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Novel Proximal Type Coincidence Point Results with Applications

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Abstract In this paper, the authors establish some intriguing coincidence best proximity point results for proximal contractions in the context of extended b-metric spaces. They give some illustrative examples from various cases to substantiate their conclusions. The findings discussed in the paper are more general, expanding and enhancing a variety of existing findings in the optimal proximity theory. The findings, which explain the proximal coincidence points for multivalued mappings, are the first of their kind in the current state of the art. In addition, the study provides benchmarks for employing the optimal proximity results once the existence and uniqueness requirements are met.

Keywords Extended b-metric space, Proximal contraction mappings, Coincidence best proximity point, Boundary value problem, Fredholm integral equation
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1 Introduction

In the large spectrum of mathematical problems, best proximity point and fixed point of a specific map provide the solution to a mathematical problem. Therefore, fixed point theory is of pronounced significance in numerous fields of mathematics and other disciplines of science and engineering. The conclusions that state a mathematical problem has a solution are referred to as fixed point results. The concept of metric space is basic in mathematics, and it plays a crucial role in comprehending and implementing topological concepts in a variety of areas. This idea has pulled a generous thought from mathematicians attributable to the idea of fixed point hypothesis in measurement spaces. Various augmentations and speculations of the thought of a metric space have been done in the literature. We refer readers to [4, 10, 32].

Fixed point theory is indispensable for various theories and application fields, such as variational and linear inequalities, approximation theory, nonlinear analysis, integral and differential equations and inclusions, dynamic system theory, fractal mathematics, mathematics economics

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(game theory, equilibrium problems and optimization problems) and mathematical modeling (see [1, 17]).

A metric on a space induces topological properties like open and closed sets, which lead to the study of more abstract topological spaces. Fréchet [12] introduced metric spaces in his work "Sur quelques points du calcul fonctionnel". Later on, in 1914, Hausdorff utilized this concept and pondered the role of point-sets within abstract set theory. Fagin et al. [11], furthermore, employed a relaxed triangle inequality in pattern matching and dubbed it non-linear elastic matching. A modified triangle inequality of a similar kind was also used to assess industry and commerce (see [9]) and ice floes (see [22]). Intrigued by similar notions and with applications in mind, Kamran et al. [18] created a new class of generalized metric spaces termed extended b-metric spaces by relaxing the triangle inequality even more. Very recently, within the context of generalized metric spaces, authors discussed some significant real-world engineering problems: "transverse oscillations of a homogeneous bar" (see [36]), "Hammerstein integral equations and fractional differential equations" (see [35]), and "ascending motion of a rocket" (see [34]).

On the other hand, it is always challenging to find the global minima of a function. Analytical methods are frequently not applicable in such cases, and the use of numerical solution strategies also becomes difficult. Applications of global minimization problems arise in fitting model parameters to experimental data in chemistry, physics, finance, and engineering. For example, one might minimize the energy function in case of structure prediction, or minimize the path length in case of traveling salesman problem and electrical circuit design. Given the above constraints, it is always interesting to establish the existence and convergence of the best proximity points as solutions to various minimization problems in optimization theory, which has recently attracted the attention of many authors (see [15, 23]).

The Banach contraction principle is the basic result that provides the existence of the solution of the fixed point equation $Tx = x$. In the case when T is not a self-mapping, the problem $Tx = x$ is unlikely to have a solution, because a solution of the preceding equation requires an element in the domain and an element in the co-domain of the mapping be equal. In such cases, it is advantageous to find an approximate value that is optimal in terms of approximation error. In other words, if $T : A \to B$ is a non-self-mapping in the context of a metric space, one wants to find an approximation solution x with the smallest error $d(x, Tx)$. When T is a non-self mapping, the quantity $\min_{x \in A} d(x, Tx)$ is simply an ideal optimum approximate solution to the formula $Tx = x$, which is unlikely to have a solution. If a solution x to the preceding non-linear programming problem meets the requirement that $d(x, Tx) = d(A, B)$ for every x, it becomes a suitable estimation with the lowest feasible error to the associated equation $Tx = x$. Consequently, such a solution x is referred to as the best proximity point of the mapping, and the findings that look into the possibility of best proximity points for non-self mappings are referred to as best proximity results. A detailed discussion on best proximity point results can be found in the notable papers [13, 33]. Anuradha and Veeramani [3] have established the presence of a best proximity point for proximal pointwise contractions. Some interesting common best proximity theorems have been discussed in [5, 27]. The best proximity point theorems for various forms of multi-valued mappings have also been obtained in [6–7]. Finding the "best proximity points"?for two mappings is another kind of generalization of "best proximity point" any $x \in A$ that satisfies $d(g(x), T(x)) = d(A, B)$; here, A and B are nonempty subsets of (X, d) and $T : A \to B$ and $g : A \to A$ is any mapping. The "coincidence best proximity point" of the mappings g and T is the point x. Every "coincidence best proximity point" will be transformed to "best proximity point" of mapping T if $g = I_A$ (identity over A). This is an extension of the best proximity point problem. There are numerous outcomes in the setup of metric spaces that deal with the proximity point problems. Some noteworthy articles, e.g., [14, 25] are suggestive in this setup.

Generalizing the contractive conditions and the underlying spaces is one of several generalizations of fixed and best proximity point theory. Additionally, researchers are attempting to investigate the optimal proximity point findings for multivalued mappings (this is not an easy task). Several authors have found the optimal proximity points for multivalued mappings (see [28] for more information). Since Suzuki, Kikkawa and Vetro [30] initially proposed the notion of the existence of the optimal proximity points in metric spaces in 2009, many academics have worked in this direction. However, we will discuss the chronological sequence of rational type contractions that connects our study effort. By using generalized rational type contractive conditions, Chandok et al. [8] set up a common fixed point in a partially ordered metric space in 2012. After that, in 2014, Radenovic [24] used rational type contractions to get coupled coincidence best proximity point findings in partially ordered metric space. Introducing hybrid rational Geraghty type contractive conditions, Zabihi and Razani [37] produced fixed point results in ordered b-metric space in the same year. Saleem et al. [28] developed a coincidence point in a b-metric space by using multivalued Suzuki type mapping in θ-contractions. Employing certain admissible contractions. In 2019, Khemphet [20] found a coincidence point in a generalized metric space. By using proximal Geraghty contractions, Khemphet et al. [21] developed the common best proximity coincidence point in a complete metric space. Applying (α, D)-proximal generalized Geraghty contractive conditions, Wiriyapongsanon et al. [31] demonstrated coincidence as the optimal proximity point in generalized (JS) metric space. By employing cyclic contractive type mappings. In 2021, Hiranmoy et al. [16] identified the best proximity point in a metric space. Using a novel form of multivalued mapping, Savanovic et al. [29] recently demonstrated coincidence point and a common fixed point in the context of b-metric space. We formulate the multivalued rational type contractive conditions in the context of extended b-metric space as follows, which are more general than those of the work already done in metric spaces and b-metric spaces. Our motivation for this work comes from the literature on work done in various metric spaces using the rational type contractive conditions mentioned above. We address different coincidence best proximity point theorems in the context of extended b-metric spaces in this paper. It is worth noting that the coincidence point results presented in this paper are novel and newfangled, as the contraction condition for multivalued mappings as well as single valued mappings in the context of extended b-metric spaces has yet to be examined in this configuration.

Stating in brief, this manuscript has two major aims, both of which are based on the aforementioned theory and driven by the work done in [13–14, 18–19, 28]. First, the purpose of this paper is to incorporate the coincidence best proximity point theorems for generalized and modified proximal contractions in the domain of extended b-metric spaces, which provide an optimal approximate solution to the equation $Tx = x$. On the other hand, we use our findings to derive sufficient conditions for the existence of solutions to nonlinear differential and integral equations. This demonstrates the new theory's utility.

It may be noted that we demonstrate the novelty of our findings by using relevant examples to show that they are true generalizations of various notable outcomes in the current state-of-the-art.

2 Preliminaries

Before heading on to the main body of the article, we recall some fundamental concepts and notations that will be useful throughout the rest of the study.

Definition 2.1 (see [18]) *Let X be a nonempty set and $\beta : X \times X \to [1, \infty)$ be a mapping. Let the function $d : X \times X \to [0, \infty)$ satisfy the underneath postulates:*

(1) $d(x_1, x_2) = 0$ *if and only if* $x_1 = x_2$,
(2) $d(x_1, x_2) = d(x_2, x_1)$,
(3) $d(x_1, x_2) \leq \beta(x_1, x_2)[d(x_1, x_3) + d(x_3, x_2)]$

for all $x_1, x_2, x_3 \in X$. Then the pair (X, d) is called an extended b-metric space.

Remark 2.1 If $\beta(x_1, x_2) = s$ for $s \geq 1$, then (X, d) reduces to a b-metric space.

Let (X, d) be an extended b-metric space, A and B be two nonempty subsets of X. We define

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\},$$

where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 2.2 (see [18]) *Let (X, d) be an extended b-metric space. The sequence $\{x_n\}$ converges to some x in X, if for each positive ϵ, there is some positive integer N_ϵ such that $d(x_n, x) < \epsilon$ for each $n \geq N_\epsilon$, it can be written as*

$$\lim_{n\to\infty} x_n = x.$$

Definition 2.3 (see [18]) *The sequence $\{x_n\}$ in extended b-metric space (X, d) is said to be a Cauchy sequence, if for every $\epsilon > 0$, $d(x_n, x_m) < \epsilon$ for all $m, n \geq N_\epsilon$, where N_ϵ is some integer.*

Definition 2.4 (see [18]) *An extended b-metric space (X,d) is said to be complete if every Cauchy sequence is convergent in X.*

Lemma 2.1 (see [2]) *Let (X,d) be a complete extended b-metric space. If d is continuous, then every convergent sequence has a unique limit.*

Definition 2.5 (see [25]) *Let (A,B) be a pair of nonempty subsets of a metric space such that A_0 is nonempty. Then a pair (A,B) has the P-property if and only if*

$$\left.\begin{aligned} d(x_1,y_1)=d(A,B)\\ d(x_2,y_2)=d(A,B)\end{aligned}\right\} \quad \text{implies } d(x_1,x_2)=d(y_1,y_2),$$

where $x_1,x_2\in A$ and $y_1,y_2\in B$.

Definition 2.6 (see [26]) *Let $\mathcal{CB}(X)$ represent the closed and bounded subsets of X. Let $\mathcal{H}$ be the Pompeiu-Hausdroff metric induced by metric d defined by*

$$\mathcal{H}(A,B)=\max\Big\{\sup_{a\in A}D(a,B),\sup_{b\in B}D(b,A)\Big\}$$

for $A,B\in\mathcal{CB}(X)$, where

$$D(a,B)=\inf\{d(a,b):b\in B\}$$

and we will denote

$$D^*(a,b)=D(a,b)-d(A,B)\quad \text{for all } a\in A \text{ and } b\in B.$$

From now onward, A and B are nonempty subsets of a complete extended b-metric space (X,d) and we define $\beta_*:X^2\to[1,\infty)$ as follows:

$$\beta_*(x,B)=\inf\{x\in A:\beta(x,y)\quad \text{for some } y\in B\}$$

and

$$\beta_*(A,B)=\inf\{\beta(x,y):x\in A \text{ and } y\in B\},$$

where $\beta:X\times X\to[1,\infty)$.

3 Proximal Type Coincidence Point Results for Multivalued Mappings

In this section, we will discuss some coincidence best proximity point theorems using the multivalued concepts on extended b-metric space.

Definition 3.1 *Let $T:A\to\mathcal{CB}(B)$ and $g:A\to A$ be mappings. The pair (g,T) is said to be an η_M-proximal contraction if there exists a real number $\eta\in[0,1)$ such that*

$$\left.\begin{aligned} D(gu,Tx)=d(A,B)\\ D(gv,Ty)=d(A,B)\end{aligned}\right\} \quad \text{implies } \mathcal{H}(Tx,Ty)\le\eta M(u,v,x,y),$$

where

$$M(u,v,x,y) = \max\Big\{d(gu,gv), \frac{D(gx,Tv) - \beta_*(gx,Tv)D(gu,Tv)}{\beta_*(gx,Tv)}, D^*(gv,Tu), \frac{D(gu,Ty) - \beta_*(gu,Ty)D(gv,Ty)}{\beta_*(gu,Ty)}\Big\}$$

for all u, v, x, y in A.

Definition 3.2 *Let $T : A \to \mathcal{CB}(B)$ and $g : A \to A$ be mappings. The pair (g,T) is said to be an η_{M^*}-proximal contraction if there exists a real number $\eta \in [0,1)$, such that*

$$\left.\begin{aligned} D(u,Tx) &= d(A,B) \\ D(v,Ty) &= d(A,B) \end{aligned}\right\} \quad \text{implies } \mathcal{H}(Tx,Ty) \le \eta M(u,v,x,y),$$

where

$$M(u,v,x,y) = \max\Big\{d(u,v), \frac{D(x,Tv) - \beta_*(x,Tv)D(u,Tv)}{\beta_*(x,Tv)}, D^*(v,Tu), \frac{D(u,Ty) - \beta_*(u,Ty)D(v,Ty)}{\beta_*(u,Ty)}\Big\}$$

for all u, v, x, y in A.

Note that, if we take $g = I_A$ (g as an identity mapping on A), then every η_M-proximal contraction reduces to an η_{M^*}-proximal contraction.

Definition 3.3 *A mapping $T : X \to \mathcal{CB}(X)$ is continuous in an extended b-metric space (X,d) at $x \in X$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$T(K(x,\delta)) \subseteq K(Tx,\epsilon),$$

where $K(x,\epsilon)$ is given as

$$K(x,\epsilon) = \{y \in X, d(x,y) < \epsilon\}.$$

Clearly, if T is continuous at x, then $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

Example 3.1 Let $X = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$. Consider the function d given as $d(x,y) = d(y,x)$ and $d(x,x) = 0$, where

d	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
$\frac{1}{2}$	0	2	1	3
$\frac{1}{3}$	2	0	4	1
$\frac{1}{4}$	1	4	0	5
$\frac{1}{5}$	3	1	5	0

Take $\beta_* : X \times X \to [1, \infty)$ to be symmetric defined as $\beta(x, y) = 28x + 38y$. It is easy to see that (X, d) is extended b-metric space. Let $A = \{\frac{1}{2}, \frac{1}{3}\}$ and $B = \{\frac{1}{4}, \frac{1}{5}\}$ be two non-empty subsets of extended b-metric space (X, d). After routine calculations, we get $d(A, B) = 1$, $A_0 = A$ and $B_0 = B$. Since $T : A \to \mathcal{CB}(B)$, we take

$$Tx = \Big\{\Big\{\frac{1}{4}, \frac{1}{5}\Big\}, \text{if } x \in \Big\{\frac{1}{2}, \frac{1}{3}\Big\}\Big\},$$

and for $g : A \to A$, we have

$$gx = \begin{cases} \dfrac{1}{2}, & \text{if } x = \dfrac{1}{3}, \\ \dfrac{1}{3}, & \text{if } x = \dfrac{1}{2}. \end{cases}$$

We show that the pair (g, T) satisfies η_M-proximal contraction

$$\mathcal{H}(Tx, Ty) \leq \eta\, M(u, v, x, y)$$

for all $u, v, x, y \in A$, where $M(u, v, x, y)$ is defined in Definition 3.1.

Now

$$\left.\begin{aligned} D\Big(g\frac{1}{3}, T\frac{1}{2}\Big) &= d(A, B) \\ D\Big(g\frac{1}{2}, T\frac{1}{3}\Big) &= d(A, B) \end{aligned}\right\} \quad \text{implies } \mathcal{H}\Big(T\frac{1}{2}, T\frac{1}{3}\Big) \leq \eta M(u, v, x, y).$$

Hence, $\mathcal{H}(T\frac{1}{2}, T\frac{1}{3}) = 0$, for every $\eta \in [0, 1)$, Definition 3.1 is satisfied.

If we take $g = I_A$ (g as an identity mapping on A), then same example will hold for the Definition 3.2.

The following theorem is based on Definition 3.1, which is more general than the results discussed in the introduction within the context of extended b-metric spaces. Example 3.2 illustrates our fact.

Theorem 3.1 *Let $T : A \to \mathcal{CB}(B)$, $g : A \to A$ and $\beta_* : X^2 \to [1, \infty)$ be mappings, where A is closed subset of an extended b-metric space (X, d). Let the pair (A, B) satisfy the P-property with $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$. If a pair of continuous mappings (g, T), where g is one-to-one, satisfies η_M-proximal contraction, and if we suppose that*

$$\lim_{n,m\to\infty} \beta(gx_n, gx_m) < \frac{1}{k}, \quad k \in (0, 1),$$

then there exists a unique coincidence best proximity point of the pair (g, T).

Proof Let x_0 be an arbitrary element in A_0. Since, $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, there exists an element x_1 in A_0 such that

$$D(gx_1, Tx_0) = d(A, B).$$

Again, since, Tx_1 is an element of $T(A_0)$ which is contained in B_0, and A_0 is contained in $g(A_0)$, it follows that there is an element x_2 in A_0 such that

$$D(gx_2, Tx_1) = d(A, B).$$

By using the property P, we acquire

$$d(gx_1, gx_2) = \mathcal{H}(Tx_0, Tx_1).$$

Since, the pair of mappings (g, T) satisfies η_M-proximal contraction, we obtain

$$d(gx_1, gx_2) \leq \eta M(x_1, x_2, x_0, x_1), \tag{3.1}$$

where

$$\begin{aligned}
&M_T(x_1, x_2, x_0, x_1)\\
&\leq \max\Big\{d(gx_1, gx_2), \frac{D(gx_0, Tx_2) - \beta_*(gx_0, Tx_2)D(gx_1, Tx_2)}{\beta_*(gx_0, Tx_2)},\\
&\qquad D^*(gx_2, Tx_1), \frac{D(gx_1, Tx_1) - \beta_*(gx_1, Tx_1)D(gx_2, Tx_1)}{\beta_*(gx_1, Tx_1)}\Big\}\\
&\leq \max\Big\{d(gx_1, gx_2), \frac{\beta_*(gx_0, Tx_2)[d(gx_0, gx_1) + D(gx_1, Tx_2)] - \beta_*(gx_0, Tx_2)D(gx_1, Tx_2)}{\beta_*(gx_0, Tx_2)},\\
&\qquad D(gx_2, Tx_1) - d(A, B),\\
&\qquad \frac{\beta_*(gx_1, Tx_1)[d(gx_1, gx_2) + D(gx_2, Tx_1)] - \beta_*(gx_1, Tx_1)D(gx_2, Tx_1)}{\beta_*(gx_1, Tx_1)}\Big\}\\
&\leq \max\Big\{d(gx_1, gx_2), \frac{\beta_*(gx_0, Tx_2)d(gx_0, gx_1)}{\beta_*(gx_0, Tx_2)}, 0, \frac{\beta_*(gx_1, Tx_1)d(gx_1, gx_2)}{\beta_*(gx_1, Tx_1)}\Big\}\\
&\leq \max\{d(gx_1, gx_2), d(gx_0, gx_1), 0, d(gx_1, gx_2)\}.
\end{aligned}$$

Hence, we have

$$M_T(x_1, x_2, x_0, x_1) \leq \max\{d(gx_1, gx_2), d(gx_0, gx_1)\}.$$

If $\max\{d(gx_1, gx_2), d(gx_0, gx_1)\} = d(gx_1, gx_2)$, then the above inequality implies

$$d(gx_1, gx_2) < \eta d(gx_1, gx_2),$$

which is a contradiction. Therefore, we conclude that

$$\max\{d(gx_1, gx_2), d(gx_0, gx_1)\} = d(gx_0, gx_1).$$

Further, by the fact that Tx_2 is a member of $T(A_0)$, which is contained in B_0, and A_0 is contained in $g(A_0)$, there exists $x_3 \in A_0$ such that

$$D(gx_2, Tx_1) = d(A, B),$$
$$D(gx_3, Tx_2) = d(A, B).$$

Again utilizing the P-property, we get

$$d(gx_2, gx_3) = \mathcal{H}(Tx_1, Tx_2).$$

Also, the pair of mappings (g, T) satisfies η_M-proximal contraction, we obtain

$$d(gx_2, gx_3) \leq \eta M(x_2, x_3, x_1, x_2),$$

where

$$\begin{aligned}
&M(x_2, x_3, x_1, x_2)\\
&\leq \max\Big\{d(gx_2, gx_3), \frac{D(gx_1, Tx_3) - \beta_*(gx_1, Tx_3)D(gx_2, Tx_3)}{\beta_*(gx_1, Tx_3)},\\
&\qquad D^*(gx_3, Tx_2), \frac{D(gx_2, Tx_2) - \beta_*(gx_2, Tx_2)D(gx_3, Tx_2)}{\beta_*(gx_2, Tx_2)}\Big\}\\
&\leq \max\Big\{d(gx_2, gx_3), \frac{\beta_*(gx_1, Tx_3)[d(gx_1, gx_2) + D(gx_2, Tx_3)] - \beta_*(gx_1, Tx_3)D(gx_2, Tx_3)}{\beta_*(gx_1, Tx_3)},\\
&\qquad D(gx_3, Tx_2) - d(A, B),\\
&\qquad \frac{\beta_*(gx_2, Tx_2)[d(gx_2, gx_3) + D(gx_3, Tx_2)] - \beta_*(gx_2, Tx_2)D(gx_3, Tx_2)}{\beta_*(gx_2, Tx_2)}\Big\}\\
&\leq \max\Big\{d(gx_2, gx_3), \frac{\beta_*(gx_1, Tx_3)d(gx_1, gx_2)}{\beta_*(gx_1, Tx_3)}, 0, \frac{\beta_*(gx_2, Tx_2)d(gx_2, gx_3)}{\beta_*(gx_2, Tx_2)}\Big\}\\
&\leq \max\{d(gx_2, gx_3), d(gx_1, gx_2), 0, d(gx_2, gx_3)\}.
\end{aligned}$$

That is

$$M(x_2, x_3, x_1, x_2) \leq \max\{d(gx_2, gx_3), d(gx_1, gx_2)\}.$$

If $\max\{d(gx_2, gx_3), d(gx_1, gx_2)\} = d(gx_2, gx_3)$, then the above inequality implies

$$d(gx_2, gx_3) < \eta d(gx_2, gx_3),$$

which is a contradiction. So we conclude that

$$\max\{d(gx_2, gx_3), d(gx_1, gx_2)\} = d(gx_1, gx_2).$$

This process can be continued further. Having chosen $x_n \in A_0$, it is clear that there exists an element $x_{n+1} \in A_0$. Since, x_{n+1} is a member of $T(A_0)$ which is contained in B_0 and A_0 is contained in $g(A_0)$ such that

$$\begin{aligned}
D(gx_n, Tx_{n-1}) &= d(A, B),\\
D(gx_{n+1}, Tx_n) &= d(A, B),
\end{aligned}$$

by using of property P, we obtain

$$d(gx_n, gx_{n+1}) = \mathcal{H}(Tx_n, Tx_{n-1})$$

and

$$d(gx_n, gx_{n+1}) \leq \eta M(x_n, x_{n+1}, x_{n-1}, x_n),$$

where

$$\begin{aligned}
&M(x_n, x_{n+1}, x_{n-1}, x_n)\\
&\leq \max\Big\{d(gx_n, gx_{n+1}), \frac{D(gx_{n-1}, Tx_{n+1}) - \beta_*(gx_{n-1}, Tx_{n+1})D(gx_n, Tx_{n+1})}{\beta_*(gx_{n-1}, Tx_{n+1})},\\
&\qquad D^*(gx_{n+1}, Tx_n), \frac{D(gx_n, Tx_n) - \beta_*(gx_n, Tx_n)D(gx_{n+1}, Tx_n)}{\beta_*(gx_n, Tx_n)}\Big\}\\
&\leq \max\Big\{d(gx_n, gx_{n+1}), D(gx_{n+1}, Tx_n) - d(A, B),\\
&\qquad \frac{\beta_*(gx_{n-1}, Tx_{n+1})[d(gx_{n-1}, gx_n) + D(gx_n, Tx_{n+1})] - \beta_*(gx_{n-1}, Tx_{n+1})D(gx_n, Tx_{n+1})}{\beta_*(gx_{n-1}, Tx_{n+1})},\\
&\qquad \frac{\beta_*(gx_n, Tx_n)[d(gx_n, gx_{n+1}) + D(gx_{n+1}, Tx_n)] - \beta_*(gx_n, Tx_n)D(gx_{n+1}, Tx_n)}{\beta_*(gx_n, Tx_n)}\Big\}\\
&\leq \max\Big\{d(gx_n, gx_{n+1}), 0, \frac{\beta_*(gx_{n-1}, Tx_{n+1})d(gx_{n-1}, gx_n)}{\beta_*(gx_{n-1}, Tx_{n+1})}, \frac{\beta_*(gx_n, Tx_n)d(gx_n, gx_{n+1})}{\beta_*(gx_n, Tx_n)}\Big\}\\
&\leq \max\{d(gx_n, gx_{n+1}), 0, d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}.
\end{aligned}$$

Hence,

$$M(x_n, x_{n+1}, x_{n-1}, x_n) \leq \max\{d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n)\}.$$

If $\max\{d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n)\} = d(gx_n, gx_{n+1})$, then the above inequality implies

$$d(gx_n, gx_{n+1}) < \eta d(gx_n, gx_{n+1}),$$

which is a contradiction. Hence, we have

$$\max\{d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n)\} = d(gx_{n-1}, gx_n).$$

Keeping with the same pattern, we assert that

$$d(gx_n, gx_{n+1}) < \eta^n d(gx_0, gx_1).$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. Since (X, d) is a complete extended b-metric space, for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
&d(gx_n, gx_m)\\
&\leq \beta(gx_n, gx_m)d(gx_n, gx_{n+1}) + \beta(gx_n, gx_m)d(gx_{n+1}, gx_m)\\
&\leq \beta(gx_n, gx_m)d(gx_n, gx_{n+1}) + \beta(gx_n, gx_m)\beta(gx_{n+1}, gx_m)d(gx_{n+1}, gx_{n+2})\\
&\quad + \beta(gx_n, gx_m)\beta(gx_{n+1}, gx_m)d(gx_{n+2}, gx_m)\\
&\leq \beta(gx_n, gx_m)d(gx_n, gx_{n+1}) + \beta(gx_n, gx_m)\beta(gx_{n+1}, gx_m)d(gx_{n+1}, gx_{n+2})\\
&\quad + \cdots + \beta(gx_n, gx_m)\beta(gx_{n+1}, gx_m)\cdots\beta(gx_{m-2}, gx_m)\beta(gx_{m-1}, gx_m)d(gx_{m-1}, gx_m)\\
&\leq \beta(gx_n, gx_m)\eta^n d(gx_0, gx_1) + \beta(gx_n, gx_m)\beta(gx_{n+1}, gx_m)\eta^{n+1} d(gx_0, gx_1)
\end{aligned}$$

$$+ \cdots + \beta(gx_n, gx_m)\beta(gx_{n+1}, gx_m) \cdots \beta(gx_{m-2}, gx_m)\beta(gx_{m-1}, gx_m)\eta^{m-1} d(gx_0, gx_1)$$
$$= d(gx_0, gx_1) \sum_{i=1}^{m-1} \eta^i \prod_{j=1}^{i} \beta(gx_i, gx_m).$$

Assume that

$$S_n = \sum_{i=1}^{n} \eta^i \prod_{j=1}^{i} \beta(gx_i, gx_m).$$

We can write

$$d(gx_n, gx_m) \leq [S_{m-1} - S_n]. \tag{3.2}$$

Using ratio test, we get

$$a_i = \eta^i \prod_{j=1}^{i} \beta(gx_i, gx_m), \quad \frac{a_{i+1}}{a_i} < \frac{1}{k}.$$

Further taking limit as $n \to \infty$ in inequality (3.2), we infer

$$\lim_{n\to\infty} d(gx_n, gx_m) = 0,$$

which implies that $\{gx_n\}$ is a Cauchy sequence in a complete extended b-metric space (X, d), hence it is convergent. Suppose that $\{gx_n\}$ converges to some x^* in A (as the set A is closed), assuring that the sequence $\{x_n\} \subseteq A_0$ since $\{x_n\} \to x^*$. Therefore, (g, T) is a pair of continuous mapping so one can write

$$d(gx^*, Tx^*) = d(A, B).$$

Therefore, x^* is a coincidence best proximity point of the pair of mapping (g, T).

Uniqueness: Suppose that there are two distinct coincidence best proximity points of (g, T) with $x \neq y$. Then $q = d(gx, gy) > 0$. Since $D(gx, Tx) = D(gy, Ty) = d(A, B)$, using the P-property, we conclude that $q = \mathcal{H}(Tx, Ty)$. Moreover the pair of mapping (g, T) satisfies η_M-proximal contraction, we obtain $q \leq \eta q$.

Hence, $\eta \geq 1$. Since $\eta \leq 1$, we conclude that $\eta = 1$, which is a contradiction again. Hence the uniqueness is certified.

Corollary 3.1 *Let $T : A \to \mathcal{CB}(B)$ and $\beta_* : X \times X \to [1, \infty)$ be mappings, where A is closed subset of an extended b-metric space (X, d) and the pair (A, B) satisfies the P-property with $T(A_0) \subseteq B_0$. If a continuous mapping T satisfies η_{M^*}-proximal contraction. Suppose that*

$$\lim_{n,m\to\infty} \beta(x_n, x_m) < \frac{1}{k}, \quad k \in (0, 1).$$

Then there exists a unique best proximity point of the mapping T.

Proof Taking into account the identity mapping $g = I_A$ (g is the identity on A), the proof can be completed on the similar lines as in Theorem 3.1.

Example 3.2 Let $X = \{0,1,2,3,4,5\}$, consider the function d given as $d(x,y) = d(y,x)$ and $d(x,x) = 0$, where

d	0	1	2	3	4	5
0	0	$\frac{1}{21}$	$\frac{1}{23}$	$\frac{1}{31}$	$\frac{1}{11}$	$\frac{1}{13}$
1	$\frac{1}{21}$	0	$\frac{1}{25}$	$\frac{1}{15}$	$\frac{1}{31}$	$\frac{1}{17}$
2	$\frac{1}{23}$	$\frac{1}{25}$	0	$\frac{1}{19}$	$\frac{1}{23}$	$\frac{1}{31}$
3	$\frac{1}{31}$	$\frac{1}{15}$	$\frac{1}{19}$	0	$\frac{1}{21}$	$\frac{1}{23}$
4	$\frac{1}{11}$	$\frac{1}{31}$	$\frac{1}{23}$	$\frac{1}{21}$	0	$\frac{1}{25}$
5	$\frac{1}{13}$	$\frac{1}{17}$	$\frac{1}{31}$	$\frac{1}{23}$	$\frac{1}{25}$	0

Take $\beta_* : X \times X \to [1, \infty)$ to be symmetric defined as $\beta(x,y) = 33x + 35y$. It is easy to see that (X, d) is an extended b-metric space. Suppose $A = \{0,1,2\}$ and $B = \{3,4,5\}$ are nonempty subsets of extended b-metric space (X, d). After simple calculation, $d(A,B) = \frac{1}{31}$ satisfies the P-property and $A_0 = A$ and $B_0 = B$. Now, consider the mappings T and g defined by

$$Tx = \begin{cases} 4, & \text{if } x = 0, \\ \{4,5\}, & \text{if } x \in \{1,2\} \end{cases}$$

and

$$gx = \begin{cases} 0, & \text{if } x = 2, \\ 1, & \text{if } x = 1, \\ 2, & \text{if } x = 0. \end{cases}$$

Clearly $T(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$. Now we have to show that the pair (g, T) satisfies η_M-proximal contraction

$$\mathcal{H}(Tx, Ty) \leq \eta M(u, v, x, y)$$

for all $u, v, x, y \in A$. Then

$$D(g1, T0) = d(A, B),$$
$$D(g1, T2) = d(A, B),$$
$$D(g0, T1) = d(A, B)$$

and

$$D(g0, T2) = d(A, B).$$

Case (i) If $D(g1,T0) = D(g1,T2) = d(A,B)$, then $u = 1, v = 1, x = 0$ and $y = 2$. After simple calculation,

$$\mathcal{H}(Tx,Ty) = \mathcal{H}(T0,T2) = 0.$$

Also

$$\begin{aligned} M(1,1,0,2) &= \max\Big\{d(g1,g1), \frac{D(go,T1) - \beta_*(go,T1)D(g1,T1)}{\beta_*(go,T1)}, \\ &\qquad D^*(g1,T1), \frac{D(g1,T2) - \beta_*(g1,T2)D(g1,T2)}{\beta_*(g1,T2)}\Big\} \\ &= \max\Big\{0, \frac{D(2,\{4,5\}) - \beta_*(2,\{4,5\})D(1,\{4,5\})}{\beta_*(2,\{4,5\})}, \\ &\qquad 0, \frac{D(1,\{4,5\}) - \beta_*(1,\{4,5\})D(1,\{4,5\})}{\beta_*(1,\{4,5\})}\Big\} \\ &= \max\Big\{0, \frac{-205}{6386}, 0, \frac{-172}{5363}\Big\} = 0, \end{aligned}$$

which shows that the pair (g,T) satisfies η_M-proximal contraction, since

$$0 \leq \eta(0).$$

Case (ii) If $D(g1,T0) = D(g0,T1) = d(A,B)$, then $u = 1, v = 0, x = 0$ and $y = 1$. Utilizing these values, we obtain

$$\mathcal{H}(Tx,Ty) = \mathcal{H}(T0,T1) = 0$$

and

$$\begin{aligned} M(1,0,0,1) &= \max\Big\{d(g1,g0), \frac{D(go,T0) - \beta_*(go,T0)D(g1,T0)}{\beta_*(go,T0)}, \\ &\qquad D^*(g0,T1), \frac{D(g1,T1) - \beta_*(g1,T1)D(g0,T1)}{\beta_*(g1,T1)}\Big\} \\ &= \max\Big\{\frac{1}{25}, \frac{D(2,4) - \beta_*(2,4)D(1,4)}{\beta_*(2,4)}, \\ &\qquad 0, \frac{D(1,\{4,5\}) - \beta_*(1,\{4,5\})D(1,\{4,5\})}{\beta_*(1,\{4,5\})}\Big\} \\ &= \max\Big\{\frac{1}{25}, \frac{-4707}{146878}, 0, \frac{-172}{5363}\Big\} = \frac{1}{25}. \end{aligned}$$

Then

$$0 \leq \eta\Big(\frac{1}{25}\Big).$$

This shows that the pair (g,T) satisfies η_M-proximal contraction.

Case (iii) If $D(g1,T0) = D(g0,T2) = d(A,B)$, then $u = 1,\ v = 0,\ x = 0$ and $y = 2$. We acquire

$$\mathcal{H}(Tx,Ty) = \mathcal{H}(T0,T2) = 0$$

and

$$\begin{aligned}M(1,0,0,2) &= \max\Big\{d(g1,g0), \frac{D(go,T0)-\beta_*(go,T0)D(g1,T0)}{\beta_*(go,T0)},\\ &\qquad D^*(g0,T1), \frac{D(g1,T2)-\beta_*(g1,T2)D(g0,T2)}{\beta_*(g1,T2)}\Big\}\\ &= \max\Big\{\frac{1}{25}, \frac{D(2,4)-\beta_*(2,4)D(1,4)}{\beta_*(2,4)}, 0, \frac{D(1,\{4,5\})-\beta_*(1,\{4,5\})D(2,\{4,5\})}{\beta_*(1,\{4,5\})}\Big\}\\ &= \max\Big\{\frac{1}{25}, \frac{-4707}{146878}, 0, \frac{-172}{5363}\Big\} = \frac{1}{25}.\end{aligned}$$

Hence, the required contraction is satisfied in this case for

$$0 \le \eta\Big(\frac{1}{25}\Big).$$

Case (iv) If $D(g1,T2) = D(g0,T1) = d(A,B)$, then $u = 1,\ v = 0,\ x = 2$ and $y = 1$. We infer

$$\mathcal{H}(Tx,Ty) = \mathcal{H}(T2,T1) = 0$$

and

$$\begin{aligned}M(1,0,2,1) &= \max\Big\{d(g1,g0), \frac{D(g2,T0)-\beta_*(g2,T0)D(g1,T0)}{\beta_*(g2,T0)},\\ &\qquad D^*(g0,T1), \frac{D(g1,T1)-\beta_*(g1,T1)D(g0,T1)}{\beta_*(g1,T1)}\Big\}\\ &= \max\Big\{\frac{1}{25}, \frac{D(0,4)-\beta_*(0,4)D(1,4)}{\beta_*(0,4)}, 0, \frac{D(1,\{4,5\})-\beta_*(1,\{4,5\})D(2,\{4,5\})}{\beta_*(1,\{4,5\})}\Big\}\\ &= \max\Big\{\frac{1}{25}, \frac{-1509}{47740}, 0, \frac{-172}{5363}\Big\} = \frac{1}{25}.\end{aligned}$$

This shows that the pair (g,T) satisfies η_M-proximal contraction for

$$0 \le \eta\Big(\frac{1}{25}\Big).$$

Case (v) If $D(g1,T2) = D(g0,T2) = d(A,B)$, then $u = 1,\ v = 0,\ x = 2$ and $y = 2$. Calculating different terms, we get

$$\mathcal{H}(Tx,Ty) = \mathcal{H}(T2,T2) = 0$$

and

$$\begin{aligned}M(1,0,2,2) &= \max\Big\{d(g1,g0), \frac{D(g2,T0)-\beta_*(g2,T0)D(g1,T0)}{\beta_*(g2,T0)},\\ &\qquad D^*(g0,T1), \frac{D(g1,T2)-\beta_*(g1,T2)D(g0,T2)}{\beta_*(g1,T2)}\Big\}\\ &= \max\Big\{\frac{1}{25}, \frac{D(0,4)-\beta_*(0,4)D(1,4)}{\beta_*(0,4)}, 0, \frac{D(1,\{4,5\})-\beta_*(1,\{4,5\})D(2,\{4,5\})}{\beta_*(1,\{4,5\})}\Big\}\\ &= \max\Big\{\frac{1}{25}, \frac{-1509}{47740}, 0, \frac{-172}{5363}\Big\} = \frac{1}{25}.\end{aligned}$$

Hence the validity of the contraction condition is certified, as

$$0 \le \eta\Big(\frac{1}{25}\Big).$$

Case (vi) If $D(g0, T1) = D(g0, T2) = d(A, B)$, then $u = 0, v = 0, x = 1$ and $y = 2$. By using the given values, we get

$$\mathcal{H}(Tx, Ty) = \mathcal{H}(T1, T2) = 0$$

and

$$\begin{aligned} M(0,0,1,2) &= \max\Big\{d(g0, g0), \frac{D(g1, T0) - \beta_*(g1, T0)D(g0, T0)}{\beta_*(g1, T0)}, \\ &\qquad D^*(g0, T0), \frac{D(g0, T2) - \beta_*(g0, T2)D(g0, T2)}{\beta_*(g0, T2)}\Big\} \\ &= \max\Big\{0, \frac{D(1,4) - \beta_*(1,4)D(2,4)}{\beta_*(1,4)}, \frac{8}{713}, \frac{D(2,\{4,5\}) - \beta_*(2,\{4,5\})D(2,\{4,5\})}{\beta_*(2,\{4,5\})}\Big\} \\ &= \max\Big\{0, \frac{-5340}{123349}, \frac{8}{713}, \frac{-205}{6386}\Big\} = \frac{8}{713}. \end{aligned}$$

This amounts to say that

$$0 \le \eta\Big(\frac{8}{713}\Big).$$

This asserts that the pair (g, T) satisfies an η_M-proximal contraction.

Consequently, $D(g1, T1) = d(A, B)$, so 1 is the unique coincidence point of the pair of mapping (g, T). Therefore, Theorem 3.1 is validated.

Remark 3.1 It may be noted that the pair of mapping (g, T) does not satisfy the proximal conditions of the main results in [14, 28] within the realm of extended b-metric spaces. Hence the results of [14, 28] cannot be applied to the pair (g, T), showing that under the context of extended b-metric spaces, our generalizations are applicable and generalizations in the true sense.

4 Proximal Type Coincidence Point Results for Single-Valued Mappings

This section is devoted to discussing some coincidence best proximity point theorems for single-valued mappings in the framework of extended b-metric spaces.

Definition 4.1 *Let $T : A \to B$ and $g : A \to A$ be mappings. The pair (g, T) is said to be an η_S-proximal contraction if there exists a real number $\eta \in [0, 1)$ such that*

$$\left.\begin{aligned} d(gu, Tx) = d(A, B) \\ d(gv, Ty) = d(A, B) \end{aligned}\right\} \quad \textit{implies} \quad d(Tx, Ty) \le \eta\, M(u, v, x, y),$$

where

$$M(u, v, x, y) = \max\Big\{d(gu, gv), \frac{d(gx, Tv) - \beta(gx, Tv)d(gu, Tv)}{\beta(gx, Tv)},$$

$$d^*(gv, Tu), \frac{d(gu, Ty) - \beta(gu, Ty)d(gv, Ty)}{\beta(gu, Ty)}\Big\}$$

for all u, v, x, y in A.

Definition 4.2 *Let $T : A \to B$ and $g : A \to A$ be mappings. The pair (g, T) is said to be an η_{S^*}-proximal contraction if there exists a real number $\eta \in [0, 1)$ such that*

$$\left.\begin{aligned} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{aligned}\right\} \quad \text{implies} \quad d(Tx, Ty) \leq \eta M(u, v, x, y),$$

where

$$M(u, v, x, y) = \max\Big\{d(u, v), \frac{d(x, Tv) - \beta(x, Tv)d(u, Tv)}{\beta(x, Tv)},$$
$$d^*(v, Tu), \frac{d(u, Ty) - \beta(u, Ty)d(v, Ty)}{\beta(u, Ty)}\Big\}$$

for all u, v, x, y in A.

Note that, if we take $g = I_A$ (g as an identity mapping on A), then every η_S-proximal contraction reduces to an η_{S^*}-proximal contraction.

Example 4.1 Let $X = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\right\}$, and consider the function d given as $d(x, y) = d(y, x)$ and $d(x, x) = 0$, where

d	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
$\frac{1}{2}$	0	2	3	1	4	3
$\frac{1}{3}$	2	0	4	2	1	5
$\frac{1}{4}$	3	4	0	5	6	1
$\frac{1}{5}$	1	2	5	0	6	5
$\frac{1}{6}$	4	1	6	6	0	3
$\frac{1}{7}$	3	5	1	5	3	0

Take $\beta_* : X \times X \to [1, \infty)$ to be symmetric defined as $\beta(x, y) = 38x + 42y$. It is easy to see that (X, d) is extended b-metric space. Suppose $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and $B = \{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}\}$ are nonempty subsets of extended b-metric space (X, d). After simple calculation, $d(A, B) = 1$, $A_0 = A$ and $B_0 = B$. As we know that $T : A \to \mathcal{CB}(B)$,

$$Tx = \begin{cases} \frac{1}{5}, & \text{if } x \in \left\{\frac{1}{4}, \frac{1}{2}\right\}, \\ \frac{1}{6}, & \text{if } x = \frac{1}{3} \end{cases}$$

and $g : A \to A$,

$$gx = \begin{cases} \dfrac{1}{2}, & \text{if } x = \dfrac{1}{4}, \\ \dfrac{1}{3}, & \text{if } x = \dfrac{1}{3}, \\ \dfrac{1}{4}, & \text{if } x = \dfrac{1}{2}. \end{cases}$$

Now we have to show that the pair (g, T) satisfies η_M-proximal contraction

$$d(Tx, Ty) \leq \eta M(u, v, x, y)$$

for all $u, v, x, y \in A$. Now,

$$\left.\begin{aligned} d\Big(g\frac{1}{4}, T\frac{1}{2}\Big) = d(A, B) \\ d\Big(g\frac{1}{2}, T\frac{1}{4}\Big) = d(A, B) \end{aligned}\right\} \quad \text{implies } d\Big(T\frac{1}{2}, T\frac{1}{4}\Big) \leq \eta M(u, v, x, y).$$

Therefore, $d(T\frac{1}{2}, T\frac{1}{4}) = 0$ for every $\eta \in [0, 1)$, Definition 4.1 is satisfied.

If we take $g = I_A$ (g as an identity mapping on A), then the same example holds for Definition 4.2.

Theorem 4.1 *Let $T : A \to B$, $g : A \to A$ and $\beta : X \times X \to [1, \infty)$ be mappings, where A is a closed subset of an extended b-metric space (X, d) and the pair (A, B) satisfies the P-property with $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$. If a pair of continuous mappings (g, T), where g is one-to-one satisfies η_S-proximal contraction and suppose that*

$$\lim_{n,m\to\infty} \beta(gx_n, gx_m) < \frac{1}{k}, \quad k \in (0, 1),$$

then there exists coincidence best proximity points of the pair (g, T).

Proof Since every single-valued mapping is a multivalued mapping, the remaining proof is the same as that of Theorem 3.1.

Corollary 4.1 *Let $T : A \to B$ and $\beta : X \times X \to [1, \infty)$ be mappings, where A is a closed subset and the pair (A, B) satisfies the P-property with $T(A_0) \subseteq B_0$. If a continuous mapping T satisfies η_{S^*}-proximal contraction. Suppose that*

$$\lim_{n,m\to\infty} \beta(x_n, x_m) < \frac{1}{k}, \quad k \in (0, 1).$$

Then there exist best proximity points of the mapping T.

Proof If we take identity mapping $g = I_A$ (g is identity on A), then the remaining proof is same as Theorem 4.1.

Corollary 4.2 *Let (X, d) be an extended b-metric space and $T : X \to X$, $\beta : X \times X \to [1, \infty)$ be mappings. If $\lim_{n,m\to\infty} \beta(x_n, x_m) < \frac{1}{k}$, where $k \in (0, 1)$, T is continuous and there exists a real number $\eta \in [0, 1)$ such that the following η_{S^*}-proximal type contraction is satisfied*

$$d(Tx, Ty) \leq \eta\, M(u, v, x, y),$$

where

$$M(u,v,x,y)=\max\Big\{d(u,v),\frac{d(x,Tv)-\beta(x,Tv)d(u,Tv)}{\beta(x,Tv)},$$
$$d^*(v,Tu),\frac{d(u,Ty)-\beta(u,Ty)d(v,Ty)}{\beta(u,Ty)}\Big\}$$

for all u,v,x,y in X, then there exists a unique fixed point of the mapping T.

Example 4.2 Consider $X=\{11,12,13,14,15,16\}$ to be a complete extended b-metric space (X,d), if

$$d(x,y)=(x-y)^2,\quad x,y\in X.$$

Also, suppose that $A=\{11,13,15\}$ and $B=\{12,14,16\}$ are nonempty subsets of X. After simple calculations, $d(A,B)=1$ satisfies the P-property, where $A_0=A$, $B_0=B$ and $\beta(x,y)=18xy+8$. Now, we define a mapping $T:A\to B$ as follows:

$$Tx=\begin{cases}12, & \text{if } x\in\{11,13\},\\ 14, & \text{if } x=15\end{cases}$$

and $g:A\to A$,

$$gx=\begin{cases}11, & \text{if } x=11,\\ 13, & \text{if } x=15,\\ 15, & \text{if } x=13.\end{cases}$$

Clearly, T$(A_0)\subseteq B_0$ and $g(A_0)\subseteq A_0$. Now, we have to show that the pair (g,T) satisfies η_S-proximal contraction

$$d(Tx,Ty)\le\eta M(u,v,x,y)\quad \text{for all } u,v,x,y\in A.$$

Since

$$d(g11,T13)=d(A,B),$$
$$d(g13,T15)=d(A,B),$$

where $u=11$, $v=13$, $x=13$ and $y=15$, after routine calculation, we obtain

$$d(T13,T15)=d(12,14)=4$$

and

$$\begin{aligned}M(11,13,13,15)&=\max\Big\{d(g11,g13),\frac{d(g13,T13)-\beta(g13,T13)d(g11,T13)}{\beta(g13,T13)},\\&\qquad d^*(g13,T11),\frac{d(g11,T15)-\beta(g11,T15)d(g13,T15)}{\beta(g11,T15)}\Big\}\\&=\max\Big\{d(11,15),\frac{d(15,12)-\beta(15,12)d(11,12)}{\beta(15,12)},\\&\qquad d^*(15,12),\frac{d(11,14)-\beta(11,14)d(15,14)}{\beta(11,14)}\Big\}\end{aligned}$$

$$= \max\left\{16, \frac{-3239}{3248}, 8, \frac{-2771}{2780}\right\} = 16.$$

Consequently, we have

$$4 \leq \eta(16)$$

for $\eta = \frac{1}{3}$, satisfying η_S-proximal contraction. Hence $d(g11, T11) = d(A, B)$. So 11 is the unique coincidence point of the pair of mapping (g, T). Therefore, all the conditions of Theorem 4.1 are contended.

5 Applications of the Obtained Results

Let $X = \mathcal{C}[a, b]$ be a set of all real valued continuous functions on $[a, b]$. Define two mappings $d : X \times X \to [0, +\infty)$ by

$$d(x, y) = \sup_{c\in[a,b]} |x(c) - y(c)|^p$$

for all $x, y \in X$ and $e(x, y) = r + x + y,\ p \geq 2,\ r > 2$. Then, (X, d) is a complete extended b-metric space. Consider the Fredholm integral equation given by

$$x(t) = \eta(t) + \lambda \int_a^b \mathcal{I}(t, s, x(s))\mathrm{d}s, \tag{5.1}$$

where $t \in [a, b]$, $|\lambda| > 0$ and $\mathcal{I} : [a, b] \times [a, b] \times X \to \Re$ and $\eta : [a, b] \to \Re$ are continuous functions. Let $T : X \to X$ be an integral operator defined by

$$Tx(t) = f(t) + \lambda \int_a^b \mathcal{I}(t, s, x(s))\mathrm{d}s. \tag{5.2}$$

Then $x(t)$ is a fixed point of T if and only if it is a solution of the Fredholm integral equation (5.1).

We now offer the following subsequent theorem to establish the existence of a solution to the Fredholm integral equation (5.1).

Theorem 5.1 *Let $T : X \to X$ be an integral operator defined in* (5.2)*. Suppose that the following assumptions hold*:

(1) *For any $x_0 \in X$, $\lim_{n,m\to\infty} \beta(T^n x_0, T^m x_0) < \frac{1}{\eta}$, where $\eta = \frac{1}{2^p}$.*

(2) *For any $x, y \in X, x \neq y$, $\mathcal{I} : [a, b] \times [a, b] \times X \to \Re$ satisfies*

$$|\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))| \leq \xi(t, s)|x(s) - y(s)|,$$

where $(s, t) \in [a, b] \times [a, b]$ and $\xi : [a, b] \times [a, b] \to \Re$ is a continuous function satisfying

$$\sup_{t\in[a,b]} \int_a^b \xi^p(t, s)|x(s) - y(s)|^p \mathrm{d}s \leq \frac{1}{2^p|\lambda|^p(b - a)^{p-1}} M(t, s, x, y), \tag{5.3}$$

where $M(t, s, x, y)$ is defined as in Definition 4.2*.*

Then, the integral operator T has a unique solution in X.

Proof Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$, $n \geq 1$. From (5.2), we obtain

$$x_{n+1} = Tx_n(t) = L(t) + \lambda \int_a^b \mathcal{I}(t, s, x_n(s))\mathrm{d}s. \tag{5.4}$$

Let $q > 1$ be a constant with $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder's inequality, we speculate that

$$\begin{aligned}
|Tx(t) - Ty(t)|^p &= \Big|\lambda \int_a^b \mathcal{I}(t, s, x(s))\mathrm{d}s - \lambda \int_a^b \mathcal{I}(t, s, y(s))\mathrm{d}s\Big|^p \\
&\leq \Big(\int_a^b |\lambda||\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))|\mathrm{d}s\Big)^p \\
&\leq \Big(\int_a^b |\lambda|^q \mathrm{d}s\Big)^{\frac{p}{q}} \Big(\Big(\int_a^b |\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))|^p \mathrm{d}s\Big)^{\frac{1}{p}}\Big)^p \\
&= |\lambda|^p (b-a)^{p-1} \Big(\int_a^b |\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))|^p \mathrm{d}s\Big) \\
&\leq |\lambda|^p (b-a)^{p-1} \int_a^b \xi^p(t, s)|x(s) - y(s)|^p \mathrm{d}s.
\end{aligned}$$

And we deduce that

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t \in [a,b]} |Tx(t) - Ty(t)|^p \\
&\leq |\lambda|^p (b-a)^{p-1} \sup_{t \in [a,b]} \Big[\int_a^b \xi^p(t, s)|x(s) - y(s)|^p \mathrm{d}s\Big] \\
&\leq \frac{1}{2^p} \mathcal{M}(t, s, x, y).
\end{aligned}$$

Setting $\eta = \frac{1}{2^p}$, we obtain that

$$d(Tx, Ty) \leq \eta\, \mathcal{M}(t, s, x, y). \tag{5.5}$$

Thus, all the conditions of Corollary 4.2 are satisfied and hence T possesses a unique fixed point in X.

5.1 An application to the solution of a second-order differential equation

The existence of a solution for the preceding second order boundary value problem is manifested in this section:

$$\begin{cases} u''(t) = W(t, u(t), s(t)), \quad t \in [0, 1], \\ u(0) = u_0, \ u(1) = u_1, \end{cases} \tag{5.6}$$

where $W : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Firstly, consider the space $X = C(\mathbb{Q})\,(\mathbb{Q} = [0, 1], \mathbb{R})$ of continuous functions defined on $\mathbb{Q}$. Obviously, this space with metric given by

$$d(u, v) = \sup_{c \in [a,b]} |u(c) - v(c)|^p$$

for all $u, v \in X$ and $e(u, v) = r + u + v,\ p \geq 2,\ r > 2$ is a complete extended b-metric space.

Theorem 5.2 *Consider problem* (5.6)*. Suppose that for any* $x_0 \in X$, $\lim_{n,m\to\infty} \beta(T^n x_0, T^m x_0) < \frac{1}{\eta}$, *where* $\eta = \frac{1}{2^p}$, *and for suitable value of* λ *if* $|\lambda|^p \le \frac{1}{2^p}$, *then, the second order boundary value problem* (5.6) *has a unique solution.*

Proof Problem (5.6) is equivalent to the second kind Fredholm integral equation

$$u(t) = L(t) + \lambda \int_0^1 U^*(t,s)u(s)\mathrm{d}s, \quad t \in [0,1], \tag{5.7}$$

in which $L(t) = u_0 + t(u_1 - u_0)$ and $U^*(t,s)$ is the Green's function, given by

$$U^*(t,s) = \begin{cases} s(1-s), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1. \end{cases} \tag{5.8}$$

Note that if $u \in C(\mathbb{Q})$ is a fixed point of T, then u is a solution of (5.7), consequently a solution of (5.6).

Let $u, v \in C(\mathbb{Q})$ and $q > 1$ be a constant with $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder's inequality, we acquire that

$$\begin{aligned}
|Tu(t) - Tv(t)|^p &= \Big|\lambda \int_0^1 U^*(t,s)u(s)\mathrm{d}s - \lambda \int_0^1 U^*(t,s)v(s)\mathrm{d}s\Big|^p \\
&\le \Big(\int_0^1 |\lambda||U^*(t,s)u(s) - U^*(t,s)v(s)|\mathrm{d}s\Big)^p \\
&\le \Big(\int_0^1 |\lambda|^q \mathrm{d}s\Big)^{\frac{p}{q}} \Big(\Big(\int_0^1 |U^*(t,s)u(s) - U^*(t,s)v(s)|^p \mathrm{d}s\Big)^{\frac{1}{p}}\Big)^p \\
&= |\lambda|^p \Big(\int_0^1 |U^*(t,s)u(s) - U^*(t,s)v(s)|^p \mathrm{d}s\Big) \\
&\le |\lambda|^p |u(t) - v(t)|^p \Big(\int_0^1 |U^*(t,s)|^p \mathrm{d}s\Big).
\end{aligned}$$

And we conclude that

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[a,b]} |Tx(t) - Ty(t)|^p \\
&\le |\lambda|^p \sup_{t\in[a,b]} |u(t) - v(t)|^p \Big(\int_0^1 |U^*(t,s)|^p \mathrm{d}s\Big) \\
&\le |\lambda|^p d(u,v), \qquad (\text{for any value of } p \text{ and utilizing (5.8)}) \\
&\le \frac{1}{2^p}\mathcal{M}(t,s,x,y).
\end{aligned}$$

Fixing $\eta = \frac{1}{2^p}$, we get

$$d(Tx, Ty) \le \eta \mathcal{M}(t,s,x,y). \tag{5.9}$$

Hence, we conclude that the proximal condition of Corollary 4.2 is satisfied authenticating T has a unique fixed point, which is the solution of integral equation (5.7). Consequently, the differential equation (5.6) has a solution.

6 Conclusion

In this work, we give an idea of some novel coincidence best proximity point results for proximal contractions in the setting of extended b-metric spaces. Also, some fixed point results as a special cases in the setting of single-valued and multivalued mappings are studied. An example is discussed to show the significance of the investigation of this paper. Some applications of nonlinear differential and integral equations are presented to illustrate the usability of the new theory. Now, we pose the following open problem.

Future scope.

• Can the results presented in this article be extended to the framework of extended b-metric spaces endowed with graphs or in graphical metric spaces?

• Can the findings shown in this article be used to generate a solution to the following semi-linear operator system of the form:

$$\begin{cases} \mathcal{U}(\alpha,\beta)=\alpha, \\ \mathcal{V}(\alpha,\beta)=\beta, \end{cases}$$

where $\mathcal{U},\mathcal{V}:P\times P\to P$ are nonlinear operators defined on a Banach space $(P,\|\cdot\|)$?

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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