \in K^+(\mathcal{PI})$.

Put $K_{\mathcal{PE}}(R) := \{X \in K(R) \mid X \text{ is pure exact}\}$. Notice that pure exact complexes are closed under homotopy equivalences, so $K_{\mathcal{PE}}(R)$ is well-defined. If $f : X \to Y$ is a cochain map between pure exact complexes, then $\mathrm{Con}(f)$ is also pure exact. Thus, $K_{\mathcal{PE}}(R)$ is a triangulated subcategory of $K(R)$. Because pure exact complexes are closed under summands, $K_{\mathcal{PE}}(R)$ is a thick subcategory of $K(R)$. Then the pure derived category is defined as $D_{\mathrm{pur}}(R) := K(R)/K_{\mathcal{PE}}(R)$. Similarly, we can define $D^*_{\mathrm{pur}}(R) := K^*(R)/K^*_{\mathcal{PE}}(R)$ for $* \in \{+, -, b\}$.

Next we recall the facts which will be used in the subsequent sections.

2.1 Let $\mathcal{C}$ be an idempotent complete triangulated category with [1] being the shift functor. Assume that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. Recall that $\mathcal{B}$ is closed under extension if for any triangle $X \to Y \to Z \to$ in $\mathcal{C}$ with $X, Z \in \mathcal{B}$, we have $Y \in \mathcal{B}$. The subcategory $\mathcal{B}$ is cosuspended (resp. suspended) if it is closed under extension and under functor $[-1]$ (resp. [1]). It is easy to see that $\mathcal{B}$ is cosuspended (resp. suspended) if and only if for any triangle $X \to Y \to Z \to$ (resp. $Z \to Y \to X \to$) in $\mathcal{B}$ with $Z \in \mathcal{B}$, one has that $X \in \mathcal{B} \Leftrightarrow Y \in \mathcal{B}$.

2.2 An object $M \in \mathcal{C}$ has a $\mathcal{B}$-resolution (resp. $\mathcal{B}$-coresolution) of length at most m ($m \geq 0$), if there are triangles $M_{i+1} \to X_i \to M_i$ (resp. $M_i \to X_i \to M_{i+1}$) with $0 \leq i \leq m$ such that $M_0 = M$, $M_{m+1} = 0$ and each $X_i \in \mathcal{B}$. In this case, we denote by $\mathcal{B}$-res.dim$(M) \leq m$ (resp. $\mathcal{B}$-cores.dim$(M) \leq m$). One may compare such notions with usual finite resolutions and coresolutions, respectively, in the module category.

2.3 Associated with a subcategory $\mathcal{B}$, we have the following notations, where $n \geq 0$ and m is an integer,

$$\begin{aligned}
(\widehat{\mathcal{B}})_n &= \{L \in \mathcal{C} \mid \mathcal{B}\text{-res.dim}(L) \leq n\},\\
(\check{\mathcal{B}})_n &= \{L \in \mathcal{C} \mid \mathcal{B}\text{-cores.dim}(L) \leq n\},\\
\widehat{\mathcal{B}} &= \{L \in \mathcal{C} \mid L \in (\widehat{\mathcal{B}})_n \text{ for some } n\},\\
\check{\mathcal{B}} &= \{L \in \mathcal{C} \mid L \in (\check{\mathcal{B}})_n \text{ for some } n\},\\
\mathcal{B}^{\perp_{\neq 0}} &= \{N \in \mathcal{C} \mid \mathrm{Hom}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\},\\
{}^{\perp_{\neq 0}}\mathcal{B} &= \{N \in \mathcal{C} \mid \mathrm{Hom}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\},\\
\mathcal{B}^{\perp_{>m}} &= \{N \in \mathcal{C} \mid \mathrm{Hom}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\},\\
{}^{\perp_{>m}}\mathcal{B} &= \{N \in \mathcal{C} \mid \mathrm{Hom}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\},\\
\mathcal{B}^{\perp_{\gg 0}} &= \{N \in \mathcal{C} \mid N \in \mathcal{B}^{\perp_{>m}} \text{ for some } m\}.
\end{aligned}$$

Note that $\mathcal{B}^{\perp_{>m}}$ (resp. ${}^{\perp_{>m}}\mathcal{B}$) is suspended (resp. cosuspended) and closed under direct summands and that $\mathcal{B}^{\perp_{\gg 0}}$ is a triangulated subcategory of $\mathcal{C}$.

2.4 The subcategory $\mathcal{B}$ is said to be semi-selforthogonal (resp. self-orthogonal) if $\mathcal{B} \subseteq \mathcal{B}^{\perp_{>0}}$ (resp. $\mathcal{B} \subseteq \mathcal{B}^{\perp_{\neq 0}}$). For instance, both subcategories of all projective and all injective R-modules are self-orthogonal in R-modules, for a ring R.

In the following of this section, we always assume that $\mathcal{B}$ is semi-selforthogonal and that $\mathcal{B}$ is additively closed.

2.5 Associated with the subcategory $\mathcal{B}$, we also have the following two useful subcategories:

$$\begin{aligned}
\mathcal{X}_{\mathcal{B}} = \{N \in {}^{\perp_{\neq 0}}\mathcal{B} \mid {}& \text{there are triangles } N_i \to B_i \to N_{i+1} \to \text{ such that } N_0 = N,\\
& N_i \in {}^{\perp_{\neq 0}}\mathcal{B} \text{ and } B_i \in \mathcal{B} \text{ for all } i \leq 0\},\\
{}_{\mathcal{B}}\mathcal{X} = \{N \in \mathcal{B}^{\perp_{\neq 0}} \mid {}& \text{there are triangles } N_{i+1} \to B_i \to N_i \to \text{ such that } N_0 = N,\\
& N_i \in \mathcal{B}^{\perp_{\neq 0}} \text{ and } B_i \in \mathcal{B} \text{ for all } i \leq 0\}.
\end{aligned}$$

Since $\mathcal{B}$ is closed under finite direct sums and direct summands, we can summarize some results on the subcategories associated with $\mathcal{B}$.

Lemma 2.2 *Let $\mathcal{B}$ be a semi-selforthogonal subcategory of a triangulated category $\mathcal{C}$ such that $\mathcal{B}$ is additively closed. Then*

(1) *The three subcategories $\check{\mathcal{B}} \subseteq \mathcal{X}_{\mathcal{B}} \subseteq {}^{\perp_{>0}}\mathcal{B}$ are cosuspended and closed under direct summands.*

(2) *The three subcategories $\widehat{\mathcal{B}} \subseteq {}_{\mathcal{B}}\mathcal{X} \subseteq \mathcal{B}^{\perp_{>0}}$ are suspended and closed under direct summands.*

(3) $\mathcal{B} = \check{\mathcal{B}} \cap \mathcal{B}^{\perp_{>0}} = \widehat{\mathcal{B}} \cap {}^{\perp_{>0}}\mathcal{B}$.

(4) $(\check{\mathcal{B}})_n = \mathcal{X}_{\mathcal{B}} \cap (\mathcal{X}_{\mathcal{B}})^{\perp_{>n}} = \mathcal{X}_{\mathcal{B}} \cap ({}^{\perp_{>0}}\mathcal{B})^{\perp_{>n}}$. *In particular, it is closed under extensions and direct summands.*

(5) $(\widehat{\mathcal{B}})_n = {}_{\mathcal{B}}\mathcal{X} \cap {}^{\perp_{>n}}({}_{\mathcal{B}}\mathcal{X}) = {}_{\mathcal{B}}\mathcal{X} \cap {}^{\perp_{>n}}(\mathcal{B}^{\perp_{>0}})$. *In particular, it is closed under extensions and direct summands.*

(6) *The following three subcategories coincide with each other*:

(i) $\langle \mathcal{B} \rangle$*: the smallest triangulated subcategory containing* $\mathcal{B}$*.*

(ii) $(\widehat{\mathcal{B}})_- := \{X \in \mathcal{C} \mid$ *there is some* $Y \in \widehat{\mathcal{B}}$ *and some* $i \leq 0$ *such that* $X = Y[i]\}$*.*

(iii) $(\widecheck{\mathcal{B}})_+ := \{X \in \mathcal{C} \mid$ *there is some* $Y \in \widecheck{\mathcal{B}}$ *and some* $i \geq 0$ *such that* $X = Y[i]\}$*.*

(7) $\widehat{\mathcal{B}} = \mathcal{B}^{\perp_{>0}} \cap \langle \mathcal{B} \rangle$*.*

(8) $\widecheck{\mathcal{B}} = {}^{\perp_{>0}}\mathcal{B} \cap \langle \mathcal{B} \rangle$*.*

Lemma 2.3 (cf. [23]) *Let* $P^\bullet \in K^-(\mathcal{PP})$ *and* $Y^\bullet$ *be an arbitrary complex. Then there is an isomorphism of abelian groups* $\mathrm{Hom}_{K(R)}(P^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P^\bullet, Y^\bullet)$*.*

Lemma 2.4 (cf. [23]) $\mathrm{Pext}^n_R(M, N) \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(M, N[n])$ *for any* $M, N \in R$-Mod *and any integer* n*.*

3 Pure Silting Complexes

In this section, we introduce the concept of pure silting complexes and obtain some basic properties of them.

Definition 3.1 *A complex* $T^\bullet$ *is said to be*

(1) *partial pure silting, if*

(i) $T^\bullet \in K^b(\mathcal{PP})$;

(ii) $\mathrm{Add}_{D_{\mathrm{pur}}(R)} T^\bullet \subseteq T^{\bullet\perp_{i>0}}$;

(2) *pure silting, if it satisfies* (i)–(ii) *and*

(iii) $K^b(\mathcal{PP}) = \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)} T^\bullet \rangle$*, that is,* $K^b(\mathcal{PP})$ *coincides with the smallest triangulated subcategory containing* $\mathrm{Add}_{D_{\mathrm{pur}}(R)} T^\bullet$*.*

Define

$$\begin{aligned} D^{\leq 0}_{\mathrm{pur}}(R) &:= (\mathcal{PP})^{\perp_{>0}} \\ &= \{X^\bullet \in D_{\mathrm{pur}}(R) \mid \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P, X^\bullet[i]) = 0 \text{ for any } P \in \mathcal{PP} \text{ and all } i > 0\}. \end{aligned}$$

For any $X^\bullet \in D^{\leq 0}_{\mathrm{pur}}(R)$, it satisfies that

$$\mathrm{H}^i \mathrm{Hom}_R(P, X^\bullet) = \mathrm{Hom}_{K(R)}(P, X^\bullet[i]) \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P, X^\bullet[i]) = 0$$

by Lemma 2.3, for any $P \in \mathcal{PP}$ and $i > 0$. It is easy to get $D^-_{\mathrm{pur}}(R) = (\mathcal{PP})^{\perp_{\gg 0}}$. Indeed, $X^\bullet \in (\mathcal{PP})^{\perp_{\gg 0}}$ if and only if there is an integer n such that $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P, X^\bullet[i]) \simeq \mathrm{Hom}_{K(R)}(P, X^\bullet) = \mathrm{H}^i \mathrm{Hom}_R(P, X^\bullet) = 0$ for $P \in \mathcal{PP}$ and $i > n$ if and only if $X^\bullet \in D^-_{\mathrm{pur}}(R)$.

Theorem 3.1 *Assume that* $T^\bullet \in D_{\mathrm{pur}}(R)$ *and* $\mathrm{Add}_{D_{\mathrm{pur}}(R)} T^\bullet \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$*. Then* $T^\bullet$ *is pure silting if and only if it satisfies the following three conditions*:

(1) $T^\bullet \in \widehat{\mathcal{PP}}$;

(2) $\mathrm{Add}_{D_{\mathrm{pur}}(R)} T^\bullet \subseteq T^{\bullet\perp_{i>0}}$;

(3) $\mathcal{PP} \subseteq \widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)} T^\bullet}$.

Proof Sufficiency. It is easy to see that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq \widehat{\mathcal{PP}}$ from $T^\bullet \in \widehat{\mathcal{PP}}$. By Lemma 2.2, we get $\widehat{\mathcal{PP}} \subseteq \langle \mathcal{PP} \rangle = K^b(\mathcal{PP})$. Then $T^\bullet \in K^b(\mathcal{PP})$ and $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq \langle \mathcal{PP} \rangle = K^b(\mathcal{PP})$. It follows that $\mathcal{PP} \subseteq \widehat{\mathrm{Add}_{D_{\mathrm{pur}}(R)}}T^\bullet \subseteq \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle$ by (3) and Lemma 2.2. Therefore, $K^b(\mathcal{PP}) = \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle$.

Necessity. We can obtain that $\mathcal{PP} \subseteq {}^{\perp_{>0}}(\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet)$ by $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D^{\leq 0}_{\mathrm{pur}}(R) = (\mathcal{PP})^{\perp_{>0}}$. Using Definition 3.1, $K^b(\mathcal{PP}) = \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle$, we have $\mathcal{PP} \subseteq \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle$. Moreover, we can get that $\mathcal{PP} \subseteq \widehat{\mathrm{Add}_{D_{\mathrm{pur}}(R)}}T^\bullet$ by Lemma 2.2. Note that $T^\bullet \in K^b(\mathcal{PP}) = \langle \mathcal{PP} \rangle$. Then we have $T^\bullet \in \langle \mathcal{PP} \rangle \cap (\mathcal{PP})^{\perp_{>0}} = \widehat{\mathcal{PP}}$. The proof is completed.

Proposition 3.1 *Suppose that $T^\bullet$ is pure silting. If $T^\bullet \oplus M^\bullet$ is also pure silting for some $M^\bullet$, then $M^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$.*

Proof Since $T^\bullet \oplus M^\bullet$ is a pure silting complex, it is obvious that $\langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}(T^\bullet \oplus M^\bullet) \rangle = K^b(\mathcal{PP}) = \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle$ and $T^\bullet \oplus M^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}(T^\bullet \oplus M^\bullet) \subseteq (T^\bullet \oplus M^\bullet)^{\perp_{>0}}$. The latter one yields that $M^\bullet \in T^{\bullet\perp_{>0}}$. So $T^\bullet \oplus M^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}(T^\bullet \oplus M^\bullet) \subseteq (\mathrm{Add}_{D_{\mathrm{pur}}(R)}(T^\bullet \oplus M^\bullet))^{\perp_{>0}} \subseteq (\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet)^{\perp_{>0}}$ by using Lemma 2.2. We can easily obtain that

$$M^\bullet \in T^{\bullet\perp_{>0}} \cap \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}(T^\bullet \oplus M^\bullet) \rangle = T^{\bullet\perp_{>0}} \cap \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle = \widehat{\mathrm{Add}_{D_{\mathrm{pur}}(R)}}T^\bullet.$$

Hence, $M^\bullet \in \widehat{\mathrm{Add}_{D_{\mathrm{pur}}(R)}}T^\bullet \cap {}^{\perp_{>0}}(\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet) = \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ by Lemma 2.2.

Proposition 3.2 *Suppose that $T^\bullet \in D_{\mathrm{pur}}(R)$ is a pure silting complex with $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$. Then*

(1) *if there are triangles $T^\bullet_{i+1} \to P_i \to T^\bullet_i \to$ with $P_i \in \mathcal{PP}$ for all $0 \leq i \leq n$ and $T^\bullet_0 = T$, $T^\bullet_{n+1} = 0$, then $\mathrm{Add}_{D_{\mathrm{pur}}(R)}\big(\bigoplus\limits_{i=0}^{n} P_i\big) = \mathcal{PP}$.*

(2) *If there are triangles $P_i \to T^\bullet_i \to P_{i+1} \to$ with $T^\bullet_i \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ for all $0 \leq i \leq m$, where $P_0 = P$, $P_{m+1} = 0$, then $\bigoplus\limits_{i=0}^{m} T^\bullet_i$ is a pure silting complex. Moreover, $\mathrm{Add}_{D_{\mathrm{pur}}(R)}\big(\bigoplus\limits_{i=0}^{m} T^\bullet_i\big) = \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$.*

Proof (1) Since $\bigoplus\limits_{i=0}^{n} P_i \in \mathcal{PP}$ is prod-semi-selforthogonal, it is easy to see that $\bigoplus\limits_{i=0}^{n} P_i$ satisfies the first and second conditions in the definition of pure silting complexes. One can check that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq \big\langle \bigoplus\limits_{i=0}^{n} P_i \big\rangle$ by Lemma 2.2. Then there is

$$\langle \mathcal{PP} \rangle = K^b(\mathcal{PP}) = \langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \rangle \subseteq \Big\langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}\Big(\bigoplus_{i=0}^{n} P_i\Big)\Big\rangle \subseteq \langle \mathcal{PP} \rangle,$$

i.e., $\big\langle \mathrm{Add}_{D_{\mathrm{pur}}(R)}\big(\bigoplus\limits_{i=0}^{n} P_i\big)\big\rangle = \langle \mathcal{PP} \rangle = K^b(\mathcal{PP})$. Hence, $\bigoplus\limits_{i=0}^{n} P_i$ satisfies the third condition in the definition of pure silting. It is obvious that $\bigoplus\limits_{i=0}^{n} P_i$ and $\bigoplus\limits_{i=0}^{n} P_i \oplus \mathcal{PP}$ are pure silting. By Proposition 3.1, we obtain that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}\big(\bigoplus\limits_{i=0}^{n} P_i\big) = \mathcal{PP}$.

(2) It is easy to verify that both $\big(\bigoplus\limits_{i=0}^{m} T_i\big) \oplus T$ and $\bigoplus\limits_{i=0}^{m} T_i$ are pure silting by Theorem 3.1. Then we can obtain that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}\big(\bigoplus\limits_{i=0}^{m} T_i\big) = \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ by Proposition 3.1.

Proposition 3.3 *Let* $T^\bullet$ *with* $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D_{\mathrm{pur}}^{\leq 0}(R)$ *be a pure silting complex and* $n \geq 0$. *Then* $T^\bullet \in (\widehat{\mathcal{PP}})_n$ *if and only if* $\mathcal{PP} \subseteq (\widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$.

Proof Let $P \in \mathcal{PP}$. We can get that $P \in (\widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_m$ for some $m > 0$ by Theorem 3.1. If $m \leq n$, then the conclusion obviously holds. Assume that $m > n$. There is a series of triangles in $D_{\mathrm{pur}}(R)$: $P_i^\bullet \to T_i^\bullet \to P_{i+1}^\bullet \to$ with $T_i^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$, where $P_0^\bullet = P$ and $P_{m+1}^\bullet = 0$ for $0 \leq i \leq m$. Note that $P_m^\bullet \simeq T_m^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$. Applying the functor $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_m^\bullet, -)$ to these triangles, one can easily obtain that

$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_m^\bullet, P_{m-1}^\bullet[1]) &\simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_m^\bullet, P_{m-2}^\bullet[2])\\ &\simeq \cdots \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_m^\bullet, P_0^\bullet[m])\\ &= \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P, P_0^\bullet[m]).\end{aligned}$$

Since $T^\bullet \in (\widehat{\mathcal{PP}})_n$ and by Lemma 2.2, we can verify that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq {}^{\perp_{>n}}(\mathcal{PP}^{\perp_{>0}})$. Therefore, $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_m^\bullet, P_{m-1}^\bullet[1]) = 0$. It follows that the triangle $P_{m-1}^\bullet \to T_{m-1}^\bullet \to P_m^\bullet \to$ splits. Then, $P_{m-1}^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ and $P \in (\widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_{m-1}$. Continuing this process, we can obtain $\mathcal{PP} \in (\widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$.

Conversely, since $T^\bullet \in D_{\mathrm{pur}}^{\leq 0}(R)$ is a partial pure silting complex, we have $T^\bullet \in K^b(\mathcal{PP}) = \langle \mathcal{PP} \rangle$. Then, $T^\bullet \in \langle \mathcal{PP} \rangle \cap \mathcal{PP}^{\perp_{>0}} = \widehat{\mathcal{PP}}$. There is a series of triangles in $D_{\mathrm{pur}}(R)$: $T_{i+1}^\bullet \to P_i \to T_i^\bullet \to$ with $T_0^\bullet = T^\bullet, P_i \in \mathcal{PP}$ and $T_{n+1}^\bullet = 0$ for $0 \leq i \leq n$. Consequently, $T_n^\bullet \simeq P_n \in \mathcal{PP}$.

Lemma 3.1 *Suppose that* $T^\bullet \in D_{\mathrm{pur}}(R)$ *with* $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \in T^{\bullet\perp_{>0}}$. *Then* $T^{\bullet\perp_{>0}} = {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$.

Proof It is clear that ${}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X} \subseteq T^{\bullet\perp_{>0}}$ by the definition of ${}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$.

Take any $M^\bullet \in T^{\bullet\perp_{>0}}$, and consider the triangle $M_1^\bullet \to T^{\bullet(\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, M^\bullet))} \xrightarrow{f} M^\bullet \to$, where f is the canonical evaluation map. Applying $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, -)$ to this triangle, then $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(M_1^\bullet, T^\bullet[i]) = 0$ for all $i > 0$, i.e., $M_1^\bullet \in T^{\bullet\perp_{>0}}$. Continuing the process, we get triangles $M_{j+1}^\bullet \to T_j^\bullet \to M_j^\bullet \to$ for all $j \geq 1$, with each $T_j^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ and $M_j^\bullet \in T^{\bullet\perp_{>0}}, M_0^\bullet = M^\bullet$. Therefore, $M^\bullet \in {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$. Consequently, $T^{\bullet\perp_{>0}} \subseteq {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$, and the conclusion holds.

Proposition 3.4 *If* $T^\bullet$ *with* $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D_{\mathrm{pur}}^{\leq 0}(R)$ *is a partial pure silting complex, then* $T^\bullet$ *is pure silting if and only if* $T^{\bullet\perp_{>0}} \subseteq D_{\mathrm{pur}}^{\leq 0}(R)$.

Proof Let $M^\bullet \in T^{\bullet\perp_{>0}}$. Since $T^\bullet$ is pure silting, we have that $P \in (\widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$ for some n and any $P \in \mathcal{PP}$. There is a series of triangles $P_i^\bullet \to T_i^\bullet \to P_{i+1}^\bullet \to$ with $P_0^\bullet = P, T_i^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ for $0 \leq i \leq n$. Applying the functor $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(-, M^\bullet)$ to these triangles with $M^\bullet \in T^{\bullet\perp_{>0}}$, since $T_n^\bullet \simeq P_n^\bullet \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ for any $j > 0$, we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P, M^\bullet[j]) &\simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_1^\bullet, M^\bullet[j+1])\\ &\simeq \cdots \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_n^\bullet, M^\bullet[j+n]) = 0.\end{aligned}$$

It is easy to see that $M^\bullet \in P^{\perp_{>0}}$. Hence $T^{\bullet\perp_{>0}} \subseteq \mathcal{PP}^{\perp_{>0}} = D_{\mathrm{pur}}^{\geq 0}(R)$.

Conversely, since $T^\bullet \in D^{\leq 0}_{\mathrm{pur}}(R)$ is a partial pure silting complex, we have $T^\bullet \in K^b(\mathcal{PP}) = \langle \mathcal{PP} \rangle$. Hence, $T^\bullet \in \langle \mathcal{PP} \rangle \cap \mathcal{PP}^{\perp_{>0}} = \widehat{\mathcal{PP}}$. There is a series of triangles in $D_{\mathrm{pur}}(R) : T^\bullet_{i+1} \to P_i \to T^\bullet_i \to$ with $T^\bullet_0 = T^\bullet, P_i \in \mathcal{PP}$ and $T^\bullet_{n+1} = 0$ for $0 \leq i \leq n$. Then, $T^\bullet_n \simeq P_n \in \mathcal{PP}$. Applying the functor $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(-, P)$ to these triangles for any $P \in \mathcal{PP}$, we have the following isomorphisms:

$$\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, P[j]) \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_1, P[j-1]) \simeq \cdots \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_n, P[j-n]) = 0$$

for $j > n$. It follows that $\mathcal{PP} \subseteq T^{\bullet\perp_{>n}}$, i.e., $\mathcal{PP}[n] \subseteq T^{\bullet\perp_{>0}}$. Since $0 \in T^{\bullet\perp_{>0}}$, we can get $\mathcal{PP} \in (\underbrace{T^{\bullet\perp_{>0}}})_n$. By Lemma 3.1 and [7, Corollary 2.8], there is a triangle in $D_{\mathrm{pur}}(R) : X^\bullet \to Y^\bullet \to P \to$ with $X^\bullet \in T^{\bullet\perp_{>0}}$ and $Y^\bullet \in (\underbrace{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$. Since $T^{\bullet\perp_{>0}} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$, we obtain that the triangle is split. It is immediate that $P \in (\underbrace{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$ for any $P \in \mathcal{PP}$, since $(\underbrace{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$ is closed under direct summands. The proof is completed.

We call that a complex $T^\bullet \in D_{\mathrm{pur}}(R)$ is n-pure silting, if it is a pure silting complex such that $\mathcal{PP} \in (\underbrace{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet})_n$.

Consider the following subcategory of $D_{\mathrm{pur}}(R)$. Let $T^\bullet \in D_{\mathrm{pur}}(R)$ and $n > 0$. We denote

$$\begin{aligned}\mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet) = \{M^\bullet \in D_{\mathrm{pur}}(R) \mid &\text{ there exist some triangles } M^\bullet_{i+1} \to T^\bullet_i \to M^\bullet_i \to \\ &\text{with } T^\bullet_i \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \text{ for all } 0 \leq i < n, \text{ where } M^\bullet_n \in D^{\leq 0}_{\mathrm{pur}}(R) \text{ and } M^\bullet_0 = M^\bullet\}.\end{aligned}$$

Observe that $\mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$ is closed under coproducts. We give the following result to show more properties about this subcategory.

Lemma 3.2 (1) *$D^{\leq 0}_{\mathrm{pur}}(R)[n] \subseteq \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$.*

(2) *If $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$, then $\mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet) \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$.*

Proof Since $0 \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ and $D^{\leq 0}_{\mathrm{pur}}(R)$ is suspended, it is not difficult to verify the conclusion by the definitions.

Proposition 3.5 *Suppose that $T^\bullet$ with $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ is an n-pure silting complex. Then $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$.*

Proof We have proved that $T^{\bullet\perp_{>0}} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ in Proposition 3.4. It is easy to obtain that $T^{\bullet\perp_{>0}} = {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$ by Lemma 3.1. Taking any $M^\bullet \in T^{\bullet\perp_{>0}}$, we can easily get from the definition of ${}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$ that there is a series of triangles in $D_{\mathrm{pur}}(R) : M^\bullet_{i+1} \to T^\bullet_i \to M^\bullet_i \to$ with $T^\bullet_i \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ for all $0 \leq i < n$, where $M^\bullet_n \in {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ and $M^\bullet_0 = M^\bullet$. Hence, $T^{\bullet\perp_{>0}} \subseteq \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$.

We prove the reverse inclusion. For any $M^\bullet \in \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$, there are triangles in $D_{\mathrm{pur}}(R) : M^\bullet_{i+1} \to T^\bullet_i \to M^\bullet_i \to$ with $M^\bullet_n \in D^{\leq 0}_{\mathrm{pur}}(R), T^\bullet_i \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ and $M^\bullet_0 = M^\bullet$ for all $0 \leq i < n$. Thus, $X^\bullet \in T^{\bullet\perp_{>n}}$ for any $X^\bullet \in D^{\leq 0}_{\mathrm{pur}}(R) = \mathcal{PP}^{\perp_{>0}}$.

Indeed, $T^\bullet \in (\widehat{\mathcal{PP}})_n$ follows from Proposition 3.3. Then there is a series of triangles in $D_{\mathrm{pur}}(R) : T^\bullet_{i+1} \to P_i \to T^\bullet_i \to$ with $T^\bullet_{n+1} = 0, P_i \in \mathcal{PP}$ and $T^\bullet_0 = T^\bullet$ for $0 \leq i \leq n$. Applying the functor $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(-, X^\bullet)$ to these triangles, we obtain the following isomorphisms

$$\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, X^\bullet[j]) \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_1, X^\bullet[j-1]) \simeq \cdots$$

$$\begin{aligned}&\simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_n, X^\bullet[j-n])\\ &= \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(P_n, X^\bullet[j-n]) = 0\end{aligned}$$

by $X^\bullet \in D^{\leq 0}_{\mathrm{pur}}(R) = \mathcal{PP}^{\perp_{>0}}$ and $T^\bullet_n \simeq P_n$ for $j > n$.

Applying $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, -)$ to the triangles $M^\bullet_{i+1} \to T^\bullet_i \to M^\bullet_i \to$ for $0 \leq i < n$, and since $M^\bullet_n \in D^{\leq 0}_{\mathrm{pur}}(R) = \mathcal{PP}^{\perp_{>0}}$, we have that

$$\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, M^\bullet[i]) \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, M^\bullet_1[i+1]) \simeq \cdots \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, M^\bullet_n[i+n]) = 0.$$

Therefore, $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, M^\bullet[i]) = 0$, i.e., $M^\bullet \in T^{\bullet\perp_{>0}}$. In a word, $\mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet) \subseteq T^{\bullet\perp_{>0}}$.

Proposition 3.6 *Assume that the left pure global dimension of R is finite, and* $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D^b_{\mathrm{pur}}(R) \cap D^{\leq 0}_{\mathrm{pur}}(R)$. *If* $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$, *then $T^\bullet$ is n-pure silting.*

Proof Since $T^\bullet \in \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet) = T^{\bullet\perp_{>0}}$ and $\mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$ is closed under coproducts, it follows that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq T^{\bullet\perp_{>0}}$. By the assumption, we have that $T^\bullet \in D^b_{\mathrm{pur}}(R)$. According to [26], one can get $T^\bullet \in K^b(\mathcal{PP}) = \langle \mathcal{PP} \rangle$. Therefore, we obtain that $T^\bullet$ is partial pure silting.

It is not difficult to obtain that $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet) \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ by the assumption and Lemma 3.2. Moreover, we can get that $T^\bullet$ is pure silting by Proposition 3.4.

Note that $T^\bullet \in \langle \mathcal{PP} \rangle \cap \mathcal{PP}^{\perp_{>0}} = \widehat{\mathcal{PP}}$ by $T^\bullet \in D^{\leq 0}_{\mathrm{pur}}(R) = \mathcal{PP}^{\perp_{>0}}$ and Lemma 2.2. Then there is an integer m such that $T^\bullet \in (\widehat{\mathcal{PP}})_m$. We can obtain that $T^\bullet$ is m-pure silting by Proposition 3.3. So, there is a series of triangles in $D_{\mathrm{pur}}(R) : T^\bullet_{i+1} \to P_i \to T^\bullet_i \to$ with $P_i \in \mathcal{PP}$ for all $0 \leq i \leq m$, where $T^\bullet_{m+1} = 0$ and $T^\bullet_0 = T^\bullet$. Applying the functor $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(-, P)$ to the triangles with $P \in \mathcal{PP}$, it is easy to see that

$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_m, P) &\simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_{m-1}, P[1]) \simeq \cdots\\ &\simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet_0, P[m])\\ &= \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, P[m]) = 0.\end{aligned}$$

Since $P[n] \in D^{\leq 0}_{\mathrm{pur}}(R)[n] \subseteq \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet) = T^{\bullet\perp_{>0}}$, we have $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(T^\bullet, P[i+n]) = 0$ for any $i > 0$. If $m \leq n$, then $T^\bullet$ is n-pure silting. If $m > n$, then the triangle $T^\bullet_m \to P_{m-1} \to T^\bullet_{m-1} \to$ splits, i.e., $T^\bullet \in (\widehat{\mathcal{PP}})_{m-1}$. Repeating the process, it follows that $T^\bullet \in (\widehat{\mathcal{PP}})_n$. Therefore, $T^\bullet$ is n-pure silting.

Note that

$$D^{[a,b]}_{\mathrm{pur}}(R) = \{M^\bullet \in D_{\mathrm{pur}}(R) \mid \mathrm{H}^i\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(M^\bullet, P[i]) = 0, \text{where } i > b \text{ and } i < a \text{ for } P \in \mathcal{PP}\}.$$

Combining Propositions 3.5 and 3.6, we have the following characterization of n-pure silting complexes.

Theorem 3.2 *Suppose that* $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet \subseteq D^{[-t,0]}_{\mathrm{pur}}(R)$ *for $t \geq 0$. Then the following are equivalent*:

(1) *$T^\bullet$ is n-pure silting.*

(2) $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{D^{\leq 0}_{\mathrm{pur}}(R)}(T^\bullet)$.

Auslander and Reiten [27] showed that there is a one-to-one correspondence between isomorphism classes of basic tilting modules and certain covariantly finite suspended subcategories. Later, Buan [28] showed that there is a one-to-one correspondence between basic tilting complexes and certain covariantly finite subcategories of the bounded derived category of an Artin algebra. In the following, we aim to extend such a result to pure silting complexes.

We need the following definition.

Definition 3.2 *Let* $\mathcal{X} \subseteq \mathcal{Y}$ *be two subcategories of* D*.*

(1) $\mathcal{X}$ *is said to be covariantly finite in* $\mathcal{Y}$ *if for any* $Y \in \mathcal{Y}$*, there is a homomorphism* $f : Y \to X$ *for some* $X \in \mathcal{X}$ *such that* $\mathrm{Hom}_D(f, X')$ *is surjective for any* $X' \in \mathcal{X}$*.*

(2) $\mathcal{X}$ *is said to be specially covariantly finite in* $\mathcal{Y}$ *if for any* $Y \in \mathcal{Y}$*, there is a triangle* $X \to Y \to U$ *with some* $X \in \mathcal{X}$ *such that* $\mathrm{Hom}_D(U, X'[1]) = 0$ *for any* $X' \in \mathcal{X}$*.*

Note that in the latter case, one has that $U \in {}^{\perp_{i>0}}\mathcal{X}$ if $\mathcal{X}$ is closed under [1].

Proposition 3.7 *Assume that* $T^\bullet \in D^{\leq 0}_{\mathrm{pur}}(R)$ *is pure silting. Then* $\widecheck{T^{\bullet\perp_{>0}}} = D^{-}_{\mathrm{pur}}(R)$ *and* $T^{\bullet\perp_{>0}} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ *is specially covariantly finite in* $D^{-}_{\mathrm{pur}}(R)$*.*

Proof It is not difficult to obtain that $T^{\bullet\perp_{>0}} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ by Proposition 3.4. Therefore, we get $\widecheck{T^{\bullet\perp_{>0}}} = D^{-}_{\mathrm{pur}}(R)$. Taking any $X^\bullet \in D^{-}_{\mathrm{pur}}(R)$, it is obvious that $X^\bullet \in T^{\bullet\perp_{i\geq m}}$ for some m, by $T^\bullet \in K^b(\mathcal{PP})$, that is, $X^\bullet[m] \in T^{\bullet\perp_{>0}}$. Since $0 \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$, we can see that $X^\bullet \in \widecheck{T^{\bullet\perp_{>0}}}$. Hence $\widecheck{T^{\bullet\perp_{>0}}} = D^{-}_{\mathrm{pur}}(R)$.

Note that $T^{\bullet\perp_{>0}} = {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$ by Lemma 3.1. For any $X^\bullet \in D^{-}_{\mathrm{pur}}(R) = \widecheck{T^{\bullet\perp_{>0}}} = {}_{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}\mathcal{X}$, by [7, Corollary 2.8], we obtain a triangle $U^\bullet \to X^\bullet \to Y^\bullet \to$ with $U^\bullet \in \widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet}$ and $Y^\bullet \in T^{\bullet\perp_{>0}}$. Since $\widecheck{\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet} \subseteq {}^{\perp_{>0}}(T^{\bullet\perp_{>0}})$, specially we get that $\mathrm{Hom}_D(M^\bullet, U^\bullet[1]) = 0$ for any $M^\bullet \in T^{\bullet\perp_{>0}}$. It follows that $T^{\bullet\perp_{>0}}$ is specially covariantly finite in $D^{-}_{\mathrm{pur}}(R)$.

Proposition 3.8 *Assume that* $\mathcal{T} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ *is specially covariantly finite in* $D^{-}_{\mathrm{pur}}(R)$ *and suspended such that* $\check{\mathcal{T}} = D^{-}_{\mathrm{pur}}(R)$*. If* $\mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ *is closed under coproducts, then there is a pure silting complex* $T^\bullet$ *such that* $\mathcal{T} = T^{\bullet\perp_{>0}}$*.*

Proof It is easy to verify that $D^{-}_{\mathrm{pur}}(R) = \check{\mathcal{T}} \subseteq ({}^{\perp_{>0}}\mathcal{T})^{\perp_{\gg 0}}$. Hence, ${}^{\perp_{>0}}\mathcal{T} \subseteq K^b(\mathcal{PP})$.

Since $\mathcal{T}$ is specially covariantly finite in $D^{-}_{\mathrm{pur}}(R)$ and suspended by the assumption. Taking any $M^\bullet \in D^{-}_{\mathrm{pur}}(R)$, there is a series of triangles $M^\bullet_j \to T^\bullet_j \to M^\bullet_{j+1} \to$ with $T^\bullet_j \in \mathcal{T}$ for all $j \geq 0$, where $M^\bullet_0 := M^\bullet$ and each $M^\bullet_j \in {}^{\perp_{>0}}\mathcal{T}$ for $j \geq 1$. It yields that $T^\bullet_j \in \mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ for all $j \geq 1$. By ${}^{\perp_{>0}}\mathcal{T} = K^b(\mathcal{PP})$ and $M^\bullet \in D^{-}_{\mathrm{pur}}(R)$, one can verify that $M^\bullet \in ({}^{\perp_{>0}}\mathcal{T})^{\perp_{>n}}$ for some n depending on $M^\bullet$. Applying the functor $\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(-, M^\bullet_{n+1})$ to the above triangles, we obtain that

$$\mathrm{Hom}_{D_{\mathrm{pur}}(R)}(M^\bullet_{n+1}, M^\bullet_n[1]) \simeq \cdots \simeq \mathrm{Hom}_{D_{\mathrm{pur}}(R)}(M^\bullet_{n+1}, M^\bullet[n+1]) = 0.$$

Thus, the triangle $M^\bullet_n \to T^\bullet_n \to M^\bullet_{n+1} \to$ is split. Moreover, $M^\bullet_n$ is a direct summand of $T^\bullet_n$. It follows that $\mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ is closed under coproducts and $\mathcal{T}$ is suspended by the assumption. Then we have that $\mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ is closed under direct summands. Therefore, $M^\bullet_n \in \mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$.

Since $\mathcal{T} \subseteq D^{\leq 0}_{\mathrm{pur}}(R) = \mathcal{PP}^{\perp_{>0}}$, we get $\mathcal{PP} \in {}^{\perp_{>0}}\mathcal{T}$. In particular, let the object $M^\bullet$ in the above be any pure projective module P. Then we can obtain the triangles $P_j \to T^{\bullet\prime}_j \to P_{j+1} \to$ with $P_j \in {}^{\perp_{>0}}\mathcal{T}$ and $T^{\bullet\prime}_j \in \mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ for all $0 \leq j \leq n$, where $P_0 = P$ and $P_{n+1} = 0$. Taking $T^\bullet = \bigoplus\limits_{j=0}^{n} T^{\bullet\prime}_j$, then we show that $T^\bullet$ is pure silting. Firstly, we can verify that $T^\bullet$ is partial pure silting since ${}^{\perp_{>0}}\mathcal{T} \subseteq K^b(\mathcal{PP})$ and $\mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ is closed under coproducts. Moreover, it also shows that $\mathcal{PP} \in \widecheck{\mathrm{Add}}_{D_{\mathrm{pur}}(R)}T^\bullet$ by the argument above. Consequently, $T^\bullet$ is pure silting.

It suffices to prove that $\mathcal{T} = T^{\bullet\perp_{>0}}$. One can easily check that $\mathcal{T} \subseteq {}^{\perp_{>0}}T^\bullet$ by $T^\bullet = \bigoplus\limits_{j=0}^{n} T^{\bullet\prime}_j$ and $T^{\bullet\prime}_j \in \mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$ for all $0 \leq j \leq n$. Taking any $N^\bullet \in T^{\bullet\perp_{>0}}$. Similar to the above discussion, we have a series of triangles $N^\bullet_j \to T^{\bullet\prime\prime}_j \to N^\bullet_{j+1} \to$ with $N^\bullet_j \in \mathcal{T}^{\perp_{>0}}, T^{\bullet\prime\prime}_0 \in \mathcal{T}$ and $T^{\bullet\prime\prime}_j \in \mathcal{T} \cap \mathcal{T}^{\perp_{>0}}$ for all $1 \leq j \leq m$, where $N^\bullet_0 = N^\bullet$ and $N^\bullet_{m+1} = 0$. Observe that all objects in these triangles are in $T^{\bullet\perp_{>0}}$. For any $L^\bullet \in \mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T}$, one can easily get that $T^\bullet \oplus L^\bullet$ is also silting. Then we have $L \in \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$ by Proposition 3.1. It follows that $\mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T} \subseteq \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$. Moreover, $\mathcal{T} \cap {}^{\perp_{>0}}\mathcal{T} = \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$. So the above triangles imply that $N^\bullet_1 \in \widecheck{\mathrm{Add}}_{D_{\mathrm{pur}}(R)}T^\bullet$ and consequently, $N^\bullet_1 \in T^{\bullet\perp_{>0}} \cap \widecheck{\mathrm{Add}}_{D_{\mathrm{pur}}(R)}T^\bullet = \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet$. We have that $N^\bullet \in \mathcal{T}$ by $\mathcal{T}$ suspended. It follows that the triangle $N^\bullet \to T^\bullet_0 \to N^\bullet_1 \to$ is split. So we can learn that $T^{\bullet\perp_{>0}} \subseteq \mathcal{T}$. This completes the proof.

By Propositions 3.7–3.8, we obtain the following desired result. Here, we say two pure complexes $M^\bullet$ and $N^\bullet$ are equivalent if $\mathrm{Add}_{D_{\mathrm{pur}}(R)}M^\bullet = \mathrm{Add}_{D_{\mathrm{pur}}(R)}N^\bullet$.

Theorem 3.3 *There is a one-to-one correspondence, given by* $u : T^\bullet \mapsto T^{\bullet\perp_{>0}}$, *between equivalence classes of pure silting complexes in* $D^{\leq 0}_{\mathrm{pur}}(R)$ *and subcategories* $\mathcal{T} \subseteq D^{\leq 0}_{\mathrm{pur}}(R)$ *which are specially covariantly finite in* $D^{-}_{\mathrm{pur}}(R)$*, suspended and closed under coproducts such that* $\check{\mathcal{T}} = D^{-}_{\mathrm{pur}}(R)$.

Proof It follows from Propositions 3.7–3.8 that the correspondence is well-defined. Note that u is surjective by Proposition 3.8. If both $T^\bullet_1$ and $T^\bullet_2$ are pure silting with $T_1^{\bullet\perp_{>0}} = T_2^{\bullet\perp_{>0}}$, then we can verify that $T^\bullet_1 \oplus T^\bullet_2$ is also pure silting. It is easy to get from Proposition 3.1 that $\mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet_1 = \mathrm{Add}_{D_{\mathrm{pur}}(R)}T^\bullet_2$, so $T^\bullet_1$ and $T^\bullet_2$ are equivalent. Hence, u is bijective.

4 2-term Pure Silting Complex

In this section, R denotes a finite dimensional k-algebra over a field k. Let $P^\bullet : P_1 \xrightarrow{d^1} P_0$ be a complex in $D^b_{\mathrm{pur}}(R)$ with $P_i \in \mathcal{P}\mathrm{proj}$ for $i = 0, 1$. Consider the subcategories of R-mod,

$$\mathcal{T}(P^\bullet) = \{X \in R\text{-mod} \mid \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X[1]) = 0\}$$

and

$$\mathcal{F}(P^\bullet) = \{Y \in R\text{-mod} \mid \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, Y) = 0\}.$$

Assume that $P^\bullet : P_1 \to P_0$ is a 2-term pure silting complex in $K^b(\mathcal{P}\mathrm{proj})$. Considering the subcategories of $D^b_{\mathrm{pur}}(R)$,

$$D^{\leq 0}_{\mathrm{pur}}(P^\bullet) = \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X^\bullet[i]) = 0 \text{ for } i > 0\}$$

and

$$D^{\geq 0}_{\mathrm{pur}}(P^\bullet) = \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X^\bullet[i]) = 0 \text{ for } i < 0\},$$

we can obtain the following facts.

Lemma 4.1 *Let $P^\bullet$ be a 2-term pure silting complex in $K^b(\mathcal{P}\mathrm{proj})$. Then*
(1) *$(D^{\leq 0}_{\mathrm{pur}}(P^\bullet), D^{\geq 0}_{\mathrm{pur}}(P^\bullet))$ is a t-structure in $D^b_{\mathrm{pur}}(R)$.*
(2) *$\mathcal{T}(P^\bullet) = D^{\leq 0}_{\mathrm{pur}}(P^\bullet) \cap R\text{-mod}$ and $\mathcal{F}(P^\bullet) = D^{\geq 1}_{\mathrm{pur}}(P^\bullet) \cap R\text{-mod}$.*

Proof We can obtain the first conclusion by [29, Section 3.3]. It is easy to get the second statement by the definitions.

Next we pay attention to the relations between 2-term pure silting complexes, t-structures and torsion pairs in module categories.

Lemma 4.2 *For any $X^\bullet \in D^b_{\mathrm{pur}}(R)$ and $n \in \mathbb{Z}$, then there is a functorial exact sequence*

$$0 \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^{n-1}(X^\bullet)[1]) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X^\bullet[n]) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^n(X^\bullet)) \to 0.$$

Proof For $X^\bullet[n] \in D^b_{\mathrm{pur}}(R)$, applying $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(-, X^\bullet[n])$ to the distinguished triangle

$$P_1 \xrightarrow{d^1} P_0 \to P^\bullet \to P_1[1],$$

there is an induced exact sequence

$$\begin{aligned} 0 \to \mathrm{Coker}(\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(d^1, X^\bullet[n-1])) &\to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X^\bullet[n]) \\ &\to \ker(\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(d^1, X^\bullet[n])) \to 0. \end{aligned}$$

Since

$$\begin{aligned} \ker(\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(d^1, X^\bullet[n])) &\cong \ker(\mathrm{Hom}_R(d^1, \mathrm{H}^n(X^\bullet))) \\ &\cong \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^n(X^\bullet)) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Coker}(\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(d^1, X^\bullet[n-1])) &\cong \mathrm{Coker}(\mathrm{Hom}_R(d^1, \mathrm{H}^{n-1}(X^\bullet))) \\ &\cong \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^{n-1}(X^\bullet)[1]), \end{aligned}$$

it is easy to get the conclusion.

Proposition 4.1 *Let $P^\bullet$ be a 2-term silting complex in $K^b(\mathcal{P}\mathrm{proj})$ and $\mathcal{H}_{\mathrm{pur}}(P^\bullet) := D^{\leq 0}_{\mathrm{pur}}(P^\bullet) \cap D^{\geq 0}_{\mathrm{pur}}(P^\bullet)$ be the heart of induced t-structure $(D^{\leq 0}_{\mathrm{pur}}(P^\bullet), D^{\geq 0}_{\mathrm{pur}}(P^\bullet))$. Assume $B = \mathrm{End}_{D^b_{\mathrm{pur}}(R)}(P^\bullet)^{\mathrm{op}}$,*

(1) *$\mathcal{H}_{\mathrm{pur}}(P^\bullet)$ is an abelian category and the short exact sequences in $\mathcal{H}_{\mathrm{pur}}(P^\bullet)$ are precisely the triangles in $D^b_{\mathrm{pur}}(R)$ all of whose vertices are objects in $\mathcal{H}_{\mathrm{pur}}(P^\bullet)$.*

(2) *For a complex $X^\bullet$ in $D^b_{\mathrm{pur}}(R)$, we have that $X^\bullet$ is in $\mathcal{H}_{\mathrm{pur}}(P^\bullet)$ if and only if $\mathrm{H}^0(X^\bullet)$ is in $\mathcal{T}(P^\bullet)$, $\mathrm{H}^{-1}(X^\bullet)$ is in $\mathcal{F}(P^\bullet)$ and $\mathrm{H}^i(X^\bullet) = 0$ for $i \neq -1, 0$.*

(3) *The functor* $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -) : \mathcal{H}_{\mathrm{pur}}(P^\bullet) \to B$-mod *is an equivalence of abelian categories.*

(4) *For any* $E \in \mathcal{P}$proj, *there is a triangle in* $D^b_{\mathrm{pur}}(R)$

$$E \xrightarrow{h} P'^\bullet \xrightarrow{f} P''^\bullet \xrightarrow{g} E[1] \tag{4.1}$$

with $P'^\bullet, P''^\bullet \in \mathrm{add}P^\bullet$.

Proof (1) We can easily obtain the statement from the fact that the pair $(D^{\leq 0}_{\mathrm{pur}}(P^\bullet), D^{\geq 0}_{\mathrm{pur}}(P^\bullet))$ is a t-structure in $D^b_{\mathrm{pur}}(R)$.

(2) By Lemma 4.2, we can obtain that

$$\begin{aligned}
D^{\leq 0}_{\mathrm{pur}}(P^\bullet) &= \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^i(X^\bullet)) = 0 \text{ for } i > 0, \\
&\qquad \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^j(X^\bullet)[1]) = 0 \text{ for } j \geq 0\} \\
&= \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{H}^i(X^\bullet) = 0 \text{ for } i > 0 \text{ and } \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^0(X^\bullet)[1]) = 0\} \\
&= \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{H}^i(X^\bullet) = 0 \text{ for } i > 0 \text{ and } \mathrm{H}^0(X^\bullet) \in \mathcal{T}(P^\bullet)\}, \\
D^{\geq 0}_{\mathrm{pur}}(P^\bullet) &= \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^i(X^\bullet)) = 0 \text{ for } i < 0, \\
&\qquad \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^j(X^\bullet)[1]) = 0 \text{ for } j < -1\} \\
&= \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{H}^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathrm{H}^{-1}(X^\bullet)[1]) = 0\} \\
&= \{X^\bullet \in D^b_{\mathrm{pur}}(R) \mid \mathrm{H}^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } \mathrm{H}^{-1}(X^\bullet) \in \mathcal{F}(P^\bullet)\}.
\end{aligned}$$

Hence, we have that $X^\bullet \in \mathcal{H}_{\mathrm{pur}}(P^\bullet)$ if and only if $\mathrm{H}^0(X^\bullet)$ is in $\mathcal{T}(P^\bullet)$, $\mathrm{H}^{-1}(X^\bullet)$ is in $\mathcal{F}(P^\bullet)$ and $\mathrm{H}^i(X^\bullet) = 0$ for $i \neq -1, 0$.

(3) The proof is similar to [8, Theorem 1.3].

(4) Let $P''^\bullet \to E[1]$ be a right $\mathrm{add}P^\bullet$-approximation of $E[1]$, then there is a triangle

$$E \to Y^\bullet \to P''^\bullet \to E[1],$$

where $Y^\bullet$ is a 2-term complex in $K^b(\mathcal{P}\mathrm{proj})$. Then, we apply the functors $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -)$ and $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(-, P^\bullet)$ to the above triangle, respectively. It is easy to obtain that

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, Y^\bullet[1]) = 0 \quad \text{and} \quad \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(Y^\bullet, P^\bullet[1]) = 0.$$

Applying $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(-, Y^\bullet)$ to the above triangle, it yields that $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(Y^\bullet, Y^\bullet[1]) = 0$. Hence, $P^\bullet \oplus Y^\bullet$ is a 2-term partial pure silting complex in $K^b(\mathcal{P}\mathrm{proj})$. By the above triangle, it follows that $E \in \langle P^\bullet \oplus Y^\bullet \rangle$. Moreover, we can obtain that $P^\bullet \oplus Y^\bullet$ is a 2-term pure silting complex. Then, we have the desired triangle (4.1).

Let $X, Y \in R$-mod and $f : X \to Y$. We denote by $\mathrm{PIm}f$ the pure image of f, that is, $\mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R(P, \mathrm{Im}f)$ remains to be epic for any $P \in \mathcal{P}$proj. Let $X \in R$-mod, and consider the canonical sequence of X,

$$0 \to tX \xrightarrow{i_X} X \to X/tX \to 0,$$

where $tX = \Sigma \mathrm{PIm}f$ with $f \in \mathrm{Hom}_R(\mathrm{H}^0(P^\bullet), X)$ such that any $g : E \to X$ factors through f, for any $E \in \mathcal{P}$proj. Then we study some properties of $\mathcal{T}(P^\bullet)$ and $\mathcal{F}(P^\bullet)$.

Lemma 4.3 *The following statements hold.*

(1) $\mathcal{T}(P^\bullet)$ *is closed under* P*-epimorphic images.*

(2) $\mathcal{F}(P^\bullet)$ *is closed under* P*-submodules.*

(3) *For any* $X \in R$-mod, $\mathrm{Hom}_R(\mathrm{H}^0(P^\bullet), i_X)$ *is an isomorphism.*

Proof (1) Let $Y \to Z$ be a pure epimorphism with $Y \in \mathcal{T}(P^\bullet)$. Then we obtain a pure exact sequence in R-mod $0 \to X \to Y \to Z \to 0$. Applying the functor $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -)$ to the sequence, there is an induced exact sequence

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, Y[1]) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, Z[1]) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X[2]).$$

It yields that $Z \in \mathcal{T}(P^\bullet)$ by $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X[2]) = 0$. The conclusion holds.

(2) Let $0 \to X \to Y \to Z \to 0$ be a pure exact sequence in R-mod with $Y \in \mathcal{F}(P^\bullet)$. Applying the functor $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -)$ to the sequence, it is easy to verify that there is an exact sequence

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, Y) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, Z).$$

Therefore, we can obtain that $X \in \mathcal{F}(P^\bullet)$. The proof is completed.

(3) It is easy to obtain the desired isomorphism by the definition of tX.

Lemma 4.4 *Let* $P^\bullet$ *be a* 2*-term pure silting complex. Then there is a triangle in* $D^b_{\mathrm{pur}}(R)$ *of the form*

$$\mathrm{H}^{-1}(P^\bullet)[1] \to P^\bullet \to \mathrm{H}^0(P^\bullet) \to \mathrm{H}^{-1}(P^\bullet)[2].$$

Proof Let $P'^\bullet := 0 \to \mathrm{Im} d^1 \to P_0 \to 0$. Then there is a pure exact sequence

$$0 \to \ker d^1[1] \to P^\bullet \to P'^\bullet \to 0$$

in $C^b(R)$. It is easy to verify that $P'^\bullet$ is pure quasi-isomorphic to $\mathrm{H}^0(P^\bullet)$. Then we obtain that $P'^\bullet \cong \mathrm{H}^0(P^\bullet)$ in $D^b_{\mathrm{pur}}(R)$. Hence, we can get the desired triangle in $D^b_{\mathrm{pur}}(R)$ of the form

$$\mathrm{H}^{-1}(P^\bullet)[1] \to P^\bullet \to \mathrm{H}^0(P^\bullet) \to \mathrm{H}^{-1}(P^\bullet)[2].$$

Lemma 4.5 *Let* $P^\bullet$ *be a* 2*-term pure silting complex. For any* $X \in R$-mod, *there is a functorial isomorphism* $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X) \cong \mathrm{Hom}_R(\mathrm{H}^0(P^\bullet), X)$ *and a monomorphism* $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(\mathrm{H}^0(P^\bullet), X[1]) \to \mathrm{Hom}_R(P^\bullet, X[1])$.

Proof Applying the functor $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(-, X)$ to the triangle in Lemma 4.4,

$$\mathrm{H}^{-1}(P^\bullet)[1] \to P^\bullet \to \mathrm{H}^0(P^\bullet) \to \mathrm{H}^{-1}(P^\bullet)[2],$$

we can obtain the required conclusion by the fact that there are non-zero negative extensions between modules.

Theorem 4.1 *Let* $P^\bullet : P_1 \to P_0$ *be a complex with* $P_i \in \mathcal{P}\mathrm{proj}$. *Then the following statements are equivalent.*

(1) $P^\bullet$ *is a* 2*-term pure silting complex in* $D^b_{\mathrm{pur}}(R)$.

(2) $\mathcal{T}(P^\bullet)\cap\mathcal{F}(P^\bullet)=0$ *and* $\mathrm{H}^0(P^\bullet)\in\mathcal{T}(P^\bullet)$.

(3) $\mathcal{T}(P^\bullet)\cap\mathcal{F}(P^\bullet)=0$ *and* $tX\in\mathcal{T}(P^\bullet), X/tX\in\mathcal{F}(P^\bullet)$ *for all* $X\in R$-mod.

(4) $(\mathcal{T}(P^\bullet),\mathcal{F}(P^\bullet))$ *is a torsion pair in* R-mod.

Proof (1) $\Leftrightarrow$ (2) It follows from Lemma 4.2 that $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,P^\bullet[i])=0$ for all $i>0$ if and only if $\mathrm{H}^0(P^\bullet)\in\mathcal{T}(P^\bullet)$. Taking any $X\in\mathcal{T}(P^\bullet)\cap\mathcal{F}(P^\bullet)$, we have $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,X[n])=0$ for all $n\in\mathbb{Z}$. Furthermore, $X=0$. On the other hand, let $X^\bullet\in D^b_{\mathrm{pur}}(R)$ with $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,X[n])=0$ for all $n\in\mathbb{Z}$. It follows by Lemma 4.2 that $\mathrm{H}^n(X^\bullet)\in\mathcal{T}(P^\bullet)\cap\mathcal{F}(P^\bullet)=0$.

(2) $\Rightarrow$ (3) Let $X\in R$-mod. Since $\mathrm{H}^0(P^\bullet)\in\mathcal{T}(P^\bullet)$, we have $tX\in\mathcal{T}(P^\bullet)$. It follows by Lemma 4.5 that

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,X/tX)\cong\mathrm{Hom}_R(\mathrm{H}^0(P^\bullet),X/tX)$$

and $\mathrm{Hom}_R(\mathrm{H}^0(P^\bullet),i_X)$ is an isomorphism. So, $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,X/tX)=0$. Moreover, $X/tX\in\mathcal{F}(P^\bullet)$.

(3) $\Rightarrow$ (4) This follows from the definition.

(4) $\Rightarrow$ (2) It is sufficient to prove $\mathrm{H}^0(P^\bullet)\in\mathcal{T}(P^\bullet)$. By Lemma 4.5, one can easily verify that $0=\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,\mathcal{F}(P^\bullet))\cong\mathrm{Hom}_R(\mathrm{H}^0(P^\bullet),\mathcal{F}(P^\bullet))$. Since $(\mathcal{T}(P^\bullet),\mathcal{F}(P^\bullet))$ is a torsion pair, we learn that $\mathrm{H}^0(P^\bullet)\in\mathcal{T}(P^\bullet)$.

Remark 4.1 Let $D_\theta=\{X\in\mathrm{Mod}R\mid\mathrm{Hom}(P_0,X)\to\mathrm{Hom}(P_1,X)\text{ is an epimorphism}\}$, where $\theta:P_1\to P_0$ is a morphism with P_1,P_0 pure projective. Then we get that the torsion pair $(\mathcal{T}(P^\bullet),\mathcal{F}(P^\bullet))$ coincides with $(D_\theta,T^{\perp_{>0}})$.

Proof Assume that $P^\bullet:P_1\xrightarrow{\theta}P_0$ is a 2-term pure silting complex in $D^b_{\mathrm{pur}}(R)$, and $T=\mathrm{H}^0(P^\bullet)=\mathrm{Coker}\theta$. Consider the distinguished triangle in $D^b_{\mathrm{pur}}(R)$

$$P_1\xrightarrow{\theta}P_0\to P^\bullet\to P_1[1].$$

Applying the functor $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(-,X)$ to the triangle for any R-module X, there is an induced exact sequence

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P_0,X)\to\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P_1,X)\to\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet[-1],X)\to 0.$$

Since $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P_i,X)\cong\mathrm{Hom}_R(P_i,X)$ with $i=0,1$, it is easy to see that $X\in D_\theta$ if and only if $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,X[1])=0$ if and only if $X\in\mathcal{T}(P^\bullet)$.

Conversely, since

$$\mathrm{Hom}_R(T,X)\cong\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(T,X)\cong\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet,X),$$

we can learn that $X\in T^{\perp_{>0}}$ if and only if $X\in\mathcal{F}(P^\bullet)$. The proof is completed.

Proposition 4.2 *Assume that* $P^\bullet$ *is a 2-term pure silting complex in* $D^b_{\mathrm{pur}}(R)$ *and* $(\mathcal{T}(P^\bullet),\mathcal{F}(P^\bullet))$ *is a torsion pair induced by* $P^\bullet$.

(1) *For any* $X\in R$-mod, $X\in\mathrm{add}\mathrm{H}^0(P^\bullet)$ *if and only if* X *is* Ext*-projective in* $\mathcal{T}(P^\bullet)$.

(2) *For any* $X\in\mathcal{T}(P^\bullet)$, *there is a pure-exact sequence* $0\to K\to T_0\to X\to 0$ *with* $T_0\in\mathrm{add}\mathrm{H}^0(P^\bullet)$ *and* $K\in\mathcal{T}(P^\bullet)$.

Proof (1) Sufficiency. Assume that X is Ext-projective in $\mathcal{T}(P^\bullet)$. Since $\mathcal{T}(P^\bullet) = \mathrm{Fac}\mathrm{H}^0(P^\bullet)$, there is a pure exact sequence

$$0 \to K \to T_0 \xrightarrow{\gamma} X \to 0, \tag{4.2}$$

where $T_0 \xrightarrow{\gamma} X$ is a right $\mathrm{add}\mathrm{H}^0(P^\bullet)$-approximation.

Necessity. Since $\mathrm{Hom}_R(\mathrm{H}^0(P^\bullet), \gamma)$ is epimorphic, it is obvious that $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \gamma)$ is an epimorphism. Applying $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -)$ to sequence (4.2), we can obtain the following exact sequence

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, T_0) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, X) \to \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, K[1]) \to 0.$$

So, $\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, K[1]) = 0$, i.e., $K \in \mathcal{T}(P^\bullet)$. By the assumption, we have that the sequence (4.2) is split. Then $X \in \mathrm{add}\mathrm{H}^0(P^\bullet)$. Using the monomorphism in Lemma 4.5, we can obtain that $\mathrm{add}\mathrm{H}^0(P^\bullet)$ is Ext-projective in $\mathcal{T}(P^\bullet)$.

(2) We can get the conclusion in the proof of (1).

Let $B = \mathrm{End}_{D^b_{\mathrm{pur}}(R)}(P^\bullet)^{op}$. We consider the following subcategories of B-mod:

$$\mathcal{X}(P^\bullet) = \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathcal{F}(P^\bullet)[1]) \quad \text{and} \quad \mathcal{Y}(P^\bullet) = \mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, \mathcal{T}(P^\bullet)).$$

They push us to state the Brenner-Butler Theorem in the version of pure silting complexes.

Theorem 4.2 *Assume that $P^\bullet$ is a 2-term pure silting complex in $D^b_{\mathrm{pur}}(R)$. Then we have that $(\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ is a torsion pair in B-*mod *and there are equivalences*

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -) : \mathcal{T}(P^\bullet) \to \mathcal{Y}(P^\bullet)$$

and

$$\mathrm{Hom}_{D^b_{\mathrm{pur}}(R)}(P^\bullet, -[1]) : \mathcal{F}(P^\bullet) \to \mathcal{X}(P^\bullet).$$

*The equivalences send pure exact sequences with terms in $\mathcal{T}(P^\bullet)$ (resp. $\mathcal{F}(P^\bullet)$) to short exact sequences in B-*mod.

Proof This follows from Proposition 4.1(1)(3), and the fact that $\mathcal{T}(P^\bullet) \bigcup \mathcal{F}(P^\bullet) \subset \mathcal{H}_{\mathrm{pur}}(P^\bullet)$.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Happel, D. and Ringel, C. M., Tilted algebras, *Trans. Amer. Math. Soc.*, **274**(2), 1982, 399–443.

[2] Colby, R. R. and Fuller, K. R., Tilting, cotilting, and serially tilted rings, *Comm. Algebra*, **18**(5), 1990, 1585–1615.

[3] Angeleri-Hügel, L. and Coelho, F. U., Infinitely generated tilting modules of finite projective dimension, *Forum Math.*, **13**(2), 2001, 239–250.

[4] Miyashita, Y., Tilting modules of finite projective dimension, *Math. Z.*, **193**(1), 1986, 113–146.

[5] Bazzoni, S., A characterization of n-cotilting and n-tilting modules, *J. Algebra*, **273**(1), 2004, 359–372.

[6] Keller, B. and Vossieck, D., Aisles in derived categories, *Bull. Soc. Math. Belg. Sér. A*, **40**(2), 1988, 239–253.

[7] Wei, J., Semi-tilting complexes, *Israel J. Math.*, **194**(2), 2013, 871–893.

[8] Hoshino, M., Kato, Y. and Miyachi, J., On t-structures and torsion theories induced by compact objects, *J. Pure Appl. Algebra*, **167**, 2002, 15–35.

[9] Koga, H., On partial tilting complexes, *Comm. Algebra*, **39**(7), 2011, 2417–2429.

[10] Buan, A. and Zhou, Y., A silting theorem, *J. Pure Appl. Algebra*, **220**, 2016, 2748–2770.

[11] Frederik, M. and Jorge, V., Silting and cosilting classes in derived categories, *J. Algebra*, **501**, 2008, 526–544.

[12] Eilenberg, S. and Moore, J. C., Foundations of relative homological algebra, *Mem. Amer. Math. Soc.*, **55**, 1965, 39 pp.

[13] Enochs, E. E. and Jenda, O. M. G., Relative Homological Algebra, De Gruyter Expositions in Mathematics, **30**, Walter de Gruyter Gmbh & Co. KG, Berlin, 2011.

[14] Auslander, M. and Solberg, Ø., Relative homology and representation theory II. Relative cotilting theory, *Comm. Algebra*, **21**(9), 1993, 3033–3079.

[15] Wei, J., A note on relative tilting modules, *J. Pure Appl. Algebra*, **214**(4), 2010, 493–500.

[16] Yan, L., Li, W. and Ouyang, B., Gorenstein cotilting and tilting modules, *Comm. Algebra*, **44**(2), 2016, 591–603.

[17] Moradifar, P. and Yassemi, S., Infinitely generated Gorenstein tilting modules, *Algebr. Represent. Theory*, **25**(6), 2022, 1389–1427.

[18] Göbel, R. and Trlifaj, J., Approximations and Endomorphism Algebras of Modules, De Gruyter Expositions in Mathematics, **41**, Walter de Gruyter GmbH & Co. KG, Berlin, 2006.

[19] Kiepiński, R. and Simson, D., On pure homological dimension, *Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys.*, **23**, 1975, 1–6.

[20] Puninski, G. and Rothmaler, P., Pure-projective modules, *J. London Math. Soc.*, **71**(2), 2005, 304–320.

[21] Simson, D., Pure-periodic modules and a structure of pure-projective resolutions, *Pacific J. Math.*, **207**(1), 2002, 235–256.

[22] Warfield, R. B., Purity and algebraic compactness for modules, *Pacific J. Math.*, **28**, 1969, 699–719.

[23] Zheng, Y. and Huang, Z., On pure derived categories, *J. Algebra*, **454**, 2016, 252–272.

[24] Rotman, J. J., An Introduction to Homological Algebra (2nd ed.), Universitext, Springer, New York, 2009.

[25] Christensen, L. W., Frankild, A. and Holm, H., On Gorenstein projective, injective and flat dimensions–a functorial description with applications, *J. Algebra*, **302**(1), 2006, 231–279.

[26] Yan, M. and Yao, H., The relation between Gorenstein derived and pure derived categories, *Bull. Iranian Math. Soc.*, **48**, 2022, 3481–3499.

[27] Auslander, M. and Reiten, I., Applications of contravariantly finite subcategories, *Adv. Math.*, **86**, 1991, 111–152.

[28] Buan, A., Subcategories of the derived category and cotilting complexes, *Colloq. Math.*, **88**(1), 2001, 1–11.

[29] Koenig, S. and Yang, D., Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras, *Doc. Math.*, **19**, 2014, 403–438.