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Non-relativistic Limit from Complex Klein-Gordon Equation to a System of Coupled Nonlinear Schrödinger Equations in 2D and 3D*

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Abstract In this paper, the authors consider the non-relativistic limit of solutions to the defocusing cubic nonlinear Klein-Gordon equations. Inspired by the work of Lei and Wu (2023), which considered the real-valued case, the authors show that the solutions to the complex Klein-Gordon equation can be described by using a system of two coupled nonlinear Schrödinger equations as the speed of light tends to infinity both in two and three dimensions. On the one hand, if the initial data belong to a lower-regularity Sobolev space $H^\alpha, \alpha \in [1, 4]$, the error is proportionate to the α order of the reciprocal of the speed of light, with a constant that grows linearly in time. On the other hand, if initial data are smoother (at least in H^4), the error is proportionate to the square of the reciprocal of the speed of light, with a constant that grows linearly in time. The error of two different regularity requirements holds globally over time.

Keywords Nonlinear Klein-Gordon equations, Nonlinear Schrödinger equations, Non-relativistic limit, Convergence rate
2020 MR Subject Classification 35L70, 35Q55, 35B25, 35B40

1 Introduction

1.1 Background

In this paper, we consider the defocusing cubic nonlinear Klein-Gordon equation, which reads

$$\frac{\hbar^2}{mc^2}\partial_{tt}u - \frac{\hbar^2}{m}\Delta u + mc^2 u + |u|^2 u = 0, \tag{1.1}$$

where $u = u(t, x) : \mathbb{R}^{1+d} \to \mathbb{C}$ is a complex-valued function, c denotes the speed of light, $\hbar$ is the Planck constant, and $m > 0$ is the mass of particles. The Klein-Gordon equation, a relativistic counterpart to the Schrödinger equation, accurately describes high-speed particles and those with spin 0. In the non-relativistic limit, the relativistic correction terms in the Klein-Gordon equation tend to zero, leaving behind the part corresponding to the Schrödinger equation. Exploring its non-relativistic limit is crucial in quantum physics and computational mathematics. In physics, it enhances the understanding of quantum mechanics, especially

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in scenarios involving spin. In computational mathematics, since the Klein-Gordon equation is relativistic, solving it numerically is more complex than that of the Schrödinger equation, and exploring the non-relativistic limit can yield more efficient numerical methods for solving quantum problems involving spinning particles.

In this paper, we focus on the convergence rate in the non-relativistic regime ($\varepsilon \to 0$) of the nonlinear Klein-Gordon equation (1.1). The initial data are given as

$$u(0,x) \triangleq u_0 = u_0(x), \quad \partial_t u(0,x) \triangleq u_1^\varepsilon = \frac{u_1}{\varepsilon^2}, \tag{1.2}$$

where u_0 and u_1 are complex-valued functions satisfying certain regularity assumptions.

The non-relativistic limit of (1.1) has attracted significant attention in mathematical analysis, as evidenced by works such as [2, 7, 9–14] and references therein. In particular, Masmoudi and Nakanishi [11] demonstrated that a broad class of solutions to (1.1) can be characterized by a system of coupled nonlinear Schrödinger equations. They established an error estimate of $\mathcal{O}(\varepsilon^{\frac{1}{2}})$ within an L^2 framework, assuming finite energy and L^2 convergence of the initial data. It is worth noting that the estimates obtained in [11] are valid for some fixed time interval. Furthermore, Bao, Lu and Zhang [2] extended the study to the three-dimensional problem and demonstrated that for any

$$(u_0, u_1) \in (H^{s+4\alpha}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))^2$$

with $s > \frac{3}{2}$, the error estimate is $\mathcal{O}((1+t)\varepsilon^2)$ for $\alpha = 1, 2$. Their estimate is valid for times up to $\mathcal{O}(\varepsilon^{-\frac{1}{2}\alpha})$.

Recently, Lei and Wu [7] achieved a significant breakthrough in the non-relativistic limit problem. They conducted a thorough exploration of the fundamental aspects of the equation, yielding superior results in terms of convergence order, minimal regularity optimization, and validity time interval. They proved that for any

$$(u_0, u_1) \in H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$$

with $1 \le \alpha \le 4$, the error estimate is $\mathcal{O}(\varepsilon^2 + (\varepsilon^2 t)^{\frac{1}{4}\alpha})$, and the estimate holds for all $t \ge 0$. Furthermore, their paper demonstrates that the convergence estimate and the regularity requirements they derived are optimal.

Despite the comprehensive and in-depth investigation into convergence speed estimates, necessary regularity of initial data, validity time intervals, and the underlying mechanism as outlined in [7], along with the provision of optimal estimates, their setup restricts the solution u to being a real-valued function. However, as shown in the following, to estimate the complex-valued function, one needs to handle more complex Schrödinger equation systems. In this paper, we will follow the method proposed in [7] to extend their results to the case where u is a complex-valued function, aiming to enhance the classical results presented in [11].

1.2 Main result

Without loss of generality, we will set the constants as $\hbar = m = 1$ and $c = \varepsilon^{-1}$ in the rest of the paper. Then (1.1) can be reduced to

$$\begin{cases} \varepsilon^2 \partial_{tt} u^\varepsilon - \Delta u^\varepsilon + \dfrac{1}{\varepsilon^2} u^\varepsilon + |u^\varepsilon|^2 u^\varepsilon = 0, \\ u^\varepsilon(0,x) \triangleq u_0^\varepsilon = u_0(x), \quad \partial_t u^\varepsilon(0,x) \triangleq u_1^\varepsilon = \dfrac{u_1}{\varepsilon^2}. \end{cases} \tag{1.3}$$

Then we apply the following decomposition to the solution of (1.3):

$$u^\varepsilon = \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+ + \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_- + r^\varepsilon. \tag{1.4}$$

The decomposition (1.4) is called the WKB decomposition, which has been widely used in non-relativistic limit studies (see [1, 11] and the references therein). The profile functions u_+, u_- are usually called modulated wave functions of u^ε. This decomposition can eliminate the time oscillations of the wave function. It is worth noting that in the case where u^ε is a complex-valued function, there is a certain difference between the WKB decompositions in the complex-valued case and the real-valued case. In the case where u^ε is a real valued function, the WKB decomposition takes the following form

$$u^\varepsilon = \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} v + \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{v} + r^\varepsilon, \tag{1.5}$$

where $v, \overline{v}$ are a pair of conjugate functions. However, in (1.4), u_+ and u_- are not necessarily required to be equal. This difference makes it difficult to directly generalize the results in the case of real valued functions to the case of complex functions, which we will further explain in the following text.

Inserting (1.4) into (1.3), we have

$$\begin{aligned}
&\varepsilon^2\partial_{tt} r^\varepsilon - \Delta r^\varepsilon + \frac{1}{\varepsilon^2} r^\varepsilon + \Phi(u_+, u_-, r^\varepsilon) + \Psi(u_+, u_-) + \Omega(\partial_{tt}u_+, \partial_{tt}u_-)\\
&+ \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}[2\mathrm{i}\partial_t u_+ - \Delta u_+ + |u_+|^2 u_+ + 2|u_-|^2 u_+]\\
&+ \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}[-2\mathrm{i}\partial_t \overline{u}_- - \Delta \overline{u}_- + |u_-|^2 \overline{u}_- + 2|u_+|^2 \overline{u}_-] = 0,
\end{aligned} \tag{1.6}$$

where

$$\begin{aligned}
\Phi(u_+, u_-, r^\varepsilon) &= 2r^\varepsilon |\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+ + \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-|^2 + \overline{r^\varepsilon}(\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+ + \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-)^2\\
&\quad + (r^\varepsilon)^2(\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_+ + \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_-) + 2|r^\varepsilon|^2(\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+ + \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-) + |r^\varepsilon|^2 r^\varepsilon,\\
\Psi(u_+, u_-) &= \mathrm{e}^{\frac{3\mathrm{i}t}{\varepsilon^2}} (u_+)^2 u_- + \mathrm{e}^{-\frac{3\mathrm{i}t}{\varepsilon^2}} (\overline{u}_-)^2 \overline{u}_+,\\
\Omega(\partial_{tt}u_+, \partial_{tt}u_-) &= \varepsilon^2[\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} \partial_{tt}u_+ + \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \partial_{tt}\overline{u}_-].
\end{aligned}$$

Formally, we may set u_+, u_- to be the $\mathbb{C}^2$-valued solutions of the following system of nonlinear Schrödinger equations:

$$\begin{cases}
2\mathrm{i}\partial_t u_+ - \Delta u_+ + |u_+|^2 u_+ + 2|u_-|^2 u_+ = 0,\\
2\mathrm{i}\partial_t u_- - \Delta u_- + |u_-|^2 u_- + 2|u_+|^2 u_- = 0,\\
u_+ = u_{+,0} = \frac{1}{2}(u_0 + \mathrm{i}u_1), \quad u_- = u_{-,0} = \frac{1}{2}(\overline{u_0} - \mathrm{i}\overline{u_1}).
\end{cases} \tag{1.7}$$

It is obvious that, in general, $u_+ \neq u_-$, if u_0 is a complex-valued function. In this case, the initial data for r^ε are

$$r^\varepsilon(0) = 0, \quad \partial_t r^\varepsilon(0) \triangleq r_1 = -\partial_t u_{+,0} - \partial_t u_{-,0}, \tag{1.8}$$

and r^ε satisfies the following equation:

$$\varepsilon^2\partial_{tt} r^\varepsilon - \Delta r^\varepsilon + \frac{1}{\varepsilon^2} r^\varepsilon + \Phi(u_+, u_-, r^\varepsilon) + \Psi(u_+, u_-) + \Omega(\partial_{tt}u_+, \partial_{tt}u_-) = 0. \tag{1.9}$$

At this juncture, our primary objective shifts towards investigating the remainder function r^ε, which tends to zero in the non-relativistic limit $\varepsilon \to 0$. We also aim to establish estimates for r^ε in terms of a power of ε, along with the necessary regularity of the initial data and the validity time interval.

As previously mentioned, the WKB decomposition of complex-valued functions and real-valued functions takes different forms, resulting in distinct structures for their corresponding Schrödinger equations. When u is a real function, the profile function satisfying (1.5) consists of conjugate functions, simplifying the equations they satisfy to either the standard defocusing cubic nonlinear Schrödinger-wave equation or the standard defocusing cubic nonlinear Schrödinger equation.

However, in the case where u is a complex function, the equation satisfied by the profile function becomes a more complex Schrödinger system (1.7). This complexity poses the most significant challenge when extending the results from [7] to the case of complex-valued functions. The standard equations satisfied by the profile functions in the case of real-valued functions have been thoroughly studied, and a complete theory of well-posedness and scattering exists. However, for complex-valued functions, it is essential to establish relevant estimates for the solutions of the Schrödinger system. Fortunately, the methods presented in [4] equip us with ample tools to develop a scattering theory for the solutions.

To estimate r^ε, we apply a scaling transformation S_ε to (1.6), where $S_\varepsilon f = \varepsilon f(\varepsilon^2 t, \varepsilon x)$. This transforms the main operator part of the equation into the standard form of the Klein-Gordon equation. Utilizing the scattering theory for the system (1.7), we can easily estimate $\Phi(u_+, u_-, r^\varepsilon)$ and $\Omega(\partial_{tt}u_+, \partial_{tt}u_-)$. To estimate $\Psi(u_+, u_-)$, under the scaling transformation and normal form transformation, $\Psi(u_+, u_-)$ is transformed into the following form:

$$\int_{t_0}^{t} \mathrm{e}^{\mathrm{i}(t-s)\langle\nabla\rangle} \langle\nabla\rangle^{-1} F(S_\varepsilon u_+, S_\varepsilon u_-) \mathrm{d}s. \tag{1.10}$$

It is worth noting that the estimate for the above integral is straightforward when its integrand is operated by an operator $P_{\geq 1}$. Frequency-localized operators will be defined in Section 2. The low-frequency component can be estimated using the technique of integration by parts.

Our final result will be proven in two steps. Firstly, we provide an estimate for r^ε for the regular case, where the initial values satisfy $(u_0, u_1) \in H^{4+s_c}$. Utilizing the results from the regular case, we can easily obtain estimates for the low-frequency part, as mentioned in (1.10). This allows us to extend the results to cases where $(u_0, u_1) \in H^\alpha$ for some $\alpha \in [1, 4]$. Our main result is outlined below.

Theorem 1.1 *Let u^ε and (u_+, u_-) be solutions to* (1.3) *and* (1.7)*, respectively. When $d = 2, 3,\ t > 0$, suppose that*

$$(u_0, u_1) \in H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d), \quad \alpha \in [1, 4].$$

Then the following error estimates

$$\|u^\varepsilon(t) - \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+(t) - \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-(t)\|_{L^2} \leq C(\varepsilon^\alpha + (\varepsilon^2 t)^{\frac{1}{4}\alpha}), \quad \text{when } \alpha \in [1, 2], \tag{1.11}$$

$$\|u^\varepsilon(t) - \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+(t) - \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-(t)\|_{L^2} \leq C(\varepsilon^2 + (\varepsilon^2 t)^{\frac{1}{4}\alpha}), \quad \text{when } \alpha \in [2, 4] \tag{1.12}$$

hold for all $t > 0$, where the generic constant C depends only on $\|(u_0, u_1)\|_{H^\alpha}$.

Remark 1.1 Compared to the results in [11], our study yields superior convergence estimates under the same regularity assumptions. Additionally, our results provide convergence estimates under more general regularity conditions. Furthermore, our findings are applicable to arbitrary time and two- or three-dimensional spaces, demonstrating a broader scope of applicability compared to previous research.

2 Preliminary

2.1 Notation

We denote $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. Let $C>0$ be a positive constant. If $f \le Cg$, we write $f \lesssim g$ or $f = \mathcal{O}(g)$.

For short, we write $L^p = L^p(\mathbb{R}^d)$, $H^s = H^s(\mathbb{R}^d)$, $L_t^q L_x^r(I) = L_t^q L_x^r(I\times\mathbb{R}^d)$ and $L_t^q H_x^s(I) = L_t^q H_x^s(I\times\mathbb{R}^d)$. We denote the norm

$$\|f\|_{X\cap Y} \triangleq \|f\|_X + \|f\|_Y.$$

We denote the norm

$$\|(f,g)\|_{X\times Y} \triangleq \|f\|_X + \|g\|_Y.$$

To simplify the notation, when two spaces X, Y are the same, we use $\|(f,g)\|_X$ instead of $\|(f,g)\|_{X\times X}$.

We use $\widehat{f}$ or $\mathcal{F}f$ to denote the Fourier transform of f:

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) \triangleq \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x.$$

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a radial, real-valued and smooth cut-off function such that $\chi \ge 0$ and χ satisfies

$$\chi(\xi) = \begin{cases} 1, & \text{when } |\xi| \le \frac{1}{2}, \\ 0, & \text{when } |\xi| \ge 1. \end{cases}$$

Moreover, we also need the following frequency cut-off operators: For any $N \in 2^{\mathbb{Z}}$,

$$\begin{aligned} &P_{\le N} f \triangleq \mathcal{F}^{-1}\Big(\chi\Big(\frac{\xi}{N}\Big)\widehat{f}(\xi)\Big), \\ &P_{>N} f \triangleq f - P_{\le N} f. \end{aligned}$$

2.2 Useful lemmas

First, we recall the following Strichartz estimates (see [6] and the references therein).

Lemma 2.1 (Strichartz estimate) *Let $I \subset \mathbb{R}$. Suppose that (q,r) and $(\widetilde{q},\widetilde{r})$ are L_x^2-admissible. Then*

$$\|\mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\psi\|_{L_t^q L_x^r(I)} \lesssim \|\langle\nabla\rangle^{\frac{d+2}{2}(\frac{1}{2}-\frac{1}{r})}\psi\|_{L_x^2} \tag{2.1}$$

and

$$\Big\|\int_0^t \mathrm{e}^{\mathrm{i}(t-s)\langle\nabla\rangle} F(s)\mathrm{d}s\Big\|_{L_t^q L_x^r(\mathbb{R})} \lesssim \|\langle\nabla\rangle^{\frac{d+2}{2}(1-\frac{1}{r}-\frac{1}{\widetilde{r}})} F(s)\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(\mathbb{R})}. \tag{2.2}$$

The Kato-Ponce inequality will be frequently used in this paper. The result was originally proved in [5] and then extended to the endpoint case in [3, 8] recently.

Lemma 2.2 (Kato-Ponce inequality) *For $s > 0$, $1 < p \le \infty$, $1 < p_1, p_3 < \infty$ and $1 < p_2, p_4 \le \infty$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p} = \frac{1}{p_3} + \frac{1}{p_4},$$

the following inequality holds:

$$\|J^s(fg)\|_{L^p} \le C(\|J^s(f)\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|J^s(g)\|_{L^{p_3}}\|f\|_{L^{p_4}}),$$

where the constant $C > 0$ depends on s, p, p_1, p_2, p_3, p_4.

We also need the following estimate for the solution of (1.7).

Lemma 2.3 (Scattering for the system of NLS) *Let $d = 2, 3$, $(u_0, u_1) \in H^s(\mathbb{R}^d)$ and let (u_+, u_-) be a solution of (1.7). Suppose that (q, r) is L^2_x-admissible. Then*

$$\|(|\nabla|^s u_+, |\nabla|^s u_-)\|_{L^q_t L^r_x(\mathbb{R})} \le C(\|(u_0, u_1)\|_{H^{d-2}})\|(u_0, u_1)\|_{\dot{H}^s} \quad \textit{for any } s \ge 0. \tag{2.3}$$

The proof of Lemma 2.3 can be followed from the method in [4].

3 Regular Case

Proposition 3.1 *Let $d = 2, 3$ and $T > 0$. Assume that there exists $\delta_0 > 0$ such that*

(1) $(u_0, u_1) \in H^{4+s_c}$;

when $d = 3$, additionally assume that

(2) $\varepsilon^2 T\|(u_0, u_1)\|_{\dot{H}^{\frac{9}{2}}} \le \delta_0$.

Let $u^\varepsilon, (u_+, u_-)$ be the solutions to (1.3) and (1.7), respectively. Then

$$\sup_{t\in[0,T]} \|u^\varepsilon(t) - \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+(t) - \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-(t)\|_{L^2_x} \le C(\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2} + \varepsilon^2 T\|(u_0, u_1)\|_{\dot{H}^4}),$$

where the constant $C > 0$ only depends on $\|(u_0, u_1)\|_{H^1}$.

Proof Firstly, we define the scaling transformation

$$S_\varepsilon f = \varepsilon f(\varepsilon^2 t, \varepsilon x).$$

Let $R(t, x) = S_\varepsilon r^\varepsilon(t, x) = \varepsilon r^\varepsilon(\varepsilon^2 t, \varepsilon x)$. Similarly, we define $h^\varepsilon = S_\varepsilon u_+$ and $g^\varepsilon = S_\varepsilon u_-$. Then by (1.9), R obeys the following equation:

$$\begin{cases} \partial_{tt} R - \Delta R + R + E(h^\varepsilon, g^\varepsilon, R) + F(h^\varepsilon, g^\varepsilon) + \varepsilon^4 G(\partial_{tt} u_+, \partial_{tt} u_-) = 0, \\ R(0, x) = 0, \quad \partial_t R(0, x) = \varepsilon^2 S_\varepsilon r_1, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} E(h^\varepsilon, g^\varepsilon, R) &= 2R|\mathrm{e}^{\mathrm{i}t} h^\varepsilon + \mathrm{e}^{-\mathrm{i}t}\overline{g^\varepsilon}|^2 + \overline{R}(\mathrm{e}^{\mathrm{i}t} h^\varepsilon + \mathrm{e}^{-\mathrm{i}t}\overline{g^\varepsilon})^2 \\ &\quad + R^2(\mathrm{e}^{-\mathrm{i}t}\overline{h^\varepsilon} + \mathrm{e}^{\mathrm{i}t} g^\varepsilon) + 2|R|^2(\mathrm{e}^{\mathrm{i}t} h^\varepsilon + \mathrm{e}^{-\mathrm{i}t}\overline{g^\varepsilon}) + |R|^2 R, \end{aligned} \tag{3.2}$$

$$F(h^\varepsilon, g^\varepsilon) = \mathrm{e}^{3\mathrm{i}t}(h^\varepsilon)^2 g^\varepsilon + \mathrm{e}^{-3\mathrm{i}t}(\overline{g^\varepsilon})^2\overline{h^\varepsilon}, \tag{3.3}$$

$$G(\partial_{tt} u_+, \partial_{tt} u_-) = \mathrm{e}^{\mathrm{i}t} S_\varepsilon(\partial_{tt} u_+) + \mathrm{e}^{-\mathrm{i}t} S_\varepsilon(\partial_{tt}\overline{u}_-). \tag{3.4}$$

Denote

$$\phi_1 \triangleq \langle\nabla\rangle^{-1}(\partial_t + \mathrm{i}\langle\nabla\rangle)R,$$
$$\phi_2 \triangleq \langle\nabla\rangle^{-1}(\partial_t - \mathrm{i}\langle\nabla\rangle)R.$$

It is easy to derive that

$$\begin{cases} R = -\dfrac{\mathrm{i}}{2}(\phi_1 - \phi_2), \\ \partial_t R = \dfrac{1}{2}\langle\nabla\rangle(\phi_1 + \phi_2). \end{cases} \tag{3.5}$$

Then we have that ϕ_j obeys the following equation:

$$(\partial_t \mp \mathrm{i}\langle\nabla\rangle)\phi_j + \langle\nabla\rangle^{-1}[E(h^\varepsilon, g^\varepsilon, R) + F(h^\varepsilon, g^\varepsilon) + \varepsilon^4 G(\partial_{tt}u_+, \partial_{tt}u_-)] = 0 \tag{3.6}$$

with the initial data

$$\phi_j(0) \triangleq \phi_{j,0} = \varepsilon^2\langle\nabla\rangle^{-1}S_\varepsilon r_1. \tag{3.7}$$

By Duhamel's formula, we obtain that

$$\begin{aligned} \phi_j(t) &= \mathrm{e}^{\pm\mathrm{i}(t-t_0)\langle\nabla\rangle}\phi_j(t_0) \\ &\quad - \int_{t_0}^t \mathrm{e}^{\pm\mathrm{i}(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}[E(h^\varepsilon, g^\varepsilon, R) + F(h^\varepsilon, g^\varepsilon) + \varepsilon^4 G(\partial_{tt}u_+, \partial_{tt}u_-)]\mathrm{d}s. \end{aligned} \tag{3.8}$$

Since the arguments to estimate $\phi_1(t)$ and $\phi_2(t)$ are the same, we only give the details of $\phi_1(t)$. First, we consider the term $\int_{t_0}^t \mathrm{e}^{\mathrm{i}(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}F(h^\varepsilon, g^\varepsilon)\mathrm{d}s$. From the definition of F in (3.3), we can split the term into three parts as follows:

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\int_{t_0}^t \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon)^2 g^\varepsilon]\mathrm{d}s \\ &+ \mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\int_{t_0}^t \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon})^2\overline{h^\varepsilon}]\mathrm{d}s \\ &+ \int_{t_0}^t \mathrm{e}^{\mathrm{i}(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}P_{>1}F(h^\varepsilon, g^\varepsilon)\mathrm{d}s. \end{aligned} \tag{3.9}$$

Note that

$$\mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle\pm 3)}P_{\le 1} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle\pm 3)})\frac{1}{-\mathrm{i}(\langle\nabla\rangle \pm 3)}P_{\le 1}. \tag{3.10}$$

Using (3.10) and integrating by parts, we have

$$\begin{aligned} &\int_{t_0}^t \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon)^2 g^\varepsilon]\mathrm{d}s \\ &= \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\frac{1}{-\mathrm{i}(\langle\nabla\rangle - 3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon)^2 g^\varepsilon]|_{s=t_0}^{s=t} \\ &\quad - 2\int_{t_0}^t \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\frac{1}{-\mathrm{i}(\langle\nabla\rangle - 3)}\langle\nabla\rangle^{-1}P_{\le 1}[h^\varepsilon\cdot\partial_s h^\varepsilon\cdot g^\varepsilon]\mathrm{d}s \\ &\quad - \int_{t_0}^t \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\frac{1}{-\mathrm{i}(\langle\nabla\rangle - 3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon)^2\cdot\partial_s g^\varepsilon]\mathrm{d}s. \end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned}
&\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon})^2\overline{h^\varepsilon}]\mathrm{d}s \\
&= \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\frac{1}{-\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon})^2\overline{h^\varepsilon}]|_{s=t_0}^{s=t} \\
&\quad - 2\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\frac{1}{-\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[\overline{g^\varepsilon}\cdot\partial_s\overline{g^\varepsilon}\cdot\overline{h^\varepsilon}]\mathrm{d}s \\
&\quad - \int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\frac{1}{-\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon})^2\cdot\partial_s\overline{h^\varepsilon}]\mathrm{d}s. \qquad (3.12)
\end{aligned}$$

Now we insert (3.11)–(3.12) into (3.9), and then we obtain that

$$\begin{aligned}
&\phi_1(t) \\
&= \mathrm{e}^{\mathrm{i}(t-t_0)\langle\nabla\rangle}\phi_1(t_0) \qquad (3.13\text{a}) \\
&\quad - \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}[E(h^\varepsilon,g^\varepsilon,R)+P_{>1}F(h^\varepsilon,g^\varepsilon)+\varepsilon^4G(\partial_{tt}u_+,\partial_{tt}u_-)]\mathrm{d}s \qquad (3.13\text{b}) \\
&\quad + \frac{\mathrm{e}^{3\mathrm{i}t}}{\mathrm{i}(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon(t))^2g^\varepsilon(t)] - \frac{\mathrm{e}^{\mathrm{i}(t-t_0)\langle\nabla\rangle+3\mathrm{i}t_0}}{\mathrm{i}(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon(t_0))^2g^\varepsilon(t_0)] \qquad (3.13\text{c}) \\
&\quad + \frac{\mathrm{e}^{-3\mathrm{i}t}}{\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon(t)})^2\overline{h^\varepsilon(t)}] - \frac{\mathrm{e}^{\mathrm{i}(t-t_0)\langle\nabla\rangle-3\mathrm{i}t_0}}{\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon(t_0)})^2\overline{h^\varepsilon(t_0)}] \qquad (3.13\text{d}) \\
&\quad - 2\mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\frac{1}{\mathrm{i}(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le 1}[h^\varepsilon\cdot\partial_s h^\varepsilon\cdot g^\varepsilon]\mathrm{d}s \qquad (3.13\text{e}) \\
&\quad - \mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle-3)}\frac{1}{\mathrm{i}(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le 1}[(h^\varepsilon)^2\cdot\partial_s g^\varepsilon]\mathrm{d}s \qquad (3.13\text{f}) \\
&\quad - 2\mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\frac{1}{\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[\overline{g^\varepsilon}\cdot\partial_s\overline{g^\varepsilon}\cdot\overline{h^\varepsilon}]\mathrm{d}s \qquad (3.13\text{g}) \\
&\quad - \mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}s(\langle\nabla\rangle+3)}\frac{1}{\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le 1}[(\overline{g^\varepsilon})^2\cdot\partial_s\overline{h^\varepsilon}]\mathrm{d}s. \qquad (3.13\text{h})
\end{aligned}$$

Second, we denote some useful norms as follows:

$$\begin{aligned}
\|f\|_{S_{s_c}(I)} &\triangleq \||\nabla|^{s_c}f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I)}, \\
\|f\|_{Y_{s_c}(I)} &\triangleq \|f\|_{S_{s_c}(I)} + \||\nabla|^{s_c}\langle\nabla\rangle^{\frac12}f\|_{L_t^\infty L_x^2(I)},
\end{aligned}$$

and

$$\begin{aligned}
\|f\|_{Z_{s_c}(I)} &\triangleq \|f\|_{L_{t,x}^{\frac43}(I)}, \quad \text{when } d=2, \\
\|f\|_{Z_{s_c}(I)} &\triangleq \||\nabla|^{\frac12}f\|_{L_t^{\frac{10}{9}}L_x^{\frac{30}{17}}(I)}, \quad \text{when } d=3.
\end{aligned}$$

For the linear term, by Lemma 2.1, we have that

$$\||\nabla|^\gamma \mathrm{e}^{\mathrm{i}(t-t_0)\langle\nabla\rangle}\phi_j(t_0)\|_{Y_{s_c}(\mathbb{R})} \lesssim \|\langle\nabla\rangle^{\frac12}|\nabla|^{\gamma+s_c}\phi_j(t_0)\|_{L^2}. \qquad (3.14)$$

- Estimate for (3.13b).

By Lemma 2.1, we have

$$\begin{aligned}\||\nabla|^\gamma(3.13\mathrm{b})\|_{Y_{s_c}(I)} \lesssim{}& \||\nabla|^\gamma E(h^\varepsilon, g^\varepsilon, R)\|_{Z_{s_c}(I)} + \||\nabla|^\gamma P_{>1}F(h^\varepsilon, g^\varepsilon)\|_{Z_{s_c}(I)}\\ &+ \varepsilon^4\||\nabla|^{\gamma+s_c}G(\partial_{tt}u_+, \partial_{tt}u_-)\|_{L_t^1L_x^2(I)}.\end{aligned} \tag{3.15}$$

First, we estimate (3.15) for the case when $d = 2$. In this case, we only need to consider the case of $\gamma = 0$. Then for the first part of (3.15) we have

$$\begin{aligned}\|E(h^\varepsilon, g^\varepsilon, R)\|_{Z_0(I)} &\lesssim \sum_{j=0}^{j=2}(\|h^\varepsilon\|^j_{S_0(I)} + \|g^\varepsilon\|^j_{S_0(I)})\|R\|^{3-j}_{S_0(I)}\\ &\lesssim \sum_{j=0}^{j=2}\|(u_+, u_-)\|^j_{S_0(\varepsilon^2 I)}\|R\|^{3-j}_{S_0(I)}.\end{aligned} \tag{3.16}$$

For the second part $P_{>1}F(h^\varepsilon, g^\varepsilon)$, we have

$$\begin{aligned}P_{>1}F(h^\varepsilon, g^\varepsilon) ={}& \mathrm{e}^{3\mathrm{i}t}[P_{\gtrsim 1}h^\varepsilon \cdot h^\varepsilon \cdot g^\varepsilon + P_{\gtrsim 1}g^\varepsilon \cdot (h^\varepsilon)^2]\\ &+ \mathrm{e}^{-3\mathrm{i}t}[P_{\gtrsim 1}\overline{g^\varepsilon} \cdot \overline{g^\varepsilon} \cdot \overline{h^\varepsilon} + P_{\gtrsim 1}\overline{h^\varepsilon} \cdot (\overline{g^\varepsilon})^2].\end{aligned} \tag{3.17}$$

Then by Hölder's inequality, we have

$$\begin{aligned}\|P_{>1}F(h^\varepsilon, g^\varepsilon)\|_{Z_0(I)} \lesssim{}& (\|P_{\gtrsim 1}h^\varepsilon\|_{S_0(I)} + \|P_{\gtrsim 1}g^\varepsilon\|_{S_0(I)})\|h^\varepsilon\|_{S_0(I)}\|g^\varepsilon\|_{S_0(I)}\\ &+ \|P_{\gtrsim 1}g^\varepsilon\|_{S_0(I)}\|h^\varepsilon\|^2_{S_0(I)} + \|P_{\gtrsim 1}h^\varepsilon\|_{S_0(I)}\|g^\varepsilon\|^2_{S_0(I)}\\ \lesssim{}& (\|P_{\gtrsim 1}h^\varepsilon\|_{S_0(I)} + \|P_{\gtrsim 1}g^\varepsilon\|_{S_0(I)})[\|h^\varepsilon\|^2_{S_0(I)} + \|g^\varepsilon\|^2_{S_0(I)}].\end{aligned} \tag{3.18}$$

Note that

$$\|P_{\gtrsim 1}f\|_{S_{s_c}(I)} \lesssim \||\nabla|^2P_{\leq 1}f\|_{S_{s_c}(I)} + \||\nabla|^2P_{>1}f\|_{S_{s_c}(I)}, \tag{3.19}$$

so we have

$$\|P_{\gtrsim 1}h^\varepsilon\|_{S_0(I)} + \|P_{\gtrsim 1}g^\varepsilon\|_{S_0(I)} \leq \||\nabla|^2(h^\varepsilon, g^\varepsilon)\|_{S_0(I)}. \tag{3.20}$$

Furthermore, by Lemma 2.3, we have

$$\|(h^\varepsilon, g^\varepsilon)\|_{S_0(I)} = \|(u_+, u_-)\|_{S_0(\varepsilon^2 I)} \leq C; \tag{3.21}$$

$$\||\nabla|^2(h^\varepsilon, g^\varepsilon)\|_{S_0(I)} \leq \varepsilon^2\||\nabla|^2(u_+, u_-)\|_{S_0(\mathbb{R})} \leq C\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2}. \tag{3.22}$$

It is worth noting that the constant C here and below depends only on $\|(u_0, u_1)\|_{H^{d-2}}$. Combining (3.18)–(3.22), we get

$$\|P_{>1}F(h^\varepsilon, g^\varepsilon)\|_{Z_0(I)} \leq C\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2}. \tag{3.23}$$

For the last term of (3.15), we have that

$$\|G(\partial_{tt}u_+, \partial_{tt}u_-)\|_{L_t^1L_x^2(I)} \lesssim |I|\|(\partial_{tt}u_+, \partial_{tt}u_-)\|_{L_t^\infty L_x^2(\mathbb{R})}. \tag{3.24}$$

From (1.7), we obtain

$$\begin{aligned}|\partial_{tt}u_+| &= O\{\Delta^2 u_+ + \Delta u_+(u_+^2+u_-^2) + u_+[(\nabla u_+)^2+(\nabla u_-)^2]\\ &\quad + \nabla u_+\nabla u_- u_- + \Delta u_- u_+ u_- + (u_+)^5 + (u_+)^3(u_-)^2 + u_+(u_-)^4\},\\ |\partial_{tt}u_-| &= O\{\Delta^2 u_- + \Delta u_-(u_+^2+u_-^2) + u_-[(\nabla u_+)^2+(\nabla u_-)^2]\\ &\quad + \nabla u_-\nabla u_+ u_+ + \Delta u_+ u_+ u_- + (u_-)^5 + (u_-)^3(u_+)^2 + u_-(u_+)^4\}.\end{aligned}$$

Therefore, by Gagliardo-Nirenberg's inequality, we obtain

$$\|(\partial_{tt}u_+,\partial_{tt}u_-)\|_{L_t^\infty L_x^2(\mathbb{R})} \le C(\|(u_+,u_-)\|_{L_t^\infty L_x^2(\mathbb{R})})\|(u_+,u_-)\|_{L_t^\infty \dot H_x^4(\mathbb{R})}.$$

Using the conservation law of mass of (1.7), we have

$$\|(u_+,u_-)\|_{L_t^\infty L_x^2(\mathbb{R})} = \|(u_{+,0},u_{-,0})\|_{L^2(\mathbb{R})} \lesssim \|(u_0,u_1)\|_{L^2(\mathbb{R})}, \tag{3.25}$$

and by the scattering theory in Lemma 2.3, we have

$$\|(u_+,u_-)\|_{L_t^\infty \dot H_x^4(\mathbb{R})} \le C\|(u_0,u_1)\|_{\dot H^4(\mathbb{R})}. \tag{3.26}$$

Combining (3.25) and (3.26), we obtain

$$\|(\partial_{tt}u_+,\partial_{tt}u_-)\|_{L_t^\infty L_x^2(\mathbb{R})} \le C\|(u_0,u_1)\|_{\dot H^4(\mathbb{R})}. \tag{3.27}$$

Together with (3.16), (3.23) and (3.27), we obtain

$$\begin{aligned}\|(3.13\text{b})\|_{Y_0(I)} &\le C_1\sum_{j=0}^{j=2}\|(u_+,u_-)\|_{S_0(\varepsilon^2 I)}^j\|R\|_{S_0(I)}^{3-j}\\ &\quad + C_2\varepsilon^2\|(u_0,u_1)\|_{\dot H^2} + C_3\varepsilon^4|I|\|(u_0,u_1)\|_{\dot H^4(\mathbb{R})},\end{aligned} \tag{3.28}$$

where the constants $C_j, j=1,2,3$ depend only on $\|(u_0,u_1)\|_{L^2}$.

Next, we estimate (3.15) when $d=3$. In this case, $\gamma\in\big[-\frac{1}{2},0\big],\ s_c=\frac{1}{2}$.

Firstly, for the first term of (3.15), by Lemma 2.2, we have

$$\begin{aligned}\||\nabla|^\gamma E(h^\varepsilon,g^\varepsilon,R)\|_{Z_{\frac12}(I)} &\lesssim \sum_{j=0}^{j=2}(\|h^\varepsilon\|_{S_{\frac12}(I)}^j + \|g^\varepsilon\|_{S_{\frac12}(I)}^j)\|R\|_{S_{\frac12}(I)}^{2-j}\||\nabla|^\gamma R\|_{S_{\frac12}(I)}\\ &\lesssim \sum_{j=0}^{j=2}\|(u_+,u_-)\|_{S_{\frac12}(\varepsilon^2 I)}^j\|R\|_{S_{\frac12}(I)}^{2-j}\||\nabla|^\gamma R\|_{S_{\frac12}(I)}.\end{aligned} \tag{3.29}$$

Similar to the case of $d=2$, we perform the decomposition (3.17) on the second term $P_{>1}F(h^\varepsilon,g^\varepsilon)$. Then by Hölder's inequality, we get

$$\begin{aligned}&\||\nabla|^\gamma P_{>1}F(h^\varepsilon,g^\varepsilon)\|_{Z_{\frac12}(I)}\\ &\lesssim (\||\nabla|^\gamma P_{\gtrsim 1}h^\varepsilon\|_{S_{\frac12}(I)} + \||\nabla|^\gamma P_{\gtrsim 1}g^\varepsilon\|_{S_{\frac12}(I)})[\|h^\varepsilon\|_{S_{\frac12}(I)}^2 + \|g^\varepsilon\|_{S_{\frac12}(I)}^2].\end{aligned} \tag{3.30}$$

Using (3.19) for (3.30), we obtain

$$\||\nabla|^\gamma P_{\gtrsim 1}h^\varepsilon\|_{S_{\frac12}(I)} + \||\nabla|^\gamma P_{\gtrsim 1}g^\varepsilon\|_{S_{\frac12}(I)} \le \||\nabla|^{2+\gamma}(h^\varepsilon,g^\varepsilon)\|_{S_{\frac12}(I)}.$$

Again by Lemma 2.3, we have

$$\||\nabla|^{2+\gamma}(h^\varepsilon,g^\varepsilon)\|_{S_{\frac12}(I)}\le\varepsilon^{2+\gamma}\||\nabla|^{2+\gamma}(u_+,u_-)\|_{S_{\frac12}(\mathbb{R})}\le C\varepsilon^{2+\gamma}\|(u_0,u_1)\|_{\dot H^{\frac52+\gamma}},\tag{3.31}$$

$$\begin{aligned}\|(h^\varepsilon,g^\varepsilon)\|_{S_{\frac12}(I)}&=\||\nabla|^{\frac12}(u_+,u_-)\|_{L^{\frac{10}{3}}_{t,x}(\varepsilon^2 I)}\le(\varepsilon^2|I|)^{\frac14}\||\nabla|(u_+,u_-)\|_{L^{20}_tL^{\frac{15}{7}}_x(\varepsilon^2 I)}\\&\le C(\varepsilon^2|I|)^{\frac14}.\end{aligned}\tag{3.32}$$

Combining (3.31) and (3.32), we obtain that

$$\||\nabla|^\gamma P_{>1}F(h^\varepsilon,g^\varepsilon)\|_{Z_{\frac12}(I)}\le C\varepsilon^{2+\gamma}\|(u_0,u_1)\|_{\dot H^{\frac52+\gamma}}\min\{1,(\varepsilon^2|I|)^{\frac12}\}.\tag{3.33}$$

For the last term, similar to (3.27) we get

$$\||\nabla|^{\gamma+\frac12}(\partial_{tt}u_+,\partial_{tt}u_-)\|_{L^\infty_tL^2_x(\mathbb{R})}\le C(\|(u_+,u_-)\|_{L^\infty_t\dot H^{\frac12}_x(\mathbb{R})})\|(u_+,u_-)\|_{L^\infty_t\dot H^{\frac92+\gamma}_x(\mathbb{R})}.$$

Using Lemma 2.3, we have

$$\begin{aligned}\|(u_+,u_-)\|_{L^\infty_t\dot H^{\frac12}_x(\mathbb{R})}&\le C\|(u_0,u_1)\|_{\dot H^{\frac12}(\mathbb{R})}\le C\|(u_0,u_1)\|_{H^1},\\\|(u_+,u_-)\|_{L^\infty_t\dot H^{\frac92+\gamma}_x(\mathbb{R})}&\le C\|(u_0,u_1)\|_{\dot H^{\frac92+\gamma}}.\end{aligned}$$

So we have

$$\||\nabla|^{\gamma+s_c}(\partial_{tt}u_+,\partial_{tt}u_-)\|_{L^\infty_tL^2_x(\mathbb{R})}\le C\|(u_0,u_1)\|_{\dot H^{\frac92+\gamma}}.\tag{3.34}$$

Combining (3.29) and (3.33)–(3.34), for $d=3$, $\gamma\in\big[-\frac12,0\big]$, we get

$$\begin{aligned}\||\nabla|^\gamma(3.13\text{b})\|_{Y_{\frac12}(I)}\le{}&C_1\sum_{j=0}^{j=2}\|(u_+,u_-)\|^j_{S_{\frac12}(\varepsilon^2 I)}\|R\|^{2-j}_{S_{\frac12}(I)}\||\nabla|^\gamma R\|_{S_{\frac12}(I)}\\&+C_2\varepsilon^{2+\gamma}\|(u_0,u_1)\|_{\dot H^{\frac52+\gamma}}\min\{1,(\varepsilon^2|I|)^{\frac12}\}\\&+C_3\varepsilon^{4+\gamma}|I|\|(u_0,u_1)\|_{\dot H^{\frac92+\gamma}},\end{aligned}\tag{3.35}$$

where the constants $C_j, j=1,2,3$ depend only on $\|(u_0,u_1)\|_{H^1}$.

• Estimate for (3.13c)–(3.13d).

Recalling the expressions of (3.13c)–(3.13d), we have

$$\begin{aligned}(3.13\text{c})&=\frac{\mathrm{e}^{3it}}{\mathrm{i}(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le1}[(h^\varepsilon(t))^2g^\varepsilon(t)]-\frac{\mathrm{e}^{\mathrm{i}(t-t_0)\langle\nabla\rangle+3it_0}}{\mathrm{i}(\langle\nabla\rangle-3)}\langle\nabla\rangle^{-1}P_{\le1}[(h^\varepsilon(t_0))^2g^\varepsilon(t_0)],\\(3.13\text{d})&=\frac{\mathrm{e}^{-3it}}{\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le1}[(\overline{g^\varepsilon(t)})^2\overline{h^\varepsilon(t)}]-\frac{\mathrm{e}^{\mathrm{i}(t-t_0)\langle\nabla\rangle-3it_0}}{\mathrm{i}(\langle\nabla\rangle+3)}\langle\nabla\rangle^{-1}P_{\le1}[(\overline{g^\varepsilon(t_0)})^2\overline{h^\varepsilon(t_0)}].\end{aligned}$$

Note that

$$\forall r\in[1,\infty],\quad\Big\|\frac{1}{\langle\nabla\rangle\pm3}\langle\nabla\rangle^{-1}f\Big\|_{L^r(\mathbb{R}^d)}\lesssim\|f\|_{L^r(\mathbb{R}^d)}.$$

Then, we have

$$\begin{aligned}&\||\nabla|^\gamma[(3.13\text{c})+(3.13\text{d})]\|_{Y_{s_c}(I)}\\\lesssim{}&\||\nabla|^\gamma(h^\varepsilon)^2g^\varepsilon\|_{S_{s_c}(I)}+\||\nabla|^{\gamma+\frac12}(h^\varepsilon)^2g^\varepsilon\|_{L^\infty_tL^2_x(I)}\\&+\||\nabla|^\gamma(g^\varepsilon)^2h^\varepsilon\|_{S_{s_c}(I)}+\||\nabla|^{\gamma+\frac12}(g^\varepsilon)^2h^\varepsilon\|_{L^\infty_tL^2_x(I)}\\\lesssim{}&[\|(|\nabla|^\gamma h^\varepsilon,|\nabla|^\gamma g^\varepsilon)\|_{S_{s_c}(I)}+\|(|\nabla|^{\gamma+\frac12}h^\varepsilon,|\nabla|^{\gamma+\frac12}g^\varepsilon)\|_{L^\infty_tL^2_x(I)}]\cdot\|(h^\varepsilon,g^\varepsilon)\|^2_{L^\infty_{t,x}}.\end{aligned}$$

Gagliardo-Nirenberg's inequality gives that

$$\|(h^\varepsilon, g^\varepsilon)\|^2_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^2)} = \varepsilon^2\|(u_+, u_-)\|^2_{L^\infty_{t,x}} \lesssim \varepsilon^2\|(u_+, u_-)\|_{L^\infty_t L^2_x}\|(u_+, u_-)\|_{L^\infty_t \dot{H}^2_x},$$
$$\|(h^\varepsilon, g^\varepsilon)\|^2_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^3)} = \varepsilon^2\|(u_+, u_-)\|^2_{L^\infty_{t,x}} \lesssim \varepsilon^2\|(u_+, u_-)\|_{L^\infty_t \dot{H}^1_x}\|(u_+, u_-)\|_{L^\infty_t \dot{H}^2_x}.$$

By Lemma 2.3, we obtain that

$$\begin{aligned}\|(h^\varepsilon, g^\varepsilon)\|^2_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^2)} &\le C\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2},\\ \|(h^\varepsilon, g^\varepsilon)\|^2_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^3)} &\le C\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2},\\ \|(|\nabla|^\gamma h^\varepsilon, |\nabla|^\gamma g^\varepsilon)\|_{S_{s_c}(I)} &\le C\varepsilon^\gamma.\end{aligned}$$

Thus, we obtain that when $d = 2$,

$$\|(3.13\text{c}) + (3.13\text{d})\|_{Y_0(I)} \le C\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2}, \tag{3.36}$$

and when $d = 3$, $\gamma \in \big[-\frac{1}{2}, 0\big]$,

$$\||\nabla|^\gamma[(3.13\text{c}) + (3.13\text{d})]\|_{Y_{\frac{1}{2}}(I)} \le C\varepsilon^{2+\gamma}\|(u_0, u_1)\|_{\dot{H}^2}. \tag{3.37}$$

• Estimate for (3.13e) and (3.13g).

Recalling the definitions of (3.13e) and (3.13g), and by Lemma 2.1, we have

$$\begin{aligned}&\||\nabla|^\gamma[(3.13\text{e}) + (3.13\text{g})]\|_{Y_{s_c}(I)}\\ \lesssim{}& \Big\||\nabla|^\gamma \frac{1}{\langle\nabla\rangle - 3}P_{\le 1}[h^\varepsilon \cdot \partial_t h^\varepsilon \cdot g^\varepsilon]\Big\|_{Z_{s_c}(I)} + \Big\||\nabla|^\gamma \frac{1}{\langle\nabla\rangle + 3}P_{\le 1}[\overline{g^\varepsilon} \cdot \partial_t \overline{g^\varepsilon} \cdot \overline{h^\varepsilon}]\Big\|_{Z_{s_c}(I)}\\ \lesssim{}& \||\nabla|^\gamma(h^\varepsilon \cdot \partial_t h^\varepsilon \cdot g^\varepsilon)\|_{Z_{s_c}(I)} + \||\nabla|^\gamma(\overline{g^\varepsilon} \cdot \partial_t \overline{g^\varepsilon} \cdot \overline{h^\varepsilon})\|_{Z_{s_c}(I)}.\end{aligned}$$

Now, we consider the case of $d = 2$ and $d = 3$ respectively.

For $d = 2$, we only need to consider $\gamma = 0$ as before. By Hölder's inequality, we have

$$\begin{aligned}&\|h^\varepsilon \cdot \partial_t h^\varepsilon \cdot g^\varepsilon\|_{Z_0(I)} + \|\overline{g^\varepsilon} \cdot \partial_t \overline{g^\varepsilon} \cdot \overline{h^\varepsilon}\|_{Z_0(I)}\\ \lesssim{}& \|h^\varepsilon\|_{S_0(I)}\|\partial_t h^\varepsilon\|_{S_0(I)}\|g^\varepsilon\|_{S_0(I)} + \|g^\varepsilon\|_{S_0(I)}\|\partial_t g^\varepsilon\|_{S_0(I)}\|h^\varepsilon\|_{S_0(I)}\\ \lesssim{}& (\|\partial_t h^\varepsilon\|_{S_0(I)} + \|\partial_t g^\varepsilon\|_{S_0(I)}) \cdot \|h^\varepsilon\|_{S_0(I)} \cdot \|g^\varepsilon\|_{S_0(I)}\\ \lesssim{}& \varepsilon^2\|(\partial_t u_+, \partial_t u_-)\|_{S_0(\varepsilon^2 I)}\|(u_+, u_-)\|^2_{S_0(\varepsilon^2 I)}.\end{aligned}$$

By (1.7) and Lemma 2.3, we obtain that

$$\begin{aligned}\|\partial_t u_\pm\|_{S_0} &\lesssim \|\Delta u_\pm\|_{S_0} + \|u_\pm\|_{S_0}(\|u_+\|^2_{L^\infty_{t,x}} + \|u_-\|^2_{L^\infty_{t,x}})\\ &\lesssim \|\Delta u_\pm\|_{S_0} + \|u_\pm\|_{S_0}\|(u_+, u_-)\|_{L^\infty_t L^2_x}\|(u_+, u_-)\|_{L^\infty_t \dot{H}^2_x}\\ &\le C\|(u_0, u_1)\|_{\dot{H}^2}.\end{aligned}$$

By (3.21), we obtain

$$\|(3.13\text{e}) + (3.13\text{g})\|_{Y_0(I)} \le 4C\varepsilon^2\|(u_0, u_1)\|_{\dot{H}^2}. \tag{3.38}$$

For $d = 3$, $\gamma \in \big[-\frac{1}{2}, 0\big]$ and $s_c = \frac{1}{2}$, by the Kato-Ponce inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\||\nabla|^\gamma(h^\varepsilon \cdot \partial_t h^\varepsilon \cdot g^\varepsilon)\|_{Z_{\frac{1}{2}}(I)} + \||\nabla|^\gamma(\overline{g^\varepsilon} \cdot \partial_t \overline{g^\varepsilon} \cdot \overline{h^\varepsilon})\|_{Z_{\frac{1}{2}}(I)} \\
\lesssim\ & (\||\nabla|^\gamma \partial_t h^\varepsilon\|_{S_{\frac{1}{2}}(I)} + \||\nabla|^\gamma \partial_t g^\varepsilon\|_{S_{\frac{1}{2}}(I)}) \cdot \|h^\varepsilon\|_{S_{\frac{1}{2}}(I)} \cdot \|g^\varepsilon\|_{S_{\frac{1}{2}}(I)} \\
\lesssim\ & \varepsilon^{2+\gamma}\||\nabla|^\gamma(\partial_t u_+, \partial_t u_-)\|_{S_{\frac{1}{2}}(\varepsilon^2 I)}\|(u_+, u_-)\|^2_{S_{\frac{1}{2}}(\varepsilon^2 I)}.
\end{aligned}$$

Using the same argument as in the case of $d = 2$, we have

$$\begin{aligned}
\||\nabla|^\gamma \partial_t u_\pm\|_{S_{\frac{1}{2}}} &\lesssim \||\nabla|^{\gamma+2} u_\pm\|_{S_{\frac{1}{2}}} + \||\nabla|^\gamma u_\pm\|_{S_{\frac{1}{2}}}(\|u_+\|^2_{L^\infty_{t,x}} + \|u_-\|^2_{L^\infty_{t,x}}) \\
&\lesssim \||\nabla|^{\gamma+2} u_\pm\|_{S_{\frac{1}{2}}} + \||\nabla|^\gamma u_\pm\|_{S_{\frac{1}{2}}}\|(u_+, u_-)\|_{L^\infty_t \dot H^1_x}\|(u_+, u_-)\|_{L^\infty_t \dot H^2_x} \\
&\le C[\|(u_0, u_1)\|_{\dot H^{\frac{5}{2}+\gamma}} + \|(u_0, u_1)\|_{\dot H^2}].
\end{aligned}$$

For the case of $d = 3$ and $\gamma \in \big[-\frac{1}{2}, 0\big]$, by (3.32), we obtain

$$\begin{aligned}
&\||\nabla|^\gamma[(3.13\text{e}) + (3.13\text{g})]\|_{Y_{\frac{1}{2}}(I)} \\
&\le C\varepsilon^{2+\gamma}[\|(u_0, u_1)\|_{\dot H^{\frac{5}{2}+\gamma}} \min\{1, (\varepsilon^2|I|)^{\frac{1}{2}}\} + \|(u_0, u_1)\|_{\dot H^2}].
\end{aligned} \tag{3.39}$$

• Estimate for (3.13f) and (3.13h).

Similarly to (3.13e) and (3.13g), we have

$$\begin{aligned}
&\||\nabla|^\gamma[(3.13\text{f}) + (3.13\text{h})]\|_{Y_{s_c}(I)} \\
\lesssim\ & \||\nabla|^\gamma[(h^\varepsilon)^2 \cdot \partial_t g^\varepsilon]\|_{Z_{s_c}(I)} + \||\nabla|^\gamma[(g^\varepsilon)^2 \cdot \partial_t h^\varepsilon]\|_{Z_{s_c}(I)}.
\end{aligned}$$

Arguing as in (3.38)–(3.39), we obtain that when $d = 2$,

$$\|(3.13\text{f}) + (3.13\text{h})\|_{Y_0(I)} \le C\varepsilon^2\|(u_0, u_1)\|_{\dot H^2}, \tag{3.40}$$

and when $d = 3$ and $\gamma \in \big[-\frac{1}{2}, 0\big]$,

$$\begin{aligned}
&\||\nabla|^\gamma[(3.13\text{f}) + (3.13\text{h})]\|_{Y_{\frac{1}{2}}(I)} \\
&\le C\varepsilon^{2+\gamma}[\|(u_0, u_1)\|_{\dot H^{\frac{5}{2}+\gamma}} \min\{1, (\varepsilon^2|I|)^{\frac{1}{2}}\} + \|(u_0, u_1)\|_{\dot H^2}].
\end{aligned} \tag{3.41}$$

Now we are ready to prove Proposition 3.1. First, we prove the case when $d = 2$.

Based on all the analysis and estimates above, we obtain that for $d = 2$ and $j = 1, 2$,

$$\begin{aligned}
\|\phi_j\|_{Y_0(I)} \le\ & C_0\|\langle\nabla\rangle^{\frac{1}{2}}\phi_j(t_0)\|_{L^2} + C_1\sum_{i=0}^{i=2}\|(u_+, u_-)\|^i_{S_0(\varepsilon^2 I)}\|R\|^{3-i}_{S_0(I)} \\
&+ C_2\varepsilon^2\|(u_0, u_1)\|_{\dot H^2} + C_3\varepsilon^4|I|\|(u_0, u_1)\|_{\dot H^4},
\end{aligned}$$

where the constants $C_j, j = 0, 1, 2, 3$ depend only on $\|(u_0, u_1)\|_{L^2}$. By Young's inequality

$$\begin{aligned}
\|\phi_j\|_{Y_0(I)} \le\ & C_0\|\langle\nabla\rangle^{\frac{1}{2}}\phi_j(t_0)\|_{L^2} + C_1\|R\|_{S_0(I)}(\|(u_+, u_-)\|^2_{S_0(\varepsilon^2 I)} + \|R\|^2_{S_0(I)}) \\
&+ C_2\varepsilon^2\|(u_0, u_1)\|_{\dot H^2} + C_3\varepsilon^4|I|\|(u_0, u_1)\|_{\dot H^4}.
\end{aligned} \tag{3.42}$$

We split $\mathbb{R}$ into several subintervals which are denoted by J_l, that is

$$\bigcup_{l=1}^{L} J_l = \mathbb{R}^+$$

with

$$J_0 = [0, t_1], \quad J_l = [t_l, t_{l+1}) \quad \text{for } l = 1, 2, \cdots, L-1, \ J_L = [t_L, +\infty),$$

such that

$$C_1 \|(u_+, u_-)\|_{S_0(J_l)}^2 \leq \frac{1}{4}. \tag{3.43}$$

Denote $I_l = \varepsilon^{-2} J_l \cap [0, T]$ and $T_l = \varepsilon^{-2} t_l$. Then by (3.42)–(3.43), we have that for any $l \leq L$,

$$\begin{aligned} \|\phi_j\|_{Y_0(I_l)} \leq &\frac{4}{3} C_0 \|\langle \nabla \rangle^{\frac{1}{2}} \phi_j(T_l)\|_{L^2} + \frac{4}{3} C_1 \|R\|_{S_0(I_l)}^3 \\ &+ \frac{4}{3} C_2 \varepsilon^2 \|(u_0, u_1)\|_{\dot{H}^2} + \frac{4}{3} C_3 \varepsilon^4 T \|(u_0, u_1)\|_{\dot{H}^4}. \end{aligned} \tag{3.44}$$

From the definition of the initial data of $\phi_j(0)$ in (3.7), it is worth noting that

$$\|\langle \nabla \rangle^{\frac{1}{2}} \phi_j(0)\|_{L^2} \leq \varepsilon^2 \|S_\varepsilon r_1\|_{L^2}.$$

Denoting

$$a_0 = (2C_0)^L [\varepsilon^2 \|S_\varepsilon r_1\|_{L^2} + 2C_2 \varepsilon^2 \|(u_0, u_1)\|_{\dot{H}^2} + 2C_3 \varepsilon^4 T \|(u_0, u_1)\|_{\dot{H}^4}],$$

choosing ε_0 suitably small such that for any $\varepsilon \in [0, \varepsilon_0]$ and finite interval $[0, T] \subset \mathbb{R}$, we have that

$$\frac{4}{3} C_1 a_0^2 \leq \frac{1}{4}. \tag{3.45}$$

Suppose that for some $0 \leq l \leq L-1$,

$$\begin{aligned} \|\langle \nabla \rangle^{\frac{1}{2}} \phi_j(T_l)\|_{L^2} \leq &(2C_0)^l \varepsilon^2 \|S_\varepsilon r_1\|_{L^2} \\ &+ \sum_{k=0}^{l-1} (2C_0)^k (2C_2 \varepsilon^2 \|(u_0, u_1)\|_{\dot{H}^2} + 2C_3 \varepsilon^4 T \|(u_0, u_1)\|_{\dot{H}^4}). \end{aligned} \tag{3.46}$$

Then we have

$$\|\langle \nabla \rangle^{\frac{1}{2}} \phi_j(T_l)\|_{L^2} \leq a_0.$$

Therefore, by (3.44) for I_l, and applying the bootstrap argument on $S_0(I_l)$ norm first, we obtain that

$$\begin{aligned} \|\phi_j\|_{S_0(I_l)} &\leq (2C_0)^{l+1} \varepsilon^2 \|S_\varepsilon r_1\|_{L^2} + \sum_{k=0}^{l} (2C_0)^k (2C_2 \varepsilon^2 \|(u_0, u_1)\|_{\dot{H}^2} + 2C_3 \varepsilon^4 T \|(u_0, u_1)\|_{\dot{H}^4}) \\ &< a_0. \end{aligned}$$

Inserting it into (3.44), we can obtain the estimate for $Y_0(I_l)$, that is

$$\|\phi_j\|_{Y_0(I_l)} \leq (2C_0)^{l+1} \varepsilon^2 \|S_\varepsilon r_1\|_{L^2} + \sum_{k=0}^{l} (2C_0)^k (2C_2 \varepsilon^2 \|(u_0, u_1)\|_{\dot{H}^2} + 2C_3 \varepsilon^4 T \|(u_0, u_1)\|_{\dot{H}^4}).$$

This proves (3.46) for $l+1$, and thus it holds for any $1 \le l \le L$. Therefore, we obtain that

$$\|\phi_j\|_{Y_0(I)} \le C(\varepsilon^2\|(u_0,u_1)\|_{\dot H^2} + \varepsilon^4 T\|(u_0,u_1)\|_{\dot H^4}).$$

Scaling back, and from the definition of ϕ_j in (3.5), we obtain when $d=2$,

$$\|r\|_{L_t^\infty L_x^2([0,T])} \le C(\varepsilon^2\|(u_0,u_1)\|_{\dot H^2} + \varepsilon^2 T\|(u_0,u_1)\|_{\dot H^4}).$$

Next, we prove Proposition 3.1 for the case when $d=3$.

Similarly to the case of $d=2$, we split $\mathbb{R}^+$ into intervals. To simplify notations, we apply the same notations as before. We have already proved that when $d=3$,

$$\begin{aligned} \||\nabla|^\gamma\phi_j\|_{Y_{\frac12}(I)} \le{}& C_0\|\langle\nabla\rangle^{\frac12}|\nabla|^\gamma\phi_j(t_0)\|_{\dot H^{\frac12}} + C_1(\|R\|^2_{S_{\frac12}(I)} + \|(u_+,u_-)\|^2_{S_{\frac12}(\varepsilon^2 I)})\||\nabla|^\gamma R\|_{S_{\frac12}(I)} \\ &+ C_2\varepsilon^{2+\gamma}[\|(u_0,u_1)\|_{\dot H^2} + \|(u_0,u_1)\|_{\dot H^{\frac52+\gamma}}\min\{1,(\varepsilon^2|I|)^{\frac12}\}] \\ &+ C_3\varepsilon^{4+\gamma}|I|\|(u_0,u_1)\|_{\dot H^{\frac92+\gamma}}, \end{aligned} \tag{3.47}$$

where the constants $C_j, j=0,1,2,3$ depend only on $\|(u_0,u_1)\|_{H^1}$. First, we treat the case of $\gamma=0$. Under the same assumption as (3.43), we choose ε_0 and δ_0 suitably small for any $\varepsilon\in[0,\varepsilon_0]$, such that

$$\varepsilon^4|I|\|(u_0,u_1)\|_{\dot H^{\frac92}} \le \delta_0, \quad \varepsilon^{\frac52}|I|^{\frac12}\|(u_0,u_1)\|_{\dot H^{\frac52}} \le \delta_0.$$

For any $1\le l\le L$, we have

$$\begin{aligned} \|\phi_j\|_{Y_{\frac12}(I_l)} \le{}& \frac43 C_0\|\langle\nabla\rangle^{\frac12}|\nabla|^{\frac12}\phi_j(t_0)\|_{L^2} + \frac43 C_1\|R\|^3_{S_{\frac12}(I_l)} \\ &+ \frac43 C_2\varepsilon^2\|(u_0,u_1)\|_{\dot H^2} + \frac43 C_3\varepsilon^4 T\|(u_0,u_1)\|_{\dot H^{\frac92}}. \end{aligned} \tag{3.48}$$

Note that

$$\|\langle\nabla\rangle^{\frac12}|\nabla|^{\frac12}\phi_j(t_0)\|_{L^2} \le \varepsilon^2\|r_1\|_{\dot H^{\frac12}},$$

and denote

$$a_1 = (2C_0)^L[\varepsilon^2\|r_1\|_{\dot H^{\frac12}} + 2C_2\varepsilon^2\|(u_0,u_1)\|_{\dot H^2} + 2C_3\varepsilon^4 T\|(u_0,u_1)\|_{\dot H^{\frac92}}].$$

Choose ε_0 and δ_0 suitably small for any $\varepsilon\in[0,\varepsilon_0]$ such that

$$\frac43 C_1 a_1^2 \le \frac14. \tag{3.49}$$

Under the same assumption as (3.43), using the same argument as above, we obtain that

$$\|R\|_{Y_{\frac12}(I)} \le C(\varepsilon^2\|(u_0,u_1)\|_{\dot H^2} + \varepsilon^2\|(u_0,u_1)\|_{\dot H^{\frac52}} + \varepsilon^2 T\|(u_0,u_1)\|_{\dot H^{\frac92}}). \tag{3.50}$$

Choose ε_0 suitably small. Then the last estimate (3.50) gives that

$$C_1\|R\|^2_{S_{\frac12}(I)} \le \frac14.$$

Inserting this estimate into (3.47), we have that for any $-\frac12 \le \gamma \le 0$,

$$\||\nabla|^\gamma\phi_j\|_{Y_{\frac12}(I)} \le C[\|\langle\nabla\rangle^{\frac12}|\nabla|^\gamma\phi_j(t_0)\|_{\dot H^{\frac12}} + \varepsilon^{2+\gamma}\|(u_0,u_1)\|_{\dot H^2} + \varepsilon^{4+\gamma}|I|\|(u_0,u_1)\|_{\dot H^{\frac92+\gamma}}].$$

Choosing $\gamma=-\frac{1}{2}$ and scaling back, we obtain that for any

$$T:\ \varepsilon^2T\|(u_0,u_1)\|_{\dot H^{\frac{9}{2}}}\le\delta_0,\quad \varepsilon T^{\frac{1}{2}}\|(u_0,u_1)\|_{\dot H^{\frac{5}{2}}}\le\delta_0,$$

it holds that

$$\|r\|_{L_t^\infty L_x^2([0,T])}\le C[\varepsilon^2\|(u_0,u_1)\|_{\dot H^2}+\varepsilon^2T\|(u_0,u_1)\|_{\dot H^4}]. \tag{3.51}$$

4 Non-regular Case

In this section, we continue to consider the case when $(u_0,u_1)\in H^\alpha$ for some $\alpha\in[1,4]$. The main result is as follows.

Proposition 4.1 *Let $T>0$. Assume that $d=2,3$, $(u_0,u_1)\in H^\alpha$ for some $\alpha\in[1,4]$. Then there exist constants $\delta_0>0$ and $\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$ satisfying $\varepsilon^2T\le\delta_0$, the following holds: When $1\le\alpha\le 2$, we have*

$$\sup_{t\in[0,T]}\|u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-(t)\|_{L^2}\le C(\varepsilon^\alpha+(\varepsilon^2T)^{\frac{1}{4}\alpha}).$$

When $\alpha\ge 2$, we have

$$\sup_{t\in[0,T]}\|u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-(t)\|_{L^2}\le C(\varepsilon^2+(\varepsilon^2T)^{\frac{1}{4}\alpha}).$$

Proof Let $N=N(\varepsilon)>0$ be a parameter to be determined later which satisfies that

$$N(\varepsilon)\to\infty,\qquad \text{as } \varepsilon\to 0.$$

Denote by (u_+^N,u_-^N) the solutions to the following system with initial data $(u_+^N(0),u_-^N(0))=(P_{\le N}u_{+,0},P_{\le N}u_{-,0})$:

$$\begin{cases}2\mathrm{i}\partial_tu_+-\Delta u_++|u_+|^2u_++2|u_-|^2u_+=0,\\ 2\mathrm{i}\partial_tu_--\Delta u_-+|u_-|^2u_-+2|u_+|^2u_-=0,\\ u_+(0,x)=P_{\le N}u_{+,0},\quad u_-(0,x)=P_{\le N}u_{-,0}.\end{cases}$$

Set $w_+=u_+-u_+^N$, $w_-=u_--u_-^N$. Then w_+ and w_- are the solutions to the following equations:

$$\begin{cases}2\mathrm{i}\partial_tw_+-\Delta w_++|u_+|^2u_++2|u_-|^2u_+-|u_+^N|^2u_+^N-2|u_-^N|^2u_+^N=0,\\ 2\mathrm{i}\partial_tw_--\Delta w_-+|u_-|^2u_-+2|u_+|^2u_--|u_-^N|^2u_-^N-2|u_+^N|^2u_-^N=0,\\ w_+(0,x)=P_{>N}u_{+,0},\quad w_-(0,x)=P_{>N}u_{-,0}.\end{cases}$$

We will complete the proof in two parts. First, we consider the case of $\alpha\in[1,2]$.

- The case of $\alpha\in[1,2]$.

By Lemma 2.3, there exists some constant $C>0$ which depends only on $\|(u_0,u_1)\|_{H^1}$ such that

$$\|(u_+,u_-)\|_{S_{s_c}(\mathbb{R})}+\|(u_+^N,u_-^N)\|_{S_{s_c}(\mathbb{R})}\le C. \tag{4.1}$$

Moreover, by Lemma 2.1, we have for any $I=[t_0,t_1]\subset\mathbb{R}$,

$$\begin{aligned}&\|(w_+,w_-)\|_{L_t^\infty L_x^2(I)}\\ &\lesssim \|(w_+(0),w_-(0))\|_{L^2}+\|(w_+,w_-)\|_{L_t^\infty L_x^2(I)}(\|(u_+,u_-)\|_{S_{s_c}(I)}^2+\|(u_+^N,u_-^N)\|_{S_{s_c}(I)}^2).\end{aligned}$$

By (4.1) and then arguing similarly to the regular case in Proposition 3.1, we obtain that

$$\|(w_+,w_-)\|_{L_t^\infty L_x^2(I)}\le C\|(P_{>N}u_0,P_{>N}u_1)\|_{L^2}\le CN^{-\alpha}\|(u_0,u_1)\|_{H^\alpha},\tag{4.2}$$

where the constant $C>0$, depends only on $\|(u_0,u_1)\|_{H^1}$. Now we write

$$\begin{aligned}u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-(t)&=u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+^N(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-^N(t)\\ &\quad-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}w_+(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{w}_-(t).\end{aligned}$$

Note that under the hypothesis in Proposition 4.1, $(P_{\le N}u_0,P_{\le N}u_1)$ satisfy the assumption in Proposition 3.1. Then we obtain that

$$\begin{aligned}&\sup_{t\in[0,T]}\|u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+^N(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-^N(t)\|_{L^2}\\ &\le C(\varepsilon^2\|(P_{\le N}u_0,P_{\le N}u_1)\|_{\dot H^2}+\varepsilon^2T\|(P_{\le N}u_0,P_{\le N}u_1)\|_{\dot H^4})\end{aligned}$$

for any T satisfying

$$\begin{aligned}&T<\infty,\qquad\text{when } d=2;\\ &\varepsilon^2TN^{\frac72}\le C(\|(u_0,u_1)\|_{H^1})\delta_0,\quad \varepsilon T^{\frac12}N^{\frac32}\le C(\|(u_0,u_1)\|_{H^1})\delta_0,\qquad\text{when } d=3.\end{aligned}\tag{4.3}$$

Bernstein's inequality gives that

$$\|(P_{\le N}u_0,P_{\le N}u_1)\|_{\dot H^2}\le N^{2-\alpha}C(\|(u_0,u_1)\|_{H^\alpha}),\tag{4.4}$$
$$\|(P_{\le N}u_0,P_{\le N}u_1)\|_{\dot H^4}\le N^{4-\alpha}C(\|(u_0,u_1)\|_{H^\alpha}).\tag{4.5}$$

Hence, it implies that

$$\sup_{t\in[0,T]}\|u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+^N(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-^N(t)\|_{L^2}\le C(\varepsilon^2N^{2-\alpha}+\varepsilon^2TN^{4-\alpha}).$$

Combining this with (4.2), we obtain that

$$\sup_{t\in[0,T]}\|u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-(t)\|_{L^2}\le C(\varepsilon^2N^{2-\alpha}+\varepsilon^2TN^{4-\alpha}+N^{-\alpha}),\tag{4.6}$$

where the constant C depends only on $\|(u_0,u_1)\|_{H^\alpha}$. When $T\le\varepsilon^2$, we choose $N=\varepsilon^{-1}$, and when $T\ge\varepsilon^2$, we choose $N=(\varepsilon^2T)^{-\frac14}$. Moreover, (4.3) holds if $\varepsilon^2T\le\delta_0$. Based on (4.6), we get the desired result in the case of $\alpha\in[1,2]$ when $\varepsilon^2T\le\delta_0$.

• The case of $\alpha\ge2$.

Repeating the above procedure, we obtain that when $\alpha\ge2$,

$$\begin{aligned}&\sup_{t\in[0,T]}\|u^\varepsilon(t)-\mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}}u_+^N(t)-\mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}}\overline{u}_-^N(t)\|_{L^2}\\ &\le C(\|(P_{\le N}u_0,P_{\le N}u_1)\|_{H^2})(\varepsilon^2+\varepsilon^2T\|(P_{\le N}u_0,P_{\le N}u_1)\|_{\dot H^4})\end{aligned}\tag{4.7}$$

for any T satisfying

$$\begin{aligned}&T<\infty, \quad \text{when } d=2;\\ &\varepsilon^2 T N^{\frac{5}{2}} \le C(\|(u_0,u_1)\|_{H^2})\delta_0, \quad \varepsilon T^{\frac{1}{2}} N^{\frac{1}{2}} \le C(\|(u_0,u_1)\|_{H^2})\delta_0, \quad \text{when } d=3.\end{aligned} \tag{4.8}$$

Combining (4.2) and (4.5), we obtain

$$\sup_{t\in[0,T]} \|u^\varepsilon(t) - \mathrm{e}^{\frac{\mathrm{i}t}{\varepsilon^2}} u_+(t) - \mathrm{e}^{-\frac{\mathrm{i}t}{\varepsilon^2}} \overline{u}_-(t)\|_{L^2} \le C(\varepsilon^2 + \varepsilon^2 T N^{4-\alpha} + N^{-\alpha}), \tag{4.9}$$

where the constant C depends only on $\|(u_0,u_1)\|_{H^\alpha}$. Choose $N=(\varepsilon^2 T)^{-\frac{1}{4}}$. Then (4.3) and (4.8) hold if $\varepsilon^2 T \le \delta_0$. Based on (4.6) and (4.9), we prove Theorem 1.1 when $\varepsilon^2 T \le \delta_0$.

Proof of Theorem 1.1 Note that we have the energy conservation law:

$$\begin{aligned}&\int \Big(\varepsilon^2|\partial_t u^\varepsilon|^2 + |\nabla u^\varepsilon|^2 + \varepsilon^{-2}|u^\varepsilon|^2 + \frac{3}{2}|u^\varepsilon|^4\Big)\,\mathrm{d}x\\ =&\int \Big(\varepsilon^2|u_1^\varepsilon|^2 + |\nabla u_0^\varepsilon|^2 + \varepsilon^{-2}|u_0^\varepsilon|^2 + \frac{3}{2}|u_0^\varepsilon|^4\Big)\,\mathrm{d}x.\end{aligned} \tag{4.10}$$

This implies that $\|u^\varepsilon\|_{L_t^\infty L_x^2(\mathbb{R})} \le C\|(u_0,u_1)\|_{H^1\times L^2}$. This yields (1.11)–(1.12) when $\varepsilon^2 T \ge \delta_0$. This finishes the proof of Theorem 1.1.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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